Comments on “On ‘hearing the shape of drums’: An experimental study using vibrating smectic films”

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Abstract. – We point out that one of a pair of isospectral shapes subjected to a certain intentionally altered building scheme would be expected to lose isospectrality because of the mathematical form of a relevant spectral function expansion which involves angles. A higher “calibration” mode which has common exactly specified eigenfrequency for all three shapes is mentioned.

A recent paper [1] by Even and Pieranski presented the results of an experimental investigation, using smectic films, of the isospectrality of pairs of shapes built from seven triangles. They also constructed an altered shape (A' in ref. [1]) related to their shape A (90°) but utilizing a different symmetry for one of the constituent isosceles rectangular triangles, and drew attention to the consequent loss of isospectrality.

We should like to point out that their *a contrario*-type experiment might *a priori* be expected to yield a different frequency spectrum because of the nature of a mathematical asymptotic spectral function expansion involving a sum over the eigenvalues λ of the negative two-dimensional Laplacian (Helmholtz equation with Dirichlet boundary conditions) which for shapes such as those under discussion takes the form [2–4]

$$\sum \exp[-\lambda s] \sim a/(4\pi s) - \ell/[8\sqrt{\pi s}] + \sum (\pi^2 - \theta^2)/(24\pi\theta) + \dots \quad (1)$$

as $s \rightarrow 0$, where a is the area of the domain, ℓ is the length of its perimeter, and the θ are the internal angles. Whilst the altered shape A' in ref. [1] does have the same area and perimeter as shape A , the internal angles are different, such that the angle sum in the 3rd term of eq. (1) is different. Thus the set of eigenvalues $\{\lambda\}$, or eigenfrequencies-squared $\{f^2\}$, could not be the same.

Shape A' was found experimentally in ref. [1] to have a different overall frequency spectrum and different mode patterns, in particular for the important 9th mode. For the isospectral pair of shapes A and B (90°), this mode involved the fundamental patterns for the basic building triangles. For unit cell length, its eigenvalue can be shown mathematically (cf. refs. [5,6]) to equal exactly $5\pi^2$.

Nevertheless, the 21st mode for the *a contrario* shape A' introduced in ref. [1], as well as for the two standard shapes A and B (angle $\alpha = 90^\circ$ in ref. [1]), does have a common eigenvalue, which can be shown mathematically to equal exactly $10\pi^2$. The corresponding eigenfunction may be written explicitly, in an appropriate coordinate system, in a factorized form which exhibits nodal lines as [6]

$$z = \sin(\pi x) \sin(\pi y) \sin[\pi(x - y)] \sin[\pi(x + y)], \quad (2)$$

and corresponds to a nodal pattern of 14 smaller congruent sub-triangles. Since the pattern of this 21st mode is easily identifiable, it may serve as a calibration standard for numerical and experimental investigations.

The 9th and 21st mode modal patterns for the standard shapes B (90°) and A (90°) may be found, for example, in fig. 2 of the paper by Wu *et al.* [5]. (Different papers may have various orientations for the shapes.) We have computed contour plots of the 9th and 21st modes for the shape A' using Matlab. The pattern of the 9th mode for A' corresponds to the experimental amplitude pictured in fig. 6 of ref. [1]. The plot for the 21st mode clearly displayed the pattern described in the preceding paragraph.

We also computed some eigenvalues for all three shapes. Agreement across all three shapes for the 21st mode was within 0.005%. The numerical procedure tended to overestimate eigenvalues, but these were within 0.02% of the known exact value $5\pi^2$ for the 9th mode of shapes B and A, and within 0.04% of the exact value $10\pi^2$ for the 21st mode of all three shapes. Concerning ratios, for shapes B and A, modes 21 and 9, the numerically computed values yielded ratios 2.0004 and 2.0005, respectively, very close to the theoretical exact value of 2. The ratio of mode 9 to mode 1 for the shape A' was computed to be 4.645, against which the experimental value 4.582 in table I of ref. [1] may now be compared.

In conclusion, we have shown by a simple geometrical argument based on a spectral expansion that the results of the *a contrario* experiment of ref. [1] using the altered shape A' would actually be expected to violate isospectrality. On the other hand, the remarkable higher “mathematical” 21st mode (2) may serve as a calibration benchmark for both theory and experiments for all three shapes.

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