Open Quantum Systems with Linear Dynamics: Optimal Unravellings for Optimal Feedback Control

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ABSTRACT

Quantum feedback control is the control of the dynamics of a quantum system by feeding back (in real time) the results of monitoring that system. For systems with linear dynamics, the control problem is amenable to exact analysis. In these cases, the quantum system is equivalent to a stochastic system of classical phase-space variables with linear drift and constant diffusion, and with a measured current (e.g. a homodyne photocurrent) linear in the system variables. However, the classical evolution is constrained in order to represent valid quantum evolution. We quantify this in terms of a linear matrix inequality (LMI) relating the drift and diffusion (a sort of zero temperature fluctuation-dissipation theorem), and another LMI for the covariance matrix of the possible conditioned states (i.e. under all possible monitoring schemes consistent with the master equation). For *manipulable* systems (i.e. where the experimenter has arbitrary control over the parameters in a Hamiltonian linear in the system variables) the covariance of the conditioned state is all that is needed to calculate the effectiveness of the feedback. In this case the double optimization problem reduces to a semidefinite program, which can be solved efficiently in general. We illustrate this with an example drawn from quantum optics.

Keywords: Quantum control, fluctuation-dissipation theorem, linear matrix inequalities, optimal control, quantum feedback, open quantum systems, quantum trajectory

1. INTRODUCTION

Classical feedback control is a large and well-established area of engineering.¹ Quantum feedback control, by contrast, is a relatively small field, but is rapidly developing (see Ref.² for a recent review of theory) especially with the notable success of recent experiments in the field.^{3,4} Just as quantum mechanics is much more difficult to model than classical mechanics, so quantum feedback control theory is in general much more difficult than its classical counterpart. There are cases however for which the quantum theory is no more difficult (although it is more constrained) than the classical theory. In particular for classical systems with linear dynamics in phase space, and with Gaussian noise, classical optimal feedback control theory can be applied to good effect to the analogous quantum systems. This formalism was first introduced by Belavkin,⁵ and Doherty and Jacobs and co-workers^{6,7} have developed its physical interpretation and application.

The theory in this paper develops this earlier work in a number of ways. First (Sec. 3.2), we consider an arbitrary number of degrees of freedom, and give a simple linear matrix inequality (LMI) to separate evolutions which are valid quantum mechanically from those that are not. Second (Sec. 2.3), we consider a very large class of “unravellings” (ways to monitor the system) compatible with the specified evolution. Third (Sec. 3.3), we determine another simple LMI to separate those long-time conditioned states that are compatible with some such unravelling from those that are not compatible with any such unravelling. Fourth (Sec. 4), we show in detail that any of our quantum evolution equations has an equivalent classical model, and that this classical model corresponds exactly to the Heisenberg equations of motion, and the Heisenberg input-output relations, for the system. Fifth (Sec. 5), we show that for some control problems, the LMIs we have introduced earlier

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can be used to provide a simple solution. More specifically, we can find a solution to the problem of how well quantum feedback control can minimize the stationary expectation value of a positive quadratic observable (such as harmonic oscillator energy), allowing optimization both of the unravelling (the way the system is monitored) and of the control “force” (in phase space) — hence the title of this paper.

2. OPEN QUANTUM SYSTEMS

2.1. Master Equations

While it is generally believed that a closed quantum mechanical system obeys Schrödinger’s equation, it is often of interest to consider a physical system that is weakly coupled to a large environment. Commonly the specific state of the environment is not of interest or not accessible to experiments and it is useful to average over the environmental states to derive an equation of motion for the system alone. In general, such an equation is not of the form of Schrödinger’s equation for a pure state $|\psi(t)\rangle$, but rather an irreversible equation for a mixed state $\rho(t)$. For example, because the system may exchange energy with the environment the equation for ρ may be dissipative. If this evolution can be represented by a Markovian evolution equation for ρ (that is, an autonomous differential equation for ρ) then the requirement that this is linear and preserves the positivity of the state matrix ρ is a strong restriction. (In fact for physical reasons the evolution is required to correspond to a *completely positive* map.⁸) These equations are termed master equations and take the general (Lindblad) form⁹

$$\dot{\rho} = -i[\hat{H}, \rho] + \mathcal{D}[\hat{\mathbf{e}}]\rho. \quad (1)$$

Here $\hat{H}$, the system Hamiltonian, is an Hermitian operator, while $\hat{\mathbf{e}}$ is a *vector* of operators $(\hat{e}_1, \dots, \hat{e}_K)^T$. (We use T to indicate the transpose of a vector.) For strict mathematical rigour¹⁰ these operators should be bounded, but in practice this is not necessary. The superoperator $\mathcal{D}$, which takes $\hat{\mathbf{e}}$ as its argument, is defined (in a generalization of the notation of Ref.¹¹) by

$$\mathcal{D}[\hat{\mathbf{e}}] \equiv \sum_{k=1}^K \mathcal{D}[\hat{e}_k] \quad (2)$$

where for a single operator $\hat{e}$, the action of $\mathcal{D}$ on an arbitrary operator ρ is defined by

$$\mathcal{D}[\hat{e}]\rho \equiv \hat{e}\rho\hat{e}^\dagger - (\hat{e}^\dagger\hat{e}\rho + \rho\hat{e}^\dagger\hat{e})/2. \quad (3)$$

Lindblad master equations may be derived as approximate descriptions of many physical systems, such as the dephasing of a nuclear magnetic moment in a nuclear magnetic resonance (NMR),¹² or the spontaneous emission of an atom or the decay of light out of a Fabry-Perot cavity.¹³

A theme of this paper is to identify the best analogy between quantum and classical systems. Since the state matrix is used to predict the statistics of measurement results, it is more analogous to a classical probability distribution than to a point in phase space. This analogy has been proposed before in a quantum control context,⁶ and has recently gained adherents in quantum foundations community (see for example Refs.^{15–17}). The Lindblad master equation is therefore analogous to the Fokker-Planck equation (or, more generally, a differential Chapman-Kolmogorov equation¹⁸) for the propagation of the probability distribution of a classical system with stochastic dynamics. We will see later that this analogy can be made exact for linear systems.

2.2. Quantum Trajectories

The most familiar measurements in quantum mechanics are projective measurements, where the distribution of measurement outcomes is given by the diagonal element of the state matrix ρ in the appropriate basis (or the sum of such elements for non-unit-rank projectors). The system state given the result is updated by applying the projection operator corresponding to the measurement outcome. In this paper we are concerned instead with projective measurements made on the *environment* to which the system is coupled. For weak couplings, such as allow the derivation of a Lindblad master equation, it is possible to make such projective measurements continually, on a time scale short compared to the characteristic dynamical times of the system, but long compared to the correlation time for the relevant environment operators.⁹ This makes it possible to obtain

information about the system, while not changing its average dynamics as described by the master equation. The new *conditional* dynamics of the system (taking into account the results obtained from the measurement on the environment) is called a quantum trajectory.¹⁹ In this paper we are concerned only with continuous trajectories, but jump-like trajectories also occur.^{11, 19} Although our treatment below is mostly formal, continuous quantum trajectories apply to realistic models of measurement on quantum systems particularly in quantum optics and atomic physics.¹⁹

The most general continuous quantum trajectory equation was derived recently by Wiseman and Diósi.²⁰ For a system obeying the master equation (1) it is an Itô stochastic differential equation known as a stochastic master equation²¹:

$$d\rho_c = -i[\hat{H}, \rho_c]dt + \mathcal{D}[\hat{s}]\rho_c dt + \mathcal{D}[\hat{c}]\rho_c dt + d\mathbf{z}^\dagger(t)\Delta_c\hat{c}\rho_c + \rho_c\Delta_c\hat{c}^\dagger d\mathbf{z}(t). \quad (4)$$

Note that here the $\dagger$ indicates transpose (T) of the vector and Hermitian adjoint of its components. Here the c subscript emphasizes that the state ρ_c for the system is *conditioned* on the continuous monitoring. The K -vector of operators $\hat{e}$ has been split into two smaller vectors, a K^u -vector $\hat{s}$ and a K^o -vector $\hat{c}$. Here $K = K^u + K^o$ and u and o stand for unobserved and observed respectively. This is because we have assumed that only the environment(s) giving rise to the irreversible term $\mathcal{D}[\hat{c}]$ are being monitored. That is why the final, conditioning, term in Eq. (4) contains only the operators $\hat{c}$. In this term we are using the following convention: for an arbitrary operator $\hat{o}$,

$$\Delta_c\hat{o} \equiv \hat{o} - \langle\hat{o}\rangle_c, \quad (5)$$

where $\langle\hat{o}\rangle_c = \text{Tr}[\rho_c\hat{o}]$. Also, we have introduced $d\mathbf{z}$, a K^o -vector $(dz_1, \dots, dz_{K^o})^T$ of infinitesimal complex Wiener increments.¹⁸ It satisfies $E[\mathbf{z}] = 0$, where E denotes expectation value, and

$$d\mathbf{z}d\mathbf{z}^\dagger = I dt, \quad (6)$$

where I is the identity matrix (here $K^o \times K^o$). If $K^u = 0$, so that all output channels are observed, the stochastic master equation (4) preserves the purity of states and so can be replaced by a stochastic Schrödinger equation.^{19, 20} The stochastic master equation (4) can be thought of as a quantum filtering equation,²² analogous to the classical Kushner-Stratonovich equation²³ (of which the Kalman filter¹ is a special case).

From this equation it is evident that the *unconditional* equation, obtained by averaging over the noise processes, is the master equation (1). There are *infinitely* many different trajectory equations, or *unravellings*, compatible with this master equation. These different unravellings are parametrized by a *symmetric* $K^o \times K^o$ matrix U which specifies the correlations

$$d\mathbf{z}d\mathbf{z}^T = U dt. \quad (7)$$

These correlations can be set independently of those in Eq. (6), apart from the restriction on U that²⁰

$$\tilde{U} = \frac{1}{2} \begin{pmatrix} I + \text{Re}[U] & \text{Im}[U] \\ \text{Im}[U] & I - \text{Re}[U] \end{pmatrix} \geq 0. \quad (8)$$

That is, $\tilde{U}$ must have all eigenvalues non-negative.

The measurement results upon which the evolution of ρ_c is conditioned is a vector of complex functions of time given by

$$\mathbf{J}^\dagger dt = \langle\hat{c}^\dagger + \hat{c}^\dagger U\rangle_c dt + d\mathbf{z}^\dagger \quad (9)$$

Since in optics measurements of this sort give rise to photocurrents, we will call $\mathbf{J}$ a current. Note that U may be time-varying, and may even depend upon past values of $\mathbf{J}$. The last case is what is called *adaptive* measurements,²⁴ and these have been realized experimentally in order to increase the accuracy of phase estimation.⁴ However we will restrict ourselves here to Markovian continuous unravellings, in which U is a constant. The Markovicity of the unravellings is a strictly stronger restriction than the Markovicity of the unconditioned evolution, the Lindblad master equation (1).

As noted above, the increments $d\mathbf{z}$ have mean zero when averaged over all the measurement outcomes, thus recovering Eq. (1). In closed loop feedback, however, the Hamiltonian is controlled to be a function of the

measurement output at earlier times. In that case the average evolution is not that of Eq. (1) and indeed it is not describable by a master equation at all in general (although see Ref.²⁵). For the purposes of feedback control it is convenient to consider a real measurement record with uncorrelated noises, as opposed to the complex current $\mathbf{J}$ with (in general) correlated noises. We therefore define a $2K^o$ -vector of real currents $\mathbf{y}(t)$ as follows

$$\mathbf{y} = \tilde{U}^{-1/2} \begin{pmatrix} \text{Re}[\mathbf{J}] \\ \text{Im}[\mathbf{J}] \end{pmatrix} = \tilde{U}^{1/2} \begin{pmatrix} 2\text{Re}[\langle \hat{c} \rangle_c] \\ 2\text{Im}[\langle \hat{c} \rangle_c] \end{pmatrix} + d\mathbf{w} \quad (10)$$

Here $d\mathbf{w}$ is a vector of *uncorrelated* real Wiener increments. That is, the only non-zero second-order moments are $d\mathbf{w}d\mathbf{w}^T = I dt$. These increments are analogous to the innovations²³ of a Kushner-Stratonovich equation which “unravels” a Fokker-Planck equation.

3. LINEAR DYNAMICS AND LINEAR MATRIX INEQUALITIES

3.1. Gaussian States

We are interested in systems of N degrees of freedoms, with the n th described by a position operator $\hat{q}_n$ and momentum operator $\hat{p}_n$. These satisfy the canonical commutation relations

$$[\hat{q}_n, \hat{p}_m] = i\delta_{nm}, \quad [\hat{q}_n, \hat{q}_m] = 0 = [\hat{p}_n, \hat{p}_m]. \quad (11)$$

Note that we are using units for which $\hbar = 1$. We can define a vector of operators

$$\hat{\mathbf{x}} = (\hat{q}_1, \hat{p}_1, \dots, \hat{q}_N, \hat{p}_N)^T \quad (12)$$

in terms of which the commutation relations may be written

$$[\hat{x}_n, \hat{x}_m] = i\Sigma_{nm} \quad (13)$$

where $\Sigma = \bigoplus_{n=1}^N \sigma_n$ is a $(2N) \times (2N)$ matrix. It is a direct sum of symplectic matrices

$$\sigma_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (14)$$

Note that Σ is unitary and real so that $\Sigma^{-1} = \Sigma^T$, and also that $\Sigma^T = -\Sigma$ so that $i\Sigma$ is Hermitian.

It is possible to define Gaussian states of quantum particles, and these arise naturally for systems with linear dynamics (as will be defined in Sec. 3.2). Like their classical counterparts, they are determined by their first and second order moments alone.⁹ In the following we review their properties. It is convenient to define a covariance matrix, which in the quantum case must be symmetrized:

$$V_{nm} = (\langle \Delta \hat{x}_n \Delta \hat{x}_m \rangle + \langle \Delta \hat{x}_m \Delta \hat{x}_n \rangle) / 2. \quad (15)$$

Here, as before, $\Delta \hat{x}_n \equiv \hat{x}_n - \langle \hat{x}_n \rangle$. Not all positive real symmetric matrices are valid quantum covariance matrices; the identity

$$V_{nm} + i\Sigma_{nm}/2 = \text{Tr}[\rho \Delta \hat{x}_n \Delta \hat{x}_m] \quad (16)$$

and positivity of ρ guarantee that V satisfies the linear matrix inequality (LMI)

$$V + i\Sigma/2 \geq 0. \quad (17)$$

As well as being necessary for V to be a valid covariance matrix it turns out that it is also sufficient. It also implies that V (if finite) must be strictly positive. This can be proven by considering the real $\mathbf{r}$ and imaginary $i\mathbf{h}$ parts of an eigenvector of $V + i\Sigma/2$. Then $(\mathbf{r}^T, \mathbf{h}^T)$ is a row-eigenvector of the matrix $\begin{pmatrix} V & -\Sigma \\ \Sigma & V \end{pmatrix}$. This matrix cannot be positive if V has a zero eigenvalue because Σ has full rank.

The LMI satisfied by quantum covariance matrices should be understood as an uncertainty principle expressing the fact that one cannot simultaneously specify the value of conjugate observables in quantum mechanics. Considering the case of single particle (and reinstating $\hbar$) this LMI implies the well known Heisenberg-Robertson inequality for the position and momentum variance $\langle (\Delta q)^2 \rangle \langle (\Delta p)^2 \rangle \geq \hbar^2/4$.

3.2. Linear Dynamics: Quantum Fluctuation-Dissipation Theorem

As indicated above, in this paper we consider linear systems and Gaussian states in order to best make the comparison to optimal control. To guarantee that Gaussian states remain Gaussian under time evolution we will restrict to Hamiltonians that are at most quadratic in the canonical position and momentum operators, $\hat{\mathbf{x}}$. To obtain a linear map between time-dependent classical inputs to the system and the measured signal the time-dependent terms in $\hat{H}$ will be required to be linear in $\hat{\mathbf{x}}$. That is,

$$\hat{H} = \frac{1}{2} \hat{\mathbf{x}}^T G \hat{\mathbf{x}} - \hat{\mathbf{x}}^T \Sigma B \mathbf{u}(t) \quad (18)$$

The input signal $\mathbf{u}(t)$ is a real $2N$ -vector. Hermiticity of the Hamiltonian is guaranteed if G is real and symmetric and B is real. The final restriction on the Lindblad master equation (1) is that the operators $\hat{\mathbf{e}}$ are also linear in the $\hat{\mathbf{x}}$. That is, there is a $(2N) \times K^u$ matrix $\tilde{S}$ such that

$$\hat{\mathbf{s}} = \tilde{S}^T \hat{\mathbf{x}}, \quad (19)$$

and a $(2N) \times K^o$ matrix $\tilde{C}$ such that

$$\hat{\mathbf{e}} = \tilde{C}^T \hat{\mathbf{x}}. \quad (20)$$

Using the (unconditioned) master equation (1) we find the equations of motion for the first and second order moments

$$\begin{aligned} \frac{d\langle \hat{\mathbf{x}} \rangle}{dt} &= A \langle \hat{\mathbf{x}} \rangle + B \mathbf{u}(t) \\ \frac{dV}{dt} &= AV + VA^T + D. \end{aligned} \quad (21)$$

Here $A = \Sigma G + A^u + A^o$ and $D = D^u + D^o$, where

$$A^u = \Sigma \text{Im} [\tilde{S} \tilde{S}^\dagger] \quad (22)$$

$$D^u = \Sigma \text{Re} [\tilde{S} \tilde{S}^\dagger] \Sigma^{-1}. \quad (23)$$

and A^o and D^o are defined with $\tilde{C}$ replacing $\tilde{S}$. It is worth noting that the drift matrix A is an arbitrary real matrix. To see this recall that G is arbitrary real symmetric, whereas $\text{Im} [\tilde{S} \tilde{S}^\dagger + \tilde{C} \tilde{C}^\dagger]$ is an arbitrary real skew-symmetric matrix, and Σ is invertible. Also, since Σ is unitary, the diffusion matrix D is a positive real symmetric matrix.

It is possible to derive a LMI relating D and A by noting that

$$\Sigma^{-1} D \Sigma + \frac{i}{2} [\Sigma^{-1} A - (\Sigma^{-1} A)^T] = \tilde{S} \tilde{S}^\dagger + \tilde{C} \tilde{C}^\dagger \geq 0. \quad (24)$$

Thus

$$D - i(A\Sigma - \Sigma^T A^T)/2 \geq 0. \quad (25)$$

As well as being a necessary condition, this LMI is also a sufficient condition on D for a given drift matrix A . That is, it guarantees that $V(t) + i\Sigma/2 \geq 0$ for all $t > 0$ provided it is true at $t = 0$. This is because the invertibility of Σ allows us to construct a Lindblad master equation explicitly from the above equations given a valid A and D matrix.

The sufficiency of this LMI also can be proven from its relation to the (more general) matrix inequality for completely positive maps (CPMs) for Gaussian states.^{26,27} If the covariance matrix of the Gaussian state transforms according to

$$V \rightarrow M^T V M + N \quad (26)$$

then this is a CPM if and only if the matrices M and N satisfy

$$2N + i\Sigma - iM^T \Sigma M \geq 0 \quad (27)$$

For a continuous CPM we have

$$M = 1 + A^T dt \quad (28)$$

$$N = D dt \quad (29)$$

Then the matrix inequality (27) becomes the LMI (25).

The LMI (25) can be interpreted as a generalized fluctuation-dissipation relation for open quantum systems. A dissipative system is one that loses energy so that the evolution is strictly stable. That is, the real parts of the eigenvalues of A must be negative. Any strictly stable A must have a non-zero value of $A\Sigma + \Sigma^T A^T$ and in this case the LMI (25) places a lower bound on the fluctuations D about equilibrium.

By contrast, it is easy to verify that the contribution to A arising from the Hamiltonian $\hat{H}$ places no restriction on D . This is because energy conserving dynamics cannot give rise to dissipation. To see this note that $\Sigma^{-1}(\Sigma G)\Sigma = G\Sigma = -(\Sigma G)^T$ and so ΣG is similar to the negative of its transpose. If λ is an eigenvalue then so is $-\lambda$, so therefore it is impossible for all the eigenvalues of ΣG to have negative real parts. That is, $A = \Sigma G$ cannot be a strictly stable system.

For systems where A is strictly stable, the system variance tends to a constant V^{ss} defined by

$$AV^{ss} + V^{ss}A^T + D = 0. \quad (30)$$

Of course the system mean will depend upon the driving $Bu(t)$, but for an undriven strictly stable system $\langle \mathbf{x} \rangle^{ss} = \mathbf{0}$.

3.3. Linear Conditional Dynamics

In terms of the matrix $\tilde{C}$ we can write the vector of measurement outputs as

$$y dt = C \langle \hat{\mathbf{x}} \rangle dt + d\mathbf{w} \quad (31)$$

where

$$C = \tilde{U}^{1/2} \begin{pmatrix} 2\text{Re}[\tilde{C}] \\ 2\text{Im}[\tilde{C}] \end{pmatrix}. \quad (32)$$

The noise process $d\mathbf{w}$ is most closely analogous to the innovation of a Kalman filter.^{1,23} Given the measurement outputs $\mathbf{y}(t)$ it is possible to propagate the quantum state ρ_c conditioned on $\mathbf{y}(t)$ using the stochastic master equation (4). Because of the linearity assumptions we have made, this conditioned state is analogous to the *a posteriori* probability distribution propagated by the Kalman filter equations for classical linear systems. We will make this analogy more quantitative in the following.

Once again it is possible to find from (4) an evolution equation for the first and second order moments. In order to compute the covariance matrix V it is necessary to use the Itô product rule.¹⁸ We find

$$d\langle \hat{\mathbf{x}} \rangle_c = [A\langle \hat{\mathbf{x}} \rangle_c + B\mathbf{u}(t)] dt + (V_c C^T + \Gamma^T) d\mathbf{w} \quad (33)$$

$$\dot{V}_c = AV_c + V_c A^T + D - (V_c C^T + \Gamma^T)(C V_c + \Gamma) \quad (34)$$

where

$$\Gamma = \tilde{U}^{1/2} \begin{pmatrix} +\text{Im}[\tilde{C}] \\ -\text{Re}[\tilde{C}] \end{pmatrix} \Sigma^{-1}. \quad (35)$$

The stochastic term in $d\langle \mathbf{x} \rangle$ and the final term in $\dot{V}_c$ describe the conditioning on the measured current $\mathbf{y}$.

Note that the equation for V_c is deterministic and independent of the measurement results. It is known as a matrix Riccati equation.²³ The final term causes a reduction in uncertainty about the system state (that is, a reduction in the eigenvalues of V_c). In the long time limit, V_c may tend to a unique steady-state. If such a steady-state exists, it follows from the positivity of the final term in Eq. (34) that

$$D + AV_c^{ss} + V_c^{ss}A^T \geq 0. \quad (36)$$

That is, for any unravelling U having a steady-state conditioned variance, that variance will satisfy Eq. (36).

This LMI was derived earlier by Wiseman and Vaccaro²⁸ as a necessary condition for V_c^{ss} to be a possible stationary conditioned covariance matrix for some unravelling. It was there also shown that, with the addition of the Gaussian state LMI (17), one has a *sufficient* condition on V_c^{ss} . Since Eq. (36) pays no heed to the distinction between monitored ($\hat{c}$) and unmonitored ($\hat{s}$) channels, this sufficient condition applies only if all channels can be monitored. It is also important to note that the construction of the proof in Ref.²⁸ does not guarantee that V_c^{ss} is a unique steady-state solution. For instance, in the complete absence of any dynamics, Eq. (36) places no restriction on V_c^{ss} , but there is no unique steady-state solution. However, the typical systems we are interested in do have a unique V_c^{ss} (for any given unravelling). For example, all strictly stable systems have this property.²⁹ We will address this question further in future work.

It is also worth noting that in general the LMI (36) for the conditioned states is strictly stronger than the LMI which says that the conditioned state must be more certain than the unconditioned state:

$$V^{ss} - V_c^{ss} \geq 0, \quad (37)$$

where here the unconditioned state obeys Eq. (30). For an example of this, see Refs.^{28,30} and also Sec. 5.1.

4. EQUIVALENT CLASSICAL SYSTEMS

4.1. Process and Measurement Noise

The existence of an equivalent classical model for a quantum system with linear dynamics (including measurements) was discussed in Ref.¹⁴ Here we develop the theory in full generality for the first time.

To begin we consider just the unconditioned moment equations (21). It is simple to verify that these also propagate the mean and covariance matrix for the following noise-driven classical system

$$d\mathbf{x} = A\mathbf{x}dt + B\mathbf{u}(t)dt + \bar{S}d\mathbf{v}_p^u(t) + \bar{C}d\mathbf{v}_p^o. \quad (38)$$

Here $\bar{S}$ is a $(2N) \times P^u$ matrix (not necessarily square) such that

$$\bar{S}\bar{S}^T = \Sigma \text{Re}[\bar{S}\bar{S}^\dagger]\Sigma^T = D^u, \quad (39)$$

and $d\mathbf{v}_p^u(t)$ is a P^u -vector of real white noise processes satisfying $d\mathbf{v}_p^u d\mathbf{v}_p^{uT} = I$. Here the subscript p indicates that this is what engineers call *process* noise.¹ Note that P^u can always be chosen to be the minimum of $2N$ and K^u . Similarly, $\bar{C}$ is a $(2N) \times P^o$ matrix such that

$$\bar{C}\bar{C}^T = \Sigma \text{Re}[\bar{C}\bar{C}^\dagger]\Sigma^T = D^o, \quad (40)$$

and $d\mathbf{v}_p^o(t)$ is a P^o -vector of uncorrelated real white noise processes. Again, P^o can be chosen to be not greater than $2N$ or K^o . Because $d\mathbf{v}_p^u(t)$ and $d\mathbf{v}_p^o(t)$ are Gaussian noise processes, the solution of Eq. (38) is a Gaussian random variable, provided the initial conditions are Gaussian. In the case of a strictly stable system, it is Gaussian in the long-time limit irrespective of the initial conditions. Thus to prove equivalence to the quantum model of (21) all that is needed is to calculate the evolution of the mean and co-variance matrix of $\mathbf{x}$, which is easily done using the Itô calculus.

Although linear dynamics of a quantum system are consistent with a noise-driven classical system, there are quantum features in the classical equations. Specifically, the noise power D cannot be specified independently of the system dynamics A since otherwise the uncertainty relations $V(t) + i\Sigma/2 \geq 0$ would not be satisfied for all times. Although we have set $\hbar = 1$, the occurrence of the matrix of commutators Σ in the LMI's (17) and (25) indicates the quantum mechanical origin of these relations.

Turning now to the conditioned equations (33) and (34), these correspond to the Kalman filter equations for a classical system with process dynamics as in Eq. (38) and with a measured output

$$\mathbf{y} = C\mathbf{x} + d\mathbf{v}_m/dt \quad (41)$$

The measurement noise $\mathbf{v}_m(t)$ is a $2K^o$ -vector of uncorrelated real white noise processes. However, there are in general correlations between the measurement noise and the process noise introduced by the measurement channels. This is unusual in classical systems, but is treated by Whittle.²³ The correlations are given by

$$\bar{C}d\mathbf{v}_p^o d\mathbf{v}_m^T = dt\Gamma^T \quad (42)$$

This claimed correspondence can be demonstrated in an elementary manner as follows. Since $d\mathbf{v}_p^o$ is not independent of $d\mathbf{v}_m$ it is necessary first to split the last noise term into

$$\bar{C}d\mathbf{v}_p^o = \Gamma^T d\mathbf{v}_m + \bar{C}'d\mathbf{v}_p', \quad (43)$$

where $\bar{C}'\bar{C}'^T = D^o - \Gamma^T\Gamma$, and where $d\mathbf{v}_p'$ is normalized process noise that is independent of the measurement noise. This allows us to rewrite Eq. (38) as

$$d\mathbf{x} = A\mathbf{x}dt + B\mathbf{u}(t)dt + \bar{S}d\mathbf{v}_p^u + \Gamma^T(\mathbf{y} - C\mathbf{x})dt + \bar{C}'d\mathbf{v}_p'. \quad (44)$$

Let the mean and covariance of $\mathbf{x}$ be $\langle \mathbf{x} \rangle$ and V . Let $\mathbf{x}_1$ be an estimate for $\mathbf{x} + d\mathbf{x}$, taking into account the dynamical effect of $\mathbf{y}$ on $\mathbf{x}$. Then using the Itô calculus one finds that this estimate is

$$\mathbf{x}_1 = \langle \mathbf{x} \rangle + A\langle \mathbf{x} \rangle dt + B\mathbf{u}(t)dt + \Gamma^T d\mathbf{w} \quad (45)$$

and that it has a covariance matrix

$$V_1 = V + AVdt + VA^Tdt + Ddt - (\Gamma^T CV + VC^T\Gamma + \Gamma^T\Gamma)dt \quad (46)$$

This updated estimate for $\mathbf{x} + d\mathbf{x}$ does not take into account the fact that $\mathbf{y}$ depends upon $\mathbf{x}$ and so yields information about it. Thus we can form another (independent) estimate

$$\mathbf{x}_2 = C^{-1}\mathbf{y} \quad (47)$$

with a covariance

$$V_2 = (C^T C dt)^{-1} \quad (48)$$

Strictly, $\mathbf{x}_2$ as defined is an estimate for $\mathbf{x}$, not $\mathbf{x} + d\mathbf{x}$. However, the infinite noise in this estimate (48) means that the distinction is irrelevant. Now, we can optimally combine these two estimates to obtain a new estimate $\mathbf{x}_3$ using the standard statistical formulae¹

$$V_3 = (V_1^{-1} + V_2^{-1})^{-1} \quad (49)$$

$$\mathbf{x}_3 = V_3(V_1^{-1}\mathbf{x}_1 + V_2^{-1}\mathbf{x}_2) \quad (50)$$

Taylor-expanding these expressions and identifying $V_3 = V + dV$ and $\mathbf{x}_3 = \langle \mathbf{x} \rangle + d\langle \mathbf{x} \rangle$ yields the Kalman Filter equations (33) and (34).

Note that this classical model, as well as having a dynamical noise process associated with any damping in the system also has a dynamical noise related to the output matrix C . This includes noise in the conjugate observable to the measurement output $\mathbf{y}$, as discussed below. The existence in this classical model of a process noise with a magnitude related to C and having correlations with the measurement noise arises from the underlying quantum mechanics. Note again that the matrix of commutators arises in both the process noise power and the noise correlations. This strange dependence of the fictional underlying classical systems on the measurement sensitivity is a manifestation of the so-called measurement back-action.

In typical engineering situations the measurement sensitivity associated with C is independent of the dynamical noise. But here in the covariance matrix equation the drift and diffusion matrices gained several terms that were associated with the measurement coupling. There are some cancellations in the expression for $\dot{V}$ when written in terms of the drift A^u and diffusion D^u for the unobserved system that may make the situation clearer. It is easy to show that

$$\begin{aligned} \dot{V} = & (\Sigma G + A^u)V + V(\Sigma G + A^u)^T + D^u - VC^T CV \\ & + \Sigma C^T C \Sigma^T + (\Sigma G')V + V(\Sigma G')^T \end{aligned} \quad (51)$$

where G' is a symmetric matrix given by

$$G' = \begin{pmatrix} \text{Re}[\tilde{C}] \\ \text{Im}[\tilde{C}] \end{pmatrix}^T \begin{pmatrix} -\text{Im}[U] & \text{Re}[U] \\ \text{Re}[U] & \text{Im}[U] \end{pmatrix} \begin{pmatrix} \text{Re}[\tilde{C}] \\ \text{Im}[\tilde{C}] \end{pmatrix}. \quad (52)$$

The first line of equation (51) is simply the matrix Riccati equation one would expect for the conditioned covariance matrix of a classical measurement with output $\mathbf{y}(t) = C\mathbf{x}(t) + d\mathbf{v}_m(t)/dt$ where the measurement noise has unit power and is uncorrelated with the process noise. The terms on the second line though are quantum mechanical in origin. The first is an extra term in the diffusion that corresponds to back-action noise in the observable complementary to the observable $C\mathbf{x}$ appearing in $\mathbf{y}(t)$. This follows from the fact that the effect of Σ [see Eq. (14)] is to swap each of the conjugate positions and momenta (up to a sign). The last two terms correspond to a modification of the Hamiltonian H since G' is symmetric. These result from the correlations between the process and measurement noise in the equivalent classical model. It is possible to choose G' to be zero for a given C but some physically relevant measurements such as homodyne detection of the light lost from a Fabry-Perot cavity do not have $G' = 0$.

4.2. Quantum Langevin Equations

In this section we show that not only do the classical equations (38) and (41) reproduce the conditional evolution of the quantum system, but in addition they can be reinterpreted physically as operator equations for the system. The Heisenberg equation of motion for a system operator for an open quantum system is often called the quantum Langevin equation. For the master equation (1) the quantum Langevin equation for an arbitrary system operator $\hat{r}$ is⁹

$$d\hat{r} = \left\{ i[\hat{H}, \hat{r}] + \hat{s}^\dagger \hat{r} \hat{s} - \frac{1}{2} \{ \hat{s}^\dagger \hat{s}, \hat{r} \} + \hat{c}^\dagger \hat{r} \hat{c} - \frac{1}{2} \{ \hat{c}^\dagger \hat{c}, \hat{r} \} - [\hat{\mathbf{b}}^{u\dagger} \hat{s} - \hat{s}^\dagger \hat{\mathbf{b}}^u, \hat{r}] - [\hat{\mathbf{b}}^{o\dagger} \hat{c} - \hat{c}^\dagger \hat{\mathbf{b}}^o, \hat{r}] \right\} dt. \quad (53)$$

Here $\mathbf{b}^u dt$ is a K^u -vector of quantum noise increments satisfying

$$[\hat{b}_j^u dt, \hat{b}_k^{u\dagger} dt] = \delta_{jk} dt. \quad (54)$$

It has zero mean and all second order moments are $o(dt)$ except those given by

$$(\mathbf{b}^u dt)(\mathbf{b}^u dt)^\dagger = Idt. \quad (55)$$

The K^o -vector $\mathbf{b}^o dt$ is constructed similarly.

From this one finds

$$d\hat{\mathbf{x}} = A\hat{\mathbf{x}}dt + B\mathbf{u}(t)dt + d\hat{\mathbf{h}}^u(t) + d\hat{\mathbf{h}}^o(t) \quad (56)$$

where A is as defined in Sec. 3.2, and we have defined vectors of Hermitian noise operators:

$$d\hat{\mathbf{h}}^u(t) = -idt[\hat{\mathbf{b}}^{u\dagger}(t)\tilde{S}^T\Sigma - \Sigma^T\tilde{S}^*\hat{\mathbf{b}}^u(t)] \quad (57)$$

$$d\hat{\mathbf{h}}^o(t) = -idt[\hat{\mathbf{b}}^{o\dagger}(t)\tilde{C}^T\Sigma - \Sigma^T\tilde{C}^*\hat{\mathbf{b}}^o(t)] \quad (58)$$

Since Eq. (56) has a solution linear in $\hat{\mathbf{h}}^u$ and $\hat{\mathbf{h}}^o$, it follows that the symmetrized moments of $\hat{\mathbf{x}}$ will be determined by the symmetrized moments of these noise sources. But these moments are

$$d\hat{h}_n^u d\hat{h}_m^u + d\hat{h}_m^u d\hat{h}_n^u = 2D_{nm}^u \quad (59)$$

$$d\hat{h}_n^o d\hat{h}_m^o + d\hat{h}_m^o d\hat{h}_n^o = 2D_{nm}^o, \quad (60)$$

precisely the same as those for the classical process noises $\tilde{S}d\mathbf{v}_p$ and $\tilde{C}d\mathbf{v}_p^o$ in the classical model (38). Thus, provided one considers symmetric moments, the quantum Langevin equation (56) is completely equivalent to the classical SDE (38).

To establish a complete correspondence between the classical model and the quantum Heisenberg picture we have to consider the measurement record. In the classical model this is given by $\mathbf{y}(t)$. In the quantum model, it is the output of the baths $\mathbf{b}^o$ that are measured. The output bath operators are given by⁹

$$\hat{\mathbf{b}}_{\text{out}}^o(t) = \hat{\mathbf{b}}^o(t) + \hat{c}(t). \quad (61)$$

A continuous current will result from any measurement of an observable (set of Hermitian operators) linear in $\hat{\mathbf{b}}_{\text{out}}^o$. As discussed in Sec. 2.2, the different possible measurements are parametrized by the symmetric matrix U . In the Heisenberg picture the operator for the current (9) is

$$\hat{\mathbf{J}}^T = \hat{\mathbf{b}}_{\text{out}}^{oT} + \hat{\mathbf{b}}_{\text{out}}^{o\dagger} U + \hat{\mathbf{a}}^\dagger R. \quad (62)$$

Here $\hat{\mathbf{a}}$ is a vector of ancillary annihilation operators, which are assumed to act on a vacuum state. They obey the same sort of commutation relations as the $\hat{\mathbf{b}}^u$ operators, namely

$$[\hat{a}_\mu, \hat{a}_\nu^\dagger] dt = \delta_{\mu\nu}. \quad (63)$$

These ancillary operators ensure that all of the components of $\hat{\mathbf{J}}$ commute with one another. This is necessary since this vector operator represents an observable quantity. Assuming R to be a symmetric matrix like U , we find

$$[\hat{J}_\mu, \hat{J}_\nu] dt = U_{\mu\nu} - U_{\nu\mu} = 0, \quad (64)$$

$$[\hat{J}_\mu, \hat{J}_\nu^\dagger] dt = \delta_{\mu\nu} - U_{\mu\lambda} U_{\lambda\nu}^* - R_{\mu\lambda} R_{\lambda\nu}^*. \quad (65)$$

Thus we require the matrix R to satisfy

$$R^* R = I - U^* U. \quad (66)$$

The right-hand side of this equation is always positive, so it is always possible to find a suitable R .

From these definitions one can show that

$$\hat{\mathbf{J}}^T = \hat{\mathbf{c}}^T + \hat{\mathbf{c}}^\dagger U + \delta \hat{\mathbf{J}}^T, \quad (67)$$

where

$$\delta \hat{\mathbf{J}}^T = \hat{\mathbf{b}}^{oT} + \hat{\mathbf{b}}^{o\dagger} U + \hat{\mathbf{a}}^\dagger R \quad (68)$$

has a zero mean. Thus the mean of $\hat{\mathbf{J}}$ is the same as that of the classical current in Sec. 2.2. Also, one finds that

$$(\delta \hat{\mathbf{J}} dt)(\delta \hat{\mathbf{J}} dt)^\dagger = I dt \quad (69)$$

$$(\delta \hat{\mathbf{J}} dt)(\delta \hat{\mathbf{J}} dt)^T = U dt, \quad (70)$$

the same correlations as the noise in Sec. 2.2.

The operator for the real measurement record $\hat{\mathbf{y}}$ is thus defined as

$$\hat{\mathbf{y}} = \tilde{U}^{-1/2} \begin{pmatrix} \text{Re} \left[\hat{\mathbf{J}} \right] \\ \text{Im} \left[\hat{\mathbf{J}} \right] \end{pmatrix}, \quad (71)$$

where here $\text{Re}[\hat{J}_\mu] \equiv (\hat{J}_\mu + \hat{J}_\mu^\dagger)/2$ and similarly for Im . We find

$$\hat{\mathbf{y}} dt = C \hat{\mathbf{x}} dt + d\hat{\mathbf{v}}_m, \quad (72)$$

where

$$d\hat{\mathbf{v}}_m = \tilde{U}^{-1/2} \begin{pmatrix} \text{Re} \left[\delta \hat{\mathbf{J}} \right] \\ \text{Im} \left[\delta \hat{\mathbf{J}} \right] \end{pmatrix} dt. \quad (73)$$

This is a vector of commuting Hermitian noise operators which have statistics identical to that of the classical measurement noise process $d\mathbf{v}_m$. Moreover, the symmetric cross-correlation between $d\hat{\mathbf{v}}_m$ and $d\hat{\mathbf{v}}_p^o$ is the same as for the corresponding classical noises.

5. CLOSING THE LOOP

Thus far we have considered monitoring the system, but not controlling it. In closed loop control $\mathbf{u}(t)$ depends on the history of the measurement record $\mathbf{y}(s)$ for $s < t$. Because of the linearity of the dynamics, this affects $\langle \mathbf{x} \rangle_c$, the vector of mean values, but does not affect V , the covariance matrix V . For the same reason, the classical analogue in Sec. 4 still holds. Thus it is possible to draw upon a large body of knowledge about control for classical linear systems.^{1, 23, 29} For example, the typical aim of control is to minimize a *cost* function, which consists of the expected value of some positive function of the phase space variables $\mathbf{x}(t)$ (over time) plus a positive function of the control signal $\mathbf{u}(t)$ (over time). In linear systems, it can be proven that the optimal control signal $\mathbf{u}(t)$ is a function only of $\langle \mathbf{x} \rangle_c(t)$ at the same time. That is, the measurement record $\mathbf{y}(s)$ for $t < s$ is irrelevant except for its input into the best estimate of the system's position in phase space, $\langle \mathbf{x} \rangle_c(t)$. For systems with nonlinear dynamics, this powerful result does not hold. If in addition the positive functions mentioned above are *quadratic* functions then it is straightforward to generate the optimal $\mathbf{u}(t)$.

Much more can be said about control problems in this context, but in this paper we will concentrate on the following case. First, we assume that the matrix B (which mediates the effect of the control signal $\mathbf{u}(t)$ on the system dynamics) is of full rank. For want of a better word, we will call such systems *manipulable*. This property of *manipulability* is stronger than that of *controllability*. The former is equivalent to the existence of a real matrix F such that

$$\Phi = A + BF \quad (74)$$

is an arbitrary real matrix; here F can be defined explicitly as $B^{-1}(\Phi - A)$. The latter is equivalent to the existence of a real matrix F such that Φ has eigenvalues that are arbitrary (apart from the complex eigenvalues' coming in conjugate pairs).²⁹ Second, we ignore any cost associated with the control function and instead aim to minimize a positive quadratic function of the phase-space variables in steady-state:

$$\Lambda = \text{Tr} [\hat{\mathbf{x}}^T P \hat{\mathbf{x}} \rho_{ss}], \quad (75)$$

where P is a positive symmetric real matrix. For our classical analogue system, the equivalent expression is

$$\Lambda = E_{ss} [\mathbf{x}^T P \mathbf{x}], \quad (76)$$

where E indicates expectation value or ensemble average.

For a manipulable system with zero-cost control, we can choose the control signal to be

$$\mathbf{u}(t) = F \langle \hat{\mathbf{x}} \rangle_c(t), \quad (77)$$

where F is such that the matrix Φ as defined above tends to $-\infty$ times the identity matrix. In this case it is clear from the equation of motion for the estimate,

$$d\langle \hat{\mathbf{x}} \rangle_c = \Phi \langle \hat{\mathbf{x}} \rangle_c + (V_c C^T + \Gamma^T) d\mathbf{w}, \quad (78)$$

that the deterministic damping will overwhelm the noise so that we can set $\langle \mathbf{x} \rangle_c = \mathbf{0}$ in steady state. Now by definition

$$E [\mathbf{x}^T P \mathbf{x}] = E \left\{ \langle \hat{\mathbf{x}} \rangle_c^T P \langle \hat{\mathbf{x}} \rangle_c + \text{tr}[P V_c] \right\}, \quad (79)$$

where here tr means the usual matrix trace. Thus in this case, the cost to be minimized is

$$\Lambda = \text{tr}[P V], \quad (80)$$

where here $V \equiv V_c^{ss}$.

5.1. Semi-Definite Program for Optimum Unravellings

If, in addition to being manipulable and having cost-free control, all outputs of the system may be monitored, the problem of minimizing Λ becomes one of minimizing the linear function of V in Eq. (80) subject to the constraints of the LMIs (36) and (17). The significance of this is that this is precisely the form of a semi-definite program,³¹ and so has an efficient numerical solution regardless of the system size. In this section we will apply this theory to a very simple (one dimensional) system so that the techniques of Ref.³¹ are not needed, but the principle will be demonstrated.

Consider the system described by the master equation

$$\dot{\rho} = -i[(\hat{q}\hat{p} + \hat{p}\hat{q})/2, \rho] + \mathcal{D}[\hat{q} + i\hat{p}]\rho, \quad (81)$$

where the output arising from the second term may be monitored. This could be realized in quantum optics as a damped cavity (harmonic oscillator in the rotating frame) containing an on-threshold parametric down converter.¹³ Here q is the anti-squeezed quadrature and p the squeezed quadrature. The monitoring of the output could be realized by techniques such as homodyne or heterodyne detection.^{11, 19}

In this case the drift and diffusion matrices evaluate to

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (82)$$

Writing the possible conditioned steady-state variance V as

$$V = \begin{pmatrix} V_{qq} & V_{qp} \\ V_{qp} & V_{pp} \end{pmatrix}, \quad (83)$$

the LMIs (36) and (17) become

$$\begin{pmatrix} 1 & -2V_{qp} \\ -2V_{qp} & 1 - 4V_{pp} \end{pmatrix} \geq 0 \quad (84)$$

$$\begin{pmatrix} 2V_{qq} & 2V_{qp} + i \\ 2V_{qp} - i & 2V_{pp} \end{pmatrix} \geq 0 \quad (85)$$

The first of these implies $V_{pp} \leq 1/4 - V_{qp}^2$, while the second implies $V_{pp}V_{qq} \geq 1/4 + V_{qp}^2$.

Now say the aim of the feedback control is to produce a state where $q = p$ as nearly as possible. That is, we minimize

$$\Lambda = \langle (\hat{q} - \hat{p})^2 \rangle_{ss} \quad (86)$$

Being of quadratic form, this fits the program defined above, with

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (87)$$

In the optics case it is simple to displace the system in its phase space by application of a coherent driving field.¹³ That is, we are justified in taking the system to be manipulable. In this case, the Λ achievable with an unravelling giving a particular steady-state conditioned variance V is

$$\Lambda = \text{tr}[PV] = V_{qq} - 2V_{qp} + V_{pp} \quad (88)$$

Now any positive quadratic function will be minimized for a pure state rather than a mixed state so we may assume that $V_{pp}V_{qq} = 1/4 + V_{qp}^2$. This gives

$$\Lambda = \frac{1 + 4V_{qp}^2}{4V_{pp}} + V_{pp} - 2V_{qp} \quad (89)$$

subject only to $V_{pp} \leq 1/4 - V_{qp}^2$. The minimum is found numerically to be $\Lambda^0 \approx 1.12$ at $V_{pp}^0 \approx 0.234$ and $V_{qp}^0 \approx 0.124$. Thus we have determined the performance of the optimal feedback control of the system, for the optimal unravelling, without explicitly determining either.

It is worth emphasizing that a different (*wrong*) answer is obtained by replacing the LMI (36) by the simpler condition that the conditioned variance be compatible with the unconditioned variance (37). In this case

$$V_{ss} = \begin{pmatrix} \infty & 0 \\ 0 & 1/4 \end{pmatrix} \quad (90)$$

So that (37) becomes

$$\begin{pmatrix} \infty & -V_{qp} \\ -V_{qp} & 1/4 - V_{pp} \end{pmatrix} \geq 0. \quad (91)$$

That is, $V_{pp} < 1/4$. In this case Eq. (89) has a infimum of 1 for $V_p \nearrow 1/4$ and $V_{qp} = 1/4$.

6. CONCLUSION

In this paper we have studied Markovian open quantum systems with linear dynamics. By this we mean both that the Heisenberg equations of motion for the positions and momenta are linear, and also that any environment observables that are monitored depend linearly upon these position and momenta. For such systems we have: a) given a complete characterization of the dynamics, both unconditioned and conditioned; b) shown that for any such quantum systems there is an equivalent classical system; c) derived linear matrix inequalities which give necessary and sufficient conditions on the properties of a classical system in order for those properties to coincide with those of a quantum system; d) shown that the solution to a certain class of feedback control problems can be found efficiently using some of these linear matrix inequalities.

These results are important, not only for their practical applicability to systems with approximately linear dynamics, but also for what they teach us about the similarities and differences between quantum and classical feedback control. Classical stochastic control theory is dominated by the study of linear systems, and there is a lot of theory that can be applied in the in the quantum context, such as robust control²⁹; indeed, some of these ideas have been applied already.³² By contrast, nonlinear stochastic control is relatively undeveloped classically, although it has already become necessary to consider control for quantum systems with dynamics that cannot be linearized; see for example Refs.^{2,3,33-35} This is clearly an area in which much remains to be done.

ACKNOWLEDGMENTS

This work was supported by the Australian Research Council. We gratefully acknowledge discussions with Hideo Mabuchi, Kurt Jacobs, Ramon van Handel, and John Stockton.

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