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On local representations of rotations on discrete configuration spaces

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Abstract.

A quantum mechanical spin-half system may be characterized abstractly as having a set of two-valued observables which generate infinitesimal rotations in three dimensions. We consider a concrete realization of such a two-level system within the formalism of ensembles on configuration space, an approach which is not only capable of describing quantum mechanical systems but allows also for theories that are generalizations of quantum theory. Such a spin-half system may be called a rotational bit or robit, to distinguish it from the standard quantum qubit. After reviewing ensembles on configuration space and examining the example of constructing representations of the Galilean Lie algebra for the free particle, we construct probabilistic models for ensembles that consist of one and two spin-half systems. In the case of a single spin-half system, there are two main requirements: the configuration space must be a discrete set, labeling the outcomes of two-valued spin observables, and these observables must provide an algebraic representation of $so(3)$. The case of a pair of spin-half systems is more complicated, in that additional physical requirements concerning locality and subsystem independence must also be taken into account and now the observables must provide an algebraic representation of $so(3) \oplus so(3)$. We compare the resulting theories to the corresponding quantum mechanical systems.

1. Introduction

Physical systems that are composed of spin-half particles, while simple enough to be tractable, display enough complexity to bring out many of the most puzzling features of quantum mechanics. Already the relatively simple case of two spin-half particles leads to remarkable results. This became apparent after Bohm [1] provided a reformulation of the EPR paradox [2] in terms of a pair of spin-half particles. This same model would later be used by Bell [3] to formulate his famous theorem. The literature on such quantum systems is enormous and their study still remains a very active area of research to this date, particularly in quantum information theory.

A quantum mechanical spin-half system may be characterized abstractly as having a set of two-valued observables which generate infinitesimal rotations in three dimensions, as discussed in more detail in section 5. Using this abstract formulation, we consider a concrete realization of physical systems of this type within the formalism of ensembles on configuration space [4], an approach which is capable of describing quantum mechanical systems and, in addition, allows also for theories which are generalizations of quantum theory. The aim is to construct

probabilistic models for ensembles that consist of one and two spin-half systems and to see whether a set of basic physical requirements still allows for generalizations of the familiar quantum systems. Thus, our starting point is not the usual formulation of spin-half systems using operators and vectors in Hilbert space [5] but the more general formulation of ensembles on configuration space [4]. We then compare the resulting theories to the corresponding quantum mechanical systems.

The essential tool that we use are representations of appropriate Lie algebras on the configuration space of the system. Thus, in the case of a single spin-half system, we consider representations of the Lie algebra $so(3)$ for a configuration space that is two-dimensional. In the case of a pair of spin-half systems, we need to deal with representations of the Lie algebra $so(3) \oplus so(3)$ for a configuration space that is four-dimensional. This case however becomes more complicated than the case of a single spin-half particle because we must take into account additional physical requirements concerning locality and subsystem independence.

In the next section, we introduce the formulation of ensembles on configuration space. Before considering the case of discrete configuration spaces, which is the one that is needed for the description of spin-half particles, we discuss the more intuitive case of ensembles on configuration space for the continuous configuration space of a single particle. In the section that follows, we consider a particular example in which we look for representations of the Galilean Lie algebra and show that non-equivalent representations can arise, leading to inequivalent theories of the free particle. The following two sections are devoted to the formulation of ensembles on configuration space on discrete configuration spaces and the description of spin-half particles. We end with some concluding remarks.

2. Ensembles on configuration space

Consider the case of a non-relativistic particle with a continuous configuration space with coordinates x . The description of physical systems in terms of ensembles on configuration space requires three basic ingredients,

- a probability density $P(x)$ on configuration space, with $P(x) \geq 0$ and $\int dx P(x) = 1$,
- a canonically conjugate quantity $S(x)$,
- an ensemble Hamiltonian $H[P, S]$,

where $H[P, S]$ is a functional of P and S that must satisfy certain requirements (these conditions, which are common to all observables, are discussed below).

The *state* of the system is fully described by the functions $P(x)$ and $S(x)$. The time evolution of the state follows Hamiltonian equations of motion determined by H which, when expressed in terms of Poisson brackets, take the form

$$\frac{\partial P}{\partial t} = \{P, H\} = \frac{\delta H}{\delta S}, \quad \frac{\partial S}{\partial t} = \{S, H\} = -\frac{\delta H}{\delta P}. \quad (1)$$

To understand why we introduce $S(x)$ as a canonically conjugate variable to $P(x)$, it is convenient to think of the formalism as resulting from a two-step process. In the first step, we introduce a configuration space with coordinates x and probabilities $P(x)$. In the second step, we introduce dynamics. We assume that the equations of motion for $P(x)$ follow from an action principle. Then, if we do this via a Hamiltonian formalism, we are forced to introduce an auxiliary field canonically conjugate to $P(x)$. This is the field $S(x)$.

Notice that this is a very general formalism that essentially describes dynamics of probabilities. A physical interpretation is only possible if an appropriate ensemble Hamiltonian is introduced together with other appropriate observables.

We consider a couple of physically relevant examples of ensemble Hamiltonians. The ensemble Hamiltonian that describes a *classical particle* subject to a force derived from a potential term $V(x)$ is given by

$$H_C[P, S] = \int dx P \left[\frac{|\nabla S|^2}{2m} + V \right]. \quad (2)$$

To see this, evaluate the equations of motion,

$$\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V = 0, \quad \frac{\partial P}{\partial t} + \nabla \cdot \left(P \frac{\nabla S}{m} \right) = 0. \quad (3)$$

The first equation is the Hamilton-Jacobi equation, the second equation is the continuity equation which ensures that probability is preserved. In the case of a particle, the probability $P(x, t)$ is interpreted as the probability that the particle is at a given location in configuration space at a given time.

The ensemble Hamiltonian that describes a *quantum particle* subject to a force derived from a potential $V(x)$ is given by

$$H_Q[P, S] = H_C[P, S] + \frac{\hbar^2}{4} \int dx \frac{1}{P} \left(\frac{|\nabla P|^2}{2m} \right). \quad (4)$$

The corresponding equations of motion are

$$\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V + \frac{\hbar^2}{2} \left(\frac{\nabla^2 \sqrt{P}}{m\sqrt{P}} \right) = 0, \quad \frac{\partial P}{\partial t} + \nabla \cdot \left(P \frac{\nabla S}{m} \right) = 0. \quad (5)$$

These equations are the Madelung equations [9] which describe a quantum particle using Madelung hydrodynamical variables P and S . To see the connection to the Schrödinger equation, introduce the complex *canonical transformation* given by $\psi = \sqrt{P} e^{iS/\hbar}$, $\bar{\psi} = \sqrt{P} e^{-iS/\hbar}$ which leads to

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \psi + V\psi, \quad (6)$$

the Schrödinger equation. Thus, in the formulation in terms of ensembles on configurations space, the wavefunction representation given in terms of $\psi, \bar{\psi}$ is related to the representation in terms of hydrodynamical variables P, S via a simple canonical transformation.

Observables are functionals $A[P, S]$ of P and S which satisfy the following basic requirements: an infinitesimal canonical transformation generated by A ,

$$P \rightarrow P + \epsilon \frac{\delta A}{\delta S}, \quad S \rightarrow S - \epsilon \frac{\delta A}{\delta P}, \quad (7)$$

must preserve the normalization and the positivity of the probability. This leads to [4]

$$A[P, S + c] = A[P, S], \quad \frac{\delta A}{\delta S} = 0 \text{ if } P(x) = 0. \quad (8)$$

The first of these conditions implies that only relative values of S have physical significance. In addition, we will require that observables be functionals which are homogeneous of degree one in P , i.e., $A[\lambda P, S] = \lambda A[P, S]$, where λ is an arbitrary positive constant [4]. No further conditions are necessary. We emphasize that *the set of observables forms a closed Lie algebra under the Poisson bracket*.

Examples of particle observables include:

- the ensemble Hamiltonians introduced previously; i.e. H_C and H_Q ,
- the position observable $X[P, S] := \int dx P x$,
- the momentum observable $\pi[P, S] := \int dx P \nabla S$.

Classical and quantum observables may be defined as averages. To define a classical observable, consider a function $f(x, p)$ defined on the standard phase space of classical mechanics and define the corresponding *classical observable* by:

$$C_f := \int dx P f(x, \nabla S). \quad (9)$$

Then, one can show that the Poisson bracket for *classical observables* is isomorphic to the *phase space* Poisson bracket. To define a quantum observable, consider a Hermitian operator $\hat{M}$ defined on Hilbert space. Define the corresponding *quantum observable* by:

$$Q_{\hat{M}} := \langle \psi | \hat{M} | \psi \rangle = \int dq dq' (PP')^{1/2} e^{i(S-S')/\hbar} \langle q' | \hat{M} | q \rangle. \quad (10)$$

Then, one can show that the Poisson bracket for *quantum observables* is isomorphic to the *commutator* on Hilbert space.

Thus, we have the non-trivial result that the algebras of each of these two classes of observables are isomorphic to the algebras that arise naturally in the phase space and Hilbert space representations of classical and quantum mechanics [4].

3. Non-relativistic free particle via a representation of the Galilean group

We now look at an example of representing the action of Lie algebras within this formalism. We consider the case of a non-relativistic free particle, which may be defined in terms of the representation of the Galilean Lie algebra on a continuous configuration space of three dimensions. We will show how non-equivalent representations of this algebra can arise, leading to inequivalent theories of the free particle [4, 6]. Among them, we will recover the classical and quantum theories of the free particle.

The Galilean Lie algebra has 10 generators,

- A_i : space displacements,
- H : time displacement,
- L_i : space rotations,
- G_i : Galilean transformations (“boosts”),

which satisfy the following Lie algebra,

$$\begin{aligned} \{H, A_i\} &= 0, & \{H, L_i\} &= 0, & \{H, G_i\} &= -A_i, \\ \{L_i, A_j\} &= \epsilon_{ijk} A_k, & \{L_i, L_j\} &= \epsilon_{ijk} L_k, & \{L_i, G_j\} &= \epsilon_{ijk} G_k, \\ \{A_i, A_j\} &= 0, & \{A_i, G_j\} &= -m\delta_{ij}, & \{G_i, G_j\} &= 0, \end{aligned} \quad (11)$$

where m is the mass of the particle.

The configuration space of a non-relativistic particle is given by Euclidean space R^3 . We will look for a representation of the generators in terms of observables. For space displacements and rotations, we set

$$A_i = \int d^3x P (\partial_i S), \quad L_i = \int d^3x P (\epsilon_{ijk} x_j \partial_k S), \quad (12)$$

which correspond to the momentum and angular momentum observables. A natural choice for the Galilean boost transformations is given by

$$G_i = \int d^3x P (mx_i - t\partial_i S). \quad (13)$$

This follows from the definition $G_i = (mX_i - tA_i)$ where $X_i = \int d^3x P x_i$ is the position observable. Finally, the remaining generator H must be a Galilean scalar that satisfies

$$\{H, G_i\} = -m \int d^3x \frac{\delta H}{\delta S} x_i = -A_i = - \int d^3x P \partial_i S. \quad (14)$$

The condition given by Eq. (14) is not a very stringent one and it leads to a family of solutions $H^{(K)}$ labeled by a functional K ,

$$H^{(K)}[P, S] = \int dx P \frac{|\nabla S|^2}{2m} + K[P, S], \quad (15)$$

where K is any observable that is invariant under translations, rotations and boosts, i.e., K is a Galilean scalar. Different choices of K lead to *different* theories of the free particle. In particular, the family of solutions includes both classical and quantum systems,

$$H_C[P, S] = \int dx P \frac{|\nabla S|^2}{2m}, \quad (16)$$

$$H_Q[P, S] = H_C[P, S] + \left[\frac{\hbar^2}{4} \int dx P \frac{|\nabla \log P|^2}{2m} \right]. \quad (17)$$

Thus, we have shown that one may represent the action of the Galilean Lie algebra in terms of an algebra of observables and that there are physically inequivalent representations of this algebra leading to distinct physical theories of the free particle.

4. Ensembles on configuration space for discrete systems

Consider now a system with a discrete configuration space. In this case, the formalism of ensembles on configuration space that results is analogous to the one that we already described for the case of continuous configuration spaces, except that we replace continuous variables by discrete variables and functionals by functions [4, 7, 8]. Thus, we need

- a probability $P = (P_1, \dots, P_n)$ where n is the number of states, with $P_i \geq 0$ and $\sum_{i=1}^n P_i = 1$,
- canonically conjugate quantities $S_1, \dots, S_n$,
- an ensemble Hamiltonian $H(P_i, S_j)$,

where $H(P_i, S_j)$ is a function of the P_i and S_j that must satisfy all the conditions that are required of observables. These are that the infinitesimal canonical transformation generated by an observable $A(P_i, S_j)$ must preserve both the normalization and the positivity of the probability and that it must satisfy the homogeneity condition $A[\lambda P_i, S_j] = \lambda A[P_i, S_j]$. The first two conditions imply

$$A(P_i, S_j + c) = A(P_i, S_j), \quad \partial A / \partial S_i = 0 \text{ if } P_i = 0, \quad (18)$$

where c is a constant [4]. The Poisson bracket for two observables $A(P_i, S_j)$ and $B(P_i, S_j)$ is defined by

$$\{A, B\} = \sum_i \left(\frac{\partial A}{\partial P_i} \frac{\partial B}{\partial S_i} - \frac{\partial A}{\partial S_i} \frac{\partial B}{\partial P_i} \right). \quad (19)$$

5. Rotational bits

We now introduce the concept of a rotational bit, or *robit*. Our starting point is the observation that a spin-half system may be characterized abstractly as having a set of two-valued observables which generate infinitesimal rotations in three dimensions. Therefore, to describe a robit, we must introduce a two-dimensional configuration space and observables L_k , with $k = 1, 2, 3$, which satisfy the $so(3)$ Lie algebra,

$$\{L_j, L_k\} = \epsilon_{jkl} L_l. \quad (20)$$

The generator of rotations about a given direction will be associated with measurements of the spin in that direction.

We will first consider the case of a single robit and then the case of a physical system that consists of two robits.

5.1. One robit

In the case of a single robit, the configuration space is two-dimensional and thus the associated phase space is four-dimensional. We introduce coordinates (P_+, P_-, S_+, S_-) on this phase space.

It is convenient to define the reduced phase space with coordinates (q_1, p_1) via the canonical transformation

$$q_0 = (P_+ + P_-)/2, \quad q_1 = (P_+ - P_-)/2, \quad p_0 = S_+ + S_-, \quad p_1 = S_+ - S_-. \quad (21)$$

Then, the coordinate q_0 is a constant of the motion, as the probability must be normalized, $(P_+ + P_-) = 2q_0 = 1$. Furthermore, observables can only depend on the difference $(S_+ - S_-)$, thus they cannot be functions of p_0 .

We now look for a representation of the Lie algebra of $so(3)$ in this reduced phase space. The identification of generators with average values immediately fixes L_3 ,

$$L_3 = \frac{1}{2}(P_+ - P_-) = q_1. \quad (22)$$

Using Eq. (20), the general solutions for L_1, L_2 , up to canonical transformations, are given by

$$L_1 = \sqrt{(1/2)^2 - q_1^2} \cos(p_1 + b(q_1)), \quad L_2 = -\sqrt{(1/2)^2 - q_1^2} \sin(p_1 + b(q_1)), \quad (23)$$

where b is an arbitrary function. However, we can always set $b(q_1) = 0$ via the simple canonical transformation

$$q_1 \rightarrow q_1, \quad p_1 \rightarrow p_1 - b(q_1). \quad (24)$$

Consider then the case $b(q_1) = 0$. We can now show that single robits are strictly equivalent to quantum mechanical qubits. To see this, we consider the operators $\hat{L}_k$ for spin-half, which satisfy $[\hat{L}_j, \hat{L}_k] = i\epsilon_{jkl} \hat{L}_l$ and may be represented in terms of Pauli matrices,

$$\hat{L}_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{L}_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{L}_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (25)$$

If we now introduce the two-component wave function

$$|\psi\rangle = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \sqrt{P_+} e^{iS_+} \\ \sqrt{P_-} e^{iS_-} \end{pmatrix}, \quad (26)$$

we find that

$$\langle \psi | \hat{L}_k | \psi \rangle = L_k. \quad (27)$$

This shows that the most general theory of single robits describes quantum mechanical qubits, leading to no predictions which differ from quantum theory.

5.2. Two robits

In the case of two robits, we need to introduce the configuration space of a pair of spin-half systems, which is four-dimensional, and spin observables that provide a representation of the Lie algebra $so(3) \oplus so(3)$. What makes this case more interesting and far more complicated is that we now have to consider additional physical requirements concerning locality and subsystem independence. In particular, to get a viable physical theory, we must require that

- no signalling is possible via a rotation of either system (evolution locality),
- no signalling is possible via a measurement of either system (update locality), and
- initially independent systems remain independent under local rotations.

We introduce coordinates $(P_{++}, P_{+-}, P_{-+}, P_{--}, S_{++}, S_{+-}, S_{-+}, S_{--})$ on this phase space. Similar to the case of the single robit, we define a six-dimensional reduced phase space with coordinates $(q_1, q_2, q_3, p_1, p_2, p_3)$ via the canonical transformation

$$\begin{aligned} q_0 &= (P_{++} + P_{+-} + P_{-+} + P_{--})/2, \\ q_1 &= (P_{++} + P_{+-} - P_{-+} - P_{--})/2, \\ q_2 &= (P_{++} - P_{+-} + P_{-+} - P_{--})/2, \\ q_3 &= (P_{++} - P_{+-} - P_{-+} + P_{--})/2, \end{aligned} \quad (28)$$

and

$$\begin{aligned} p_0 &= (S_{++} + S_{+-} + S_{-+} + S_{--})/2, \\ p_1 &= (S_{++} + S_{+-} - S_{-+} - S_{--})/2, \\ p_2 &= (S_{++} - S_{+-} + S_{-+} - S_{--})/2, \\ p_3 &= (S_{++} - S_{+-} - S_{-+} + S_{--})/2. \end{aligned} \quad (29)$$

The same as in the previous case, q_0 is a constant of the motion, as the probability must be normalized. Furthermore, observables can only depend on the differences of the conjugate variables $S_{\alpha\beta}$ with $\alpha, \beta = +, -$, thus they cannot be functions of p_0 .

We now look for representations of $so(3) \oplus so(3)$ in this reduced phase space. They are given in terms of six generators M_j, N_k that satisfy

$$\{M_j, M_k\} = \epsilon_{jkl} M_l, \quad \{N_j, N_k\} = \epsilon_{jkl} N_l, \quad \{M_j, N_k\} = 0, \quad (30)$$

for $j, k = 1, 2, 3$.

Before giving explicit representations of the generators, we introduce definitions of marginal and conditional probabilities. The marginal probabilities are given by

$$\begin{aligned} P_{\pm}^{(1)} &:= P_{\pm++} + P_{\pm+-} = (q_0 \pm q_1) = (1/2 \pm q_1), \\ P_{\pm}^{(2)} &:= P_{+ \pm} + P_{- \pm} = (q_0 \pm q_2) = (1/2 \pm q_2), \end{aligned} \quad (31)$$

and the conditional probabilities by

$$P_{\alpha|\beta}^{(1)} = \frac{P_{\alpha\beta}}{P_{\beta}^{(2)}}, \quad P_{\alpha|\beta}^{(2)} = \frac{P_{\alpha\beta}}{P_{\beta}^{(1)}}. \quad (32)$$

These will be needed in the expressions that we give below.

The identification of generators with average values immediately fixes the forms of M_3 and N_3 . The average value of spin measurements in the z -direction is given by

$$M_3 = \frac{1}{2}(P_{+}^{(1)} - P_{-}^{(1)}) = q_1, \quad N_3 = \frac{1}{2}(P_{+}^{(2)} - P_{-}^{(2)}) = q_2. \quad (33)$$

To fix the functional form of the other generators, we need to consider the physical requirements concerning locality and subsystem independence. We first consider evolution and update locality requirements, i.e., that no signalling is possible via a rotation of either system (evolution locality) and no signalling is possible via a measurement of either system (update locality).

Evolution locality is guaranteed by $\{M_j, N_k\} = 0$, thus it is already satisfied if, as assumed, the generators satisfy Eq. (30).

Enforcing update locality requires substantially more effort. One can show that it requires that M_j and N_k are of the form [4]

$$M_j = P_+^{(2)} L_j(q_+^{(1)}, p_+^{(1)}) + P_-^{(2)} L_j(q_-^{(1)}, p_-^{(1)}) := M_j^+ + M_j^-, \quad (34)$$

$$N_k = P_+^{(1)} L_k(q_+^{(2)}, p_+^{(2)}) + P_-^{(1)} L_k(q_-^{(2)}, p_-^{(2)}) := N_k^+ + N_k^-, \quad (35)$$

where the L_k are *single-robit generators* and the $q_{\pm}^{(1)}, q_{\pm}^{(2)}, p_{\pm}^{(1)}, p_{\pm}^{(2)}$ are some appropriate functions of the reduced phase space coordinates which will have to be determined. To put M_3 and N_3 in the form of Eqs (34-35), first write them in the form

$$M_3 = \frac{1}{2}(P_+^{(1)} - P_-^{(1)}) = P_+^{(2)} \frac{P_{+|+} - P_{-|+}}{2} + P_-^{(2)} \frac{P_{+|-} - P_{-|-}}{2}, \quad (36)$$

$$N_3 = \frac{1}{2}(P_+^{(2)} - P_-^{(2)}) = P_+^{(1)} \frac{P_{+|+} - P_{+|-}}{2} + P_-^{(1)} \frac{P_{-|+} - P_{-|-}}{2}. \quad (37)$$

This leads immediately to expressions for $M_3^{\pm}$ and $N_3^{\pm}$,

$$M_3^{\pm} = \frac{q_1 \pm q_3}{2}, \quad N_3^{\pm} = \frac{q_2 \pm q_3}{2}. \quad (38)$$

The generators $M_j^{\pm}$ and $N_k^{\pm}$ for $j, k = 1, 2$ are derived using $M_3^{\pm}$ and $N_3^{\pm}$ with the same type of computations that were carried out for the case of a single robit.

The most general solution is then of the form [4]

$$M_j = M_j^+ + M_j^-, \quad N_k = N_k^+ + N_k^-, \quad (39)$$

where

$$\begin{aligned} M_1^{\pm} &= \sqrt{\left(\frac{1/2 \pm q_2}{2}\right)^2 - \left(\frac{q_1 \pm q_3}{2}\right)^2} \cos((p_1 \pm p_3) + b_M^{\pm}) \\ M_2^{\pm} &= -\sqrt{\left(\frac{1/2 \pm q_2}{2}\right)^2 - \left(\frac{q_1 \pm q_3}{2}\right)^2} \sin((p_1 \pm p_3) + b_M^{\pm}) \\ M_3^{\pm} &= \frac{q_1 \pm q_3}{2}, \end{aligned} \quad (40)$$

with similar expressions for the $N_k^{\pm}$ which follow from the $M_k^{\pm}$ via $(q_1, p_1) \rightarrow (q_2, p_2)$.

It turns out that the functional forms of $b_M^{\pm}$ and $b_N^{\pm}$ can be further specified using various symmetries which come from requiring invariance under particular types of relabeling and subsystem independence [4], leading to

$$b_M^{\pm} = b\left(\frac{q_1 \pm q_3}{1/2 \pm q_2}\right), \quad b_N^{\pm} = b\left(\frac{q_2 \pm q_3}{1/2 \pm q_1}\right), \quad (41)$$

where b is an arbitrary function. However, all of the requirements that we have considered so far do not allow us to fix the form of b *uniquely*.

5.3. Two robits vs. two qubits: the spin correlation matrix

We now examine how a pair of robits is related to a pair of quantum mechanical qubits. In quantum mechanics, the generators of $so(3) \oplus so(3)$ are represented by 4×4 matrix operators in the tensor product space,

$$\hat{M}_1 = \frac{1}{2} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \hat{M}_2 = \frac{1}{2} \begin{pmatrix} 0 & -i\mathbf{1} \\ i\mathbf{1} & 0 \end{pmatrix}, \quad \hat{M}_3 = \frac{1}{2} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},$$

and

$$\hat{N}_1 = \frac{1}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \hat{N}_2 = \frac{1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad \hat{N}_3 = \frac{1}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}.$$

When $b(q) = 0$, we have

$$\langle \psi | \hat{M}_j | \psi \rangle = M_j, \quad \langle \psi | \hat{N}_k | \psi \rangle = N_k. \quad (42)$$

Thus *equivalence to a pair of quantum mechanical qubits is achieved for the choice $b = 0$.*

It is natural then to turn to the question of whether there are any further conditions that may force $b = 0$, in which case the theory of a pair of robits would be physically equivalent to the theory of a pair of quantum mechanical qubits, or whether there are choices of $b \neq 0$ which lead to viable theories that are physically distinct from quantum mechanics.

One possibility is to enlarge the algebra of observable beyond the generators M_j and N_k by adding other physically relevant observables, to see if the algebra of this larger set of observables brings any additional conditions that may lead to $b = 0$.

This can be done by considering a more general situation than the one that we have considered so far. Up to now, we have derived generators for the robits using a fixed coordinate system common to both of them, one in which the Cartesian coordinates of the first robit are aligned with the Cartesian coordinates of the second robit, thus leading to a situation where both robits lie essentially on the same axis. We now consider a more general physical situation, one in which

- the z axis for the first system is aligned with the unit vector $\vec{m}$,
- the z axis for the second system is aligned with the unit vector $\vec{n}$.

Then, one can show (see Appendix A) that

$$P_{jk}(\vec{m}, \vec{n}) = \frac{1}{4} [1 + j M_3(\vec{m}) + k N_3(\vec{n}) + jk C(\vec{m}, \vec{n})]. \quad (43)$$

where $j, k = \pm 1$ and C is the correlation matrix. As the $P_{jk}(\vec{m}, \vec{n})$ are in principle measurable, and furthermore exhaust all measurable quantities, we need to consider the consequences that this has for the algebra of observables.

As $P_{jk}(\vec{m}, \vec{n})$ depends on $C(\vec{m}, \vec{n})$, it will be sufficient to focus on the correlation matrix. The crucial observation is that the correlation matrix C can be expressed solely in terms of *Poisson brackets* which depend on the generators M_j , N_k and q_3 . If we write it down explicitly, we find

$$C = \begin{pmatrix} \{\{q_3, M_2\}, N_2\} & -\{\{q_3, M_2\}, N_1\} & -\{q_3, M_2\} \\ -\{\{q_3, M_1\}, N_2\} & \{\{q_3, M_1\}, N_1\} & \{q_3, M_1\} \\ -\{q_3, N_2\} & \{q_3, N_1\} & q_3 \end{pmatrix}$$

This suggests that we *enlarge* the algebra, which until now was assumed to be generated by the closed set of observables (M_j, N_k) , to include at a minimum the set of observables (M_j, N_k, q_3) plus the other generators that correspond to the elements of the correlation matrix.

We must however require that this enlarged algebra also be closed, otherwise it will generate an ever larger set of observables which will include, in addition to the $P_{jk}(\vec{m}, \vec{n})$ that we can measure, other quantities that are *not* measurable. A straightforward but tedious calculation shows that the more general algebra generated by the M_j , N_k and the elements of C only closes if $b = 0$.

Thus the more general situation that we have just considered leads to the condition $b = 0$ and thus to the equivalence of the theory of a pair of robits to that of a pair of quantum mechanical qubits. One can show that the generators of this larger algebra turn out to be linear combinations of $SU(4)$ generators, as one would expect by its equivalence to the quantum theory.

6. Concluding remarks

We have carried out a derivation of probabilistic models of ensembles consisting of one and two spin-half particles, known as robits to distinguish them from the standard quantum mechanical qubits. Our starting point was not the usual formulation of spin-half systems using operators and vectors in Hilbert space but the more general formulation of ensembles on configuration space. Thus none of the conditions used in the derivation of the models were formulated using the standard building blocks of quantum mechanics; e.g., operators acting on states in a complex Hilbert space, unitary transformations, etc. The models are based on a minimal set of assumptions:

- the configuration space is a discrete set, labelling the outcomes of two-valued spin observables,
- the spin observables provide a representation of $so(3)$ for the case of a single robit or a representation of $so(3) \oplus so(3)$ for the case of two robits,
- for the case of a pair of robits, additional physical requirements concerning locality and subsystem independence must be taken into account,
- for the case of a pair of robits, the algebra of generators must include all observables and no additional generators that are not measurable.

We have shown that this set of assumptions leads to theories which are equivalent to the quantum theory of one and two qubits. Since it is not obvious that this result is a general one that will hold when such an analysis is applied to more complicated systems, it is of interest to extend the approach to composite systems that have more particles and/or particles of higher spin.

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Appendix A. Derivation of $P_{jk}(\vec{m}, \vec{n})$

To prove Eq. (43) notice that for arbitrary directions $\vec{m}$ and $\vec{n}$ we have

$$\begin{aligned} M_3(\vec{m}) &:= P_+^{(1)}(\vec{m}) - P_-^{(1)}(\vec{m}) \\ &= P_{++}(\vec{m}, \vec{n}) + P_{+-}(\vec{m}, \vec{n}) - P_{-+}(\vec{m}, \vec{n}) - P_{--}(\vec{m}, \vec{n}), \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} N_3(\vec{n}) &:= P_+^{(2)}(\vec{n}) - P_-^{(2)}(\vec{n}) \\ &= P_{++}(\vec{m}, \vec{n}) - P_{+-}(\vec{m}, \vec{n}) + P_{-+}(\vec{m}, \vec{n}) - P_{--}(\vec{m}, \vec{n}). \end{aligned} \quad (\text{A.2})$$

Furthermore, the correlation is defined by

$$C(\vec{m}, \vec{n}) := P_{++}(\vec{m}, \vec{n}) - P_{+-}(\vec{m}, \vec{n}) - P_{-+}(\vec{m}, \vec{n}) + P_{--}(\vec{m}, \vec{n}). \quad (\text{A.3})$$

Using the additional relation

$$1 = P_{++}(\vec{m}, \vec{n}) + P_{+-}(\vec{m}, \vec{n}) + P_{-+}(\vec{m}, \vec{n}) + P_{--}(\vec{m}, \vec{n}), \quad (\text{A.4})$$

leads to

$$P_{jk}(\vec{m}, \vec{n}) = \frac{1}{4} [1 + j M_3(\vec{m}) + k N_3(\vec{n}) + jk C(\vec{m}, \vec{n})], \quad (\text{A.5})$$

as desired.

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