Realization of a weak generalized spin measurement

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Abstract

The weak Zeno effect for a driven two-level system examined by Cresser *et al.* [Opt. Commun. **264**, 352 (2006)] was based on a particular form of weak measurement. Here we show how such a weak measurement can be realized by entangling the spin to be measured with a pointer spin.

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I. INTRODUCTION

In [1] a measurement master equation was derived describing the evolution of a quantum system subjected to a sequence of measurements. This master equation was based on the assumption that effect of the measurements on the system could be described by specific elements of a probability operator measure (POM) [2] and by specific Kraus operators [3] given below. Application of this equation to a two-level system driven by an coherent electromagnetic field revealed a range of interesting trajectories from Rabi precession through to quantum Zeno type suppressed evolution [4] as the type of observation was varied from weak, inaccurate, measurements to strong frequent accurate measurements. For strong measurements at frequent intervals a random telegraph results, with the atomic inversion remaining at $+1$ or -1 for nearly all the time with measurement-induced sudden quantum jumps between the two values. In between the two extremes is an unusual and interesting phenomenon referred to in [1] as the weak Zeno effect. This shows a random telegraph behavior with many aborted jumps between successful jumps.

The physical mechanism of a measuring procedure that would give the required POM elements and Kraus operators leading to these interesting effects was not given in [1], but it was indicated that a possible mechanism might involve coupling the spin under examination to one of its neighbours. In this paper, we explore this idea in detail and show that such a procedure can, at least in principle, give the precise form of the required POM elements and Kraus operators.

II. KRAUS OPERATORS

In the derivation [1] of the master equation describing the evolution of a driven two-level quantum system, or qubit, subjected to a sequence of imperfect generalized measurements of the energy level, it was assumed that these measurements could be represented by the following elements of a probability operator measure (POM):

$$\hat{\pi}_1 = (1 - p)|1\rangle\langle 1| + p|2\rangle\langle 2| \quad (1)$$

$$\hat{\pi}_2 = (1 - p)|2\rangle\langle 2| + p|1\rangle\langle 1|. \quad (2)$$

Here $|2\rangle$ and $|1\rangle$ are the upper and lower energy states of the qubit and the sum of the two POM elements is the unit operator as required [2]. Two limiting cases are for $p = 0$ and

$p = 1/2$, representing a very accurate, or strong, measurement and a very indeterminate, or weak, measurement respectively. In the latter case the POM elements are proportional to the unit operator and the measurement tells us nothing about the state of the qubit. Intermediate values of p can be regarded as the probability that there is an error, for example, the probability that an attempted measurement of the energy of state $|2\rangle$ yields the energy of state $|1\rangle$. To find the measurement master equation it was necessary to know the state of the qubit after each measurement. This can be found in general by use of the effect, or Kraus, operators $\hat{A}_i$ [Kraus], which change the pre-measurement density operator $\hat{\rho}$ to

$$\frac{\hat{A}_i \hat{\rho} \hat{A}_i^\dagger}{\text{Tr}(\hat{\rho} \hat{\pi}_i)} \quad (3)$$

after the measurement. The Kraus operators are such that

$$\hat{\pi}_i = \hat{A}_i^\dagger \hat{A}_i \quad (4)$$

but this property alone is insufficient to determine them uniquely. In [1] the forms

$$\hat{A}_1 = \sqrt{(1-p)}|1\rangle\langle 1| + \sqrt{p}|2\rangle\langle 2| = \hat{A}_1^\dagger \quad (5)$$

$$\hat{A}_2 = \sqrt{(1-p)}|2\rangle\langle 2| + \sqrt{p}|1\rangle\langle 1| = \hat{A}_2^\dagger \quad (6)$$

were chosen, from the infinite number of possible Kraus operators, on the basis of their simplicity. In the limit of a very weak measurement, $p = 1/2$ and the action of these operators does not affect the state of the system. Kraus operators of a similar form were also employed in [5].

The question arises as to what sort of measurement scheme can possibly yield the POM elements (1) and (2) and the Kraus operators (5) and (6). A quantum optical way in which similar operators might be realized by means of cavity quantum electrodynamics was suggested in [6]. The system we wish to study here is where the qubit is represented by a spin-half system in a magnetic field in the z -direction. In this case the POM elements for a direct accurate measurement of the z -component of spin $\hat{I}_z$ would be $|1\rangle\langle 1|$ and $|2\rangle\langle 2|$. A property of (5) and (6) is that if the system is initially in state $|1\rangle$, say, then even an inaccurate measurement does not affect the state of the system. This excludes modeling an inaccurate measurement by a mechanism involving measuring the z' -component of $\hat{I}_z$, where z' is not accurately aligned with z . In this paper we develop a physical model that realizes the required POM elements and Kraus operators by a more indirect measurement.

Here we couple for a short time another spin-half particle representing a “pointer” spin to the spin-half particle representing the qubit to be measured. After this, the pointer spin is measured.

III. POINTER SPIN

The problem of accurately measuring the orientation of a single spin, while very important for the development of a practical quantum computer, is not the problem we address here. Here we assume that this can be done and the problem we address is that of performing a controllable generalized measurement of spin described by the Kraus operators (5) and (6). We study this in terms of a model that couples an ancillary spin-half particle to the qubit to be measured with the ancillary spin being subsequently measured. In this way, the ancillary spin acts as a pointer. For the interaction with the pointer not to affect the spin under examination if it is in the state $|1\rangle$ or $|2\rangle$, these states must be eigenstates of the interaction Hamiltonian.

Consider two spin-half nuclei, such as a proton and a carbon-13 nucleus, with different gyromagnetic ratios. The two spins will have a scalar coupling with interaction energy $2\pi\hat{\mathbf{I}}\cdot\hat{\mathbf{S}}$, where $\hat{\mathbf{I}}$ and $\hat{\mathbf{S}}$ are the spin vectors for the proton and carbon nucleus respectively and J depends on the distance between the spins [7]. A strong magnetic field is applied in the z -direction and a radio-frequency field is resonant with the proton. We wish to use the carbon-13 nucleus as the pointer spin. In a strong field the precessional frequencies of the two spins are very different so only the contribution to $\hat{\mathbf{I}}\cdot\hat{\mathbf{S}}$ from $\hat{I}_z\hat{S}_z$ is static with the other terms $\hat{I}_x\hat{S}_x$ and $\hat{I}_y\hat{S}_y$ being averaged to zero very quickly. The term $\hat{I}_z\hat{S}_z$ can also be averaged quickly to zero by dressing the carbon nucleus with a strong, reasonably broad-band noise-modulated radiofrequency field centred on the carbon resonance frequency but well removed from the proton resonance frequency. The resultant rapid carbon spin flipping averages the value of $\hat{I}_z\hat{S}_z$ to zero, effectively switching off the coupling entirely. This decoupling procedure works well and is a standard technique in nuclear magnetic resonance [8].

We shall work in the doubly rotating reference frame [9] by transforming from the laboratory frame with the unitary transformation $\exp(-i\hat{I}_z\omega_p t)\exp(-i\hat{S}_z\omega_c t)$ where ω_p and ω_c are the resonance frequencies of the proton and carbon spins. In order to save using subscripts on the kets, we shall label the proton eigenstates of $\hat{I}_z$ with numbers, that is $|1\rangle$ and $|2\rangle$,

and use letters, for example $|\pm y\rangle$, to label the pointer states in the doubly rotating frame. In this frame with the coupling switched off, that is, with the decoupling field switched on, the proton undergoes Rabi precession between states $|1\rangle$ and $|2\rangle$. To approximate an instantaneous measurement, we let the Rabi precession be sufficiently slower than the motion induced by the coupling for us to ignore the Rabi precession for the brief time that the coupling is switched on and then off again. We assume that we can prepare the initial state of the carbon (pointer) spin, which will be $|-y\rangle$, in a very short time by first measuring the z -component of this spin and then applying a hard $\pi/2$ pulse about the axis appropriate to the result obtained. The procedure is to switch on the coupling, quickly put the pointer spin in the initial state $|-y\rangle$, allow time for the pointer states to become entangled with the proton states, measure $\hat{S}_x$ of the pointer and then switch the coupling off again.

A. Accurate measurement

We consider firstly the simplest case where the measurement is perfect. When the coupling is switched on, the Hamiltonian in the doubly-rotating frame is $2\pi J\hat{I}_z\hat{S}_z$ where, as mentioned above, we are allowing the Rabi precession to be sufficiently slow to be ignored for the period τ that the coupling is switched on. From the time-dependent Schrödinger equation, the unitary time evolution operator for the interaction period τ is given by

$$\exp(-i2\pi J\hat{I}_z\hat{S}_z\tau) = \cos(\pi J\tau/2) - i4\hat{I}_z\hat{S}_z \sin(\pi J\tau/2). \quad (7)$$

This result can be obtained by expanding the exponential series and remembering that $\hat{I}_z^2 = \hat{J}_z^2 = 1/4$ for spin-half particles [9]. For a perfect measurement we choose the interaction time to be $\tau = (2J)^{-1}$.

We let the initial state of the pointer spin be $|-y\rangle$ and the proton be in a general superposition state $\alpha|2\rangle + \beta|1\rangle$. After the interaction time $(2J)^{-1}$ the combined state will have evolved to $\alpha|A\rangle + \beta|B\rangle$, where

$$|A\rangle = 2^{-1/2}[1 - i4\hat{I}_z\hat{S}_z]|2\rangle|-y\rangle \quad (8)$$

$$|B\rangle = 2^{-1/2}[1 - i4\hat{I}_z\hat{S}_z]|1\rangle|-y\rangle. \quad (9)$$

Using the relations $|\pm y\rangle = 2^{-1/2}(|z\rangle \pm i|-z\rangle)$ and $|\pm x\rangle = 2^{-1/2}(|z\rangle \pm |-z\rangle)$ [11], we obtain

$$|A\rangle = \exp(-i\pi/4)|2\rangle 2^{-1/2}(|z\rangle + |-z\rangle)$$

$$= \exp(-i\pi/4)|2\rangle|x\rangle \quad (10)$$

$$\begin{aligned} |B\rangle &= \exp(i\pi/4)|1\rangle 2^{-1/2}(|z\rangle - |-z\rangle) \\ &= \exp(i\pi/4)|1\rangle|-x\rangle. \end{aligned} \quad (11)$$

The final combined state is thus the entangled state

$$|F\rangle = \alpha \exp(-i\pi/4)|2\rangle|x\rangle + \beta \exp(i\pi/4)|1\rangle|-x\rangle. \quad (12)$$

We note that this result is in accord with the result of an ideal quantum measurement procedure described by Wigner [10]. If, for example, the proton is initially in the state $|2\rangle$, that is if $\alpha = 0$, then the interaction does not change this state but does change the state of the pointer in a way dependent on the proton state. The next step is to make a strong measurement of $\hat{S}_x$, the x -component of the pointer spin. As mentioned earlier, we shall assume that this can be done and so will not discuss this further here except to say that if it is easier to measure $\hat{S}_z$, the component in the direction of the strong magnetic field, then we could apply a suitable $\pi/2$ pulse to the pointer spin immediately preceding the measurement. A measurement of $\hat{S}_x$ has two possible results, corresponding to measurement events with POM elements $|-x\rangle\langle -x|$ and $|x\rangle\langle x|$. As the proton is not measured, the associated POM contains just one element $\hat{1}_p$, the unit operator acting on the proton state space. The probability for the first of the two possible pointer spin measurement events is thus

$$\text{Tr}(|F\rangle\langle F|\hat{1}_p|-x\rangle\langle -x|) = \text{Tr}_p(\langle -x|F\rangle\langle F|-x\rangle) \quad (13)$$

where Tr is the trace over both the proton and pointer states and Tr_p is the trace over the proton states. From (12) we then find that the probability of the first pointer spin measurement event is

$$\text{Tr}_p(|\beta|^2|1\rangle\langle 1|) = \text{Tr}_p[(\alpha|2\rangle + \beta|1\rangle)(\alpha^*\langle 2| + \beta^*\langle 1|)|1\rangle\langle 1|], \quad (14)$$

so the effective POM element $\hat{\pi}_1$ for this event is $|1\rangle\langle 1|$, which is expression (1) with $p = 0$. Likewise we find that the effective POM element $\hat{\pi}_2$ for the second measurement event is $|2\rangle\langle 2|$. The resulting probabilities for these measurement events are precisely the same as we would obtain if we were to measure the z -component of spin of the proton in state $\alpha|2\rangle + \beta|1\rangle$ directly.

Depending on whether the measurement projects the state $|F\rangle$ onto the pointer state $|-x\rangle$ or $|+x\rangle$, the unnormalized state of the proton after the measurement will be either

$$\langle -x|F\rangle = \beta \exp(i\pi/4)|1\rangle \quad (15)$$

or

$$\langle x|F\rangle = \alpha \exp(-i\pi/4)|2\rangle. \quad (16)$$

If we wished to, we could eliminate the phase factors $\exp(\pm i\pi/4)$ by applying a $\pi/2$ pulse about the x -axis to the pointer spin before measurement. However, there is no need to do this as these factors cancel in calculating the post-measurement density operator as in [1]. Then appropriate Kraus operators for use with the density operator are

$$\hat{A}_1 = |1\rangle\langle 1| = \hat{A}_1^\dagger \quad (17)$$

$$\hat{A}_2 = |2\rangle\langle 2| = \hat{A}_2^\dagger, \quad (18)$$

which can be easily checked. For example the action of $\hat{A}_1$ on the state $\alpha|2\rangle + \beta|1\rangle$ produces the state (15) without the phase factor. These Kraus operators are in agreement with (5) and (6) with $p = 0$.

B. Weak measurement

To model an inaccurate measurement, we let the interaction time τ deviate from $(2J)^{-1}$ so that now

$$\pi J\tau/2 = \pi/4 - \epsilon \quad (19)$$

where ϵ can be positive or negative. The time evolution operator is now

$$\begin{aligned} \exp(-i\pi\hat{I}_z\hat{S}_z)\exp(i4\hat{I}_z\hat{S}_z\epsilon) &= (\cos\epsilon + i4\hat{I}_z\hat{S}_z\sin\epsilon) \\ &\times \exp(-i\pi\hat{I}_z\hat{S}_z). \end{aligned} \quad (20)$$

so the state $|A\rangle$ given by (10) is now replaced by

$$|A'\rangle = (\cos\epsilon + i4\hat{I}_z\hat{S}_z\sin\epsilon)|A\rangle. \quad (21)$$

Using $2\hat{S}_z|\pm x\rangle = |\mp x\rangle$ gives, from (10),

$$|A'\rangle = \exp(-i\pi/4)|2\rangle(\cos\epsilon|x\rangle + i\sin\epsilon|-x\rangle). \quad (22)$$

Similarly we find

$$|B'\rangle = \exp(i\pi/4)|1\rangle(\cos\epsilon| -x\rangle - i\sin\epsilon|x\rangle) \quad (23)$$

with the final entangled state $|F'\rangle$ being $\alpha|A'\rangle + \beta|B'\rangle$. A measurement of $\hat{S}_x$ then projects the final state onto $|\mp x\rangle$, leaving the proton either in the state

$$\begin{aligned} \langle -x|F'\rangle &= \exp(i\pi/4)\beta\cos\epsilon|1\rangle + i\exp(-i\pi/4)\alpha\sin\epsilon|2\rangle \\ &= \exp(i\pi/4)(\beta\cos\epsilon|1\rangle + \alpha\sin\epsilon|2\rangle) \end{aligned} \quad (24)$$

or in the state

$$\langle x|F'\rangle = \exp(-i\pi/4)(\alpha\cos\epsilon|2\rangle + \beta\sin\epsilon|1\rangle). \quad (25)$$

The probability with which the first measurement event occurs is given by (13) with $|F\rangle$ replaced by $|F'\rangle$, which we can write as

$$\text{Tr}_p[(\alpha|2\rangle + \beta|1\rangle)(\alpha^*\langle 2| + \beta^*\langle 1|)\hat{\pi}_1] \quad (26)$$

where

$$\hat{\pi}_1 = \cos^2\epsilon|1\rangle\langle 1| + \sin^2\epsilon|2\rangle\langle 2|. \quad (27)$$

Similarly we find that the effective POM element for the second measurement event is

$$\hat{\pi}_2 = \sin^2\epsilon|1\rangle\langle 1| + \cos^2\epsilon|2\rangle\langle 2|. \quad (28)$$

The probability for the first event is $|\alpha|^2\sin^2\epsilon + |\beta|^2\cos^2\epsilon$ and for the second event is $|\alpha|^2\cos^2\epsilon + |\beta|^2\sin^2\epsilon$. Clearly these POM elements are of the required form (1) and (2) with $p = \sin^2\epsilon$.

In determining the Kraus operators we can ignore the phase factors $\exp(\pm i\pi/4)$ as before and obtain

$$\hat{A}_1 = \cos\epsilon|1\rangle\langle 1| + \sin\epsilon|2\rangle\langle 2| \quad (29)$$

$$\hat{A}_2 = \cos\epsilon|2\rangle\langle 2| + \sin\epsilon|1\rangle\langle 1|. \quad (30)$$

Again, these can be easily checked, for example the action of $\hat{A}_1$ in (29) on the state $\alpha|2\rangle + \beta|1\rangle$ produces the unnormalised state (24) without the phase factor. These Kraus operators are in agreement with (5) and (6) with $p = \sin^2\epsilon$. It is clear that if $\epsilon = \pi/4$ then the Kraus operators are proportional to the unit operator and so the measurement does not affect the state. From (19) this is so when the coupling time τ is zero so no entanglement has built up

between the proton and the pointer spin. We have the same situation when $\epsilon = -3\pi/4$. This is when the coupling time is such that the entanglement first builds up and then reduces to zero.

IV. CONCLUSION

We have shown that the POM elements and Kraus operators necessary to give the interesting effects, such as the weak Zeno effect, described in [1] can be achieved, at least in principle, by coupling an ancillary spin to the main spin, that is, the spin under examination. This ancillary spin acts as a pointer indicating the state of the main spin. By adjusting the coupling time appropriately, the measurement can be varied from a strong von Neumann measurement to a very weak measurement that barely affects the main spin.

As viewed from the doubly rotating reference frame, the measurement process has a straightforward interpretation. The pointer spin initially points along the negative y -axis of this frame. From (10) and (11), after a time $\tau = (2J)^{-1}$, the pointer rotates to point in the positive x -direction if the main spin is in the “up” state $|2\rangle$ or in the negative x -direction if the main spin is in the down state $|1\rangle$. If the main spin is in a superposition state, the states of the two spins become entangled and a particular outcome of the measurement leaves the main spin in the associated state. The unimportant phase factors in (10) and (11) have a simple interpretation in terms of the rotating pointer. If the coupling is allowed to remain on for four times the above period, the pointer will have rotated through 2π back to the negative y -axis and will have returned to its original state except for a phase factor of $\exp(\pm i\pi) = -1$. This is due to the spinor nature of the pointer. If we had used a spin-1 particle, for example, as the pointer, the phase factor would be different.

A coupling time of $\tau = (2J)^{-1}$ as above leads to a strong, von Neumann type of measurement. To obtain a controllable weak measurement, we simply reduce this period so the main spin and pointer states are not so strongly entangled. It is worth noting that a weak measurement does not require a weak coupling. Indeed the stronger the coupling the better for the purposes used in [1], because this allows a more rapid measurement to be made. Reduction of the coupling time leads to measurements described by (1), (2), (5) and (6) as required in [1] where p is determined by the chosen coupling time. Overall, the method presented here is a reasonably straightforward way of realizing, at least in principle,

a controllable weak measurement.

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