1 **Nonlinear progressive waves in water of finite depth -** 2 **an analytic approximation**

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7 **Abstract**

8 An analytical solution using homotopy analysis method is developed to describe the non-
9 linear progressive waves in water of finite depth. The velocity potential of the wave is ex-
10 pressed by Fourier series and the nonlinear free surface boundary conditions are satisfied by
11 continuous mapping. Unlike the perturbation method, the present approach is not dependent
12 on small parameters. Thus solutions are possible for steep waves. Furthermore, a significant
13 improvement of the convergence rate and region is achieved by applying Homotopy-Padé
14 Approximants. The calculated wave characteristics of the present solution agree well with
15 previous numerical and experimental results.

16 *Key words:* progressive waves, nonlinear, finite water depth, homotopy analysis method

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17 **1 Introduction**

18 Since the early development of wave theory describing steadily progressive wave
19 motion using perturbation method Stokes (1847), many nonlinear wave solutions in
20 infinite/finite water depth have been obtained, both analytically and numerically, in
21 the last century (e.g., Skjelbreia and Hendrickson, 1960; Schwartz, 1974; Cokelet,
22 1977; Rienecker and Fenton, 1981). Stokes (1880) calculated waves in deep water
23 to $O(a_1^5)$ (a_1 was chosen as the first Fourier coefficient of amplitude by Stokes) and
24 waves in water of finite depth to $O(a_1^3)$. Nearly 80 years later, De (1955) and Skjel-
25 breia and Hendrickson (1960) obtained the fifth-order solutions for waves propa-
26 gating in water of finite depth. Later, Schwartz (1974) and Cokelet (1977) obtained
27 solutions of higher-order expansions, which made extensive use of computation by
28 digital computer, and showed that high-order expansions indeed improve the accu-
29 racy. For the high-order solutions for Stokes expansions to be rapidly convergent,
30 the expansion parameter must be small, leading to the well-known restrictions to
31 Stokes approach, namely, not too high a wave and not too shallow water depth.

32 Expansions are also made in series in terms of the shallowness d/λ (see, for exam-
33 ple, Chakrabarti (1987), where d - mean water depth and λ - wave length), giving
34 rise to cnoidal wave solutions. Since the modulus of the elliptic functions occur-
35 ring throughout the cnoidal wave theory becomes small for waves in deep water,
36 for the cnoidal wave expansions to give accurate results, the wave height parameter
37 H/d (H - the wave height) must be small (Rienecker and Fenton, 1981). Therefore,
38 cnoidal wave solutions are only applicable to waves which are not too high and in
39 relatively shallow water conditions. Although the series expansions of Skjelbreia
40 and Hendrickson (1960) and Fenton (1979) give good results for overall wave train
41 parameters, such as, the wave speed; the details of the flow field, such as the fluid

42 particle velocities are not always satisfactory (Rienecker and Fenton, 1981).

43 In order to overcome the failure of Stokes approach to yield higher waves due to
44 the chosen perturbation parameter, Schwartz (1974) extended the Stokes theory
45 to very high order by using the wave height as the independent parameter instead.
46 The ensuing series converged well for waves up to 97% of the highest. However, for
47 higher waves extrapolation was necessary for a variety of water depths. Also, the
48 convergence of the Schwartz's series deteriorated for very high waves and for very
49 shallow depths (Cokelet, 1977). Longuet-Higgins (1975) used a different perturba-
50 tion parameter to calculate deep water waves, and found that each of the integral
51 quantities exhibited maxima at a wave height slightly lower than the highest wave.
52 Similar finding was also reported by Longuet-Higgins and Fenton (1974) for the
53 solitary wave. Cokelet (1977) extended Schwartz's work and calculated the speed,
54 momentum, energy and other integral properties of waves in water of arbitrary uni-
55 form depth. He also found that all calculated wave properties reach maxima at wave
56 height slightly lower than the highest wave.

57 It is noted that all the above mentioned approaches were based on perturbation
58 method such that the higher-order solutions are an order of magnitude smaller than
59 the immediate lower one. These conditions place severe restrictions on the wave
60 heights in shallow water and thus the Stokes theory and subsequent extensions to
61 the theory are not generally applicable to shallow water. In addition, an inverse
62 plane was used, making the approaches very complex for practical engineering
63 applications.

64 Parallel to Stokes theory extension, series attempts have been made to develop a
65 method which is equally valid for deep and shallow water and which is capable of
66 direct application. Based on the use of truncated Fourier expansions for field quan-

67 titles, (solving stream function instead of velocity potential) Chappellear (1961)
68 and Dean (1965) developed a ‘Fourier approximation method’. By choosing the
69 expansions to satisfy the governing equation and bottom boundary condition, the
70 problem is reduced to solving a set of nonlinear equations for each of the Fourier
71 coefficients, and subsequently the wave properties, such as, the surface elevation
72 and wave speed. As pointed out by Rienecker and Fenton (1981), several aspects of
73 Fourier approximation methods beyond just the truncation of the series may have
74 inhibited its widespread use. These include the limitation to the attainable accu-
75 racy introduced by the use of a Simpson’s rule integration in the solution process.
76 Chaplin (1980) applied the Schmidt orthogonalisation process as an alternative to
77 Dean’s method, and obtained improved results. However, since the stream func-
78 tion expansions contain hyperbolic functions, neither of the above stream function
79 approaches can be applied for waves in deep water.

80 Following the rapid development in computing facility, considerable research ef-
81 forts on the development of progressive wave theory have been made using nu-
82 merical approaches. Rienecker and Fenton (1981) directly solved the coefficients
83 by Newton’s method, which made the solution process much simpler. Their results
84 agreed well with the numerical results for arbitrary uniform water depth of Cokelet
85 (1977) and those in shallow water of Vanden-Broeck and Schwartz (1979), and the
86 results of fluid velocities agreed well with the experimental data of Le Méhauté
87 et al. (1968). This method was further simplified by Fenton (1988) and applied to
88 waves in both deep and shallow water conditions. However, a shortcoming of the
89 method was identified as it could occasionally converge to a wrong solution for very
90 long waves. It was reported that such a problem could be overcome by using a 2nd-
91 order cnoidal or 3rd-order Stokes solution as the initial estimate (Fenton, 1999). By
92 expressing the series as fractional Fourier approximations, Lukomsky and Gandzha

93 (2003) successfully calculated steep sharp-crested waves in deep water.

94 Clamond (1999, 2003) applied an analytic transformation, the so-called renormal-
95 isation, to obtain accurate solutions of progressive waves in waters of finite depth
96 and deep water from the first-order cnoidal wave solution of KDV equation. It was
97 demonstrated that the solutions satisfy the Laplace equation exactly after renormal-
98 isation. However, the Bernoulli equation was satisfied exactly only at two points.
99 Further, the solution was found to be not as accurate but only qualitatively consis-
100 tent for very steep waves (Clamond, 2003). Other relevant works could be found in
101 Fenton (1999).

102 Recently, a new analytic approach named homotopy analysis method (HAM) has
103 seen rapid development. It has been successfully applied to many nonlinear prob-
104 lems and logically contains Lyapunov's small parameter method, the δ -expansion
105 method, and Adomian's decomposition method (Liao, 2004). Without depending
106 on a small parameter such as in a perturbation approach, the HAM has a particular
107 advantage in solving strong nonlinear problems. Other advantages associated with
108 the HAM over the perturbation technique include greater flexibility in the selection
109 of a proper set of basis functions for the solution and a much simpler method in
110 the control of the convergence rate and region. In contrast to Schwartz's approach
111 (Schwartz, 1974), this further leads to the formulation of the boundary-value prob-
112 lem in the physical plane and gives rise to a general solution approach for a larger
113 class of nonlinear problems (Liao and Cheung, 2003). The HAM was first applied
114 to the fluid mechanics problem by Liao (1995) and has been systematically de-
115 scribed by Liao (2004). Liao and Cheung (2003) successfully applied HAM in
116 fully analytical way to waves propagating in deep water.

117 In this paper, the HAM method is extended to nonlinear progressive waves in wa-

ters of finite depth. The wave characteristics are calculated and compared with previously published results. The method described in this paper could be easily applied to practical engineering applications, as the solutions are functions of the wave physical parameters directly, namely, water depth d , wave height H , and wave number k .

2 Theoretical Consideration

2.1 The basic idea behind HAM

For a nonlinear equation:

$$\mathcal{N}[f(\mathbf{x}, t)] = 0, \quad (1)$$

where $\mathcal{N}$ is a nonlinear operator, $f(\mathbf{x}, t)$ is the function to be solved, and $\mathbf{x}$ and t are spatial and temporal independent variables respectively.

A homotopy can be constructed as:

$$(1 - q)\mathcal{L}[F(\mathbf{x}, t; q) - f_0(\mathbf{x}, t)] = q\hbar\mathcal{N}[F(\mathbf{x}, t; q)], \quad q \in [0, 1], \quad (2)$$

where $F(\mathbf{x}, t; q)$ is the mapping function of $f(\mathbf{x}, t)$, $f_0(\mathbf{x}, t)$ is an initial estimate of $f(\mathbf{x}, t)$, $\hbar$ is a nonzero auxiliary parameter, and $\mathcal{L}$ is a linear auxiliary operator with the property of $\mathcal{L}[0] = 0$.

It is obvious

$$F(\mathbf{x}, t; 0) = f_0(\mathbf{x}, t), \quad (3)$$

$$\mathcal{N}[F(\mathbf{x}, t; 1)] = 0. \quad (4)$$

Therefore, as the embedding parameter q varies from 0 to 1, $F(\mathbf{x}, t; q)$ maps continuously from the initial estimate of $f_0(\mathbf{x}, t)$ to the exact solution $f(\mathbf{x}, t)$.

By Taylor's theorem, $F(\mathbf{x}, t; q)$ can be expanded with respect to the embedding parameter q as

$$F(\mathbf{x}, t; q) = f_0(\mathbf{x}, t) + \sum_{m=1}^{+\infty} f_m(\mathbf{x}, t) q^m, \quad (5)$$

where

$$f_m(\mathbf{x}, t) = \frac{1}{m!} \left. \frac{\partial^m F(\mathbf{x}, t; q)}{\partial q^m} \right|_{q=0}. \quad (6)$$

Differentiating the zeroth-order deformation equation (2) m -times with respect to q at $q = 0$ and then dividing it by $m!$, we have the following m th-order deformation equation

$$\mathcal{L}[f_m(\mathbf{x}, t) - \chi_m f_{m-1}(\mathbf{x}, t)] = \hbar R_m(\mathbf{x}, t), \quad (7)$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}, \quad (8)$$

$$R_m(\mathbf{x}, t) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}[F(\mathbf{x}, t; q)]}{\partial q^{m-1}} \right|_{q=0}. \quad (9)$$

142 If the series (5) converges at $q = 1$, we have

$$f(\mathbf{x}, t) = f_0(\mathbf{x}, t) + \sum_{m=1}^{+\infty} f_m(\mathbf{x}, t). \quad (10)$$

143 The auxiliary parameter $\hbar$ provides us a convenient way to control and adjust the
 144 rate and region of the convergence. More detailed description of HAM can be found
 145 in Liao (2004).

146 2.2 HAM for the wave problem

147 2.2.1 The description of wave propagation problem

148 Consider a two-dimensional steady gravity wave propagating in water of uniform
 149 depth. The fluid is assumed to be inviscid, incompressible and irrotational. The co-
 150 ordinate system (x, z) is set to move with the wave at a phase speed, C , where the
 151 x -axis is positive in the direction of wave propagation and the z -axis is positive
 152 vertically up from the still water level. The quantities $\phi(x, z)$ and $\zeta(x)$ are defined
 153 as the velocity potential and the wave elevation respectively. The fluid motion de-
 154 scribed by the velocity potential is governed by the Laplace equation:

$$\nabla^2 \phi(x, z) = 0 \quad -\infty < x < +\infty, -d < z < \zeta(x), \quad (11)$$

The velocity potential $\phi(x, z)$ is subject to the free surface boundary conditions:

$$C^2 \phi_{xx} + g \phi_z + \frac{1}{2} (\nabla \phi) \cdot \nabla [(\nabla \phi) \cdot (\nabla \phi)] - 2C (\nabla \phi) \cdot (\nabla \phi_x) = 0, \quad z = \zeta(x), \quad (12)$$

$$\zeta(x) = \frac{1}{g} \left[C\phi_x - \frac{1}{2}(\nabla\phi) \cdot (\nabla\phi) + K \right], \quad z = \zeta(x), \quad (13)$$

155 and the condition at sea bottom:

$$\phi_z = 0, \quad z = -d, \quad (14)$$

156 where g is the gravitational acceleration, K is the Bernoulli constant, and the sub-
157 scripts x and z denote partial derivatives in the respective directions.

158 2.2.2 The solution forms

159 By satisfying Laplace equation (11) and sea bottom condition (14), the velocity
160 potential $\phi(x, z)$ can be expressed by a set of base functions

$$\left\{ \frac{\cosh[nk(z+d)]}{\cosh(nkd)} \sin(nkx) \mid n \geq 1 \right\}, \quad (15)$$

161 in the form

$$\phi(x, z) = \sum_{n=1}^{+\infty} \alpha_n \frac{\cosh[nk(z+d)]}{\cosh(nkd)} \sin(nkx), \quad (16)$$

162 where wave number k is defined as $k = 2\pi/\lambda$, α_n is a coefficient. This provides us
163 with the *rule of solution expression* for $\phi(x, z)$ (Liao, 2004). Accordingly, the wave
164 elevation $\zeta(x)$ can be expressed by a set of base functions

$$\{\cos(nkx) \mid n \geq 1\}, \quad (17)$$

165 in the form

$$\zeta(x) = \sum_{n=1}^{+\infty} \beta_n \cos(nkx). \quad (18)$$

166 which provides us with the *rule of solution expression* for $\zeta(x)$.

167 2.2.3 $\mathcal{N}$ and $\mathcal{L}$ operators

168 Now we introduce HAM to satisfy the two nonlinear free surface boundary con-
 169 ditions (12) and (13). This is achieved by setting up the mappings $\phi(x, z) \rightarrow$
 170 $\Phi(x, z; q)$, $\zeta(x) \rightarrow \eta(x; q)$, $C \rightarrow \Lambda(q)$, and $K \rightarrow \Theta(q)$, which leads to vary
 171 $\Phi(x, z; q)$, $\eta(x; q)$, $\Lambda(q)$ and $\Theta(q)$ continuously from the initial estimates to the
 172 exact solutions given by $\phi(x, z)$, $\zeta(x)$, C and K respectively, as the embedding
 173 parameter q increases from 0 to 1.

174 Define two nonlinear operators $\mathcal{N}_1$ and $\mathcal{N}_2$ by the free surface boundary conditions
 175 Eqs. (12) and (13) respectively as:

$$\begin{aligned} \mathcal{N}_1[\Phi(x, z; q), \Lambda(q)] &= \Lambda^2(q) \Phi_{xx}(x, z; q) + g \Phi_z(x, z; q) \\ &+ \frac{1}{2} [\nabla \Phi(x, z; q)] \cdot \nabla \{ [\nabla \Phi(x, z; q)] \cdot [\nabla \Phi(x, z; q)] \} \quad (19) \\ &- 2\Lambda(q) [\nabla \Phi(x, z; q)] \cdot [\nabla \Phi_x(x, z; q)], \end{aligned}$$

$$\begin{aligned} &\mathcal{N}_2[\eta(x; q), \Phi(x, z; q), \Lambda(q), \Theta(q)] \\ &= \eta(x; q) - \frac{1}{g} \left\{ \Lambda(q) \Phi_x(x, z; q) - \frac{1}{2} [\nabla \Phi(x, z; q)] \cdot [\nabla \Phi(x, z; q)] + \Theta(q) \right\}. \quad (20) \end{aligned}$$

176 Two auxiliary operator $\mathcal{L}_1$ and $\mathcal{L}_2$ can be chosen as:

$$\mathcal{L}_1[\Phi(x, z; q), \Lambda(q)] = \Lambda^2(q)\Phi_{xx}(x, z; q) + g\Phi_z(x, z; q), \quad (21)$$

$$\mathcal{L}_2[(\bullet)] = (\bullet). \quad (22)$$

177 2.2.4 zeroth-order deformation equations

178 We construct the two zeroth-order deformation equations for the two nonlinear free
179 surface conditions (12) and (13) as

$$(1 - q)\mathcal{L}_1[\Phi(x, z; q) - \phi_0(x, z), \Lambda(q)] = q\hbar\mathcal{N}_1[\Phi(x, z; q), \Lambda(q)], \quad z = \eta(x; q), \quad (23)$$

$$(1 - q)\mathcal{L}_2[\eta(x; q) - \zeta_0(x)] = q\hbar\mathcal{N}_2[\eta(x; q), \Phi(x, z; q), \Lambda(q), \Theta(q)], \quad z = \eta(x; q), \quad (24)$$

180 where $\phi_0(x, z)$ and $\zeta_0(x, z)$ are the initial estimates of the velocity potential and the
181 free surface elevation respectively.

182 2.2.5 Initial estimates

183 The incident wave velocity potential and dispersion relation

$$\phi_0(x, z) = a_{0,1} \frac{\cosh[k(z + d)]}{\cosh(kd)} \sin(kx), \quad (25)$$

$$\Lambda(0) = \sqrt{g \tanh(kd)/k} = C_0, \quad (26)$$

are chosen as the initial estimates of $\phi(x, z)$ and C respectively, where $a_{0,1}$ is a coefficient.

In order to simplify the formulation of the free surface boundary conditions and the solution procedure, we choose:

$$\zeta_0(x) = 0, \quad (27)$$

$$\Theta(0) = 0 = K_0. \quad (28)$$

2.2.6 m th-order deformation equations

Define

$$\Phi_m(x, z; q) = \frac{1}{m!} \frac{\partial^m \Phi(x, z; q)}{\partial q^m}, \quad (29)$$

$$\phi_m(x, z) = \frac{1}{m!} \left. \frac{\partial^m \Phi(x, z; q)}{\partial q^m} \right|_{q=0}, \quad (30)$$

$$\zeta_m(x) = \frac{1}{m!} \left. \frac{\partial^m \eta(x; q)}{\partial q^m} \right|_{q=0}, \quad (31)$$

$$C_m = \frac{1}{m!} \left. \frac{d^m \Lambda(q)}{dq^m} \right|_{q=0}, \quad (32)$$

$$K_m = \frac{1}{m!} \left. \frac{d^m \Theta(q)}{dq^m} \right|_{q=0}, \quad (33)$$

and the following vectors:

$$\vec{\phi}_n = \{\phi_0(x, z), \phi_1(x, z), \phi_2(x, z), \dots, \phi_n(x, z)\}, \quad (34)$$

$$\vec{\zeta}_n = \{\zeta_0(x), \zeta_1(x), \zeta_2(x), \dots, \zeta_n(x)\}, \quad (35)$$

$$\vec{C}_n = \{C_0, C_1, C_2, \dots, C_n\}, \quad (36)$$

$$\vec{K}_n = \{K_0, K_1, K_2, \dots, K_n\}. \quad (37)$$

Similar to Eq. (7), the m th-order deformation equations are obtained by differentiating the zeroth-order deformation equations (23) and (24) m -times with respect to q at $q = 0$ and then dividing by $m!$, respectively. Due to the simple form of $\mathcal{L}_2$, the m th-order solution of elevation $\zeta_m(x)$ could be simply solved as

$$\zeta_m(x) = \chi_m \zeta_{m-1}(x) + \hbar R_m^2(\vec{\zeta}_{m-1}, \vec{\phi}_{m-1}, \vec{C}_{m-1}, \vec{K}_{m-1}). \quad (38)$$

Usually, $R_m^i (i = 1, 2)$ is decided by Eq. (9) in HAM. However, Considering that the zeroth-order deformation equations (23) and (24) are defined at the unknown boundary $z = \eta(x; q)$, $R_m^i (i = 1, 2)$ are defined as the following form instead

$$R_m^i(\bullet) = \frac{1}{(m-1)!} \left. \frac{D^{m-1} \mathcal{N}_i[\bullet; q]}{Dq^{m-1}} \right|_{q=0} \quad (i = 1, 2). \quad (39)$$

where the differential operator D^m/Dq^m at $z = \eta(x; q)$ can be determined as (see Appendix for details)

$$\frac{1}{m!} \frac{D^m \Phi(x, z; q)}{Dq^m} = \Phi_m(x, z; q) + \mathcal{T}_m[\Phi(x, z; q), \Lambda(q)], \quad (40)$$

Noting that $R_m^i (i = 1, 2)$ is only dependent on results up to the $(m-1)$ th order, thus, $\zeta_m(x)$ can be calculated directly from Eq. (38).

The m th-order solution for velocity potential $\phi_m(x, z)$ is much more complicated due to the form of $\mathcal{L}_1$ and D^m/Dq^m . From Eq. (7), we have

$$\mathcal{L}_1(\phi_m, C_m) = \chi_m \mathcal{L}_1(\phi_{m-1}, C_{m-1}) + \hbar R_m^1(\vec{\phi}_{m-1}, \vec{C}_{m-1}), \quad (41)$$

204 Substituting Eqs. (21), (30) and (40), the left-hand side of Eq. (41) can be written
 205 as:

$$\begin{aligned}
 & \mathcal{L}_1(\phi_m, C_m) \\
 &= \frac{1}{m!} \sum_{i=0}^m \binom{m}{i} \frac{D^i [\Lambda^2(q)]}{Dq^i} \bigg|_{q=0} \frac{D^{m-i} \Phi_{xx}(x, z; q)}{Dq^{m-i}} \bigg|_{q=0} + g \frac{1}{m!} \frac{D^m \Phi_z(x, z; q)}{Dq^m} \bigg|_{q=0} \\
 &= \sum_{i=0}^m \sum_{j=0}^i C_j C_{i-j} \left\{ \frac{\partial^2 \phi_{m-i}(x, z)}{\partial x^2} + \mathcal{T}_{m-i}[\Phi_{xx}(x, z; q), \Lambda(q)]|_{q=0} \right\} \\
 &+ g \left\{ \frac{\partial \phi_m(x, z)}{\partial z} + \mathcal{T}_m[\Phi_z(x, z; q), \Lambda(q)]|_{q=0} \right\},
 \end{aligned} \tag{42}$$

206 then Eq. (41) can be expressed for brevity as:

$$C_0^2 \frac{\partial^2 \phi_m(x, z)}{\partial x^2} + g \frac{\partial \phi_m(x, z)}{\partial z} = S_m(x, \vec{\phi}_{m-1}, \vec{\zeta}_m, \vec{C}_m), \quad z = \zeta_0(x), \tag{43}$$

207 in which

$$\begin{aligned}
 & S_m(x, \vec{\phi}_{m-1}, \vec{\zeta}_m, \vec{C}_m) \\
 &= \chi_m \mathcal{L}_1(\phi_{m-1}, C_{m-1}) + \hbar R_m^1(\vec{\phi}_{m-1}, \vec{C}_{m-1}) \\
 &- C_0^2 \mathcal{T}_m[\Phi_{xx}(x, z; q), \Lambda(q)]|_{q=0} - g \mathcal{T}_m[\Phi_z(x, z; q), \Lambda(q)]|_{q=0} \\
 &- \sum_{i=1}^m \sum_{j=0}^i C_j C_{i-j} \left\{ \frac{\partial^2 \phi_{m-i}(x, z)}{\partial x^2} + \mathcal{T}_{m-i}[\Phi_{xx}(x, z; q), \Lambda(q)]|_{q=0} \right\}.
 \end{aligned} \tag{44}$$

208 The right-hand side of Eq. (43) is dependent on terms up to the $(m - 1)$ th-order
 209 approximation, as well as C_m and ζ_m . Until now, C_m is still unknown, thus an

210 additional algebraic equation should be introduced to make the problem closed.

211 **3 Solution method**

212 Considering Eqs. (38) and (43), we obtain the expressions of velocity potential and
213 wave profile as

$$\phi_m(x, z) = \sum_{n=1}^{2m+1} a_{m,n} \frac{\cosh[nk(z+d)]}{\cosh(nkd)} \sin(nkx), \quad (45)$$

$$S_m(x, \vec{\phi}_{m-1}, \vec{\zeta}_m, \vec{C}_m) = \sum_{n=1}^{2m+1} b_{m,n}(\vec{C}_m) \sin(nkx), \quad (46)$$

$$\zeta_m(x) = \sum_{n=1}^{2m} c_{m,n} \cos(nkx). \quad (47)$$

214 Expanding Eq. (38), we have

$$\zeta_m(x) = \sum_{n=0}^{2m} c_{m,n} \cos(nkx) - \frac{\hbar}{g} K_{m-1}. \quad (48)$$

215 Noting Eq. (47), K_{m-1} is obtained as

$$K_{m-1} = \frac{g}{\hbar} c_{m,0}. \quad (49)$$

216 Substituting Eq. (46) into Eq. (43) and considering $\zeta_0(x) = 0$, we have

$$\sum_{n=1}^{2m+1} \left\{ \left[-C_0^2 n^2 k^2 + g(nk) \tanh(nkd) \right] a_{m,n} - b_{m,n}(\vec{C}_m) \right\} \sin(nkx) = 0, \quad (50)$$

217 where

$$a_{m,n} = \frac{b_{m,n}(\vec{C}_m)}{-C_0^2 n^2 k^2 + g(nk) \tanh(nkd)} \quad \text{for } 2 \leq n \leq 2m+1. \quad (51)$$

218 Considering the initial estimate for the phase speed in Eq. (26), we have

$$b_{m,1}(\vec{C}_m) = 0 \quad \text{for } m \geq 1, \quad (52)$$

219 which determines C_m as

$$C_m = -\frac{\gamma_m(\vec{C}_{m-1})}{\delta_m(\vec{C}_{m-1})}, \quad (53)$$

220 where $\gamma_m(\vec{C}_{m-1})$ and $\delta_m(\vec{C}_{m-1})$ are coefficients.

221 To this point, only the coefficient $a_{m,1}$ is still unknown. In order to solve $a_{m,1}$, an
 222 additional equation is introduced by relating the solution of the free surface to the
 223 wave height H :

$$\Gamma_m = \zeta_{m+1}(0) - \zeta_{m+1}(\lambda/2) = \begin{cases} H & \text{for } m = 0 \\ 0 & \text{for } m \geq 1 \end{cases}. \quad (54)$$

224 which gives

$$a_{m,1} = -\frac{\mu_m}{\tau_m}, \quad (55)$$

225 where μ_m and τ_m are coefficients.

226 The value of $a_{0,1}$ in the initial estimate $\phi_0(x, z)$ can be determined by this expres-
 227 sion directly:

$$a_{0,1} = -\frac{gH}{2\hbar k C_0} = -\frac{gH}{2\hbar \sqrt{gk \tanh(kd)}}. \quad (56)$$

228 It is noted that if C_0 is assigned with values other than Eq. (26), Eq. (51) is still
 229 valid even when $n = 1$, and C_m can be determined after Eq. (55) is solved.

230 After all the coefficients $a_{m,n}$ are determined, it is straightforward to obtain solu-
 231 tions of $\phi_m(x, z)$, ζ_m , C_m and K_m to all orders. As long as the series converge,
 232 the exact solutions of $\phi(x, z)$, $\zeta(x)$, C and K can be obtained. The present analytic
 233 solution is obtained by performing symbolic computation using *Mathematica 5* and
 234 verified by a Fortran program.

235 4 Results and Discussion

236 According to Eq. (10), the M th-order approximations of solutions are:

$$\phi(x, z) \approx \phi_0(x, z) + \sum_{m=1}^M \phi_m(x, z), \quad (57)$$

$$\zeta(x) \approx \zeta_0(x) + \sum_{m=1}^M \zeta_m(x), \quad (58)$$

$$C \approx C_0 + \sum_{m=1}^M C_m. \quad (59)$$

237

238 Wave characteristics such as velocity potential $\phi(x, z)$, surface elevation $\zeta(x)$, and
 239 the phase speed C are dependent on the wave steepness. Considering the dispersion
 240 relation between the phase speed C and the wave height H , it is also the tradition

241 that the phase speed C is often chosen as the first basis for comparisons among
242 wave theories. This quantity is also used to examine the accuracy of the results of
243 the present HAM solutions for waves propagating in water of finite depth.

244 The convergence rate and region are controlled by the auxiliary parameter $\hbar$ in
245 HAM. As $\hbar$ approaches 0, the convergence region enlarges accompanied by low
246 convergence rate indicating that increasing number of terms are required in the ap-
247 proximation to maintain the same level of accuracy. However, the accuracy and
248 convergence can be improved by the homotopy-Padé technique. As demonstrated
249 by Liao and Cheung (2003), the homotopy-Padé approximants give converging re-
250 sults over a considerably large region at the same order of approximation. Similar
251 technique is applied here and the improvement to the convergence and accuracy
252 is more pronounced in the present study for wave propagating in waters of finite
253 depth. Fig. 1 shows that results quickly converge to the accurate solution over a
254 large convergence region ($\hbar \leq -2/5$) for $H/d = 0.1729974$.

255 [Fig. 1 about here.]

256 Table 1 shows that only the first few terms of the series provided an accurate ap-
257 proximation of the solution for the above value of H/d . The homotopy-Padé ap-
258 proximants achieved convergence to five digits at [3,3] and six digits at [6,6]. How-
259 ever, the present study indicates that more terms are required for convergence as
260 the wave steepness increases depending on the choice of $\hbar$.

261 Bernoulli constant K in Eq. (13) is important for locating the still water level ex-
262 actly at $z = 0$, which is a result of the *Galilean* transformation to a non-inertial
263 moving coordinate system (Morse and Ingard, 1968). It can be omitted for wave
264 propagation on deep water. However, it can lead to a difference of actual water
265 depth with $\bar{\zeta}$ in cases of finite water depth. The mean displacement of the free

266 surface $\bar{\zeta}$ can be calculated as

$$\bar{\zeta} = \frac{1}{\lambda} \int_0^\lambda \zeta(x) dx = \sum_{m=1}^M c_{m,0}, \quad (60)$$

267 then the mean water depth is

$$d = d_0 + \bar{\zeta}, \quad (61)$$

268 where d_0 is the initial water depth.

269 The effect of neglecting the Bernoulli constant K in Eq. (13) for shallow water
 270 waves generally leads to a negative $\bar{\zeta}$ in HAM, making the actual water depth shal-
 271 lower. In order to demonstrate the effect of Bernoulli constant K on the computa-
 272 tion of progressive waves on finite water depth, phase speed are calculated for both
 273 with and without K included in Eq. (13) and results are shown in Table 1. It is seen
 274 that even very small nondimensional displacement of the free surface $k\bar{\zeta}$ resulted
 275 from neglecting K can lead a distinct difference in KC^2/g .

276 [Table 1 about here.]

277 Schwartz (1974) and Liao and Cheung (2003) presented perturbation solution and
 278 HAM solutions respectively for waves propagating in infinite water depth. In order
 279 to further validate the current HAM solutions of waves propagating in finite water
 280 depth, the nondimensional phase speed squared, kC^2/g , is computed, and the re-
 281 sults are compared with those of Schwartz (1974) and Liao and Cheung (2003). As
 282 can be seen from Table 2, in deep water, such as, $kd = 10^4$, and for the wave steep-
 283 ness of up to $H/\lambda = 0.140$, the present homotopy-Padé approximation of finite
 284 depth agrees well with the corresponding deep water solutions of Schwartz (1974)

285 and Liao and Cheung (2003). Table 2 also shows that the present homotopy-Pad
286 approximation converges rapidly with only a few number of terms (a maximum
287 of 22nd-order for $H/\lambda = 0.140$) giving identical or very similar results over the
288 range of wave steepness considered. This is a further demonstration of the excellent
289 convergence rate in the present homotopy-Pad technique.

290 [Table 2 about here.]

291 A detailed comparison between the present HAM solutions of waves in finite water
292 depth and results of Cokelet (1977) and Rienecker and Fenton (1981) is presented
293 in Table 3. Results are given in terms of the nondimensional phase speed squared
294 kC^2/g at various values of H/d , for a constant value of $\exp(-kQ/C) = 0.5$ (Q
295 is the unit span denoting the total volume rate of flow underneath the steady wave
296 per unit length in a direction normal to the x, z plane), corresponding to a wave
297 length to water depth ratio $\lambda/d \approx 9$. In order to make such comparison, the present
298 solutions are calculated based on the physical parameters, e.g., the wave height H
299 and the water depth d which are the results of the previous investigation of Cokelet
300 (1977) and Rienecker and Fenton (1981). The present homotopy-Pad approxima-
301 tion gave highly accurate results for waves all the way up to their maximum height.
302 The convergence orders of the Homotopy-Pad approximants of the series are given
303 in the brackets. Again, rapid convergence rate is shown in Table 3 with the 56th-
304 order approximation (given by the [28, 28]) of the solution series, kC^2/g . Similar
305 to the results of Cokelet (1977) and Rienecker and Fenton (1981), the present HAM
306 solutions give the maximum wave speed at a value of $H/d = 0.672143$ which is
307 the well-known characteristics of the integral quantities reaching maxima before
308 the highest wave is reproduced as stated earlier.

309 [Table 3 about here.]

310 Le Méhauté et al. (1968) conducted experimental investigation in a wave tank,
 311 and the result was presented as the variation of the horizontal fluid velocity with
 312 depth under the wave crest. As pointed out by Rienecker and Fenton (1981) the
 313 appropriate frame of reference for comparison with physical wave tank experiments
 314 is the one in which the mass flux is zero since the experiments were performed in
 315 a closed wave tank implying no net mass transport under the waves. On the other
 316 hand, the frame of reference in the present approach is a mean Eulerian velocity of
 317 zero. In order to compare with experiment, the difference of these two frames of
 318 reference, the so-called *Stokes drift* need to be taken into account using the formula
 319 suggested by Hudspeth (2006)

$$u_m \approx u_e - gH^2/(8C). \quad (62)$$

320 where u_m and u_e represent horizontal fluid velocity relative to the frames of refer-
 321 ence corresponding to zero net mass flux and zero mean Eulerian velocity, respec-
 322 tively. Since the *Stokes drift* is a weakly nonlinear quantity, Eq. (62) is calculated to
 323 the second-order of wave height. For the higher-order approximations, however, the
 324 zero mass flux description turns out to require a much more elaborate derivation.
 325 Such a derivation has never been carried out to higher than third-order (Svendsen,
 326 2006).

327 Fig. 2 shows the velocity profiles for different values of wave height and wave pe-
 328 riod from the present HAM solutions along with the experimental measurements of
 329 Le Méhauté et al. (1968), and the Fourier approximation of Rienecker and Fenton
 330 (1981). The results presented in Figure 2 are for different values of nondimensional
 331 wave height $H^* = H/d$ and period $T^* = T\sqrt{g/d}$. The agreement in Fig. 2 be-
 332 tween the HAM solutions presented here and the experimental data and the Fourier

333 approximation is very good, especially for higher waves.

334 [Fig. 2 about here.]

335 Finally, in order to demonstrate the applicability of the present theory to an engi-
336 neering problem of steep wave in finite depth, an example is chosen of $H = 50$
337 m, $\lambda = 400$ m and $d = 100$ m. The profile for this wave by the present theory is
338 given in Fig. 3 and the nondimensional wave properties in terms of particle velocity
339 $u^* = u(k/g)^{1/2}$ and dynamic pressure $p^* = pk/\rho g$ under the wave crest with depth
340 below the still water level (SWL) are plotted in Fig. 4. The HAM terms used to
341 achieve convergence are [20,20]. The presence of high nonlinearity in the wave is
342 evident in the results.

343 [Fig. 3 about here.]

344 [Fig. 4 about here.]

345 5 Conclusions

346 An analytical approximation of steep nonlinear waves propagating in water of fi-
347 nite depth is derived using the homotopy analysis method. A high convergence
348 rate over a considerably large convergence region can be achieved by applying the
349 homotopy-Padé expansion to the solution series. Comparison with previous numer-
350 ical and experimental results demonstrates that the present method not only gives
351 highly accurate results for all the integral quantities, but also produces excellent
352 results for the fluid velocity. The solution is valid for progressive waves in all con-
353 stant water depths, and approaches the deep water solution accurately. The final
354 results can be expressed as functions of physical parameters of importance, which

355 is directly applicable engineering problems. The method presented in this paper can
 356 be applied to solve many nonlinear problems in water of finite depth.

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364 **Appendix A**

365 Expanding $\Phi(x, z; q)$ using a Taylor series around $q = 0$:

$$\Phi(x, z; q) = \sum_{m=0}^{+\infty} \frac{1}{m!} \left. \frac{D^m \Phi(x, z; q)}{Dq^m} \right|_{q=0} \cdot q^m, \quad (\text{A.1})$$

366 and about the free surface at $z = \eta(x; 0)$:

$$\Phi(x, z; q) = \sum_{r=0}^{+\infty} \sum_{s=0}^{+\infty} \frac{1}{r!} \left. \frac{\partial^r \Phi_s(x, z; q)}{\partial z^r} \right|_{q=0} [\eta(x; q) - \eta(x; 0)]^r \cdot q^s, \quad (\text{A.2})$$

367 and the above two expressions for $\Phi(x, z; q)$ are equivalent. By invoking Eq. (5),
 368 we have

$$\sum_{m=0}^{+\infty} \frac{1}{m!} \frac{D^m \Phi(x, z; q)}{Dq^m} \Big|_{q=0} \cdot q^m = \sum_{r=0}^{+\infty} \sum_{s=0}^{+\infty} \frac{1}{r!} \frac{\partial^r \Phi_s(x, z; q)}{\partial z^r} \Big|_{q=0} \left[\sum_{t=1}^{+\infty} q^t \cdot \zeta_t(x) \right]^r \cdot q^s. \quad (\text{A.3})$$

369 Equating the coefficients of q^m on the left- and right-hand sides, one can obtain Eq.
 370 (40) for the operator D^m/Dq^m . The process can be carried out in *Mathematica 5.0*
 371 along with the other formulations.

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Table 1
Convergence of the series for $H/d_0 = 0.1729974$.

order m	Eq. (13) ($\bar{\zeta} = 0$)		Eq. (13) without K ($\bar{\zeta} \neq 0$)			
	kC^2/g	Padé	kC^2/g	Padé	$k\bar{\zeta}$	Padé
1	0.604405		0.602222		-3.41861E-03	
2	0.615791	0.601117	0.614982	0.601845	-1.22195E-03	-2.08127E-03
3	0.616722		0.616364		-8.76574E-04	
4	0.615093	0.615014	0.614646	0.614508	-8.53624E-04	-8.54799E-04
5	0.614817		0.614249		-8.94905E-04	
6	0.615033	0.615059	0.614468	0.614507	-9.12465E-04	-9.14133E-04
7	0.615099		0.614565		-9.07905E-04	
8	0.615068	0.615062	0.614538	0.614516	-9.03381E-04	-9.04352E-04
9	0.615052		0.614515		-9.03580E-04	
10	0.615057	0.615060	0.614517	0.614522	-9.04629E-04	-9.04563E-04
11	0.615061		0.614523		-9.04760E-04	
12	0.615060	0.615059	0.614523	0.614522	-9.04534E-04	-9.04521E-04
13	0.615059		0.614522		-9.04466E-04	
14	0.615059	0.615059	0.614521	0.614522	-9.04510E-04	-9.04522E-04
15	0.615059		0.614522		-9.04534E-04	
16	0.615059	0.615059	0.614522	0.614522	-9.04526E-04	-9.04522E-04

Table 2

Comparison of results for kC^2/g in water of infinite depth.

H/λ	Schwartz (1974)	Liao and Cheung (2003)			Present HAM solution ($kd = 10^4$)		
		[8,8]	[10,10]	[11,11]	[8,8]	[10,10]	[11,11]
0.040	1.01592	1.01592	1.01592	1.01592	1.01592	1.01592	1.01592
0.070	1.04955	1.04955	1.04955	1.04955	1.04955	1.04955	1.04955
0.100	1.10367	1.10367	1.10367	1.10367	1.10367	1.10367	1.10367
0.120	1.15182	1.15184	1.15182	1.15181	1.15184	1.15182	1.15181
0.130	1.17820	1.17834	1.17821	1.17821	1.17834	1.17821	1.17821
0.135	1.18996	1.19061	1.19003	1.19003	1.19061	1.19003	1.19003
0.140	1.1930	1.19833	1.19369	1.19385	1.19833	1.19369	1.19385

Table 3

Nondimensional phase speed squared kC^2/g at different H/d .

H/d	Rienecker and Fenton (1981)	Cokelet (1977)	Present HAM solution		
0.1729974	0.615059	0.615059	0.615059	0.615059	0.615059
			[6,6]	[7,7]	[8,8]
0.2526308	0.631112	0.631112	0.631112	0.631112	0.631112
			[6,6]	[7,7]	[8,8]
0.3802643	0.666501	0.666501	0.666501	0.666501	0.666501
			[9,9]	[10,10]	[11,11]
0.4944549	0.706443	0.706443	0.706443	0.706443	0.706443
			[12,12]	[13,13]	[14,14]
0.602447	0.748230	0.748230	0.748231	0.748230	0.748230
			[24,24]	[25,25]	[26,26]
0.651251	0.764403	0.764403	0.76440	0.76440	0.76440
			[24,24]	[25,25]	[26,26]
0.672143	0.76776	0.767748	0.76775	0.76777	0.76776
			[25,25]	[26,26]	[27,27]
0.6832	0.76703	0.76707	0.7668	0.7668	0.7668
			[24,24]	[25,25]	[26,26]
0.6908	0.7630	0.7660	0.7627	0.7628	0.7630
			[26,26]	[27,27]	[28,28]

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449		(b) $H^* = 0.420$, $T^* = 15.87$; (c) $H^* = 0.389$, $T^* = 22.49$;	
450		(d) $H^* = 0.433$, $T^* = 27.24$; (e) $H^* = 0.499$, $T^* = 8.59$; (f)	
451		$H^* = 0.522$, $T^* = 15.87$; (g) $H^* = 0.492$, $T^* = 22.49$; (h)	
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Fig. 1. Influence of $\hbar$ on convergence

Fig. 2. Horizontal fluid velocity under the wave crest: comparison between the present HAM solutions, the experimental results of Le Méhauté et al. (1968) and the Fourier approximation of Rieneker and Fenton (1981) ($z = 0$ at seabed). (a) $H^* = 0.434$, $T^* = 8.59$; (b) $H^* = 0.420$, $T^* = 15.87$; (c) $H^* = 0.389$, $T^* = 22.49$; (d) $H^* = 0.433$, $T^* = 27.24$; (e) $H^* = 0.499$, $T^* = 8.59$; (f) $H^* = 0.522$, $T^* = 15.87$; (g) $H^* = 0.492$, $T^* = 22.49$; (h) $H^* = 0.548$, $T^* = 27.27$.

Fig. 3. Example wave profile for a steep wave of $H = 50$ m, $\lambda = 400$ m and $d = 100$ m

Fig. 4. Nondimensional water particle properties under the crest for a steep wave of $H = 50$ m, $\lambda = 400$ m and $d = 100$ m