1 **Scaled boundary FEM solution of short-crested wave** 2 **diffraction by a vertical cylinder**

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6 **Abstract**

7 The scaled boundary finite-element method (SBFEM) is a novel semi-analytical method
8 developed in the elasto-statics and elasto-dynamics areas that has the advantages of com-
9 bining the finite-element method with the boundary-element method. The SBFEM method
10 weakens the governing differential equation in the circumferential direction and solves the
11 weakened equation analytically in the radial direction. It has the inherent advantage of
12 solving the unbounded fluid dynamic problem. In this paper, the boundary-value problem
13 composed of short-crested waves diffracted by a vertical circular cylinder is solved by
14 SBFEM. Only the cylinder boundary is discretized with curved surface finite-elements on
15 the circumference of the cylinder, while the radial differential equation is solved completely
16 analytically. The computation of the diffraction force based on the present SBFEM solu-
17 tion demonstrates a high accuracy achieved with a small number of surface finite-elements.
18 The method can be extended to solve more complex wave-structure interaction problems
19 resulting in direct engineering applications.

20 *Key words:* scaled boundary finite-element method, short-crested wave, wave diffraction,
21 unbounded domain, vertical cylinder

22 **1 Introduction**

23 Studies on ocean surface waves are generally focused on two-dimensional (2D)
24 waves. In reality, however, the ocean waves are more complex. The short-crested
25 wave is a common 3D wave model which describes waves generated by winds
26 blowing across the surface of the ocean. It also commonly arises, for example, from
27 the oblique interaction of two travelling plane waves or intersecting swell waves,
28 from the reflection of waves at non-normal incidence off a vertical seawall or a
29 breakwater, as well as from diffraction about the surface boundaries of a structure
30 of finite length [6]. Such waves are of paramount importance in an engineering
31 design. Unlike the plane waves propagating in a single direction, and the standing
32 waves fluctuating vertically in a confined region, short-crested waves can be doubly
33 periodic in two horizontal directions, one in the direction of propagation and the
34 other normal to it [15].

35 The interaction problem of water waves with a large vertical circular cylinder has
36 been widely investigated both numerically and experimentally in the past due to its
37 theoretical and practical importance, especially to ocean engineers. Diffraction ef-
38 fects become more important when the dimension of the structure is large compared
39 to the wave length. As one of the pioneers in wave-structure interaction research,
40 MacCamy and Fuchs [10] presented an analytical solution for linear plane waves
41 diffracted by a large vertical cylinder in intermediate water depths and their so-
42 lution was later validated by Chakrakarti and Tam's [2] experiment. Chakrabarti
43 and Tam [2] revealed that the linear diffraction solution is reasonably accurate at
44 least for $H/h \leq 0.25$ (H is wave height and h is water depth) and a range of ka

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45 (k is wave number and a is cylinder radius) between 0 and 3. Since then, many
46 numerical and experimental results have been reported on the plane wave diffrac-
47 tion [1, 7, 9, 12–14]. However, studies on the short-crested wave diffraction are very
48 limited in the literature.

49 Zhu [17] solved the diffraction problem for a circular cylinder in short-crested
50 waves using linear potential flow theory and found that the pressure distribution
51 and water run-up on the cylinder were quite different from those of plane incident
52 waves. Their patterns become very complex as ka becomes large. The hydrody-
53 namic forces on the cylinder become larger as the incident waves become less short-
54 crested. Based on the boundary-element method solution, Zhu and Moule [18] ex-
55 amined the dependence of the wave diffraction force for a given cylinder on the
56 wave number and found that the wave loads induced by short-crested waves can
57 be larger than those induced by plane waves for the same total wave number. Zhu
58 and Satravaha [19] further presented the second-order diffraction solution of short-
59 crested waves. However, the full analytical results are very complex and need vali-
60 dation.

61 Recently, a new semi-analytical method, called scaled boundary finite-element method
62 (SBFEM) has been successfully applied to soil-structure interaction problems. Com-
63 bining the advantages of finite-element and boundary-element methods, the ap-
64 proach discretizes only the boundary with surface finite-elements. It then trans-
65 forms the governing partial differential equations to ordinary differential equations,
66 and solves them analytically. The method represents singularities and unbounded
67 domains accurately and efficiently compared to the complete finite-element method
68 and needs no fundamental solution as for the boundary-element method. Fewer ele-
69 ments are required to obtain very accurate results [16]. Deeks and Cheng [4] applied
70 SBFEM to an uniform potential flow around obstacles and revealed its inherent ad-

71 vantages to model unbounded fluid problems as well as the singular points in the
72 near field of bluff obstacles. Li, et al [8] attempted to solve the problem of plane
73 wave diffraction by a vertical cylinder using SBFEM, with only limited success
74 in obtaining semi-analytical solution for high frequency waves. For low frequency
75 waves, other numerical methods, such as, the Runge-Kutta scheme needs to be im-
76 plemented to solve the radial differential equation. Such compromised approach
77 significantly diminishes, to a large extend, the advantages of SBFEM exhibited in
78 dealing with bounded fluid domain problems.

79 In this paper, linear short-crested wave diffraction by a large vertical circular cylin-
80 der in an unbounded domain is solved by SBFEM. The radial differential equation
81 is solved fully analytically. Only a few finite elements discretized on the circum-
82 ference of the cylinder are shown to be sufficient to obtain accurate results. Excel-
83 lent agreements between the present SBFEM results and the analytical solutions of
84 Zhu [17], including wave run-up, diffraction force and other hydrodynamic force
85 coefficients are achieved. The method described in this paper can be easily extended
86 to apply to more complex practical engineering problems due to its excellent com-
87 putational efficiency and accuracy, which appears to be the significant limitation
88 associated with most of the existing numerical models for wave-structure interac-
89 tion.

90 **2 Theoretical Consideration**

91 *2.1 Problem definition*

92 Consider a monochromatic short-crested wave train propagating in the direction
93 of the positive x axis. A fixed vertical cylinder extends from the sea bottom to

Fig. 1. Problem definition.

94 above the free surface of the ocean along z axis. The origin is placed at the centre
 95 of the cylinder on the mean water surface (Fig. 1). The total velocity potential,
 96 the velocity potential of incident wave, the velocity potential of scattered wave,
 97 total wave number, wave number in x direction, wave number in y direction, wave
 98 frequency, water depth, the amplitude of incident wave, the cylinder radius, the
 99 time, and the gravitational acceleration are denoted respectively as Φ , Φ_I , Φ_S , k ,
 100 k_x , k_y , ω , h , A , a , t , and g .

101 Assuming the fluid to be inviscid, incompressible and the flow to be irrotational,
 102 the fluid motion can be described by a velocity potential Φ satisfying the Laplace
 103 equation

$$\nabla^2 \Phi = 0 \quad \text{in } \Omega, \quad (1)$$

104 subject to the linearized combined free surface boundary condition

$$\Phi_{,tt} + g\Phi_{,z} = 0 \quad \text{at } z = 0, \quad (2)$$

105 and the bottom condition

$$\Phi_{,z} = 0 \quad \text{at } z = -h, \quad (3)$$

106 where Ω is the fluid domain, and comma in the subscript designates partial deriva-
107 tive with respect to the variable following the comma.

108 To simplify the problem, the velocity potentials can be decomposed as

$$\Phi(x, y, z, t) = \phi(x, y)Z(z)e^{-i\omega t}, \quad (4)$$

$$\Phi_I(x, y, z, t) = \phi^I(x, y)Z(z)e^{-i\omega t}, \quad (5)$$

$$\Phi_S(x, y, z, t) = \phi^S(x, y)Z(z)e^{-i\omega t}, \quad (6)$$

109 where

$$Z(z) = \frac{\cosh k(z + h)}{\cosh kh}. \quad (7)$$

110 This procedure leads to the sea bottom condition being automatically satisfied, and
111 the linear free surface boundary condition is satisfied using the following dispersion
112 relationship

$$\omega^2 = gk \tanh kh. \quad (8)$$

Thus, the problem reduces to two-dimensional at the free surface. The function $\phi^S(x, y)$ is governed by the Helmholtz equation with the boundary condition at the interface of fluid and structure, and the radiation condition at infinity, the so-called Sommerfeld condition:

$$\nabla^2 \phi^S + k^2 \phi^S = 0, \quad (9)$$

$$\phi_{,n}^S = -\phi_{,n}^I \quad \text{on} \quad r = a, \quad (10)$$

$$\lim_{kr \rightarrow \infty} (kr)^{1/2} (\phi_{,r}^S - ik\phi^S) = 0, \quad (11)$$

where r is the radial axis, $i = \sqrt{-1}$, and n denotes the normal to the boundary.

The velocity potential of the linear short-crested incident wave travelling in the positive x direction is given by the real part of [5]:

$$\Phi_I = -\frac{igA}{\omega} Z(z) e^{i(k_x x - \omega t)} \cos(k_y y), \quad (12)$$

and the relationship of the total velocity potential, with the scattered, and the incident wave velocity potentials is

$$\Phi = \Phi_I + \Phi_S, \quad \phi = \phi^I + \phi^S. \quad (13)$$

Equations (9)-(11) constitute the governing equation and boundary conditions for the diffraction of short-crested waves by a vertical cylinder. After obtaining ϕ^S and Φ by solving the above boundary-value problem, the velocity, free surface elevation and the dynamic pressure can be calculated respectively from

$$\mathbf{v} = \nabla \Phi, \quad (14)$$

$$\eta = \frac{i\omega}{g}\phi, \quad (15)$$

$$p = -\rho\Phi_{,t}, \quad (16)$$

where ρ is the mass density of water.

2.2 Scaled boundary finite-element method

As shown in Fig. 1, (9) is valid in the whole fluid domain Ω with the fluid-structure interface Γ_c and infinity boundary Γ_∞ . If the velocity boundary is defined by Γ_v , we have

$$\phi_{,n}^S = \bar{v}_n, \quad \text{on } \Gamma_v, \quad (17)$$

where the overbar denotes a prescribed value. To derive a finite-element approximation, the weighted residual technique is applied by multiplying a weighting function w to (9) and (17), and integrating over the domain and the boundary.

Fig. 2. The coordinate definition of SBFEM.

$$\int_{\Omega} \nabla^T w \nabla \phi^S d\Omega - \int_{\Omega} w k^2 \phi^S d\Omega - \oint_{\Gamma} w \bar{v}_n d\Gamma = 0. \quad (18)$$

SBFEM defines the domain Ω by the scaling of a single piecewise-smooth curve S relative to a scaling centre (x_0, y_0) , which is chosen at the cylinder centre in this case. The circumferential coordinate s is anticlockwise along the curve S and the normalized radial coordinate ξ is a scaling factor, defined as 1 at curve S and 0 at the scaling centre. The whole solution domain Ω is in the range of $\xi_0 \leq \xi \leq \xi_1$ and $s_0 \leq s \leq s_1$. The two straight sections $s = s_0$ and $s = s_1$ are called side-faces. They coincide, if the curve S is closed. For bounded domain, $\xi_0 = 0$ and $\xi_1 = 1$; whereas, for unbounded domain, $\xi_0 = 1$ and $\xi_1 = \infty$. So the Cartesian coordinates are transformed to the scaled boundary coordinate ξ and s with the scaling equations

$$x = x_0 + \xi x_s(s), \quad y = y_0 + \xi y_s(s). \quad (19)$$

By employing SBFEM, an approximate solution of ϕ^S is sought as

$$\phi_A(\xi, s) = \mathbf{N}(s)\mathbf{a}(\xi), \quad (20)$$

where $\mathbf{N}(s)$ is the shape function, the vector $\mathbf{a}(\xi)$ is analogous to the nodal values same as in FEM. The radial function $a_j(\xi)$ represents the variation of the scattered wave potential in the radial axis ξ at each node j , and the shape function interpolates between the nodal potential values in the circumferential axis s .

By performing scaled boundary transformation, the operator ∇ can be expressed as [4, 16]

$$\nabla = \mathbf{b}_1(s) \frac{\partial}{\partial \xi} + \frac{1}{\xi} \mathbf{b}_2(s) \frac{\partial}{\partial s}, \quad (21)$$

where $\mathbf{b}_1(s)$ and $\mathbf{b}_2(s)$ are dependent only on the boundary definition

$$\mathbf{b}_1(s) = \frac{1}{|J|} \begin{Bmatrix} y_s(s),s \\ -x_s(s),s \end{Bmatrix}, \quad \mathbf{b}_2(s) = \frac{1}{|J|} \begin{Bmatrix} -y_s(s) \\ x_s(s) \end{Bmatrix}, \quad (22)$$

and $|J|$ is the Jacobian at the boundary

$$|J| = x_s(s)y_s(s),s - y_s(s)x_s(s),s. \quad (23)$$

From (14) and (21), the approximate velocity can be expressed as

$$\mathbf{v}_A(\xi, s) = \mathbf{B}_1(s)\mathbf{a}(\xi),_\xi + \frac{1}{\xi} \mathbf{B}_2(s)\mathbf{a}(\xi), \quad (24)$$

where

$$\mathbf{B}_1(s) = \mathbf{b}_1(s)\mathbf{N}(s), \quad \mathbf{B}_2(s) = \mathbf{b}_2(s)\mathbf{N}(s),_s. \quad (25)$$

155 The weighting function w can be chosen as the same shape function as (20) by

156 Galerkin approach

$$w(\xi, s) = \mathbf{N}(s)\mathbf{w}(\xi) = \mathbf{w}(\xi)^T\mathbf{N}(s)^T. \quad (26)$$

157 Using (20), (21), (25) and (26), (18) becomes

$$\begin{aligned} & \int_{\Omega} \left[\mathbf{B}_1(s)\mathbf{w}(\xi)_{,\xi} + \frac{1}{\xi}\mathbf{B}_2(s)\mathbf{w}(\xi) \right]^T \left[\mathbf{B}_1(s)\mathbf{a}(\xi)_{,\xi} + \frac{1}{\xi}\mathbf{B}_2(s)\mathbf{a}(\xi) \right] d\Omega \\ & - \int_{\Omega} k^2 \mathbf{w}(\xi)^T \mathbf{N}(s)^T \mathbf{N}(s) \mathbf{a}(\xi) d\Omega - \oint_{\Gamma} \mathbf{w}(\xi)^T \mathbf{N}(s)^T \bar{v}_n d\Gamma = 0, \end{aligned} \quad (27)$$

158 where the incremental volume is [4, 16]

$$d\Omega = |J|\xi d\xi ds. \quad (28)$$

159 Introducing the coefficient matrices

$$\mathbf{E}_0 = \int_S \mathbf{B}_1(s)^T \mathbf{B}_1(s) |J| ds, \quad (29)$$

$$\mathbf{E}_1 = \int_S \mathbf{B}_2(s)^T \mathbf{B}_1(s) |J| ds, \quad (30)$$

$$\mathbf{E}_2 = \int_S \mathbf{B}_2(s)^T \mathbf{B}_2(s) |J| ds, \quad (31)$$

$$\mathbf{M}_0 = \int_S \mathbf{N}(s)^T \mathbf{N}(s) |J| ds, \quad (32)$$

$$\mathbf{F}_s(\xi) = \mathbf{N}(s_0)^T (-\bar{v}_n(\xi, s_0)) |J(s_0)| + \mathbf{N}(s_1)^T (-\bar{v}_n(\xi, s_1)) |J(s_1)|. \quad (33)$$

160 The above integrals (29)-(32) can be computed element by element and assembled
 161 together for the entire boundary. Expanding (27) and integrating the terms contain-
 162 ing $\mathbf{w}(\xi)_{,\xi}$ by parts with respect to ξ using Green's theorem leads to

$$\begin{aligned}
 & \mathbf{w}(\xi_1)^T \left[\mathbf{E}_0 \xi_1 \mathbf{a}(\xi_1)_{,\xi} + \mathbf{E}_1^T \mathbf{a}(\xi_1) - \int_S \mathbf{N}(s)^T (\bar{v}_n(\xi_1, s)) \xi_1 ds \right] \\
 & - \mathbf{w}(\xi_0)^T \left[\mathbf{E}_0 \xi_0 \mathbf{a}(\xi_0)_{,\xi} + \mathbf{E}_1^T \mathbf{a}(\xi_0) + \int_S \mathbf{N}(s)^T (\bar{v}_n(\xi_0, s)) \xi_0 ds \right] \\
 & - \int_{\xi_0}^{\xi_1} \mathbf{w}(\xi)^T \left[\mathbf{E}_0 \xi \mathbf{a}(\xi)_{,\xi\xi} + (\mathbf{E}_0 + \mathbf{E}_1^T - \mathbf{E}_1) \mathbf{a}(\xi)_{,\xi} - \mathbf{E}_2 \frac{1}{\xi} \mathbf{a}(\xi) + k^2 \xi \mathbf{M}_0 \mathbf{a}(\xi) - \mathbf{F}_s(\xi) \right] d\xi \\
 & = 0.
 \end{aligned} \tag{34}$$

163 To satisfy all sets of weighting function $\mathbf{w}(\xi)$, the following conditions must be
 164 satisfied:

$$\mathbf{E}_0 \xi_1 \mathbf{a}(\xi_1)_{,\xi} + \mathbf{E}_1^T \mathbf{a}(\xi_1) = \int_S \mathbf{N}(s)^T (\bar{v}_n(\xi_1, s)) \xi_1 ds, \tag{35}$$

$$\mathbf{E}_0 \xi_0 \mathbf{a}(\xi_0)_{,\xi} + \mathbf{E}_1^T \mathbf{a}(\xi_0) = - \int_S \mathbf{N}(s)^T (\bar{v}_n(\xi_0, s)) \xi_0 ds, \tag{36}$$

$$\mathbf{E}_0 \xi^2 \mathbf{a}(\xi)_{,\xi\xi} + (\mathbf{E}_0 + \mathbf{E}_1^T - \mathbf{E}_1) \xi \mathbf{a}(\xi)_{,\xi} - \mathbf{E}_2 \mathbf{a}(\xi) + k^2 \xi^2 \mathbf{M}_0 \mathbf{a}(\xi) = \xi \mathbf{F}_s(\xi). \tag{37}$$

165 The equation (37) is the so-called scaled boundary finite-element equation. By in-
 166 troducing the shape function, the Helmholtz equation has been weakened in the
 167 circumferential direction, so that the governing partial differential equation is trans-
 168 formed to an ordinary matrix differential equation in radial direction. The rank of
 169 matrices $\mathbf{E}_0$, $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{M}_0$ and vector $\mathbf{a}(\xi)$ is m (where m is the number of nodes in
 170 the curve S). In the present study, the side-faces coincide so that the flow across the
 171 side-faces is equal and opposite, thus the term $\mathbf{F}_s(\xi)$ vanishes. Therefore, the final

governing equation (37) is a homogeneous second-order ordinary matrix differential equation of rank m .

The boundary conditions (10) and (11) are weakened in the form of (36) and (35), indicating the relation of integrated nodal flow on the boundary and the velocity potentials of the nodes. For the wave diffraction problem, $\xi_0 = 1$ on the boundary of cylinder and $\xi_1 = \infty$ at infinity.

2.3 Solution technique

For a vertical circular cylinder, we have,

$$x_s(s) = a \cos(s/a), \quad y_s(s) = a \sin(s/a). \quad (38)$$

From (19), (22), (23), (25) and (29)-(32), $x_s(s)_{,s}$, $y_s(s)_{,s}$, $\mathbf{b}_1(s)$, $\mathbf{b}_2(s)$, $|J|$, $\mathbf{B}_1(s)$, $\mathbf{B}_2(s)$, $\mathbf{E}_0$, $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{M}_0$ can be calculated accordingly. The following relations hold:

$$\mathbf{E}_1 = 0 \cdot \mathbf{I}, \quad \mathbf{E}_0^{-1} \mathbf{M}_0 = a^2 \mathbf{I}, \quad (39)$$

where $\mathbf{I}$ is the identity matrix of rank m .

Using (39), pre-multiplying both sides of (37) by $\mathbf{E}_0^{-1}$ and simplifying, we have

$$\zeta^2 \mathbf{a}(\zeta)_{,\zeta\zeta} + \zeta \mathbf{a}(\zeta)_{,\zeta} - \mathbf{E}_0^{-1} \mathbf{E}_2 \mathbf{a}(\zeta) + \zeta^2 \mathbf{a}(\zeta) = 0, \quad (40)$$

where

$$\zeta = ka\xi. \quad (41)$$

186 The equation (40) is the matrix form of Bessel's differential equation. Considering
 187 the Sommerfeld radiation condition (11), it is logical to select $H_{r_j}(\zeta)\mathbf{T}_j$ as a base
 188 solution of (40).

189 The solution form of $\mathbf{a}(\zeta)$ is then expressed as

$$\mathbf{a}(\zeta) = \sum_{j=1}^m c_j H_{r_j}(\zeta) \mathbf{T}_j, \quad (42)$$

190 where $\mathbf{T}_j$ are vectors of rank m , c_j are coefficients, $H_{r_j}(\zeta)$ are the Hankel functions
 191 of the first kind.

192 Substituting (42) into (40), and using the following properties of Hankel function

$$\zeta^2 H_{r_j}''(\zeta) = -\zeta^2 H_{r_j}(\zeta) + \zeta H_{r_j+1}(\zeta) - r_j H_{r_j}(\zeta) + r_j^2 H_{r_j}(\zeta), \quad (43)$$

$$\zeta H_{r_j}'(\zeta) = -\zeta H_{r_j+1}(\zeta) + r_j H_{r_j}(\zeta), \quad (44)$$

193 where the prime denotes the derivative with respect to ζ , and the double prime
 194 represents the second derivative with respect to ζ , we have

$$\sum_{j=1}^m (\mathbf{E}_0^{-1} \mathbf{E}_2 - r_j^2 \mathbf{I}) \mathbf{T}_j \cdot c_j H_{r_j}(\zeta) = 0. \quad (45)$$

195 For any $c_j H_{r_j}(\zeta)$, (45) yields

$$(\mathbf{E}_0^{-1} \mathbf{E}_2 - r_j^2 \mathbf{I}) \mathbf{T}_j = 0. \quad (46)$$

196 Let λ_j be the eigenvalues of $\mathbf{E}_0^{-1}\mathbf{E}_2$, then $r_j = \sqrt{\lambda_j}$, and $\mathbf{T}_j$ are the eigenvectors of
 197 $\mathbf{E}_0^{-1}\mathbf{E}_2$.

198 Since the Sommerfeld radiation condition (11) or (35) has been satisfied by the
 199 Hankel function, we now only consider the body boundary condition (36) of the
 200 circular cylinder

$$\mathbf{E}_0 ka \sum_{j=1}^m c_j H'_{r_j}(ka) \mathbf{T}_j = - \left[\int_S \mathbf{N}(s)^T \mathbf{N}(s) ds \right] \bar{\mathbf{v}}_n^S, \quad (47)$$

201 where $\bar{\mathbf{v}}_n^S$ is the vector of nodal normal velocity of scattered wave on the body
 202 boundary.

203 For a circular cylinder,

$$\mathbf{E}_0 = \frac{1}{a} \int_S \mathbf{N}(s)^T \mathbf{N}(s) ds. \quad (48)$$

204 Define

$$\mathbf{T} = [\mathbf{T}_1, \mathbf{T}_2, \dots, \mathbf{T}_m], \quad (49)$$

$$\mathbf{C} = [c_1, c_2, \dots, c_m]^T, \quad (50)$$

$$\mathbf{H} = \text{diag}[H_{r_1}(ka\xi), H_{r_2}(ka\xi), \dots, H_{r_m}(ka\xi)], \quad (51)$$

$$\mathbf{H}_d = \text{diag}[H'_{r_1}(ka), H'_{r_2}(ka), \dots, H'_{r_m}(ka)], \quad (52)$$

205 where “diag” denotes a diagonal matrix with the elements in the square brackets on
 206 the main diagonal; then $\mathbf{C}$ can be solved from (47) as

$$\mathbf{C} = -\frac{1}{k}\mathbf{H}_d^{-1}\mathbf{T}^{-1}\bar{\mathbf{v}}_n^S. \quad (53)$$

The solution of $\mathbf{a}(\zeta)$ is obtained as

$$\mathbf{a}(\zeta) = \mathbf{THC} = -\frac{1}{k}\mathbf{TH}_h\mathbf{T}^{-1}\bar{\mathbf{v}}_n^S, \quad (54)$$

where $\mathbf{H}_h = \text{diag}[H_{r_1}(ka\xi)/H'_{r_1}(ka), H_{r_2}(ka\xi)/H'_{r_2}(ka), \dots, H_{r_m}(ka\xi)/H'_{r_m}(ka)]$.

Using (4), (5), (10), (12) and (17), $\bar{\mathbf{v}}_n^S$ can be easily determined on the body boundary. From (20) and (54), the approximation of scattered velocity potential can be obtained in the whole domain. The total velocity potential can then be calculated by (6) and (13). All the other physical properties such as elevation, pressures can be calculated accordingly.

In this paper, the boundary Γ_c is discretized into N three-noded quadratic elements in circumferential direction. Due to the symmetry of the physical problem, only half of the circumference is discretized.

The total force, per unit length in the direction of wave propagation is

$$\frac{dF_x}{dz} = -a \int_0^{2\pi} p \cdot \cos(\theta) d\theta = 2\pi a R(k_x, k_y, k, a) \cdot \rho g A \cdot Z(z) e^{-i\omega t}, \quad (55)$$

where the function $R(k_x, k_y, k, a)$ is a dimensionless parameter of $\frac{dF_x}{dz}$ without the term $\rho g A \cdot Z(z) e^{-i\omega t}$.

The function $R(k_x, k_y, k, a)$ determines the first-order total horizontal force in x direction on the cylinder, F_x , which can be obtained by integrating Eq. (55) with respect to z ,

$$F_x = \int_{-h}^0 \frac{dF_x}{dz} dz = 2\pi a R(k_x, k_y, k, a) \cdot \rho g A e^{-i\omega t} \cdot \tanh(kh)/k. \quad (56)$$

223 Following traditional concept, the component of force per unit of wave height in
 224 phase with the particle acceleration of the incident waves is called an effective
 225 inertia coefficient C_M and that in phase with the particle velocity is termed an
 226 effective linear drag coefficient C_D . Then the force is written as follows:

$$\text{Re} \left(\frac{dF_x}{dz} \right) = \rho \pi a^2 \left(C_M \dot{U} + \omega C_D U \right), \quad (57)$$

227 where U is the velocity of the incident waves at the origin of the cylinder in its
 228 absence.

229 Using (12), (55) and (57), we have

$$C_M = -\frac{2R_i}{k_x a}, \quad C_D = \frac{2R_r}{k_x a}, \quad (58)$$

230 where R_r and R_i are the real and imaginary parts of $R(k_x, k_y, k, a)$.

231 **3 Results and Discussion**

232 The present SBFEM scheme is verified against (1) analytical solutions for plane
 233 waves [11]; and (2) analytical solutions for short-crested waves [17]. Figs. 3-6 are
 234 the comparisons of wave run-up between the present SBFEM results and the ana-
 235 lytical solutions shown in [11]. As can be seen in the figures, for small ka ($= 0.5$),
 236 only two elements discretized over the cylinder boundary are sufficient to yield
 237 good agreement between the present SBFEM result and the analytical solution. As

Fig. 3. The run-up of progressive wave on a circular cylinder extending the entire sea depth ($ka = 0.5$).

238 ka increases from 0.5 to 5.0, the convergence of the SBFEM scheme is clearly seen
 239 as the number of elements is increased. Even at $ka = 5.0$, satisfactory numerical
 240 results were obtained when merely 6 elements were used.

241 The run-up on the circular cylinder due to short-crested waves is shown in Figs. 7-
 242 11. In these examples the total incident wave number is the same. Again, excellent
 243 agreement with the solutions of Zhu [17] is found where the SBFEM results with
 244 only 4 elements gave almost identical solutions to the analytical. Fig. 7 shows that
 245 the largest run-up occurs near the front face of the cylinder ($\theta/\pi \approx 1$) when the
 246 incident waves have infinite wavelength ($k_y = 0$) in the direction perpendicular
 247 to the wave propagation, i.e., plane waves. The other extreme case, shown in Fig.
 248 11, represents standing waves when the incident waves have infinite wavelength
 249 ($k_x = 0$) in the direction of propagation. In order to show the difference of the
 250 run-up on the cylinder between the plane waves and short-crested waves, Fig. 12 is

Fig. 4. The run-up of progressive wave on a circular cylinder extending the entire sea depth ($ka = 1.0$).

Fig. 5. The run-up of progressive wave on a circular cylinder extending the entire sea depth ($ka = 3.0$).

Fig. 6. The run-up of progressive wave on a circular cylinder extending the entire sea depth ($ka = 5.0$).

an overlay of the plots of the present SBFEM results with 4 finite elements shown in Figs. 7-11. It is observed in Fig. 12 that the elevation on the cylinder surface is quite different when the incident waves become short-crested. As the incident wave becomes more short-crested, the elevation near the front face of the cylinder ($\theta/\pi \approx 1$) decreases monotonically, however, no similar attribute is observed on the lee side of the cylinder ($\theta/\pi \approx 0$).

Table 1 shows the comparisons of the effective inertia, and the effective linear drag coefficients and total forces. Excellent computational efficiency and accuracy of the present numerical scheme are further demonstrated by examining the hydrodynamic forces. It is apparent that only four elements can lead to quite accurate results while six elements produce almost identical results as the analytical solution [17]. As pointed out by Zhu [17], both the effective inertia coefficient, C_M , and the effective drag coefficient, C_D , are generally functions of k_x and k_y . However,

Fig. 7. Comparison of short-crested wave run-up on a cylinder with radius $a = 1.0 \text{ m}$ and total incident wave number $k = \sqrt{2} \text{ m}^{-1}$ ($k_x = \sqrt{2} \text{ m}^{-1}$, $k_y = 0 \text{ m}^{-1}$).

Fig. 8. Comparison of short-crested wave run-up on a cylinder with radius $a = 1.0 \text{ m}$ and total incident wave number $k = \sqrt{2} \text{ m}^{-1}$ ($k_x = 1.2 \text{ m}^{-1}$, $k_y = \sqrt{0.56} \text{ m}^{-1}$).

Fig. 9. Comparison of short-crested wave run-up on a cylinder with radius $a = 1.0 \text{ m}$ and total incident wave number $k = \sqrt{2} \text{ m}^{-1}$ ($k_x = 1.0 \text{ m}^{-1}$, $k_y = 1.0 \text{ m}^{-1}$).

Fig. 10. Comparison of short-crested wave run-up on a cylinder with radius $a = 1.0 \text{ m}$ and total incident wave number $k = \sqrt{2} \text{ m}^{-1}$ ($k_x = \sqrt{0.56} \text{ m}^{-1}$, $k_y = 1.2 \text{ m}^{-1}$).

Fig. 11. Comparison of short-crested wave run-up on a cylinder with radius $a = 1.0$ m and total incident wave number $k = \sqrt{2} \text{ m}^{-1}$. ($k_x = 0 \text{ m}^{-1}$, $k_y = \sqrt{2} \text{ m}^{-1}$)

Fig. 12. Wave run-up on a cylinder with radius $a = 1.0$ m and total incident wave number $k = \sqrt{2} \text{ m}^{-1}$: Plane waves vs. Short-crested waves.

264 as can be seen in Table 1, for a fixed ka , C_M and C_D are clearly unchanged for all
265 the cases with the variation of k_x and k_y . It should be noted that the digits with the
266 superscript $*$ are reversed in [17] and the digits with the superscript $\#$ are erroneous
267 in [17].

Table 1

Comparison of effective inertia, effective drag coefficients and total forces.

k_x (m ⁻¹)	k_y (m ⁻¹)	k (m ⁻¹)	a (m)	C_M			C_D			$ 2\pi a R $ (m)		
				Zhu, 1993	4 elements	6 elements	Zhu, 1993	4 elements	6 elements			
1.0	1.0	$\sqrt{2}$	1.0	0.8824	0.8825	0.8824	0.2271	0.2271	0.2271	2.8626*	2.8627	2.8626
$\sqrt{0.56}$	1.2	$\sqrt{2}$	1.0	0.8824	0.8825	0.8824	0.2271	0.2272	0.2271	2.1421*	2.1423	2.1422
1.2	$\sqrt{0.56}$	$\sqrt{2}$	1.0	0.8824	0.8824	0.8824	0.2271	0.2271	0.2271	3.4351	3.4351	3.4351
$\sqrt{2}$	0.0	$\sqrt{2}$	1.0	0.8824	0.8825	0.8824	0.2271	0.2271	0.2271	4.0483	4.0485	4.0483
1.0	1.0	$\sqrt{2}$	2.0	0.2354 [#]	0.2359	0.2354	-0.2398	-0.2390	-0.2398	4.2228*	4.2203	4.2228
$\sqrt{0.56}$	1.2	$\sqrt{2}$	2.0	0.2354 [#]	0.2362	0.2354	-0.2398	-0.2386	-0.2398	3.1601*	3.1573	3.1601
1.2	$\sqrt{0.56}$	$\sqrt{2}$	2.0	0.2354 [#]	0.2348	0.2354	-0.2398	-0.2406	-0.2398	5.0674	5.0702	5.0674
$\sqrt{2}$	0.0	$\sqrt{2}$	2.0	0.2354 [#]	0.2359	0.2354	-0.2398	-0.2390	-0.2398	5.9720	5.9685	5.9720

Further calculations revealed that C_M and C_D per unit length are invariants for a fixed ka . To demonstrate this, the variation of C_M and C_D are plotted against the variation of the ratio k_y/k_x in Figs. 13 and 14, for four different values of k_x , while $k_x a$ held as a constant, corresponding to the following cases: 1) $k_x = 0.8 \text{ m}^{-1}$, $a = 2.5 \text{ m}$; 2) $k_x = 1.0 \text{ m}^{-1}$, $a = 2.0 \text{ m}$; 3) $k_x = 1.6 \text{ m}^{-1}$, $a = 1.25 \text{ m}$; 4) $k_x = 2.0 \text{ m}^{-1}$, $a = 1.0 \text{ m}$. As can be clearly seen in the figures, four curves coincide with each other. This indicates that coefficients C_M and C_D are invariants of ka . This attribute was also shown in [17], but with the misprint of k_x shown as k in the text as well as in the legend. This is not a surprising result as a short-crested wave can occur from the oblique interaction of two travelling plane-waves, thus linear forces from short-crested wave interaction with a cylinder can be effectively obtained by the superposition of two plane-waves. Therefore coefficients C_M and C_D are physically invariants of ka . However, the total force $|2\pi a R|$ for the above four cases are different, as shown in Fig. 15, the total force decreases as the incident wave becomes more short-crested. It is noted that all the SBFEM results presented in Figs. 13-15 are calculated with 6 discretized elements on the cylinder boundary.

Figs. 16-21 show the co-amplitude and co-phase lines for long- and short-crested incident waves diffracted by a circular cylinder with a radius of 1.0 m. The amplitude is nondimensionalized as $|\eta|/A$ and the phases are between $[-\pi \sim +\pi]$. k_x is fixed as $k_x = 1.0 \text{ m}^{-1}$, and k_y has different values: 0, 0.5, to 1.0 m^{-1} respectively. It is apparent that the wave patterns of diffracted short-crested waves are more complicated. The thick lines in all the co-phase plots show the sudden change of the phase values from $-\pi$ to π . From these results, one can easily draw the conclusion that as the incident waves become more short-crested, the amplitude of the diffracted waves in the rear region become smaller and the region of the co-amplitude line upstream reduces. The amphidromic points come forth for short-

Fig. 13. Variation of the effective inertial coefficient C_M vs the ratio k_y/k_x at $k_x a = 2$.

Fig. 14. Variation of the effective drag coefficient C_D vs the ratio k_y/k_x at $k_x a = 2$.

294 crested incident waves and the region like plane wave area shrinks as the incident
 295 waves become more short-crested.

Fig. 15. Variation of the total force vs the ratio k_y/k_x at $k_x a = 2$.

It is worth noting that, the computation times (recorded on a 2GHz Pentium IV PC and MATLAB 7.1) of the scaled boundary finite-element method solutions are very small. For all the cases presented in this paper, accurate results are obtained in less than 3 s, a clear demonstration that it significantly outperforms any current finite-element or boundary-element method for similar problems. Such computational efficiency and accuracy ensure great potential of direct application of the present method to many engineering problems especially in ocean engineering.

4 Conclusions

The semi-analytical scaled boundary finite-element method has been successfully applied to solve the diffraction of short-crested waves incident on a circular cylinder. In contrast to the conventional boundary-element method, which has been widely applied to wave-structure interaction problem in unbounded domain, the

Fig. 16. The curve of equal amplitude (co-amplitude) for the incident waves with longitudinal and lateral wave numbers $k_x = 1.0 \text{ m}^{-1}$ and $k_y = 0.0 \text{ m}^{-1}$, respectively.

308 new technique requires no help from any fundamental solutions. A reduction of
 309 one in the spatial dimension is achieved with this procedure, since only the body
 310 boundary is discretized with surface finite-elements. Excellent computational effi-
 311 ciency and accuracy of the present scaled boundary finite-element method has been
 312 demonstrated, as the governing equations are solved analytically in the radial di-
 313 rection. In solving the short-crested wave problem, the present numerical method
 314 is shown to reproduce the analytical solution for all the physical properties includ-
 315 ing wave run-up, effective inertia and drag force coefficients, and total force very
 316 accurately and at very low computational cost. The method holds promise in solv-
 317 ing more practical ocean engineering problems.

Fig. 17. The curve of equal phase (co-phase) for the incident waves with longitudinal and lateral wave numbers $k_x = 1.0 \text{ m}^{-1}$ and $k_y = 0.0 \text{ m}^{-1}$, respectively.

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322 References

- 323 [1] M.C. Au, C.A. Brebbia, Diffraction of water waves for vertical cylinders using
 324 boundary elements, Appl. Math. Model. 7 (2) (1983) 106-114.
- 325 [2] S.K. Chakrabarti, W.A. Tam, Interaction of waves with large vertical cylinder, J. Ship

Fig. 18. The curve of equal amplitude (co-amplitude) for the incident waves with longitudinal and lateral wave numbers $k_x = 1.0 \text{ m}^{-1}$ and $k_y = 0.5 \text{ m}^{-1}$, respectively.

- 326 Res. 19 (1) (1975) 23-33.
- 327 [3] S.K. Chakrabarti, Steady drift force on vertical cylinder - viscous vs. potential, Appl.
328 Ocean Res. 6 (2) (1984) 73-82.
- 329 [4] A.J. Deeks, L. Cheng, Potential flow around obstacles using the scaled boundary finite-
330 element method, Int. J. Numer. Meth. Fluids 41 (2003) 721-741.
- 331 [5] R.A. Fuchs, On the theory of short-crested oscillatory waves, Gravity Waves, National
332 Bureau of Standards Circular No. 521, Department of Commerce, USA, (1952) 187-
333 200.
- 334 [6] D.R. Fuhrman, P.A. Madsen, Short-crested waves in deep water: a numerical
335 investigation of recent laboratory experiments, J. Fluid Mech. 559 (2006) 391-411.
- 336 [7] D. Lesnic, L. Elliott, D.B. Ingham, Boundary element methods for determining the

Fig. 19. The curve of equal phase (co-phase) for the incident waves with longitudinal and lateral wave numbers $k_x = 1.0 \text{ m}^{-1}$ and $k_y = 0.5 \text{ m}^{-1}$, respectively.

- 337 fluid velocity in potential flow, Eng. anal. bound. elem. 11 (3) (1993) 203-213.
- 338 [8] B. Li, L. Cheng, A.J. Deeks, Wave diffraction by vertical cylinder using the scaled
339 boudary finite element method, WCCM VI & APCOM'04, Beijing, China, 2004.
- 340 [9] P.A. Martin, A.G. Dixon, The scattering of regular surface waves by a fixed, half-
341 immersed, circular cylinder, Appl. Ocean Res. 5 (1) (1983) 13-23.
- 342 [10] R.C. MacCamy, R.A. Fuchs, Wave forces on piles: a diffraction theory, Tech. Memo.
343 No. 69, U.S. Army Board, U.S. Army Corp. of Eng. 1954.
- 344 [11] C.C. Mei, The applied dynamics of ocean surface waves, World Scientific, Singapore,
345 1989.
- 346 [12] S. Neelamani, V. Sundar, C.P. Vendhan, Dynamic pressure distribution on a cylinder
347 due to wave diffraction, Ocean Eng. 16 (4) (1989) 343-353.

Fig. 20. The curve of equal amplitude (co-amplitude) for the incident waves with longitudinal and lateral wave numbers $k_x = 1.0 \text{ m}^{-1}$ and $k_y = 1.0 \text{ m}^{-1}$, respectively.

- 348 [13] V. Sundar, S. Neelamani, C.P. Vendhan, Diffracted wave field and dynamic pressures
349 around a vertical cylinder, *Ocean Eng.* 17(1-2) (1990) 125-154.
- 350 [14] N.J. Shankar, T. Balendra, C.E. Soon, Wave loads on large vertical cylinders: A design
351 method, *Ocean Eng.* 11 (1) (1984) 65-85.
- 352 [15] C.P. Tsai, D.S. Jeng, J.R.C. Hsu, Computations of the almost highest short-crested
353 waves in deep water, *Appl. Ocean Res.* 16 (6) (1994) 317-326.
- 354 [16] J.P. Wolf, *The scaled boundary finite element method*, John Wiley & Sons Ltd,
355 Chichester, England, 2003.
- 356 [17] S. Zhu, Diffraction of short-crested waves around a circular cylinder, *Ocean Eng.* 20
357 (4) (1993) 389-407.
- 358 [18] S. Zhu, G. Moule, Numerical calculation of forces induced by short-crested waves on

Fig. 21. The curve of equal phase (co-phase) for the incident waves with longitudinal and lateral wave numbers $k_x = 1.0 \text{ m}^{-1}$ and $k_y = 1.0 \text{ m}^{-1}$, respectively.

359 a vertical cylinder of arbitrary cross-section, Ocean Eng. 21 (7) (1994) 645-662.

360 [19] S. Zhu, P. Satravaha, Second-order wave diffraction forces on a vertical circular
361 cylinder due to short-crested waves, Ocean Eng. 22 (2) (1995) 135-189.