First Order Wavelike Phenomena with Electromagnetic and Acoustic Applications

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Abstract—The relationship between Maxwell's equations and the wave equation is exploited to show how problems involving the latter can be converted from a second order differential equation into one of the first order. As an immediate consequence it becomes possible to solve the wave equation without resorting to double differentiation or integration. An example of application is given utilising the Maxwellian equations to solve both an electromagnetic and an equivalent acoustic wave problem.

Index Terms—Acoustic waves, electromagnetic propagation, Maxwell equations.

I. INTRODUCTION

WAVELIKE phenomena in their simplest form involve the propagation from one place to another, through a uniform medium at constant velocity and without loss of shape, of a departure from equilibrium value in some physical quantity. Two applications for this simple behaviour which have become indispensable over the past 200 years are firstly the transmission of the electric power which provides the majority source of energy to maintain the function of modern industrialised cities, and secondly the transmission of data in the electronic form currently providing the mechanism through which most information is exchanged in both commercial and domestic contexts. Without sufficient mastery of these electromagnetic phenomena the current modern industrial global society into which we have been born, ceases to function properly. This is seen occasionally in practice when a major hurricane or earthquake disrupts the electric power and electronic communication services.

Electromagnetic phenomena are well described by Maxwell's equations [1] of electric and magnetic fields. Maxwell's equations can be written in various forms as befits whatever mathematical framework is adopted. Since Heaviside [2]–[4], it has been customary to cast the equations as a set of four coupled first order partial differential equations, using Gibbs's vector calculus [5], [6] for the spatial quantities. This form of Maxwell's equations can be reduced in uniform media further into a set of six uncoupled simultaneous partial differential equations in terms of the individual spatial components of electric and magnetic field. The cost of this conversion is that the differential equations now become second order. The benefit is that each (scalar) spatial component is decoupled and can be treated separately.

Since Clifford [7], an alternative approach has been available whereby Maxwell's equations are cast into a single first

order ordinary differential equation using Clifford's geometric algebra [8]. The benefit is that the differential equation remains first order. The cost is the need to adopt an algebra which is (at first) unfamiliar and treats electromagnetic fields in terms of non-scalar (rather than scalar) multi-dimensional entities¹.

When weaned on vector calculus and familiar with its nature it is difficult to see any need or advantage to admit the use of any other technique, although ultimately it is the application which is of paramount importance and the criterion for choice of technique should be its effectiveness and not its parentage. Of course it is difficult to make a sensible choice without at least some appreciation for the nature and capability of the various alternatives.

The role of this work is to introduce Clifford's alternative in the context of a one-dimensional example, simple enough to avoid unnecessary complication, familiar enough to provide reassurance, yet broad enough to cover particular applications in power transmission and electronic communication.

In section II an elementary derivation is provided which confirms that the electric and magnetic fields do indeed travel as waves of arbitrary shape with constant velocity. It would be surprising if this were not the case. However, the solution here is based on the first order equations, with the result emerging directly and inevitably from the analysis without any assumptions or suppositions regarding the nature of the eventual outcome.

In section III it is shown how the first order Cliffordian formulation can be applied to all phenomena adhering to the simplest second order form of the wave equation, by casting them into the mould of the Maxwellian differential equations. In doing this it is seen that second order differential equations are not necessary for any such phenomena and first order differential equations are entirely sufficient.

Numerical examples of application of the first order Cliffordian formulation to largely equivalent scattering problems of both electromagnetic and acoustic waves are given in section IV.

II. MAXWELLIAN WAVES

In three dimensions in a region of uniform material properties free from either moving or stationary electric charges, Maxwell's equations can be written using Clifford algebra in terms of a first order differential operator [9]:

$$D = \frac{\partial}{\partial x} e_1 + \frac{\partial}{\partial y} e_2 + \frac{\partial}{\partial z} e_3 - \frac{i}{c} \frac{\partial}{\partial t} e_0 \quad (1)$$

¹In time both of these apparent disadvantages become advantages.

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Manuscript submitted October 2017.

and an electromagnetic field:

$$F = \sqrt{\mu}H\sigma - i\sqrt{\epsilon}Ee_0 \quad (2)$$

in the form:

$$DF = 0 \quad (3)$$

Here $H = H_x e_1 + H_y e_2 + H_z e_3$ and $E = E_x e_1 + E_y e_2 + E_z e_3$ are the magnetic and electric fields respectively, μ and ϵ the material's values of electric permeability and electric permittivity, $1/c = \sqrt{\mu\epsilon}$, i is the (scalar) imaginary unit with property $i^2 = -1$, e_0, e_1, e_2, e_3 are the (non-scalar) Clifford units [8] with properties:

$$\begin{cases} e_p e_q = -e_q e_p \\ e_p e_p = -1 \end{cases} \quad (4)$$

and $\sigma = -e_1 e_2 e_3$. The Clifford units play the role of the Cartesian unit vectors, with the first property in equation 4 taking the role of the vector cross product $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, and the second property in equation 4 taking the role of the dot product $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$, albeit with a minus sign. The compound entity $\sigma = -e_1 e_2 e_3$ plays the role of the signed unit volume.

The right hand side of equation 3 is zero because the region of space under immediate purview is free from sources of electric and magnetic fields. Fields neither originate nor terminate in the region; they simply pass through it.

The derivation is simplified by taking the particular case shown in figure 1 where the magnetic field $H = H_z e_3$ has non-zero component only parallel to the z -direction, the electric field $E = E_y e_2$ has non-zero component only in the y -direction, and both fields vary only in the x -direction. Maxwell's equations in this case can be written in terms of the reduced operator:

$$D = \frac{\partial}{\partial x} e_1 - \frac{i}{c} \frac{\partial}{\partial t} e_0 \quad (5)$$

and field:

$$F = \sqrt{\mu}H_z e_3 \sigma - i\sqrt{\epsilon}E_y e_2 e_0 \quad (6)$$

Maxwell's equations 3 then lead by multiplication and application of the properties of Clifford units in equation 4 to the expression:

$$\begin{aligned} & - \left(\sqrt{\mu} \frac{\partial H_z}{\partial x} + \frac{1}{c} \sqrt{\epsilon} \frac{\partial E_y}{\partial t} \right) e_2 \\ & - \left(i\sqrt{\epsilon} \frac{\partial E_y}{\partial x} + \frac{i}{c} \sqrt{\mu} \frac{\partial H_z}{\partial t} \right) e_0 e_1 e_2 = 0 \end{aligned} \quad (7)$$

Fig. 1. Simple case where magnetic and electric fields each have non-zero components only in z and y -directions, and vary only in the x -direction.

The magnetic and electric fields E_y and H_z are related to one another throughout the entire universe of space-time coordinates $u = u(x, t)$. In terms of these coordinates, the derivatives of the magnetic and electric fields $H_z = H_z(u(x, t))$ and $E_y = E_y(u(x, t))$ take the form:

$$\begin{cases} \frac{\partial H_z}{\partial x} = H'_z \frac{\partial u}{\partial x} & ; & \frac{\partial H_z}{\partial t} = H'_z \frac{\partial u}{\partial t} \\ \frac{\partial E_y}{\partial x} = E'_y \frac{\partial u}{\partial x} & ; & \frac{\partial E_y}{\partial t} = E'_y \frac{\partial u}{\partial t} \end{cases} \quad (8)$$

where $H'_z = \frac{dH_z(u)}{du}|_{u=u(x,t)}$ and $E'_y = \frac{dE_y(u)}{du}|_{u=u(x,t)}$ are ordinary rather than partial differentials. Maxwell's equations 7 now reduce to:

$$\begin{aligned} -DF &= \left(\sqrt{\mu} H'_z \frac{\partial u}{\partial x} + \frac{1}{c} \sqrt{\epsilon} E'_y \frac{\partial u}{\partial t} \right) e_2 \\ &+ \left(i\sqrt{\epsilon} E'_y \frac{\partial u}{\partial x} + \frac{i}{c} \sqrt{\mu} H'_z \frac{\partial u}{\partial t} \right) e_0 e_1 e_2 = 0 \end{aligned} \quad (9)$$

The two linearly independent components, Ae_2 and $Be_0 e_1 e_2$, must both evaluate to zero simultaneously to ensure equality:

$$\begin{pmatrix} \sqrt{\mu} \frac{\partial u}{\partial x} & \frac{1}{c} \sqrt{\epsilon} \frac{\partial u}{\partial t} \\ \frac{i}{c} \sqrt{\mu} \frac{\partial u}{\partial t} & i\sqrt{\epsilon} \frac{\partial u}{\partial x} \end{pmatrix} \begin{pmatrix} H'_z \\ E'_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (10)$$

The only non-trivial solution is when the determinant of the matrix is zero:

$$i\sqrt{\mu\epsilon} \left(\frac{\partial u}{\partial x} \right)^2 - \frac{i\sqrt{\mu\epsilon}}{c^2} \left(\frac{\partial u}{\partial t} \right)^2 = 0 \quad (11)$$

Solution of the quadratic equation 11 gives:

$$\frac{\partial u}{\partial x} = \pm \frac{1}{c} \frac{\partial u}{\partial t} \quad (12)$$

and the ratio of the partial derivatives $(\frac{\partial u}{\partial x})/(\frac{\partial u}{\partial t}) = \pm 1/c$ is constant, depending on neither x nor t .

Using either the first or second row of equation 10 along with the result in equation 12, the ratio of the derivatives of the electric and magnetic fields is found as:

$$\frac{E'_y}{H'_z} = -\frac{\sqrt{\mu} \frac{\partial u}{\partial x}}{\frac{1}{c} \sqrt{\epsilon} \frac{\partial u}{\partial t}} = -\frac{i\sqrt{\mu} \frac{\partial u}{\partial t}}{i\sqrt{\epsilon} \frac{\partial u}{\partial x}} = \pm \sqrt{\frac{\mu}{\epsilon}} \quad (13)$$

where $\eta = \sqrt{\mu/\epsilon}$ is the characteristic impedance of the medium supporting the propagation. Apart from constants of integration the same relationship applies to the fields themselves, as already well known in the far field.

The two-dimensional function $u(x, t)$ can be expanded in some region as a two-dimensional power series:

$$u(x, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} x^m t^n \quad (14)$$

with unknown coefficients a_{mn} laid out in the form of a grid in figure 2. Under the condition given in equation 12 the series expansions for the partial derivatives:

$$\begin{cases} \frac{\partial u}{\partial x} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m a_{m,n-1} x^{m-1} t^{n-1} \\ \frac{\partial u}{\partial t} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n a_{m-1,n} x^{m-1} t^{n-1} \end{cases} \quad (15)$$

lead to sets of equations between the unknown coefficients a_{mn} for $m+n=1, 2, 3, 4$, etc. Each set corresponds to one of

Fig. 2. Some coefficients a_{mn} of function $u(x, t)$.

the diagonals parallel to the leading diagonal in figure 2. The sets are disjoint and can be treated independently to solve for some of the coefficients in terms of the others.

For the set $m+n=3$, shown by the shaded diagonal in figure 2:

$$u_3(x, t) = a_{03}x^0t^3 + a_{12}x^1t^2 + a_{21}x^2t^1 + a_{30}x^3t^0 \quad (16)$$

giving:

$$\begin{cases} \frac{\partial u}{\partial x} = a_{12}t^2 + 2a_{21}xt + 3a_{30}x^2 \\ \frac{\partial u}{\partial t} = 3a_{03}t^2 + 2a_{12}xt + a_{21}x^2 \end{cases} \quad (17)$$

and choosing the negative sign in equation 12:

$$\left(\frac{\partial u}{\partial x}\right) / \left(\frac{\partial u}{\partial t}\right) = \frac{a_{12}t^2 + 2a_{21}xt + 3a_{30}x^2}{3a_{03}t^2 + 2a_{12}xt + a_{21}x^2} = -\frac{1}{c} \quad (18)$$

To honour the equality the coefficients of each term in the power series of the numerator and denominator must take the same ratio of $-1/c$.

$$\begin{cases} a_{21} = -3ca_{30} \\ a_{12} = -ca_{21} = 3c^2a_{30} \\ a_{03} = -\frac{1}{3}ca_{12} = -c^3a_{30} \end{cases} \quad (19)$$

The part of the solution governed by the set of coefficients $m+n=3$ is therefore:

$$\begin{aligned} u_3(x, t) &= -c^3a_{30}t^3 + 3c^2a_{30}xt^2 - 3ca_{30}x^2t + a_{30}x^3 \\ &= a_{30}(x-ct)^3 \end{aligned} \quad (20)$$

Similar results hold for the other sets of coefficients $m+n=k$:

$$u_k(x, t) = a_{k0}(x-ct)^k \quad (21)$$

and the full solution is the sum of all the parts:

$$u(x, t) = \sum_{k=0}^{\infty} u_k(x, t) = \sum_{k=0}^{\infty} a_{k0}(x-ct)^k = u(x-ct) \quad (22)$$

The third member of equation 22 is a power series in variable $x-ct$ with arbitrary coefficients a_{k0} . By appropriate choice of the coefficients it is possible to construct any arbitrary function u in variable $x-ct$, as in the rightmost member of equation 22.

Returning to the fields, it now follows that they are functions of the arbitrary coordinate function $u(x-ct)$:

$$\begin{cases} H_z = H_z(u(x, t)) = H_z(u(x-ct)) \\ E_y = E_y(u(x, t)) = E_y(u(x-ct)) \end{cases} \quad (23)$$

As functions of an arbitrary function with argument $x-ct$, the fields themselves take the role of arbitrary functions with that argument:

$$\begin{cases} H_z = H_z(x-ct) \\ E_y = E_y(x-ct) \end{cases} \quad (24)$$

and as such exhibit the behaviour of waves of arbitrary shape travelling at velocity c . The same outcome can be achieved by noting that any function, including the identity function, can be chosen in place of u .

Although arbitrary, the shapes of the magnetic and electric fields are not independent, being related through their derivatives by the condition given in equation 13. Choosing the positive sign in equation 12 gives a second solution as waves of arbitrary shape travelling at velocity $-c$.

III. OTHER WAVES

The simplest form of the wave equation, for regions with one spatial dimension in which the material properties are uniform, can be written:

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = 0 \quad (25)$$

If the objective is to solve the wave equation of this second order form using a first order approach, it is sufficient to make the identifications:

$$\begin{cases} H \Leftrightarrow f \\ E \Leftrightarrow \eta f \end{cases} \quad (26)$$

The first order formulation of equations 5 and 6 then reduces to:

$$D = \frac{\partial}{\partial x}e_1 - \frac{i}{c} \frac{\partial}{\partial t}e_0 \quad (27)$$

and:

$$F = \frac{1}{\sqrt{c}}f e_3\sigma - \frac{i}{\sqrt{c}}f e_2e_0 = \frac{1}{\sqrt{c}}f [e_3\sigma - ie_2e_0] \quad (28)$$

The solution to the Maxwellian problem $DF=0$ also solves the second order wave problem:

$$D^2F = D(DF) = D(0) = 0 \quad (29)$$

which reduces on expansion to the coordinate dependent form:

$$\begin{aligned} \left(\frac{\partial}{\partial x}e_1 - \frac{i}{c} \frac{\partial}{\partial t}e_0\right)^2 \left(\frac{1}{\sqrt{c}}f [e_3\sigma - ie_2e_0]\right) &= 0 \\ \left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) f &= 0 \end{aligned} \quad (30)$$

Equation 30 retrieves precisely the scalar wave equation 25 in one dimension. These results also apply in two and three dimensions.

The second differentiation in the wave equation admits an additional component into the solution not present in the Maxwellian problem. It is easy to verify by substitution into equation 30 that the linear function:

$$f = Ax + Bt + Cxt \quad (31)$$

is a non-travelling solution.

For the wave equation it is usual to ignore the additional component, although any numerical solution of equation 30 does provide for its presence. If it is desired to explicitly exclude the non-travelling solution at the outset then it may be preferable to adopt the Maxwellian first order approach, which is guaranteed to be free from the additional linear terms.

IV. EXAMPLES

The theory of sections II and III is demonstrated by applying the first order Maxwellian solution to both electromagnetic and acoustic wave scattering problems. Firstly, mathematical equivalence between the two physical phenomena is established (see table I) by identifying the acoustic wave velocity and characteristic impedance with their electromagnetic counterparts. Inspection of equation 26 with reference to the equations governing acoustic phenomena then identifies the electric field as playing the role of local deviation in pressure, and the magnetic field taking the role of local particle velocity.

TABLE I
EQUIVALENT ELECTROMAGNETIC AND ACOUSTIC QUANTITIES.

Electromagnetic	Acoustic
H magnetic field	v local particle velocity
E electric field	p deviation in pressure
c wave velocity	c wave velocity
η characteristic impedance	z_o characteristic specific acoustic impedance

The scattering object chosen for the examples is a single simple plano-convex lens, with expected behaviour known *a priori* from its geometrical shape. The material properties inside and outside (see table II) have been chosen to give a relative wave velocity $c_{in}/c_{out} = 0.8$, the same for both electromagnetic and acoustic lens, but different values of relative impedance; $\eta_{in}/\eta_{out} = 0.8$ and $z_{in} = z_{out} = 1.487$. In the electromagnetic case the external region is free space and the lens is a glass with refractive index 1.25. In the acoustic case the external region is air and the lens is a mixture of argon and krypton chosen especially to give the same value of refractive index as the glass. In addition the temperature is $15^\circ C$, the pressure is $101.325 kPa$, γ is the adiabatic index, m is average molecular mass, ρ is the density, and K is the bulk modulus. Under these conditions non-dispersive linear ideal gas behaviour is taken to apply.

TABLE II
NUMERICAL VALUES FOR MATERIAL PARAMETERS IN ELECTROMAGNETIC AND ACOUSTIC WAVE SOLUTIONS.

	outside	inside
Electromagnetic	vacuum	glass
μ	$4\pi \times 10^{-7} H/m$	$4\pi \times 10^{-7} H/m$
ϵ	$8.8524 \times 10^{-12} F/m$	$1.3835 \times 10^{-11} F/m$
$c = 1/(\mu\epsilon)^{1/2}$	$2.9979 \times 10^8 m/s$	$2.3983 \times 10^8 m/s$
$\eta = (\mu/\epsilon)^{1/2}$	376.7Ω	301.4Ω
Acoustic	air	68.34%Ar+31.66%Kr
γ	1.401	5/3
m	28.96 u	53.8 u
ρ	$1.225 kg/m^3$	$2.277 kg/m^3$
c	$340.4 m/s$	$272.2 m/s$
z_o	$417.0 Pa/(m/s)$	$620.1 Pa/(m/s)$
$\mu \Leftrightarrow z_o/c = \rho$	$1.225 kg/m^3$	$2.277 kg/m^3$
$\epsilon \Leftrightarrow 1/z_o c = 1/K$	$7.044 \times 10^{-6} 1/Pa$	$5.922 \times 10^{-6} 1/Pa$

Having the same relative velocity, the geometrical behaviour of the two lenses, as dictated by geometrical optics or acoustics, is identical. Both lenses convert a point source of radiation

at the focal point into plane waves on the other side of the lens. However, the different values of relative impedance give different values of reflection coefficient; $\Gamma = (Z_{in} - Z_{out}) / (Z_{in} + Z_{out})$ where Z is either z or η as required. For the electromagnetic case the reflection coefficient takes a negative value -0.111 , whereas for the acoustic case it has a positive value $+0.196$. This affects the sign of the waves reflected from the front face of the lens.

Figures 3 and 4 show the results for both electromagnetic and acoustic wave problems. The numerical method used is a first order boundary method known as the Clifford-Cauchy-Dirac (CCD) technique, which is similar to but distinct from the EFIE, MFIE and CFIE second order boundary methods. Details of the CCD method are found in [10], with numerical examples in three, one and two dimensions, found respectively in the papers by Chantaveerod and Seagar [11]–[13].

For both problems a point source of radiation, electromagnetic or acoustic, is located on the left hand side slightly out of frame in figure 3. Diverging waves pass through the lens, standing vertically in centre frame (see figure 4) with flat rear face on the right hand side and convex front face extending from image centre towards the left. Whereas figure 3 shows the total field (incident plus internally transmitted plus externally scattered) figure 4 shows only the externally scattered field.

Inside the lens the diverging waves are formed into plane waves of slightly shorter wavelength. The plane waves persist for some distance beyond the back face. Most of the radiation passes through the lens although there is some divergent reflection from the front face, best seen in figure 4.

The reflection coefficient Γ is 76.6% greater in magnitude for the acoustic case and of opposite sign than for the electromagnetic case. As a consequence somewhat more radiation is reflected from the front face of the acoustic lens and, unlike the electromagnetic case, the reflected wave is not inverted. In figure 4 the displacements of the waves reflected from the front of the two lenses are clearly in opposite directions.

V. CONCLUSION

The main point to note here is that all phenomena with wave equations which adhere to the simple second order form of equation 25, in one or more dimensions, can be solved by a first order method. For these kinds of waves, first order differential equations are sufficient and second order differential equations are not necessary. Using the identification in section III some problems involving waves that are not electromagnetic can be solved with methods commonly used for electromagnetic waves.

Given that generally less is involved in solving a first order than a second order differential equation, it is likely when the choice exists to be more efficient using a first order approach. That should lead to advantages in terms of computational efficiency and speed with the option of solving larger problems in less time, perhaps even with higher accuracy. Where possible, embedding the wave equation into the Maxwellian framework should provide these advantages for computations involving wavelike phenomena currently solved as second order differential equations.

Fig. 3. Total field for electromagnetic (a) and acoustic (b) problems, both solved by same first order method.

Fig. 4. Externally scattered waves for electromagnetic (a) and acoustic (b) problems, showing inversion of wave reflected from left face of lens.

It is worth observing that the wavelike propagation of electromagnetic phenomena emerges naturally from their differential equation as the complete general solution, whereas such a simple direct elementary analysis is nowhere evident for the equivalent second order equation. That $f = (x - ct)$ is indeed a partial solution to the second order wave equation can be argued directly from the first order solution by the squaring of the Maxwellian problem, however that is not the complete general solution. Any direct argument from the second order equation is incomplete unless it discovers the non-travelling terms of equation 31 as well as the travelling terms.

Although the analysis here only covers the simplest possible case of the second order wave equation it is worth noting that a wide class of elliptic operators (of which the Helmholtz operator is but one) admit ‘square roots’ which can be shown to be properly defined and well behaved (see [14]–[16]). It therefore seems likely that the first order method of solution adopted here can be recast and applied also to wave equations more general than in this particular work.

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