Highlights

- Two linear and nonlinear two-stage data envelopment analysis models are compared.
- A relationship between these two models is developed.
- It is shown that the linear model is more computationally efficient.
- The linear model excludes the estimation error of the nonlinear model.
- The linear and nonlinear models are compared with real and simulated data.

Integrated data envelopment analysis: linear vs. nonlinear model

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Abstract

This paper develops a relationship between two linear and nonlinear data envelopment analysis (DEA) models which have previously been developed for the joint measurement of the efficiency and effectiveness of decision making units (DMUs). It will be shown that a DMU is overall efficient by the nonlinear model if and only if it is overall efficient by the linear model. We will compare these two models and demonstrate that the linear model is an efficient alternative algorithm for the nonlinear model. We will also show that the linear model is more computationally efficient than the nonlinear model, it does not have the potential estimation error of the heuristic search procedure used in the nonlinear model, and it determines global optimum solutions rather than the local optimum. Using 11 different data sets from published papers and also 1,000 simulated sets of data, we will explore and compare these two models. Using the data set that is most frequently used in the published papers, it is shown that the nonlinear model with a step size equal to 0.00001, requires running 1,955,573 linear programs (LPs) to measure the efficiency of 24 DMUs compared to only 24 LPs required for the linear model. Similarly, for a very small data set which consists of only 5 DMUs, the nonlinear model requires running 7,861 LPs with step size equal to 0.0001, whereas the linear model needs just 5 LPs.

Keywords: Data envelopment analysis; Efficiency; Effectiveness; Linear programming; Nonlinear programming.

1. Introduction

Simultaneous measurement of efficiency and effectiveness has been a recent research topic in data envelopment analysis (DEA) with so-called ‘integrated’ DEA models having been developed to measure the overall performance of decision making units (DMUs) as combination of their efficiency and effectiveness. Thus, Chiou et al. (2010) developed an integrated DEA model to jointly evaluate the efficiency and effectiveness of a group of intercity bus companies in Taiwan. Within this model, the transport service is known as a non-storable commodity, and the overall performance depends not only on the ability of the DMU to transform resources and inputs to production, but also on its ability to transform production to consumption. This unified measurement of the efficiency and effectiveness is important since it is more informative in identifying the sources of the poor activity performance when compared to simply measuring the efficiency or the effectiveness of the process (Chiou et al., 2010). Other studies which also suggest decomposing the overall performance into its efficiency and effectiveness components include: Karlaftis (2004), Chiou and Chen (2006), Keh et al. (2006), Lan and Lin (2006), Chiou et al. (2007), Yu and Lin (2008), Yu (2008), Yu and Lee (2009), Yu and Fan (2009), Hsieh and Lin (2010), Wu et al. (2013), Huang et al. (2014) and Wong et al. (2015).

Figure 1 is taken from Chiou et al. (2010) and represents the joint measurement of efficiency and effectiveness. Within this model, the technical efficiency is the ratio of the weighted sum of production to the weighted sum of inputs. Service effectiveness is defined as the ratio of the weighted sum of consumption to the weighted sum of production, and technical effectiveness is the ratio of weighted sum of consumption to the weighted sum of inputs.

Figure 1. Joint measurement of the efficiency and effectiveness Chiou et al. (2010, p.478)

The main focus of our paper is not on the application of this conceptual model but rather on the mathematical model developed by Chiou et al. (2010) and later improved by Lim and Zhu (2013). In this respect, a nonlinear model was developed by Chiou et al. (2010) to measure overall performance as the sum of the technical efficiency ratio and the service effectiveness ratio. Optimal multipliers associated with the input, output and consumption factors are determined by the nonlinear model. Chiou et al. (2010) provided a proof that their nonlinear model is a convex optimization problem, and that the values of the multipliers determined by their model are the global optimum and not the local optimum. However, Lim and Zhu (2013) have argued that Chiou et al. (2010) have not used the Hessian matrix correctly in their proof and, as a result, their nonlinear model is actually a non-convex optimization problem. Usability of the model has, therefore, been called into question due to lack of an efficient algorithm to solve it.

In order to solve this nonlinear model, Lim and Zhu (2013) converted the Chiou et al. (2010) model into a parametric linear program (as they called it), and then used a heuristic technique to approximate the global solution. However, the resultant model suggested by Lim and Zhu (2013) is also a nonlinear model involving two variables multiplied while also requiring the user to undertake many steps and run a sequence of multiple linear programs (LPs). Furthermore, this model is not free of estimation error. In summary, in the approach of Lim and Zhu (2013): (i) three different linear programs are run first to determine the upper and lower bounds of a parameter (called h) in the parametric linear model, (ii) a formula is then used to determine other values of the parameter between the upper and lower bounds (iii) a sequence of linear programs associated with each parameter is then run, and (iv) the best heuristic solution is finally selected from the solutions of the parametric model. These steps are explained in more detail in the next section.

On the other hand, there is a linear DEA model developed by Mahdiloo et al. (2015) to simultaneously maximize two ratios of the technical and environmental efficiencies. This linear model was further developed to maximize two efficiency ratios of the stages 1 and 2 of a two-stage activity (Mahdiloo et al., 2016). The same model was also developed in Khodakarami et al. (2016) to simultaneously measure and maximize the efficiency and effectiveness ratios.

The purpose of this paper is, therefore, to develop a mathematical relationship between these two sets of linear (Mahdiloo et al., 2015; Mahdiloo et al., 2016;

Khodakarami et al., 2016) and nonlinear (Lim and Zhu, 2013) models. For simplicity, and to be consistent with Lim and Zhu (2013), the term “nonlinear” henceforth refers to the model developed by Chiou et al. (2010) and “parametric linear” refers to the nonlinear model developed by Lim and Zhu (2013). We will show that the linear model can solve the same problem discussed by Chiou et al. (2010) and Lim and Zhu (2013), but with only one step (rather than following the multiple steps of Lim and Zhu (2013)), with only one linear program for each DMU (rather than using a sequence of linear programs as in Lim and Zhu (2013)), and without the estimation error of the heuristic search procedure of the parametric model.

We also argue that the simultaneous maximization of the efficiency and effectiveness is a special class of the sum of linear ratios (SLR) problem discussed in optimization theory. The objective function of the model in this problem can be defined as the sum of two linear ratios, and can be shown as:

$$\max\{\text{Efficiency ratio} + \text{Effectiveness ratio}\} \quad (1)$$

As discussed by Yanjun et al. (2005), optimization of SLR is a special case of fractional programming which is a focus of research in the area of nonlinear optimization. Similar to some other areas in optimization such as nonlinear fractional programming (Tsai, 2005) and signomial geometric programming (Lin and Tsai, 2012), interest in researching SLR has been growing in light of the large variety of application domains as well as the computational challenges that are faced. Different solution approaches include the “parametric simplex method, outer approximation method, image space method, unified monotonic approach, interior point algorithm, heuristic method, simplicial branch and bound duality-bounds algorithm, concave minimization method, branch and cut technique and branch and bound algorithm” (Jiao and Liu, 2015, p. 723).

In summary, the current paper has some distinctive contributions to the literature. We will develop a mathematical relationship between the linear and the parametric linear models and propose the linear model as an alternative efficient model to the parametric linear model. In doing so, it is shown that both linear and the parametric linear models classify the same DMUs as efficient and inefficient. It is also proved that a DMU is overall efficient by the nonlinear model if and only if it is overall efficient by the linear model. Unlike the parametric linear model offered by Lim and Zhu (2013),

the solutions of the linear model are determined by running only one linear program for each DMU, and the model is free of the estimation error of the heuristic search.

To achieve the above aim, the rest of this paper is organized as follows: In Section 2, the nonlinear and parametric linear models are introduced. In Section 3 the linear model is introduced and its relation with the parametric model is developed. In Section 4.1, a data set published by Chen et al. (2012) is used to compare the linear and the parametric models. In section 4.2, 10 more data sets from the published papers are used to compare the models. Section 4.3, uses 1,000 randomly generated data sets to do the final comparisons. Section 5 concludes the paper.

2. Nonlinear and parametric linear model

We start this section with introducing the nonlinear model of Chiou et al. (2010) and the parametric model of Lim and Zhu (2013).

Model (2) is the nonlinear model developed by Chiou et al. (2010) to simultaneously optimize (maximize) the technical efficiency and service effectiveness of the DMU under evaluation. Optimization is achieved by choosing the best possible values for the decision variables v_j , u_r , and w_s which are the weights of the inputs, production, and consumption, respectively. The objective function of the model is the sum of the technical efficiency ratio $(\frac{\sum_{r=1}^R u_r y_{kr}}{\sum_{j=1}^J v_j x_{kj}})$ and service effectiveness ratio $(\frac{\sum_{s=1}^S w_s z_{ks}}{\sum_{r=1}^R u_r y_{kr}})$, and it has a value between 0 and 2. The DMU is said to be efficient and effective if: $\frac{1}{2} \frac{\sum_{r=1}^R u_r^* y_{kr}}{\sum_{j=1}^J v_j^* x_{kj}} + \frac{\sum_{s=1}^S w_s^* z_{ks}}{\sum_{r=1}^R u_r^* y_{kr}} = 1$. Note that x_{kj} , y_{kr} , and z_{ks} denote the j^{th} input, the r^{th} output, and the s^{th} consumption associated with the DMU_k, respectively. These are the same notations used by Chiou et al. (2010) and Lim and Zhu (2013).

$$\begin{aligned}
 \max \quad & \frac{\sum_{r=1}^R u_r y_{kr}}{\sum_{j=1}^J v_j x_{kj}} + \frac{\sum_{s=1}^S w_s z_{ks}}{\sum_{r=1}^R u_r y_{kr}} \\
 \text{s. t.} \quad & \sum_{r=1}^R u_r y_{ir} - \sum_{j=1}^J v_j x_{ij} \leq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{s=1}^S w_s z_{is} - \sum_{r=1}^R u_r y_{ir} \leq 0 \quad i = 1, 2, \dots, I, \\
 & v_j \geq 0 \quad j = 1, 2, \dots, J, \\
 & w_s \geq 0 \quad s = 1, 2, \dots, S, \\
 & u_r \geq 0 \quad r = 1, 2, \dots, R.
 \end{aligned} \tag{2}$$

This nonlinear model can also be rewritten in the following form (Chiou et al., 2010):

$$\begin{aligned}
 \max \quad & (\sum_{r=1}^R u_r y_{kr}) (\sum_{r=1}^R u_r y_{kr}) + (\sum_{s=1}^S w_s z_{ks}) (\sum_{j=1}^J v_j x_{kj}) \\
 \text{s. t.} \quad & (\sum_{j=1}^J v_j x_{kj}) (\sum_{r=1}^R u_r y_{kr}) = 1 \\
 & \sum_{r=1}^R u_r y_{ir} - \sum_{j=1}^J v_j x_{ij} \leq 0 & i = 1, 2, \dots, I, \\
 & \sum_{s=1}^S w_s z_{is} - \sum_{r=1}^R u_r y_{ir} \leq 0 & i = 1, 2, \dots, I, \\
 & v_j \geq 0 & j = 1, 2, \dots, J, \\
 & w_s \geq 0 & s = 1, 2, \dots, S, \\
 & u_r \geq 0 & r = 1, 2, \dots, R.
 \end{aligned} \tag{3}$$

This model is run once for each DMU, and the efficiency and effectiveness of the DMUs are measured. Chiou et al. (2010) provided a proof that the values of v_j , u_r and w_s determined by the nonlinear Programs (2) or (3) are the global optimum and not the local optimum. Since all the constraints of the Model (2) are linear, Chiou et al. (2010) only investigated the concavity or convexity of the nonlinear objective function of the Model (2). They used the bordered Hessian matrix and argued that the objective function is concave and that it satisfies the requirements for the global solution of the nonlinear model. In other words, they proved that Model (2) is a convex optimization problem. It is important to note that each local optimum is globally optimal in convex optimization, and efficient and reliable algorithms are found in the literature to solve such problems (Lim and Zhu, 2013).

However, Lim and Zhu (2013) have strongly argued against this proof. These authors suggest that Chiou et al. (2010) have not used the Hessian matrix correctly and that Model (2) is actually a non-convex optimization problem. Usability of the Model (2) has, therefore, been called into question due to the lack of an efficient algorithm to solve it. In order to solve this nonlinear model, Lim and Zhu (2013) converted the model into a different nonlinear model involving two variables which are multiplied, and then solved the resultant model as a parametric linear program. A heuristic technique is used to approximate the global solution.

Model (4) is the parametric linear model of Lim and Zhu (2013). This model is obtained after some transformation steps of the nonlinear Model (2), where h is a positive real number.

$$\begin{aligned}
 \max \quad & \sum_{r=1}^R u_r y_{kr} + \sum_{s=1}^S w_s z_{ks} \\
 \text{s. t.} \quad & \sum_{r=1}^R u_r y_{ir} - \sum_{j=1}^J v_j x_{ij} \leq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{s=1}^S w_s z_{is} - h \sum_{r=1}^R u_r y_{ir} \leq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{j=1}^J v_j x_{kj} = 1 \\
 & h \sum_{r=1}^R u_r y_{kr} = 1 \\
 & v_j \geq 0 \quad j = 1, 2, \dots, J \\
 & w_s \geq 0 \quad s = 1, 2, \dots, S \\
 & u_r \geq 0 \quad r = 1, 2, \dots, R
 \end{aligned} \tag{4}$$

Lim and Zhu (2013) suggested a systematic approach to identify a suitable value for h . Lower and upper bounds of h are determined by running three different linear programs, $h \in [h^L, h^U]$. The Model (4) is, thus, solved several times with different values of the parameter h . In the initial run, Model (4) is solved for $h = h^U$. The value of h is then decreased by the small positive number ε (say $\varepsilon = 0.01$) each step, and the model is run with the new h . This process is continued until the model is run with $h = h^L$. After several runs of the model with different values of h , the largest objective function value is determined as the best heuristic solution of the model. It should be noted, however, that this sort of solution procedure is likely to have estimation errors, although with smaller values of ε the solution of the heuristic search will be improved (Liang et al., 2006). Apart from the estimation error, in order to approximate the global optimal solution, it is necessary to go through multiple steps and run a sequence of multiple linear programs. We show the steps of the parametric approach in detail below. Note that this detailed algorithmic presentation of the approach is not offered in Lim and Zhu (2013).

Step 1. Solve the following Model (5) which maximizes the service effectiveness of the DMU under evaluation:

$$\begin{aligned}
 \max \quad & e^* = \sum_{s=1}^S w_s z_{ks} \\
 \text{s. t.} \quad & \sum_{r=1}^R u_r y_{kr} = 1 \\
 & \sum_{s=1}^S w_s z_{is} - \sum_{r=1}^R u_r y_{ir} \leq 0 \quad i = 1, 2, \dots, I, \\
 & w_s \geq 0 \quad s = 1, 2, \dots, S, \\
 & u_r \geq 0 \quad r = 1, 2, \dots, R.
 \end{aligned} \tag{5}$$

Step 2. Given the optimal objective function value of Model (5), solve the following Model (6) to determine the upper bound of h :

$$\begin{aligned}
 h^U = \min \quad & \sum_{j=1}^J v_j x_{kj} \\
 \text{s. t.} \quad & \sum_{r=1}^R u_r y_{ir} - \sum_{j=1}^J v_j x_{ij} \leq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{s=1}^S w_s z_{is} - \sum_{r=1}^R u_r y_{ir} \leq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{r=1}^R u_r y_{kr} = 1 \\
 & \sum_{s=1}^S w_s z_{ks} = e^* \\
 & v_j \geq 0 \quad j = 1, 2, \dots, J, \\
 & w_s \geq 0 \quad s = 1, 2, \dots, S, \\
 & u_r \geq 0 \quad r = 1, 2, \dots, R.
 \end{aligned} \tag{6}$$

Model (6) optimizes the technical efficiency of the DMU subject to keeping the service effectiveness of the DMU unchanged.

Step 3. Solve Model (7) to obtain the lower bound h , i.e. h^L :

$$\begin{aligned}
 \frac{1}{h^L} = \max \quad & \sum_{r=1}^R u_r y_{kr} \\
 \text{s. t.} \quad & \sum_{j=1}^J v_j x_{kj} = 1 \\
 & \sum_{r=1}^R u_r y_{ir} - \sum_{j=1}^J v_j x_{ij} \leq 0 \quad i = 1, 2, \dots, I, \\
 & v_j \geq 0 \quad j = 1, 2, \dots, J, \\
 & u_r \geq 0 \quad r = 1, 2, \dots, R.
 \end{aligned} \tag{7}$$

Step 4. Let $t = 0$.

Step 5. Let $h_t = h^U - \varepsilon t$ where ε is a small positive number, like 0.01.

Step 6. Solve Model (4) with $h = h_t$ and let $e(t)$ be the related optimal objective value (note that $0 < e(t) \leq 2$).

Step 7. If $h_t \geq h^L$, then let $t = t + 1$ and go to Step 5.

Step 8. Let $e(k) = \max\{e(t) | \forall t\}$. The suitable value for h is $h_k = h^U - k\varepsilon$ and the overall efficiency score of DMU_o is $e(k)$.

Now, we deal with the computational complexity issue of the aforementioned algorithm which involves solving some linear programming problems. We assume that the well-known simplex algorithm is utilized for solving the models. Empirically, it is suggested that, on average in most instances, the simplex method roughly requires in the order of $3 \times m$ iterations to solve a linear program in the standard form involving m constraints and n decision variables. Each iteration needs $m(n - m) + n + 1$ multiplications and $m(n - m + 1)$ additions or totally $2mn - 2m^2 + n - 2$ operations. Since the number of operations per iteration is $O(mn)$, the average empirical complexity of the simplex method is $O(m^2n)$. For a deeper discussion on the

computational complexity issue in linear programming and DEA, the readers are referred to Bazaraa et al. (2010), Toloo and Jalili (2014) and Toloo et al. (2015) .

As a result, we investigate the complexity of the simplex method to solve all linear programs in algorithm proposed by Lim and Zhu (2013):

Step 1 (Model (5)). The standard form of Model (5) involves $(I + 1)$ constraints and $(R + S + I)$ decision variables and hence, the related average complexity of the simplex method is $O(I^2(R + S + I))$. Note that the number of operations in each iteration is $2(I + 1)(R + S + I) - s(I + 1)^2 + (R + S + I) - 2$.

Step 2 (Model 6). The average complexity of the simplex method is $O(I^2(R + S + J + I))$.

Step 3 (Model (7)). The average complexity of the simplex method is $O(I^2(R + J + I))$.

Step 6 (Model (4)). The average complexity of the simplex method is $O(I^2(R + S + J + I))$. It should be noted that Model (4) must be solved $\left\lceil \frac{h^U - h^L}{\varepsilon} \right\rceil$ times and hence the total number of times that Model (4) is solved is closely related to the largeness of the width of the interval $[h^L, h^U]$ as well the smallness of the step size ε .

In the next section, the linear model for the simultaneous maximization of the efficiency and effectiveness is introduced. Unlike the parametric model suggested by Lim and Zhu (2013), the linear model does not require the completion of the multiple steps described above, rather only one linear program is run for each DMU. Consequently, the computational complexity is significantly decreased.

3. The linear model

Optimizing the sum of linear ratios has been a popular research topic in nonlinear optimization and in fractional programming. Extensive literature in fractional programming with a focus on the SLR problem argues that, with the sum of concave–convex ratios, there may be several local optima which are not the global optimal solutions (Yanjun et al., 2005; Benson, 2010). This makes SLR one of the most difficult problems in fractional programming (Schaible and Shi, 2003). Furthermore, the problem of the simultaneous maximization of the technical efficiency and service effectiveness is a special class of SLR problem where the numerator of the first ratio and the denominator of the second ratio are identical. The objective function of the

Model (2) is the sum of two ratios $\frac{\sum_{r=1}^R u_r y_{kr}}{\sum_{j=1}^J v_j x_{kj}}$ and $\frac{\sum_{s=1}^S w_s z_{ks}}{\sum_{r=1}^R u_r y_{kr}}$.

To solve this special SLR problem and to develop a linear formulation to the parametric Model (4), a short review of some related works is required. It is shown by Li and Reeves (1999), among many other authors, that a basic DEA model with an objective function to maximize the efficiency score of the DMU under evaluation can be equivalently shown as a model with an objective function which minimizes the inefficiency score of the DMU. Therefore, in the objective function, the model minimizes a deviation variable like d as the inefficiency score instead of maximizing the weighted sum of outputs as an indicator of the efficiency score of the DMUs. This is, however, demonstrated by Li and Reeves (1999) for a DEA model with one ratio to be optimized. Using the same idea, Mahdiloo et al. (2015) simultaneously measured and maximized two ratios of the technical and environmental efficiencies by minimizing two inefficiency deviation variables. To solve a similar type of problem, but in a different context, Mahdiloo et al. (2016) measured the efficiencies of the activities (DMUs) with a two-stage structure by minimizing the inefficiency deviation variables of the stages 1 and 2 of the DMU's activities.

The problem solved by Chiou et al. (2010) and Lim and Zhu (2013) has in fact a two-stage underlying structure, as also noted by Lim and Zhu (2013). This means that the approach used in Mahdiloo et al. (2016) can be adopted to solve the simultaneous measurement of the efficiency and effectiveness. This approach is not, however, used by Chiou et al. (2010) and Lim and Zhu (2013) which has led to development of a nonlinear model in Chiou et al. (2010) and a parametric model in Lim and Zhu (2013). As discussed by Lim and Zhu (2013), the solution of the nonlinear model may be local optima (not global) and it is difficult to check if the solution of the nonlinear model is global. Our paper argues that the alternative parametric model suggested by Lim and Zhu (2013) is computationally complex and may also have potential estimation error.

We, therefore, adopt the approach used by Mahdiloo et al. (2015) and Mahdiloo et al. (2016) and demonstrate that the joint measurement of the efficiency and effectiveness can be solved by a linear model, with less computational complexity and without the potential estimation error. We develop the model in two steps: In step 1, the problem discussed by Chiou et al. (2010) and Lim and Zhu (2013) is mapped into the linear model suggested by Mahdiloo et al. (2016), and in step 2, a clear and rigorous mathematical proof is provided to establish the relationship between the developed linear model and the parametric model of Lim and Zhu (2013). This very important relationship is not noted by Mahdiloo et al. (2016) and therefore is not proved. Thus, it

will be a significant contribution to demonstrate a link between a computationally complex parametric model involving a potential estimation error with a linear model.

Model (8) jointly maximizes the efficiency and the effectiveness scores of the DMU under evaluation by minimizing two deviation variables of d_k and d'_k . This is similar to the approach suggested by Mahdiloo et al. (2015) to simultaneously maximize the technical and ecological efficiencies, and Mahdiloo et al. (2016) which maximizes the efficiencies of the stages 1 and 2 of a two-stage activity.

$$\begin{aligned}
 \min \quad & d_k + d'_k \\
 \text{s. t.} \quad & \sum_{j=1}^J v_j x_{ij} - \sum_{r=1}^R u_r y_{ir} \geq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{j=1}^J v_j x_{kj} - \sum_{r=1}^R u_r y_{kr} - d_k = 0 \\
 & \sum_{r=1}^R u_r y_{ir} - \sum_{s=1}^S w_s z_{is} \geq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{r=1}^R u_r y_{kr} - \sum_{s=1}^S w_s z_{ks} - d'_k = 0 \\
 & \sum_{r=1}^R u_r y_{kr} = 1 \\
 & v_j \geq 0 \quad j = 1, 2, \dots, J, \\
 & w_s \geq 0 \quad s = 1, 2, \dots, S, \\
 & u_r \geq 0 \quad r = 1, 2, \dots, R, \\
 & d_k \geq 0 \\
 & d'_k \geq 0.
 \end{aligned} \tag{8}$$

The problem of the simultaneous maximization of $\frac{\sum_{r=1}^R u_r y_{kr}}{\sum_{j=1}^J v_j x_{kj}}$ and $\frac{\sum_{s=1}^S w_s z_{ks}}{\sum_{r=1}^R u_r y_{kr}}$ as the technical efficiency and service effectiveness ratios can be seen as a multiple objective problem. The purpose of Model (8) is to maximize $\sum_{s=1}^S w_s z_{ks} - \sum_{j=1}^J v_j x_{kj}$ which is equivalent to maximizing the ratios $\frac{\sum_{r=1}^R u_r y_{kr}}{\sum_{j=1}^J v_j x_{kj}}$ and $\frac{\sum_{s=1}^S w_s z_{ks}}{\sum_{r=1}^R u_r y_{kr}}$ together under the normalization constraint $\sum_{r=1}^R u_r y_{kr} = 1$. The normalization constraint $\sum_{r=1}^R u_r y_{kr} = 1$ prevents the model from the trivial optimal solution $(v^*, w^*, u^*, d_k, d'_k) = (0_J, 0_S, 0_R, 0, 0)$. In the ideal condition, $\frac{\sum_{r=1}^R u_r^* y_{kr}}{\sum_{j=1}^J v_j^* x_{kj}}$ and $\frac{\sum_{s=1}^S w_s^* z_{ks}}{\sum_{r=1}^R u_r^* y_{kr}}$ are equal to 1. The DMU under evaluation is efficient and effective if:

$$\begin{cases} \sum_{j=1}^J v_j^* x_{kj} - \sum_{r=1}^R u_r^* y_{kr} = 0 \\ \sum_{r=1}^R u_r^* y_{kr} - \sum_{s=1}^S w_s^* z_{ks} = 0 \end{cases} \tag{9}$$

We define $d_k = \sum_{j=1}^J v_j^* x_{kj} - \sum_{r=1}^R u_r^* y_{kr}$ as the deviation from full technical efficiency score of 1, and $d'_k = \sum_{r=1}^R u_r^* y_{kr} - \sum_{s=1}^S w_s^* z_{ks}$ as the deviation from full

service effectiveness of 1. The objective function of the Model (8) should, therefore, minimize $d_k + d'_k$. The objective function of the model minimizes the sum of the deviations from the full technical efficiency score of 1 and the full service effectiveness of 1. The common term between the technical efficiency ratio and service effectiveness ratio, i.e. $\sum_{r=1}^R u_r y_{kr}$, is set equal to 1. In order to minimize d_k , the model searches for the solutions to minimize $\sum_{j=1}^J v_j x_{kj} - 1$. Also to minimize d'_k , the model searches for the solutions which minimize $1 - \sum_{s=1}^S w_s z_{ks}$. In other words, $d_k + d'_k$ is minimized by minimizing $\sum_{j=1}^J v_j x_{kj}$ and maximizing $\sum_{s=1}^S w_s z_{ks}$. This linear model does not require completion of the 8 steps of the Lim and Zhu (2013) parametric linear program (discussed in Section 2) and, in addition, running a sequence of multiple linear programs is no longer required. In addition, this model is free of the estimation error of the heuristic search used by Lim and Zhu (2013). Therefore, we propose Model (8) as a straightforward and efficient alternative to Model (4).

However, it is important to mathematically prove the relationship between the linear Model (8) and the parametric Model (4). Therefore, this section considers this very important task. As inspection makes it clear, Model (8) is equivalent to the following linear program which was also proposed by Khodakarami et al. (2016).

$$\begin{aligned}
 \max \quad & \sum_{s=1}^S w_s z_{ks} - \sum_{j=1}^J v_j x_{kj} \\
 \text{s. t.} \quad & \sum_{r=1}^R u_r y_{ir} - \sum_{j=1}^J v_j x_{ij} \leq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{s=1}^S w_s z_{is} - \sum_{r=1}^R u_r y_{ir} \leq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{r=1}^R u_r y_{kr} = 1 \\
 & v_j \geq 0 \quad j = 1, 2, \dots, J, \\
 & w_s \geq 0 \quad s = 1, 2, \dots, S, \\
 & u_r \geq 0 \quad r = 1, 2, \dots, R,
 \end{aligned} \tag{10}$$

In order to prove the relation between Model (10) and Model (4), we first state and prove the following Lemma:

Lemma 1: Let (v^*, u^*, w^*) be the optimal solution of Model (10), then $\sum_{j=1}^J v_j^* x_{kj} = v^* x_k \in [h^L, h^U]$.

Proof.

Let $(\bar{v}^*, \bar{u}^*)$ be an optimal solution for Model (7). It is clear on inspection that $(\frac{v^*}{v^* x_k}, \frac{u^*}{v^* x_k})$ is a feasible solution for Model (7) where $\frac{v^*}{v^* x_k} = (\frac{v_1^*}{v^* x_k}, \dots, \frac{v_m^*}{v^* x_k})$ and

$\frac{\mathbf{u}^*}{\mathbf{v}^* \mathbf{x}_k} = \left(\frac{u_1^*}{v^* x_k}, \dots, \frac{u_s^*}{v^* x_k} \right)$ and hence its objective value $\frac{\mathbf{u}^*}{\mathbf{v}^* \mathbf{x}_k} \mathbf{y}_k = \frac{\mathbf{u}^* \mathbf{y}_k}{\mathbf{v}^* \mathbf{x}_k} = \frac{1}{v^* x_k}$ is less than or equal to the optimal objective value $\frac{1}{h^L}$ or equivalently $\mathbf{v}^* \mathbf{x}_k \geq h^L$.

Now, let $(\bar{\mathbf{v}}^*, \bar{\mathbf{u}}^*, \bar{\mathbf{w}}^*)$ be an optimal solution for Model (6). Obviously, $(\bar{\mathbf{v}}^*, \bar{\mathbf{u}}^*, \bar{\mathbf{w}}^*)$ is a feasible solution for Model (10) and hence $w^* z_k - v^* x_k \geq \bar{w}^* z_k - \bar{v}^* x_k$. Taking $\bar{v}^* x_k = h^U$ into consideration, we have $w^* z_k - \bar{w}^* z_k \geq v^* x_k - h^U$. Note that $(\bar{\mathbf{u}}^*, \bar{\mathbf{w}}^*)$ and $(\mathbf{u}^*, \mathbf{w}^*)$ are optimal and feasible solutions for Model (5), respectively, and hence $w^* z_k \leq \bar{w}^* z_k$ which implies $v^* x_k \leq h^U$. $\square$

Now, taking Lemma 1 into consideration, we prove the following essential theorem:

Theorem 1: Let $(\mathbf{v}^*, \mathbf{u}^*, \mathbf{w}^*)$ be the optimal solution of Model (10), then the model is equivalent to Model (4) when $h = \mathbf{v}^* \mathbf{x}_k$.

Proof.

Substituting $h = \mathbf{v}^* \mathbf{x}_k$ converts Model (10) to the following equivalent LP model:

$$\begin{aligned}
 \max \quad & \sum_{s=1}^S w_s z_{ks} \\
 \text{s. t.} \quad & \sum_{r=1}^R u_r y_{ir} - \sum_{j=1}^J v_j x_{ij} \leq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{s=1}^S w_s z_{is} - \sum_{r=1}^R u_r y_{ir} \leq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{r=1}^R u_r y_{kr} = 1 \\
 & \sum_{j=1}^J v_j x_{kj} = h \\
 & v_j \geq 0 \quad j = 1, 2, \dots, J, \\
 & w_s \geq 0 \quad s = 1, 2, \dots, S, \\
 & u_r \geq 0 \quad r = 1, 2, \dots, R,
 \end{aligned} \tag{11}$$

It should be noted that $h \in [h^L, h^U]$ by Lemma 1.

Now by taking the change of variables $\bar{u}_r = \frac{u_r}{h} \forall r$ and $\bar{v}_i = \frac{v_i}{h} \forall i$, we obtain

$$\begin{aligned}
 \max \quad & \sum_{s=1}^S w_s z_{ks} \\
 \text{s. t.} \quad & \sum_{r=1}^R \bar{u}_r y_{ir} - \sum_{j=1}^J \bar{v}_j x_{ij} \leq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{s=1}^S w_s z_{is} - \sum_{r=1}^R h \bar{u}_r y_{ir} \leq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{j=1}^J \bar{v}_j x_{kj} = 1 \\
 & \sum_{r=1}^R h \bar{u}_r y_{kr} = 1 \\
 & \bar{v}_j \geq 0 \quad j = 1, 2, \dots, J \\
 & w_s \geq 0 \quad s = 1, 2, \dots, S \\
 & \bar{u}_r \geq 0 \quad r = 1, 2, \dots, R
 \end{aligned} \tag{12}$$

Clearly, adding a constant to the objective function does not change the optimal solution, hence, we have the following equivalent LP model:

$$\begin{aligned}
 \max \quad & \frac{1}{h} + \sum_{s=1}^S w_s z_{ks} \\
 \text{s. t.} \quad & \sum_{r=1}^R \bar{u}_r y_{ir} - \sum_{j=1}^J \bar{v}_j x_{ij} \leq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{s=1}^S w_s z_{is} - \sum_{r=1}^R \bar{h} u_r y_{ir} \leq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{j=1}^J \bar{v}_j x_{kj} = 1 \\
 & \sum_{r=1}^R h \bar{u}_r y_{kr} = 1 \\
 & \bar{v}_j \geq 0 \quad j = 1, 2, \dots, J \\
 & w_s \geq 0 \quad s = 1, 2, \dots, S \\
 & \bar{u}_r \geq 0 \quad r = 1, 2, \dots, R
 \end{aligned} \tag{13}$$

From the constant $\sum_{r=1}^R h \bar{u}_r y_{kr} = 1$ we have $\frac{1}{h} = \sum_{r=1}^R \bar{u}_r y_{kr}$ which leads to the following model

$$\begin{aligned}
 \max \quad & \sum_{r=1}^R \bar{u}_r y_{kr} + \sum_{s=1}^S w_s z_{ks} \\
 \text{s. t.} \quad & \sum_{r=1}^R \bar{u}_r y_{ir} - \sum_{j=1}^J \bar{v}_j x_{ij} \leq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{s=1}^S w_s z_{is} - \sum_{r=1}^R \bar{h} u_r y_{ir} \leq 0 \quad i = 1, 2, \dots, I, \\
 & \sum_{j=1}^J \bar{v}_j x_{kj} = 1 \\
 & \sum_{r=1}^R h \bar{u}_r y_{kr} = 1 \\
 & \bar{v}_j \geq 0 \quad j = 1, 2, \dots, J \\
 & w_s \geq 0 \quad s = 1, 2, \dots, S \\
 & \bar{u}_r \geq 0 \quad r = 1, 2, \dots, R
 \end{aligned} \tag{14}$$

This fact that the aforementioned manipulations are reversible, completes the proof.

□

Theorem 2: A DMU is overall efficient by the nonlinear Model (4) if and only if it is overall efficient by the LP Model (10).

Proof.

Let $(\mathbf{v}^*, \mathbf{u}^*, \mathbf{w}^*, h^*)$ be the optimal solution of Model (4) and DMU_k is overall efficient. Hence, from the feasibility conditions we have $h^* \mathbf{u}^* \mathbf{y}_k = 1$, and $\mathbf{v}^* \mathbf{x}_k = 1$ and the optimality condition leads to $\mathbf{u}^* \mathbf{y}_k + \mathbf{w}^* \mathbf{z}_k = 2$ (or equivalently $\mathbf{u}^* \mathbf{y}_k = \mathbf{w}^* \mathbf{z}_k = 1$). It is easy to verify that $(h^* \mathbf{v}^*, h^* \mathbf{u}^*, \mathbf{w}^*)$ is a feasible solution of Model (10) and its objective function value is $\mathbf{w}^* \mathbf{z}_k - \mathbf{v}^* \mathbf{x}_k = 0$ which is the maximum value for the objective function $\mathbf{w} \mathbf{z}_k - \mathbf{v} \mathbf{x}_k \leq 0$. As a result, DMU_k is overall efficient by the LP Model (10).

Now, assume that $(\mathbf{v}^*, \mathbf{u}^*, \mathbf{w}^*)$ is an optimal solution of Model (10) and DMU_k is overall efficient. Hence, $\mathbf{u}^* \mathbf{y}_k = 1$ and $\mathbf{w}^* \mathbf{z}_k - \mathbf{v}^* \mathbf{x}_k = 0$, i.e. $\mathbf{w}^* \mathbf{z}_k = \mathbf{v}^* \mathbf{x}_k$, follow from the feasibility and optimality conditions, respectively. An easy computation points out that $(\mathbf{v}^*, \mathbf{u}^*, \mathbf{w}^*, h^*) = (\mathbf{v}^*, \mathbf{u}^*, \mathbf{w}^*, \mathbf{v}^* \mathbf{x}_k)$ is a feasible solution of Model (4) and its

objective function value is $\mathbf{u}^* \mathbf{y}_k + \mathbf{w}^* \mathbf{z}_k = 1 + \mathbf{v}^* \mathbf{x}_k = 2$ which is the maximum value for the objective function $\mathbf{u} \mathbf{y}_k + \mathbf{w} \mathbf{z}_k \leq 2$. Consequently, DMU_k is overall efficient by the NLP Model (10). $\square$

4. Application

4.1. Sustainable design

The linear Model (8) or (10) and the parametric linear model of Lim and Zhu (2013) are applied in a data set used by Chen et al. (2012, p. 355). This data set was then used by Mahdiloo et al. (2016, p. 324). The data set is related to 23 car lines of an American company, and is used to evaluate the sustainable design performances of the carlines. Different carlines lead to the production of cars with different designs. Each design can, therefore, be considered as a DMU to be evaluated. Each design is made of two internal modules, namely: the “industrial design process” and the “bio-design process”. The overall performance of each design is measured as the combination of the industrial design process performance and bio-design process performance.

The industrial design process performance is evaluated in terms of the engineering specifications and product attributes, and is based on the ratio of the weighted sum of the product attributes to the weighted sum of the engineering specifications. The bio-design process is evaluated as the ratio of the weighted sum of the environmental factors to the weighted sum of the product attributes. Product attributes are in the numerator of the industrial design process ratio and in the denominator of the bio design process ratio. Therefore, the aim is to jointly maximize these two ratios, and it will be seen that this represents a similar problem to that of simultaneously maximizing efficiency and effectiveness discussed above. In this example, we apply the parametric linear and linear models to this data set and jointly measure the overall performance of the carlines or the designs.

The engineering specifications considered in the evaluation of the industrial design process are the cubic inch displacement (cid), rated horsepower (rhp), compression ratio (cmp) and axle ratio (axle). The equivalent test weight (etw) and fuel economy (mpg) are the product attributes considered in both the industrial design process and bio design process ratios. The hydrocarbon emissions (hc), carbon monoxide emissions (CO), carbon dioxide emissions (CO₂), and nitrogen oxide emissions (no_x) are the factors used in the numerator of the bio-design process ratio. In

order to make the data fit the DEA approach, Chen et al. (2012) suggested that the reciprocal of the emissions level should be used. The same applies to the cubic inch displacement, rated horsepower, compression ratio, and equivalent test weights where their reciprocals are also used.

Table 1 shows the data set, and the results are shown in Table 2 where e_1 , e_2 , and e represent the efficiency of the industrial design process, bio-design process, and the overall performance of each DMU, respectively. We have reported the v^*x_k values for all DMUs in the last column of Table 2. The results of the parametric model are based on following the 8 steps of the model introduced in Section 2 and the running of Model (4). The results of the linear model are obtained simply by running Model (8) or (10). Unlike the linear model, the parametric model requires running a sequence of many models. Models (5), (6), and (7) are used first to determine the upper and lower bounds of h and the results are shown in Table 3. $h_t = h^U - \varepsilon \times t$ is then used to calculate the other values of h . Model (4) is finally run with each parameter h . The final best values of h which result in the best efficiency scores of the DMUs are shown in the fifth column of Table 2.

The detailed results of the parametric model for two sample DMUs 17 and 23 are presented in Tables 4 and 5, respectively. As can be seen from the Table 4, Model (4) is run 44 times associated with 44 parameters h for DMU₁₇. The optimal solution of the model is obtained with $h = 1.2198$ where it gives the maximum overall performance of the DMU by $e = 0.5334$. The overall performance of the same DMU (DMU₁₇) calculated with the linear model is also equal to 0.5334. However, the linear model attains the same result by running only 1 linear program compared with 47 linear programs (3 linear programs to calculate the upper and lower bounds of h , and 44 linear programs to run Model (4) with different h parameter values) of the parametric linear approach.

A similar analysis can be undertaken for DMU₂₃ with the results presented in Table 5. There are 37 h values for DMU₂₃ and Model (4) is, therefore, run 37 times for this DMU. The best solution of the model is $e = 0.5836$ where $h = 1.1136$. The efficiency score obtained for this DMU with the linear model is 0.5845 which is slightly larger than the parametric model. The solution quality of the linear model is, therefore, slightly better than the parametric model since it does not have the estimation error of

the parametric search procedure. This is also the case with DMU_{16} which has $e = 0.9438$ with the parametric model and $e = 0.9440$ with the linear model.

To consider the complexity of the parametric model from a different angle, the number of LPs and the iterations of the simplex are shown in the sixth and the seventh columns of Table 2. As can be seen from the table, in order to complete the analysis with the parametric model, 351 LPs with 2,970 iterations are run to get the efficiency scores of the 23 DMUs. This is a relatively larger number of LPs to be solved compared to only 23 LPs with the linear model.

The above discussion is based on the requirement to run 351 LPs with an assumed the step size (ε) value equal to 0.01. However, we have investigated the required number of LPs with different ε values and the results are shown graphically in Figure 2. As suggested by Liang et al. (2006), smaller values of ε can improve the solution of the heuristic search. Considering this and also the arbitrariness of the ε value, it would be possible to use very small value like $\varepsilon = 0.0001$ in $h_t = h^U - \varepsilon \times t$. However, as can be seen from the Figure 2, this will increase the number of LPs to 2,520 which is significantly a larger number of LPs compared to running 23 LPs with Model (10).

Table 1. Data set for the carlines*

Carline (DMU)	cid (x_1)	rhp (x_2)	cmp (x_3)	axle (x_4)	etw (y_1)	mpg (y_2)	Hc (z_1)	CO (z_2)	CO ₂ (z_3)	NO _x (z_4)
1	268.31	247.23	9.84	3.31	3923	19.32	0.0216	0.4108	329.77	0.0054
2	140	160	9.7	4.04	3583	25.83	0.0067	0.1667	343.67	0.0033
3	182	217	10	3.46	3750	22.3	0.024	0.18	397	0.01
4	182	215.5	10	3.46	3938	20.85	0.0075	0.0425	107.5	0.005
5	281	250	9.4	3.27	4500	15.5	0.04	0.57	485.5	0.005
6	161	205.5	10	3.82	3688	25.55	0.0055	0.145	157	0.005
7	140	155	9.7	2	4250	37.4	0.008	0.23	237	0.02
8	140	155	9.7	2	4000	45.2	0.007	0.19	196	0.01
9	244	210	9.7	3.64	5250	16.1	0.028	0.53	283	0.005
10	280	292	9.8	3.55	5250	16.75	0.026	0.36	531.5	0.015
11	330	310	9.8	3.31	6000	15.27	0.04	0.7467	522.33	0.01
12	280	262.67	9.4	3.69	5556	16.9	0.0062	0.1333	117	0.0011
13	280	292	9.8	3.55	5000	17.9	0.026	0.21	495	0.01
14	330	310	9.8	3.31	6000	17.1	0.039	0.945	517.5	0.01
15	330	310	9.8	3.15	5375	17.85	0.0215	0.445	250.5	0.005
16	330	310	9.8	3.53	5900	16.92	0.0132	0.228	105	0.002
17	330	310	9.8	3.31	6000	11.6	0.042	0.35	532	0.01
18	330	310	9.8	3.15	5375	12.9	0.0145	0.235	244.5	0.005
19	330	310	9.8	3.53	5900	12.22	0.007	0.084	102.8	0.004
20	140	171	9.7	4.13	3625	28.3	0.011	0.29	314	0.01
21	140	171	9.7	4.09	3875	23.75	0.0235	0.335	374	0.01
22	140	143	9.7	4.01	3594	24.28	0.0225	0.2325	366.75	0.01
23	244	207	9.7	3.85	4250	17.6	0.032	0.5767	334	0.0033

*The data shown here are the original data. The reciprocals of the cid, rhp, cmp, etw, hc, CO, CO₂ and no_x are used to run the parametric linear and linear models.

Table 2. Results with the parametric linear and linear models

DMU	Parametric linear model						Linear model			
	e_1	e_2	e	h	#LPs	iterations	e_1	e_2	e	v^*x_k
1	1.0000	0.2382	0.6191	1.0000	27	255	1.0000	0.2382	0.6191	1.0000
2	0.9984	0.5968	0.7976	1.0016	14	148	0.9984	0.5968	0.7976	1.0016
3	1.0000	0.2525	0.6263	1.0000	5	41	1.0000	0.2525	0.6263	1.0000
4	0.9523	1.0000	0.9761	1.0501	5	33	0.9523	1.0000	0.9761	1.0501
5	0.9130	0.1905	0.5518	1.0953	40	306	0.9130	0.1905	0.5518	1.0953
6	1.0000	0.7483	0.8741	1.0000	5	45	1.0000	0.7483	0.8741	1.0000
7	0.9412	0.5928	0.7670	1.0625	5	19	0.9412	0.5928	0.7670	1.0625
8	1.0000	0.6377	0.8188	1.0000	5	17	1.0000	0.6377	0.8188	1.0000
9	0.7374	0.3465	0.5419	1.3562	5	41	0.7374	0.3465	0.5419	1.3562
10	0.9319	0.2230	0.5775	1.0731	10	120	0.9319	0.2230	0.5774	1.0731
11	0.8795	0.2034	0.5415	1.1370	13	140	0.8795	0.2034	0.5415	1.1370
12	0.8276	1.0000	0.9138	1.2084	5	28	0.8276	1.0000	0.9138	1.2084
13	0.9851	0.2897	0.6374	1.0151	11	102	0.9851	0.2897	0.6374	1.0151
14	0.9488	0.1749	0.5618	1.0539	21	191	0.9488	0.1749	0.5618	1.0539
15	1.0000	0.3701	0.6851	1.0000	14	125	1.0000	0.3701	0.6850	1.0000
16	0.9231	0.9645	0.9438	1.0833	12	87	0.9259	0.9622	0.9440	1.0800
17	0.8198	0.2470	0.5334	1.2198	48	378	0.8198	0.2470	0.5334	1.2198
18	0.9038	0.4371	0.6705	1.1065	44	433	0.9152	0.4246	0.6699	1.0927
19	0.8337	1.0000	0.9169	1.1994	5	39	0.8337	1.0000	0.9169	1.1994
20	0.9981	0.3634	0.6808	1.0019	6	56	0.9981	0.3634	0.6808	1.0019
21	0.9232	0.1916	0.5574	1.0832	5	48	0.9232	0.1916	0.5574	1.0832
22	0.9954	0.2109	0.6031	1.0047	5	45	0.9954	0.2109	0.6031	1.0047
23	0.8980	0.2692	0.5836	1.1136	41	273	0.8956	0.2733	0.5845	1.1166

Table 3. Upper and lower bounds of h

Carline (DMU)	h^L	h^U
1	1.0000	1.2207
2	1.0016	1.1121
3	1.0000	1.0000
4	1.0501	1.0501
5	1.0953	1.4529
6	1.0000	1.0000
7	1.0625	1.0625
8	1.0000	1.0000
9	1.3554	1.3562
10	1.0731	1.1331
11	1.1370	1.2435
12	1.2084	1.2084
13	1.0151	1.0791
14	1.0539	1.2435
15	1.0000	1.1140
16	1.0610	1.1554
17	1.2198	1.6531
18	1.0927	1.4865
19	1.1994	1.2228
20	1.0019	1.0133
21	1.0832	1.0832
22	1.0047	1.0047
23	1.1107	1.4735

Table 4. Results of the heuristic search procedure of the parametric linear model for DMU₁₇

t	$h_t = h^U - \varepsilon t$	e_1	e_2	e
0	1.6531	0.6049	0.2950	0.4500
1	1.6431	0.6086	0.2948	0.4517
2	1.6331	0.6123	0.2945	0.4534
3	1.6231	0.6161	0.2942	0.4552
4	1.6131	0.6199	0.2940	0.4569
5	1.6031	0.6238	0.2937	0.4587
6	1.5931	0.6277	0.2934	0.4606
7	1.5831	0.6317	0.2931	0.4624
8	1.5731	0.6357	0.2928	0.4642
9	1.5631	0.6398	0.2923	0.4660
10	1.5531	0.6439	0.2918	0.4678
11	1.5431	0.6481	0.2913	0.4697
12	1.5331	0.6523	0.2908	0.4716
13	1.5231	0.6566	0.2903	0.4735
14	1.5131	0.6609	0.2899	0.4754
15	1.5031	0.6653	0.2894	0.4773
16	1.4931	0.6698	0.2889	0.4793
17	1.4831	0.6743	0.2884	0.4813
18	1.4731	0.6788	0.2879	0.4834
19	1.4631	0.6835	0.2874	0.4855
20	1.4531	0.6882	0.2866	0.4874
21	1.4431	0.6930	0.2854	0.4892
22	1.4331	0.6978	0.2843	0.4910
23	1.4231	0.7027	0.2831	0.4929
24	1.4131	0.7077	0.2819	0.4948
25	1.4031	0.7127	0.2807	0.4967
26	1.3931	0.7178	0.2796	0.4987
27	1.3831	0.7230	0.2784	0.5007
28	1.3731	0.7283	0.2772	0.5027
29	1.3631	0.7336	0.2760	0.5048
30	1.3531	0.7391	0.2749	0.5070
31	1.3431	0.7446	0.2737	0.5091
32	1.3331	0.7501	0.2725	0.5113
33	1.3231	0.7558	0.2713	0.5136
34	1.3131	0.7616	0.2701	0.5159
35	1.3031	0.7674	0.2690	0.5182
36	1.2931	0.7733	0.2678	0.5206
37	1.2831	0.7794	0.2666	0.5230
38	1.2731	0.7855	0.2654	0.5255
39	1.2631	0.7917	0.2643	0.5280
40	1.2531	0.7980	0.2623	0.5301
41	1.2431	0.8045	0.2577	0.5311
42	1.2331	0.8110	0.2531	0.5320
43	1.2231	0.8176	0.2485	0.5331
44	1.2198	0.8198	0.2470	0.5334 (best solution)

Table 5. Results of the heuristic search procedure of the parametric linear model for DMU₂₃

t	$h_t = h^U - \varepsilon t$	e_1	e_2	e
0	1.4735	0.6786	0.3285	0.5036
1	1.4635	0.6833	0.3278	0.5055
2	1.4535	0.6880	0.3271	0.5075
3	1.4435	0.6927	0.3263	0.5095
4	1.4335	0.6976	0.3256	0.5116
5	1.4235	0.7025	0.3249	0.5137
6	1.4135	0.7074	0.3241	0.5158
7	1.4035	0.7125	0.3234	0.5179
8	1.3935	0.7176	0.3227	0.5201
9	1.3835	0.7228	0.3219	0.5223
10	1.3735	0.7280	0.3212	0.5246
11	1.3635	0.7334	0.3204	0.5269
12	1.3535	0.7388	0.3197	0.5293
13	1.3435	0.7443	0.3190	0.5316
14	1.3335	0.7499	0.3182	0.5341
15	1.3235	0.7555	0.3175	0.5365
16	1.3135	0.7613	0.3166	0.5390
17	1.3035	0.7671	0.3155	0.5413
18	1.2935	0.7731	0.3143	0.5437
19	1.2835	0.7791	0.3132	0.5461
20	1.2735	0.7852	0.3120	0.5486
21	1.2635	0.7914	0.3109	0.5511
22	1.2535	0.7977	0.3097	0.5537
23	1.2435	0.8042	0.3086	0.5564
24	1.2335	0.8107	0.3074	0.5590
25	1.2235	0.8173	0.3063	0.5618
26	1.2135	0.8240	0.3051	0.5646
27	1.2035	0.8309	0.3039	0.5674
28	1.1935	0.8378	0.3028	0.5703
29	1.1835	0.8449	0.3010	0.5729
30	1.1735	0.8521	0.2969	0.5745
31	1.1635	0.8594	0.2928	0.5761
32	1.1535	0.8669	0.2887	0.5778
33	1.1435	0.8745	0.2846	0.5795
34	1.1335	0.8822	0.2805	0.5814
35	1.1235	0.8900	0.2764	0.5832
36	1.1135	0.8980	0.2692	0.5836 (best solution)
37	1.1107	0.9003	0.2640	0.5822

Figure 2. Number of linear programs (LPs) solved to complete the steps of the parametric linear approach with different step sizes (ϵ)

4.2. Experiments with data sets from the published papers

To test if Model (4) can be approximated to model (10), this section uses another 10 different data sets which were used in the related literature as shown in Table 6. The number of DMUs, number of inputs, number of intermediated measures and number of outputs of these data sets are also reported in Table 6. Since the purpose of this section is to only demonstrate the usefulness of the linear model, the details of the problems being discussed in the papers and their context are not explored here, and the interested readers are referred to the papers. The last column of the table shows the required number of LPs to complete the steps of the parametric model (Model 4) with each data set.

Table 6. Data sets from the published papers and number of LPs

#Dataset	Paper	DMU	# x	# y	# z	# LPs
1	Wang et al. (1997), Chen and Zhu (2004)	27	3	1	2	108
2	Kao and Hwang (2008), Chen et al. (2010)	24	2	2	2	19,640
3	Chen et al. (2016)	30	2	2	1	12,285
4	Liu and Wang (2009)	17	3	2	1	14,626
5	Yang et al. (2011), An et al. (2016)	17	3	2	2	18,341
6	Liang et. al (2006) ²	10	3	3	2	4,198
7	Li et al. (2012) ³	30	3	2	4	16,539
8	Chen et al. (2013)	5	2	2	2	802
9	Chiou and Chen (2006)	15	3	2	2	3,009
10	Cao and Yang (2011)	40	3	1	1	160

Table 7 reports the mean absolute deviation (MAD), Pearson and Spearman rank correlations between e_1 , e_2 and e results of Model (10) and Model (4). A similar analysis was used by Ruggiero (1998), Syrjänen (2004) and Estelle et al. (2010) to explore the results of their simulated data.

Table 7. Correlation coefficients and MADs between Models (4) and (10) results

#Dataset	e_1			e_2			e		
	Pearson	Spearman	MAD	Pearson	Spearman	MAD	Pearson	Spearman	MAD
1	1.0000	0.9995	0.0000	1.0000	0.9997	0.0000	1.0000	1.0000	0.0000
2	0.8463	0.9774	0.0229	0.9110	0.9591	0.0260	0.9986	0.9974	0.0016
3	0.9913	0.9755	0.0058	0.9973	0.9986	0.0080	0.9998	0.9996	0.0344
4	1.0000	0.9386	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000
5	0.9950	0.9877	0.0083	0.9941	0.9926	0.0130	0.9994	0.9926	0.0024
6	0.8716	0.8788	0.0328	0.8531	0.7792	0.0366	0.9991	1.0000	0.0019
7	0.9459	0.9454	0.0159	0.9670	0.9804	0.0195	0.9990	0.9951	0.0018
8	0.9963	1.0000	0.0108	0.9539	0.9747	0.0181	0.9967	1.0000	0.0036
9	0.8609	0.8927	0.0056	0.9800	0.9910	0.0060	1.0000	0.9991	0.0002
10	1.0000	0.9992	0.0000	1.0000	1.0000	0.0000	1.0000	1.0000	0.0000
Mean	0.9507	0.9595	0.0102	0.9656	0.9675	0.0127	0.9993	0.9984	0.0046

The results, clearly suggest that a strong relationship exists between Model (4) and Model (10) in each of the three measures of technical efficiency (e_1), service effectiveness (e_2) and the overall efficiency (e) and thus we would argue that Model (10) can approximate the efficiency scores. Specifically, the p values produced for all the Pearson and Spearman rank correlations suggest that the relationship between Models (4) and (10) is significant at the 0.05 level. We have used the MAD to consider the difference between efficiency scores produced by Models (4) and (10). MAD has

² The data used in Liang et. al (2006) is a special problem of a two-stage DEA problem since there is one output in the first stage that does not enter the second stage. Also one of the inputs of the second stage is not an output of the first stage. Therefore, to run the parametric linear and linear models, we have assumed that all outputs of the first stage are the inputs of the second stage, and the input of the second stage which is not the output of the first stage is removed from our analysis.

³ The data in Li et al. (2012) also includes one input in the second stage which is additional to the outputs of the first stage. This variable is removed from our analysis.

been often used in many DEA studies with simulated data and in this regard Giokas (2002, p. 267) notes that: “MAD expresses the average of the absolute values of the errors and measures the overall accuracy of the results. It provides an indication of how large, in absolute terms, the error is”. Thus, higher correlation coefficients and lower MADs indicate a closer approximation (Ruggiero, 1998; Estelle et al., 2010).

As it is reported in Table 7, all the MADs are relatively low where the *maximum* MADs for e_1 , e_2 and e are 0.0328, 0.0366 and 0.0344, respectively, and the *mean* MADs for e_1 , e_2 and e are 0.0102, 0.0127 and 0.0046, respectively. By reviewing many studies which have used this measure (see for example Ruggiero (1998) who classified MADs varying from 0.04 to 0.11 as “relatively low” in his study) we can conclude that Model (10) is a good approximate of Model (4) in measuring e_1 , e_2 and e .

These experiments lead to some interesting results. As can be seen from the table, with all data sets from these published papers, Pearson correlation coefficient between the overall efficiency scores of the linear and parametric linear approaches and the Spearman rank correlation between the rankings of DMUs with these two approaches are almost equal to 1 (the mean Pearson correlation coefficient is equal to 0.9993 and the mean Spearman correlation coefficient is equal to 0.9983).

We also look more closely at the data set used in Kao and Hwang (2008) which is probably the most popular data set being used in many other related papers (see for example Chen et al. (2009), Chen et al. (2010), Cook et al. (2010) and Chen et al. (2013)). The data consists of 24 non-life insurance companies as the DMUs, two inputs, two intermediate measures and two outputs. The overall efficiency scores Pearson and Spearman correlation coefficients between the linear and parametric linear models are almost equal to 1 (0.9986 and 0.9974). However, the parametric linear model requires 19,640 LPs to complete evaluation of these 24 DMUs with $\varepsilon = 0.001$. Figure 3 shows the number of LPs for the same problem with different value of ε . As can be seen, when $\varepsilon = 0.00001$, parametric linear model's steps are completed with running 1,955,573 LPs. Note that Liang et al. (2006) suggested using smaller values for ε to improve the solution of the heuristic search of the parametric linear model. With $\varepsilon = 0.00001$, the Pearson and Spearman correlation coefficients between the linear model and parametric linear model are still equal to 0.9986 and 0.9974. As another example, consider the data set presented by Chen et al (2013) which has only 5 DMUs. The parametric linear approach requires 802 LPs to complete the analysis with $\varepsilon = 0.001$, while it increases

to 7,861 LPs with $\varepsilon = 0.0001$. Compared to only 5 LPs of the linear model, clearly, this is a very significant difference.

Figure 3. Number of LPs solved to complete the steps of the parametric linear approach for Kao and Hwang (2008) data sets with different epsilon

4.3. Weighted models

At this section, we put our focus on the weighted models of Lim and Zhu (2013). In doing so, we ran Models (4) and (10) with the new weighted objective functions as defined by (15) and (16) but with the same set of constraints used before. In this approach, (15) is the weighted objective function of Model (4), and (16) is the weighted objective function of Model (10).

$$\max \quad \alpha \sum_{r=1}^R u_r y_{kr} + (1 - \alpha) \sum_{s=1}^S w_s z_{ks} \quad (15)$$

$$\max \quad (1 - \alpha) \sum_{s=1}^S w_s z_{ks} - \alpha \sum_{j=1}^J v_j x_{kj} \quad (16)$$

The summary of the results of the weighted models is reported in Table 8. The same criteria that were used before, i.e. MAD, Pearson and Spearman rank correlations are used once more to review the approximation of Model (4) results with Model (10). We have tested the weighted models using the 10 real data sets used in subsection 4.2

and the value of α is set to be 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8 and 0.9. The results shown in Table 8 are the average results from the 10 real data sets for each α . The smaller values of MAD and larger values of Pearson and Spearman correlation coefficients indicate a closer approximation of Model (4) results with Model (10). This was also the case with these measures for Models (4) and (10) without weights. The Pearson and Spearman correlation coefficient values are slightly smaller than the models without weights but still, about 94 percent of the correlation coefficients are significant at the 0.05 level. Note that, for the weighted models including nine different values of α and 10 different data sets, we have calculated 540 Pearson and Spearman correlation coefficients. The MAD results are slightly larger than the models without weights but they are still relatively small (smaller than the range mentioned by Ruggiero, 1998 who considered the MADs varying from 0.04 to 0.11 as relatively low). These, therefore, confirm approximating weighted Model (4) results with weighted Model (10).

Table 8. Correlation coefficients and MADs between weighted Models (4) and (10)

Alpha	e_1			e_2			E		
	Pearson	Spearman	MAD	Pearson	Spearman	MAD	Pearson	Spearman	MAD
0.1	0.6876	0.6975	0.1015	0.8148	0.8129	0.0789	0.9086	0.9102	0.0321
0.2	0.7357	0.7376	0.0802	0.8149	0.8189	0.0735	0.9445	0.9333	0.0234
0.3	0.7893	0.8594	0.0540	0.8182	0.8249	0.0609	0.9645	0.9526	0.0141
0.4	0.8939	0.9264	0.0244	0.8516	0.8793	0.0334	0.9860	0.9790	0.0052
0.5	0.9507	0.9613	0.0102	0.9656	0.9675	0.0127	0.9993	0.9984	0.0013
0.6	0.9534	0.9606	0.0054	0.9827	0.9842	0.0068	0.9990	0.9987	0.0009
0.7	0.9119	0.9144	0.0201	0.8727	0.8504	0.0293	0.9851	0.9836	0.0055
0.8	0.8903	0.8810	0.0367	0.8435	0.8045	0.0511	0.9696	0.9586	0.0120
0.9	0.7523	0.7206	0.0691	0.8227	0.7810	0.0686	0.9480	0.9407	0.0212
Mean	0.8406	0.8510	0.0446	0.8652	0.8582	0.0461	0.9672	0.9617	0.0128

4.4. Experiments with simulated data

To provide a more comprehensive analysis of the question of whether Model (4) can be approximated to Model (10), this section considers the comparison of linear and parametric linear models using simulated data. For this purpose, we have used the dimensions and number of DMUs found in the papers reported in Table 6 to randomly generate 100 data sets for each. The data are uniformly distributed between 10 and 30 for all inputs, intermediate measures, and outputs. This has led to 1,000 samples with 21,500 observations. All the computations for generating the data and running the models are completed using a computer code in Matlab software. The parametric linear model is run with $\varepsilon = 0.001$. The mean Pearson correlation coefficient and Spearman

correlation coefficient for 1,000 samples are equal to 0.9831 and 0.9782, respectively. The largest number of LPs belongs to trial 89 of the data set with 30 DMUs, two inputs, two intermediate measures, and one output where 27,426 LPs were required for the parametric linear model. The results of this data set with the linear model and the parametric linear model are significantly correlated with Pearson and Spearman correlation coefficients of 0.9909 and 0.9902, respectively.

One very important result from the simulation is to confirm Theorem 2. Out of 21,500 observations, 282 were identified as overall efficient ($e = 1$) with both linear and parametric linear models. Although, there are some inefficient DMUs which have different scores with linear and parametric linear models (there are only 3 DMUs out of 21,500 observations that the difference between their e value obtained from these two models is greater than 0.2), the frontier DMUs in these two models are exactly the same. Therefore, both models classify exactly the same DMUs as efficient and inefficient. In another words, there is no DMU which was identified as efficient with the parametric linear model and not with the linear model and vice versa. This confirms Theorem 2. Table 9 reports a summary of the results. As the table suggests, the mean Pearson and mean Spearman correlation coefficients are relatively high for all efficiency measures (e_1 , e_2 and e) and the mean MADs are relatively low for all efficiency measures. This, consequently, supports the idea that Model (10) can closely approximate Model (4).

Table 9. Summary of correlation coefficients and MADs; simulated data

	e_1	e_2	e
Mean Pearson	0.9393	0.8741	0.9831
Mean Spearman	0.9090	0.8631	0.9782
Mean MAD	0.0261	0.0398	0.0068

Applying these results and the associated model to the practical scenarios that managers face, it will be appreciated that there are multiple occasions in which decision makers are faced with a choice of multiple DMUs from which they wish to select the best. In such a situation, decision makers are less interested in the relative inefficiency (i.e. the inefficiency score) of a particular DMU. Rather, they wish to understand which DMUs are efficient and which are inefficient. For instance, in the case of supplier selection problems, procurement managers typically want to select the best suppliers given, say, two key criteria. Thus the identity of inefficient suppliers is not of major

importance, rather they wish to know which suppliers are efficient as these can then, for example, form a short list of companies from which materials can be purchased (Farzipoor Saen, 2007). A similar situation applies in other contexts such as technology selection (Wu et al. 2016), market selection (Farzipoor Saen, 2011), and container selection (Shabani et al., 2011). In each case, the key is to identify the efficient DMUs and discard those that are inefficient.

5. Concluding remarks

In this paper, we have argued that simultaneously maximizing the efficiency and effectiveness ratios of DMUs is a special class of the SLR problem. Two existing nonlinear and parametric linear DEA models developed by Chiou et al. (2010) and Lim and Zhu (2013) to solve this problem are investigated, and a mathematical relation between the parametric linear model with a linear model developed by Mahdiloo et al. (2015), Mahdiloo et al. (2016) and Khodakarami et al. (2016) is proved. This relation is very important since the linear model does not have the computational complexity and potential estimation error of the parametric linear model and, unlike the nonlinear model, it determines the global optimal solution. We therefore considered that the use of the linear model (rather than the nonlinear and parametric linear models) in maximizing the sum of two linear ratios of the efficiency and effectiveness has clear benefits.

It should also be noted that, whilst the discussions made in this paper offers clear improvements to the theory of DEA, it is recommended that further developments be undertaken on the linear model, for example we suggest extending the linear model in order to consider weight restrictions, imprecise, and fuzzy data.

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Integrated data envelopment analysis: linear vs. nonlinear model

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