RESEARCH ARTICLE

An Element-wise Scheme to Analyse Local Mechanical Anisotropy in Fibre-reinforced Composites

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ABSTRACT

Herein, the general constitutive equation of bi-phasic materials equipped with orientation tensor is presented in direct notation. The formulation is refined by some correction factors specific to natural fibre-reinforced composites; then, a planar case is derived. The necessity of local information is emphasised through the introduction of auxiliary maps, which included volume fraction and orientation data. A semi-analytical homogenisation method is introduced through finite element analysis. Auxiliary maps are shown to be a better alternative to the overall orientation of fibres. Global calculations are insensitive to local variations whilst appropriate auxiliary maps offer refined results. Considering the multi-disciplinary application of orientation tensors, the proposed scheme can be used in all areas where local information cannot be disregarded.

KEYWORDS

Orientation tensor; transverse isotropy; natural fibre composite; computational mechanics

1. Introduction

Statistical distribution of orientation. Statistical measurement of orientation in space results in spherical data [1]. The ‘direction’ and ‘orientation’ terms are distinguished from each other since a vector designates the former whereas a line or axis is adequate for the latter. Therefore, a unit-length vector ($\mathbf{d}$) contains the information for both direction and orientation. The loci of the head of all such vectors is a unit 2-sphere, i.e., $\mathbb{S}^2 = \{\mathbf{d} : \|\mathbf{d}\| = 1\}$, where $\|\square\|$ denotes the Frobenius norm. The notion of direction in 3D requires the whole sphere whereas orientation is expressed via the hemisphere, i.e., it suffices for illustration of centro-symmetric statistical distributions. The orientation probability density function (ODF) $\Psi(\mathbf{d})$ provides the probability of finding a fibre in the direction $\mathbf{d}$. In 2D, similar arguments hold for a unit 1-sphere (unit circle).

Although such distributions can be captured by ODFs, tensors are used as more compact alternatives. They are approximations that mathematically provide more flexibility. For instance, their invariants can be used to develop constitutive laws while objectivity (material frame indifference) is preserved. Moreover, other properties such

as the clustering index [2] can be determined from orientation tensors that might be used to detect inter-region deviation of fibre orientation. Although this statistical index might show more sensitivity in detecting local fibre distribution (in comparison to the eigenvalues of the orientation tensor), it lacks the flexibility provided by a tensorial measure [3].

Terminology. ‘Orientation tensors’, ‘fabric tensors’, ‘moment tensors’ or ‘structure tensors’ are different terms used to indicate quantities that capture the directional dependency of a microstructure. Although all these terms refer to the same mathematical quantity, the use in the literature has been more context-based. Often, orientation tensors are used to acquire the orientation of short fibres in injection moulding [4] or generally fibre-reinforced composites [5]; fabric tensors are used to characterise the direction of granular materials [6] or porous materials such as snow [7] and bone [8] or quantify the distribution of damage [9]; and structure tensors are used for fibrous biological tissue such as cartilage [10]. Herein, the term ‘orientation tensor’ will be used as it attempts to quantify the orientation of the meso-structure that could contain any of the aforementioned heterogeneities. In the literature, it is mostly used to point to either the first or third kinds of Cowin’s fabric tensors.

Kinds of orientation tensors. Kanatani [11] introduced the idea of approximating any distribution (more specifically ODFs) using spherical harmonics [12] as bases for expansion in 3D; three kinds of orientation tensors were introduced to approximate an ODF:

- (1) The fabric tensor of first kind of order k (moment tensor) is the average of tensor products of directions, see Eq. (11). Herein, this kind is called the ‘averaged orientation tensor’ and it is the building block of other types.
- (2) The fabric tensors of second kind of order k are the coefficient tensors contracted with the direction vectors as basis of the expansion. They are calculated from the first kind but the coefficients of higher order depend on the lower order ones, which results in an inflexible approximation [11].
- (3) The fabric tensor of third kind of order k is calculated from its first kind counterpart by a subtraction; it is equivalent to the traceless symmetric component in the irreducible decomposition of the fabric tensor of first kind.

The term ‘fabric tensor’ was used by Cowin as the second best measure of meso-structure for a porous material after porosity [13]. He generalised the previously-discussed idea of fabric ellipsoid [14] to fabric tensors, i.e., all second-order positive-definite symmetrical tensors characterising ‘local architectural anisotropy’ or ‘fabric’. Cowin’s fundamental assumption was that the direction-dependency of mechanical response follows that of the fabric. Namely, anisotropy could be incorporated into an isotropic elastic tensor via the fabric tensor [15]. Based on this idea, the orientation tensor has been used to create anisotropic models from the isotropic ones, see the alternative model in [16]. Cowin also developed tensorial polynomial failure criteria based on the fabric tensors of first kind [17].

Spectral analysis of orientation tensor. Cowin’s idea of using orientation tensors to describe material anisotropy manifests itself in spectral analysis, i.e., principal directions of orientation tensor coincides with those of the material. The most-commonly used second-order orientation tensor with three, two, and one distinct eigenvalues

denotes orthotropy, transverse-isotropy, and isotropy, respectively. Identifying lower orders of symmetry require higher order tensors. For instance, second-order orientation tensors cannot identify a cubic symmetry and just simply reduces to isotropy [13]. Nevertheless, spectral analysis of the 2nd-order orientation tensor is sufficient for estimating the principal direction of the material [18].

Auxiliary Maps. Measures like the orientation tensor, which could provide meso-structural material properties, refine one’s understanding about material behaviour; they are called ‘auxiliary maps (fields)’ herein. In the context of fibre-reinforced composites (FRCs), volume fraction and orientation of fibres are omnipresent in the literature—but very often discussed in a global sense. Thus, they might be suggested as the first two auxiliary maps for FRCs. Although the availability of local data as auxiliary maps might be limited, there are few numerical methods in the literature that take advantage of them anyway.

Embedded elements. Discrete numerical simulation of meso-structure is possible but computationally limited. For example in FRCs, modelling fibres as discrete elements implicitly incorporates auxiliary maps into the finite element model but it is computationally expensive. To reduce the cost, often smaller samples of the material are analysed, e.g., in the form of representative elements [19,20], which might lead to apparent mechanical properties rather than the true effective ones. Such artefacts can be related to the wall effect of boundary conditions and the sample size that link the obtained properties more to the structure of the realisations rather than those of the ensemble (material).

Multi-phase elements. ‘Multi-phase element’ methods extract integration point properties from auxiliary maps without using any embedded elements. However, they often require additional number of integration points to properly assign the properties from the meso-structure to elements [22]. An additional obstacle is incorporating the anisotropy of the microstructure, which requires an ODF or its discretised version in the form of orientation tensors; the latter is often calculated in finite regions, e.g., layers of soft tissue [23] or arbitrary layers in injection moulding [24]. However, depending on the manufacturing process, e.g., in sheet moulding compound, the orientation of fibres along thickness becomes insignificant while in-plane dispersed or aggregated texture is created for discontinuous fibres [25]. Or in injection moulding, the orientation of the core layer is more planar than the two boundary layers [4]. Lack of out-of-plane orientation distribution is quantitatively backed up due to vanishing the minimum eigenvalue along thickness, see the provided results in [24] for a layer-wise approach.

Natural fibre composites. Although analytical homogenisation schemes might provide a solid base for calculating effective parameters, localisation calculations can be detrimental, e.g., in strength calculation of brittle inclusions [21]. Most of the analytical formulations are implemented numerically in a global fashion, i.e., the orientation tensor is used to obtain the overall direction-dependency rather than a local one [26]. For the case of natural fibre-reinforced composites (NFRCs), inherent variation of material and geometrical properties, e.g., length-dependent strength [27], length-wise non-uniformity [28], irregular cross-section [28], and sensitivity to hygroscopic effects [29], adds to the complications of the analysis. Such statistical variations also motivate for

a closer look at the homogenisation process in order to obtain a more refined smeared response.

Aim. The aim is to obtain a computationally-inexpensive approach by reducing the anisotropic and homogeneity conditions from the global scale to local by means of auxiliary maps. By this method, instead of embedded fibre elements, their homogenised properties are used in a local sense while preserving their dominant orientation. Moreover, the global ergodicity condition is released, and thus randomly-oriented fibres can be used within the provided framework and the failure of elements can be considered independent of each other. Namely, the RV for the whole structure is replaced by element-wise RVs. A semi-analytical procedure is introduced within the framework of finite element (FE) method to carry out the calculations. Moreover, damage is indirectly modelled by deactivating failed elements.

Outline. The outline of the current work is as follows: in Section 2, the theoretical background is reviewed, i.e., the mathematical framework for orientation tensors is reviewed, a constitutive law is introduced in its general form, and the required special case is derived. In section 3, the details of the numerical scheme regarding creating the prototype is described. Section 4 discusses the results along with the visual maps illustrating the local anisotropy of the samples. Finally in Section 5, concluding remarks are presented along with an overview towards future studies.

Notation. Herein, the intrinsic notation is used for the sake of brevity and with the hope of a better representation of notions. However, the equivalent indicial notation is presented to avoid any ambiguity. Namely, Einstein's summation convention is adopted only for subscripts while other indices, like superscripts, are merely used for elaboration. For the former case, Latin indices run through the values 1, 2, and 3. Therefore, 3D Cartesian basis vectors are denoted in this abbreviated form $\{\mathbf{e}_i\} \equiv \{\mathbf{e}_i\}_{i=1}^3$.

Tensors of zeroth order (or scalars) are symbolized by italic letters, e.g., a , lowercase bold italic letters denote first-order tensors, e.g., $\mathbf{a} = a_i \mathbf{e}_i$, second-order tensors are designated by italic uppercase bold letters, e.g., $\mathbf{A} = A_{lm} \mathbf{e}_l \otimes \mathbf{e}_m$, and fourth-order tensors are symbolized by italic uppercase bold calligraphic letters, e.g., $\mathcal{A} = A_{stuv} \mathbf{e}_s \otimes \mathbf{e}_t \otimes \mathbf{e}_u \otimes \mathbf{e}_v$. To emphasise on the order of a tensor, specially for higher order tensors, italic uppercase bold calligraphic letters are used along with a prefix indicating the order, e.g., ${}^6\mathcal{A} = A_{ijklmn} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \otimes \mathbf{e}_m \otimes \mathbf{e}_n$. Finally, non-serif bold letters are used to denote matrix quantities.

In the current study, the following operations are used for the aforementioned Carte-

sian tensors in terms of orthonormal bases:

$$\text{dyadic product: } \mathbf{a} \otimes \mathbf{b} := a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j, \quad (1a)$$

$$\text{tensor/vector composition: } \mathbf{A} \cdot \mathbf{a} := A_{ij} a_j \mathbf{e}_i, \quad (1b)$$

$$\text{tensor/tensor composition: } \mathbf{A} \cdot \mathbf{B} := A_{ij} B_{jk} \mathbf{e}_i \otimes \mathbf{e}_k, \quad (1c)$$

$$\text{double scalar product: } \mathbf{A} : \mathbf{B} := A_{ij} B_{ij}, \quad (1d)$$

$$\begin{aligned} k\text{-th tensor-product power: } \mathbf{A}^{\otimes k} &:= \underbrace{\mathbf{A} \otimes \dots \otimes \mathbf{A}}_{(k-1)\text{-times dyadic product}}, & \forall k \in \mathbb{N}^+, \\ & & (1e) \end{aligned}$$

$$\text{first conjugation product: } \mathbf{A} \underline{\otimes} \mathbf{B} := A_{ik} B_{jl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \quad (1f)$$

$$\text{second conjugation product: } \mathbf{A} \overline{\otimes} \mathbf{B} := A_{il} B_{jk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \quad (1g)$$

$$\text{symmetric product: } \mathbf{A} \underline{\underline{\otimes}} \mathbf{B} := 1/2(\mathbf{A} \underline{\otimes} \mathbf{B} + \mathbf{A} \overline{\otimes} \mathbf{B}), \quad (1h)$$

and the operators are defined over the appropriate operand $\square$ as

$$\text{symmetric operator: } \square^{\text{sym}} := 1/2(\square + \square^T), \quad (2a)$$

$$\text{trace operator: } \text{tr}(\square) := \square : \mathbf{I}, \quad (2b)$$

where $\mathbf{I}$ is the 2nd-order unit tensor. Finally, 4th-order projectors are represented in terms of their 2nd-order counterpart:

$$\mathcal{I} = \mathbf{I} \underline{\otimes} \mathbf{I}, \quad (3a)$$

$$\mathcal{I}^{\text{sym}} = \mathbf{I} \underline{\underline{\otimes}} \mathbf{I}, \quad (3b)$$

$$\mathcal{I}^{\text{vol}} = \frac{1}{3} \mathbf{I} \otimes \mathbf{I}. \quad (3c)$$

2. Theoretical Background

2.1. Discrete Evaluation of Oriented Functions

The PDF of orientation $\mathbf{d}$, denoted by $\psi(\mathbf{d})$, quantifies the probability of finding fibres in that direction. The *directional average* of a direction-dependent (oriented) quantity such as $f \equiv f(\mathbf{d})$ is defined as [30]

$$\langle\langle f \rangle\rangle := \int_{\mathbb{S}^2} \psi(\mathbf{d}) f(\mathbf{d}), \quad (4)$$

in which the differential element is dropped for the sake of generality. The orientation PDF ψ is normalised with respect to unit sphere by enforcing $\langle\langle 1 \rangle\rangle \stackrel{!}{=} 1$. In the $\mathbb{S}^2$ unit sphere, the polar angle (ϑ) and the azimuthal angle (φ) are used to parametrise the orientation vector:

$$\mathbf{d} \equiv \mathbf{d}(\vartheta, \varphi) = \sin \vartheta \cos \varphi \mathbf{e}_1 + \sin \vartheta \sin \varphi \mathbf{e}_2 + \cos \vartheta \mathbf{e}_3, \quad (5)$$

where the $\vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ ranges of the parameters along with the Cartesian basis vectors $\{\mathbf{e}_i\}$ completely characterise the space, see Fig. 1.

Figure 1. Representation of direction vectors in $\mathbb{S}^2$

Generally, the continuous orientation PDF can be approximated in the case of N discrete non-zero values of the oriented function $f(\mathbf{d})$ by an empirical distribution density in the form [11]

$$\psi(\mathbf{d}) = \frac{1}{N} \sum_{j=1}^N \delta(\mathbf{d} - \mathbf{d}^{(j)}), \quad (6)$$

where by means of the Dirac delta generalised function (δ), the existence of a fibre along a particular direction in $\mathbb{S}^2$ is captured:

$$\delta(\mathbf{d} - \mathbf{d}^{(j)}) = \frac{1}{\sin \vartheta^{(j)}} \delta(\varphi - \varphi^{(j)}) \delta(\vartheta - \vartheta^{(j)}). \quad (7)$$

Thus, the average value obtained from a continuous oriented function in Eq. (4) can be replaced by N contributions from a discretised counterpart:

$$\langle\langle f \rangle\rangle = \frac{1}{N} \sum_{j=1}^N f(\mathbf{d}^{(j)}), \quad (8)$$

which is equivalent to Kanatani's orientation tensor of first kind [11]. Alternative versions are available in literature; for instance in the context of fibre-reinforced composites, a weighted average is used such that the volume of fibres (Ω_f) is the scaling parameter [24]:

$$\langle\langle f \rangle\rangle' = \frac{1}{\Omega} \sum_{j=1}^N \Omega_f f(\mathbf{d}^{(j)}), \quad (9)$$

where Ω is the total volume of fibres in the sample.

2.2. Averaged Orientation Tensor

Defining the *orientation tensor* as

$$\mathbf{A} := \mathbf{d} \otimes \mathbf{d}, \quad (10)$$

results in an *averaged orientation tensor* of order $2k$ ¹ or orientation tensor of first kind (moment tensor) [11]:

$$\mathcal{H}_k := \langle\langle \mathbf{A}^{\otimes k} \rangle\rangle. \quad (11)$$

Since $\mathbf{A}$ has no odd powers, it is symmetric, and thus its average $\mathcal{H}_k$ is also symmetric [11,31]. Note that this formulation automatically rules out the trivial odd-ordered combinations of orientation vectors. Moreover, lower order averaged orientation tensors are obtained by contracting higher ones:

$$\mathcal{H}_{k-1} = \mathcal{H}_k : \mathbf{I}, \quad \forall k \in \mathbb{N} \geq 2, \quad (12)$$

and therefore they are not independent [4]. This is another incentive to use the orientation tensor of the third kind of various orders since they are independent from each other [11]. Both types are similar, and thus share the same eigenvalues and eigenvectors. Therefore, either one can be used in spectral analysis.

The second-order orientation tensor $\mathbf{H}_1 = \langle\langle \mathbf{A} \rangle\rangle$ is frequently used in the literature; for simplicity, it will be denoted by $\mathbf{H}$ henceforth. The fourth-order orientation tensor $\mathcal{H}_2 = \langle\langle \mathbf{A} \otimes \mathbf{A} \rangle\rangle$ (denoted by $\mathcal{H}$) is also available in the literature, see [32]. In either case, symmetry reduces the number of independent parameters. For instance in [23], a ‘generalised structure tensor’ is introduced to capture the orientation of collagen fibres in soft tissue (equivalent to the second-order averaged orientation tensor). As mentioned earlier, the orientation PDF is centro-symmetric, and thus $\psi(-\mathbf{d}) = \psi(\mathbf{d})$. An additional symmetry axis $\mathbf{d}_0$ could also exist if $\psi(\mathbf{Q} \cdot \mathbf{d}_0) = \psi(\mathbf{d}_0)$, where $\mathbf{Q}$ belongs to the symmetry group of all proper orthogonal tensors about $\mathbf{d}_0$. If it is assumed that the polar axis coincides with $\mathbf{d}_0$, the averaged orientation tensor can be represented by a single dispersion parameter. In the absence of such symmetry in an n -dimensional space, evaluation of the $2k$ -order average orientation tensor requires the calculation of n^{2k} integrals over $\mathbb{S}^2$ (without considering its universal symmetry). A k -th order averaged orientation tensor can be represented by k scalars provided that a transversely isotropic orientation PDF exists [30].

Transverse isotropy of the orientation PDF does not exist in most practical cases, at least not globally. For instance in a short FRC, no symmetry groups might be found for the whole specimen or at the other extreme, even a complete isotropy might be detected for completely random samples; either case might be unsophisticated characterisations of meso-structure. On the other hand, a symmetry axis might just exist locally where only a few fibres (ideally one) exist(s) whereas demanding such a condition for the whole sample is a very strong one. The remedy might be an auxiliary map with enough resolution so that our assumption of transverse isotropy carries small errors. In the current study, the resolution of the auxiliary maps is set according to the finite elements of composite. Obviously, other finer/coarser arrangements are possible.

2.3. Strength-of-Material Homogenisation

Enhanced rule of mixture. Consider a transversely isotropic material with the axis of symmetry along the 1-direction and the 2-3 isotropy plane. From the strength-of-material point of view, homogenisation formulas for unidirectional composites, i.e., the rule of mixture (RoM) and the inverse rule of mixture (IRoM), are based on the

¹Mathematically, it is shown by ${}^{2k}\mathcal{H} \equiv \langle\langle \mathbf{d}^{\otimes 2k} \rangle\rangle$, and thus it is a tensor of order $2k$.

iso-strain and iso-stress assumptions, respectively:

$$E_1 = \zeta_f E_f + \zeta_m E_m, \quad (13a)$$

$$\frac{1}{E_2} = \frac{\zeta_f}{E_f} + \frac{\zeta_m}{E_m}, \quad (13b)$$

where ζ is the volume fraction and E shows the elastic modulus. Subscripts of ‘f’ and ‘m’ are used to denote fibre and matrix phases, respectively. The elastic moduli E_1 and E_2 set the upper and lower bounds of stiffness and are in accordance with the longitudinal and transverse stiffnesses, respectively. An efficiency parameter η is used to further adjust these bounds by altering the contribution of fibres. The result is the enhanced rule of mixture (En-RoM) [33]

$$E'_1 = \eta \zeta_f E_f + \zeta_m E_m, \quad (14a)$$

$$\frac{1}{E'_2} = \frac{\zeta_f}{\kappa E_f} + \frac{\zeta_m}{E_m}, \quad (14b)$$

with $\eta := \kappa \eta_d \eta_l \eta_o$ where κ is the fibre area correction factor (FACF), η_d is the fibre diameter distribution factor (FDDF), η_l is the fibre length distribution factor (FLDF), and η_o is the fibre orientation distribution factor (FODF), respectively. Well-characterised fibre diameters assign a unit value to FDDF; a unit value is also assigned to FLDF since continuous fibres are the focus of current study. The FODF can be estimated by Krenchel’s formula [34]

$$\eta_o = \sum_n a_n \cos^4 \theta, \quad (15)$$

where a_n is the fraction of fibres making a θ angle with the applied load. The FACF is suggested by [28] and is used to rectify the underestimation of stiffness and strength due to an overestimation in measuring the cross-section area (CSA) of fibres. For jute fibres, a typical value of 1.42 was suggested in the same study.

In the same spirit, the relation for shear modulus (μ'_{12}) is established

$$\frac{1}{\mu'_{12}} = \frac{\zeta_f}{\kappa \mu_f} + \frac{\zeta_m}{\mu_m}, \quad (16a)$$

The Poisson’s ratio in anisotropic elasticity could be defined as $\nu_{ij} := -\varepsilon_j/\varepsilon_i$ and for an orthotropic material, due to the symmetry of the compliance matrix, the relation $\nu_{ij}/E_i = \nu_{ji}/E_j$ is established. The Poisson’s ratio of the composite ν_{12} is obtained from

$$\nu_{12} = \zeta_f \nu_f + \zeta_m \nu_m. \quad (17)$$

Finally, the transversely isotropic elastic compliance matrix $\mathbf{C}^{-1}$ for the composite along the 1-axis, i.e., when 2-3 is the plane of symmetry, is established using contracted

Voigt notation:

$$\mathbf{C}^{-1} = \begin{bmatrix} \frac{1}{E'_1} & -\frac{\nu'_{21}}{E'_2} & -\frac{\nu'_{21}}{E'_2} & 0 & 0 & 0 \\ & \frac{1}{E'_2} & -\frac{\nu'_{23}}{E'_2} & 0 & 0 & 0 \\ & & \frac{1}{E'_2} & 0 & 0 & 0 \\ & \text{sym} & & \frac{1}{\mu'_{23}} & 0 & 0 \\ & & & & \frac{1}{\mu'_{12}} & 0 \\ & & & & & \frac{1}{\mu'_{12}} \end{bmatrix}, \quad (18)$$

where $\mu'_{23} = \frac{E'_2}{2(1+\nu'_{23})}$.

Equations (14a) to (17) are generally used globally and in order to apply them locally, the efficiency parameter and the volume fraction must be localised. The volume fraction is readily calculated from its auxiliary map and all of the parameters involved in calculation of efficiency parameter η can be assumed constant except for the FODF since its spatial variation is not negligible—specially in the cases where fibre clustering or resin-rich volumes exist. Thus, the second auxiliary map would be that of the orientation tensor.

Using the obtained material properties, the elastic tensor for a transversely isotropic material is in hand. The axis of symmetry is obtained from the spectral analysis of the orientation tensor, i.e., the principal direction of the material is denoted by the eigenvector of the maximum eigenvalue of the orientation tensor. This tensor is used to calibrate the more general formulation discussed in the following section.

2.4. Constitutive Material Model

In [26], the formulation of elastic moduli of composites is carried out by setting up a quadratic free energy function in which the invariants of the orientation tensor $\mathbf{A}$ and infinitesimal strains tensor $\boldsymbol{\epsilon}$ are incorporated. The result is rearranged in the following (minor and major) symmetric form for the averaged elastic tensor:

$$\langle\langle \mathbf{C} \rangle\rangle = 3\lambda \mathbf{I}^{\text{vol}} + 2\mu_{\text{T}} \mathbf{I}^{\text{sym}} + 2\alpha(\mathbf{H} \otimes \mathbf{I})^{\text{sym}} + 4(\mu_{\text{L}} - \mu_{\text{T}})(\mathbf{H} \underline{\otimes} \mathbf{I})^{\text{sym}} + \beta \mathcal{H}, \quad (19)$$

where out of the five constants, λ , α , and β are functions of elastic properties of an aligned FRC and μ_{L} and μ_{T} are its shear moduli in the longitudinal and transverse directions. All these parameters are calculated based on the relations of the previous section. The orientation tensor adjusts the deviation from this perfectly aligned status. Overall, the symmetry group of this constitutive tensor is all orthogonal transformations about the axis of symmetry. This form is identical to what is proposed in [4] or more generally in [13]. Herein, the rearranged representation concisely illustrates the individual symmetrical terms in direct notation. Furthermore, the 4th-order averaged orientation tensor $\mathcal{H}$ can be replaced by means of various closure approximations; see [35,36].

Considering the fact that the invariant $\text{tr}(\mathbf{H})$ is unit, the model in Eq. (19) is calibrated for the principal material directions. This is the case where symmetry reduces the the number of independent parameters to one. Namely, the only non-zero component of $\mathbf{H}$ becomes its maximum eigenvalue. Thereby, the constitutive equation of transversely isotropic material becomes available for any arbitrary orientation tensor. Accordingly, the material parameters of Eq. (19) can be readily related to the Voigt

components of the elastic tensor in principal direction—which is obtained by inverting Eq. (18):

$$\beta = C_{11} + C_{22} - 2C_{12} - 4C_{66}, \quad (20a)$$

$$\alpha = C_{12} - C_{23}, \quad (20b)$$

$$\mu_L - \mu_T = C_{66} + \frac{1}{2}(C_{23} - C_{22}), \quad (20c)$$

$$\lambda = C_{23}, \quad (20d)$$

$$\mu_T = \frac{1}{2}(C_{22} - C_{23}), \quad (20e)$$

where $\mu_T \equiv \mu_{23}$ and $\mu_L \equiv \mu_{12}$ retain their physical meanings.

After incorporating the averaged orientation tensor, an explicit formulation of the final elasticity tensor for any arbitrary orientation can be obtained from Eq. (19). By assuming $\nu_{23} = 0$ and a planar distribution of fibres on the 1-2 plane, the non-zero components of the elastic tensor become

$$C_{11} = \lambda + 2\mu_T + 4(\frac{1}{2}\alpha + \mu_L - \mu_T)H_{11} + \beta H_{11}^2, \quad (21a)$$

$$C_{12} = \lambda + \alpha(H_{11} + H_{22}) + \beta H_{11}H_{22}, \quad (21b)$$

$$C_{13} = \alpha H_{11} + \lambda, \quad (21c)$$

$$C_{16} = \alpha H_{11} + (\alpha + 2\mu_L - 2\mu_T)H_{12} + \beta H_{11}H_{12}, \quad (21d)$$

$$C_{22} = \lambda + 2\mu_T + 4(\frac{1}{2}\alpha + \mu_L - \mu_T)H_{22} + \beta H_{22}^2, \quad (21e)$$

$$C_{23} = \alpha H_{22} + \lambda, \quad (21f)$$

$$C_{26} = \alpha H_{22} + (\alpha + 2\mu_L - 2\mu_T)H_{12} + \beta H_{22}H_{12}, \quad (21g)$$

$$C_{33} = \lambda + 2\mu_T, \quad (21h)$$

$$C_{36} = \alpha H_{12}, \quad (21i)$$

$$C_{44} = (\mu_L - \mu_T)H_{22} + \mu_T, \quad (21j)$$

$$C_{45} = (\mu_L - \mu_T)H_{12}, \quad (21k)$$

$$C_{55} = \beta H_{12}^2(\mu_L - \mu_T)(H_{11} + H_{22}) + \mu_T, \quad (21l)$$

$$C_{66} = \mu_T + (\mu_L - \mu_T)(H_{11} + H_{22}) + \beta H_{12}^2. \quad (21m)$$

The plane stress case governs the behaviour of thin laminates and is of interest in the current study. Therefore, the reduced stiffness elastic tensor ($\bar{\mathbf{C}}_{2D}$) [37,38] can be calculated from the evaluated components:

$$\bar{\mathbf{C}}_{2D} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ & \bar{C}_{22} & \bar{C}_{26} \\ \text{sym} & & \bar{C}_{66} \end{bmatrix}, \quad (22)$$

with

$$\bar{C}_{ij} := C_{ij} - \frac{C_{i3}C_{3j}}{C_{33}}, \quad \forall i, j \in \{1, 2, 6\}. \quad (23)$$

3. Methodology

Experimental values. Experimental data of jute-fibre reinforced epoxy tensile samples tested in [27,39] are used as a study case. The composite samples were made from a $150 \times 25 \times 3.52 \text{ mm}^3$ plate with an average volume fraction of $\zeta_f = 18.88\%$. The average properties of tensile samples were deemed to represent those of the laminate, and thus only the plate properties were used in numerical procedures. The statistical distribution of the fibre orientation, length, and CSA were available in the mentioned studies, see Table 1 for a summary. The computational analyses were facilitated by means of the interaction between an in-house computer program and a commercial FE package.

Table 1. Summary of the case study material properties based on the experimental data of [27,40]

Material	Parameter	Value
Jute fibre	Axial elastic modulus	27.9 GPa
	Axial-radial Poisson's ratio	0.35
	Average axial fibre strength*	298.13 MPa
	Fibre length distribution	Log-normal (mean=4.183, SD=0.976)
	CSA distribution	Log-normal (mean=7.55, SD=0.52)
	Orientation distribution	Gaussian (mean=7.36, STD=17.95)
	Mean fibre orientation factor	0.81 ± 0.06
Epoxy	Elastic modulus	2.65 GPa
	Poisson's ratio	0.35
	Failure strength	61 MPa

* The average axial strength corresponds to a fibre with average length in the distribution.

Python pre-processor. In order to avoid modelling discrete fibres, the derived analytical formulation in the previous section is combined with finite element analyses [41]. The object-oriented programming paradigm is used within a Python script to generate pseudo-random long fibres that follow certain distributions. The auxiliary maps were obtained from the generated objects using a python script with multi-processing capability to increase computational efficiency. The FE prototypes were generated using the same Python script through an automated process that managed the parametric model generation from the prototype and controlled the job submissions [20].

FORTRAN processor. The semi-open MARC/MENTAT 2019 FE package is used to take advantage of its customised subroutine capability [42]. Namely, the material constitutive behaviour and element strength checking were carried out by means of several FORTRAN90 subroutines. The required auxiliary maps were passed from the Python script to FORTRAN subroutines by means of several data files. The maximum stress failure criterion is used to check the failure of the homogenised medium, i.e., a simultaneous failure of matrix and fibre is simulated. To this end, at the end of every increment, the mean stress of an element was calculated by averaging the stresses at its integration points and compared with the average strength of the element. In the case of failure, the respective element is removed to implicitly incorporate the stiffness reduction effect of damage progression into the FE prototype. The load increments continued until sample failure, which is marked by the solver losing convergence.

Python post-processor. The Python script interacted with the MENTAT pre-processor for model generation and then submitted the job to the MARC solver for analysis where FORTRAN subroutines were involved. Afterwards, the script gains the control back and the post-processing result are exported to Excel files.

Fibres. Rather than using discrete fibres in the FE model, the auxiliary maps for the local volume fraction and local orientation tensors were used in constitutive modelling. The maps were extracted based on the generated fibre data, i.e., fibre length distribution, fibre CSA distribution, and fibre orientation distribution. The technical jute fibres were assumed to be straight entities with a constant CSA per fibre that were generated from a log-normal distribution of the ‘true’ fibre CSA [43]. The quasi-unidirectional fibre orientation distribution was achieved by following a Gaussian distribution of angles and allowing fibres to penetrate the soft boundary of the model to create a periodic structure. The properties of the jute fibres follow the data in Table 1. It is worth mentioning that although the length variation of fibres is considered, the variation of length-dependent properties were ignored. For instance, the elastic modulus of fibres was taken to be the mean of the 785 jute samples tested in [27], i.e., a mean value of 27.9 GPa, which is sensibly independent of fibre length. Furthermore, the strength of the fibres was assumed to be length-independent.

Matrix. The epoxy matrix was modelled by 2D plane stress elements (type 3) as an isotropic material. Constitutive material behaviour of the four-node isoparametric quadrilateral elements was replaced by Eq. (22) and linear interpolation functions were used. A perfect bond between the fibres and matrix was assumed, and thus any interphase interactions were disregarded for simplicity.

FE prototype. The FE prototype consisted of anisotropic plane stress elements, which represented the homogenised properties of the composite. In order to focus on the effects of orientation, the standard deviation (SD) of the Gaussian orientation distribution was varied from 0 to 20 according to Table 2—the SD=17.95 was included for validation against experimental results. Five realisations were created for each SD and the average value of the results were reported in the aforementioned table. In addition to the experimental value, a global method was included. This benchmarking method used the global orientation of the fibres from spectral analysis and calculated the effective elastic modulus by Eq. (15). Moreover, the failure of the homogenised elements were checked by means of the maximum failure stress criterion and the failed elements removed from the analysis at each increment. The FE analyses halted when achieving convergence became infeasible due to the loss of positive-definiteness of the stiffness matrix.

4. Results and Discussion

Mesh density and map resolution. Often, the transfer of data between the FE prototype and the available auxiliary maps is carried out irrespective of the required resolution. Herein, the resolution of these auxiliary maps is linked to the size of elements, i.e., higher mesh densities demand higher resolution from the auxiliary maps. Although in practical cases the resolution might be limited to the accuracy of test apparatus, the approximations above this precision level can be established by the user.

Obviously, numerically-generated data can provide the required resolution more easily. In any case, such tensorial fields provide additional information of the meso-structure to obtain better approximations of the response of the heterogeneous medium.

Figure 2. Mesh sensitivity of a local analysis for one realisation (SD=20)

Mesh sensitivity. Since element removal has an effect similar to integrating damage into the constitutive material model, mesh dependency inherently exists. Thus, to alleviate this localised deficiency, the elastic modulus of an FE prototype with the highest discrepancy (SD=20) was regarded. Obviously, convergence of other cases with lower SD is obtained more easily. Mesh density is defined as [19]

$$\gamma := \frac{1}{\ell_{\text{mesh}}}, \quad (24)$$

where (ℓ_{mesh}) is the mesh length. It is varied over a range of 0.1 to 2 to monitor the convergence behaviour. In the same manner, the auxiliary maps are also refined accordingly. As illustrated in Figure 2, the effective (or apparent) elastic modulus of the composite gains convergence quickly with γ over 1. In the current study, a mesh density of 1 is adopted in the analyses considering the trade-off between accuracy and efficiency.

Refining auxiliary maps. In Figure 3, a fibre volume fraction auxiliary map is shown. The local volume fraction is calculated as a volume average over each element, using Eq. (9), and is depicted at its corresponding centre. Thus, the corner of each pixel in the graphs correspond to the centre of an element. The mesh density controls both the refinement of finite elements and the related auxiliary maps, i.e., finer mesh requires higher resolution auxiliary maps. All the illustrated maps correspond to the same realisation with a fibre volume fraction $\zeta_f = 18.88\%$. It is clear that by increasing the mesh density, a wider range of variation is captured over the domain and a more realistic visualisation of fibre distribution is obtained.

A similar argument can be laid out for the fibre orientation distribution. In Fig. 4, the volume-averaged orientation tensor (weighted by fibre volume) is calculated over each element and depicted along its principal direction. Moreover, the length of each fibre is scaled by the first eigenvalue of the averaged orientation tensor to show the

intensity of the anisotropy. In the figures, the angle is scaled 6 times for a better presentation. The mean angle of the ODF is 7.36° and SD of 20. It is evident that increasing the resolution of orientation auxiliary map reveals more local fluctuations since the range of angles increase from $7.49^\circ \leq \varphi \leq 8.17^\circ$ for $\gamma = 0.1$ to $3.16^\circ \leq \varphi \leq 13.58^\circ$ for $\gamma = 1.5$.

Figure 3. Refining the volume fraction auxiliary map according to the mesh density for one realisation (SD=20)

Figure 4. Refining the orientation tensor auxiliary map according to the mesh density for one realisation (SD=20); the principal directions are graphed while the angles are scaled 6 times for clarification

Parametric study. The parametric study is carried out by varying the SD of the ODF, which is a Gaussian distribution as reported in [27]. Setting the SD to zero for the orientation distribution results in the special case of aligned fibres along the mean angle of the distribution (7.36°). Namely, the global orientation corresponds to the local one for each element. Therefore, the first point in Fig. 5 represents the global calculation that results in a mean stiffness of 8965.86 MPa. Similarly, the following five points, all below $SD=5$, result in values close to the initial perfectly aligned case. The obtained results from the local analysis do not show a significant difference compared to the global value. A rather sharp drop in stiffness is observed for values over SD of 5. Moreover, the results seems to be more spread out for higher values. Interestingly, the experimental value was recorded for the SD of 17.95, which is best estimated by the local FE realisations in the range of 15–17.5. Note that the experimental value is quite distant with respect to the global value and even the local values with low SD. More specifically in Table 2, global and low-variation local values results show a relative errors of around 9% whereas the local analysis of $SD=17.95$ contains half as much error. One could argue that based on the local FE results, the estimation of the ODF in [27] by the SDs in the range of 15–17.95 seems to be plausible.

Figure 5. Parametric result of all realisations along with SD error bars; the experimental value of 8190 was reported for $SD=7.95$ in [27]

As a measure of global behaviour, Krenchel’s formulation is used assuming that all the fibres are oriented along the mean angle, see Table 2. The angle is obtained from the eigenvector of the principal direction, which is denoted by the maximum eigenvalue. The behaviour of this analytical model, along with the angle obtained from the FE analysis clearly shows that global behaviour is insensitive to local fibre orientation. This global value overestimates the experimental value by approximately 15%.

From the micromechanical point of view, focusing on local computation has a secondary benefit: the ergodicity hypothesis remains valid for each elements. Thus, what is practically done is introducing an RV for each element rather than the whole structure. Inexpensive computational cost makes this approach practical. This assumption allowed to remove an element when the scale separation hypothesis collapses due to failure—without affecting other elements.

Obviously, the discussion is based on the few number of realisations. More accurate results could be obtained by increasing the number of realisations to better approximate the ensemble value. Namely, more realisations may decrease the variation of the

output but it would not violate the overall observed behaviour.

Table 2. Experimental and numerical results for the effective elastic modulus of the composite (E_c , in MPa), and relative error in percentage

Composite property		Standard deviation of angle distribution										
		0	1	2	3	4	5	10	15	17.5	17.95	20
E_{Local}	Mean (MPa)	8965.86	8958.61	8897.15	8940.65	8903.21	8905.26	8567.98	8074.07	7795.04	7805.69	7403.08
	SD (MPa)	23.33	53.76	74.20	26.65	42.32	120.40	135.48	98.19	381.70	215.77	264.99
	Relative error (%)	9.47	9.38	8.63	9.17	8.71	8.73	4.62	1.42	4.82	4.69	10.13
θ_{Global}	Mean	7.36	7.41	7.59	7.17	7.29	7.41	7.27	7.39	7.56	6.79	8.23
	SD	0.00	0.25	0.40	0.13	0.24	0.17	0.97	0.42	1.81	1.56	1.41
E_{Global}	Krenchel (MPa)	9386.08	9382.99	9370.84	9398.16	9390.46	9382.87	9392.08	9383.88	9372.93	9421.69	9325.91
	Relative error (%)	14.60	14.57	14.42	14.75	14.66	14.56	14.68	14.58	14.44	15.04	13.87

5. Conclusion

The focus of this work was on developing a simple element-wise semi-numerical scheme to homogenise aligned NFRCs through a cost-effective approach. To this end, after reviewing the concept of orientation tensor and presenting the pertaining constitutive equation, the following points were motivated to consider auxiliary maps:

- Local anisotropy (more specifically transverse isotropy for the case of long fibres) and its global counterpart were distinguished from each other. In the former, the elastic stiffness acquires a point-wise transverse symmetry for which the axis of symmetry is a function of location whereas in the latter insensitivity to the local variation is evident.
- Accepting the ergodic hypothesis for a material requires statistical homogeneity of its micro-structure (even when random heterogeneities are present). Fulfilling such a condition for the whole structure is quite strong but demanding it element-wisely relaxes the requirements.

The effective elastic modulus of NFRCs of an experimental study was predicted. Mesh dependency of the results was quickly removed by refining the elements while the auxiliary maps were scaled down to the same extent. It was observed that increasing the SD of ODF over values of 5 highly reduces the effective stiffness of the sample. While lower values of SD overestimated the experimental result, higher values (around the reported SD of the ODF) give plausible estimates within 5% error range in average. Analytical approach that considers the global average orientation, remained insensitive to local variation.

Based on the obtained results, the following points are suggested:

- (1) Instead of the more common global or layer-wise calculation of the orientation tensor, an element-wise approach should be adopted (where possible). The proposed procedure is in accordance with subdividing the whole structure to finite elements as auxiliary maps followed the same refinement. In addition, auxiliary maps of volume fraction and orientation tensor enabled setting up independent RVs for each element.
- (2) It was shown that mesh-independent results were obtained using auxiliary maps and through a 2D simple analysis. Namely, projecting all layers onto a single

2D layer does not include much loss of information in regard to orientation. This could be a good recommendation that as a result of specific manufacturing techniques, 3D computational is bypassed.

By implementing the suggestions of the current study, adequate local morphological information was captured and a more accurate, yet computationally inexpensive, simulation of mechanical response was carried out.

Finally, it is worth mentioning that the incentive behind presenting the concept of ‘auxiliary maps’ was the fact that many layers of data regarding a composite sample could be made available, e.g., from Computed tomography scans (μ CT) or 3D mapping by depth-sensing indentation. Nevertheless, the presence of such pieces of data are limited to the availability of the equipment, the accuracy of the equipment, the required precision, and the target output among other factors. Therefore, the user could decide how many layers of data, i.e., how many auxiliary maps, are required to refine the intended output. In the presented work, only the volume fraction and fibre orientation maps were used for the stiffness predictions. For other aspects such as failure strength or damage, additional maps would be necessary, e.g., a porosity map, or fibre clustering data. Note that in the absence of experimental local data, one could generate synthetic statistical data, e.g., by the Monte-Carlo methods or other pseudo-random generators. A numerical method could be useful to generate a synthetic map when only statistical estimates of a map is available.

Future work. Auxiliary map(s) of clustering, minimum inter-fibre distance, and/or damage distribution might be used in conjunction with the introduced maps of the current study to elaborate the understanding of material response in future works. A comparison between the local and global failure criteria should be carried out to challenge the proposed method in strength calculations—specially when element removal is involved.

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References

- [1] Fisher NI, Lewis T, Embleton BJJ. Statistical analysis of spherical data. Cambridge: Cambridge University Press; 1987.
- [2] Ranganathan S, Advani SG. Characterization of orientation clustering in short-fiber composites. *Journal of Polymer Science Part B: Polymer Physics*. 1990;28(13):2651–2672.
- [3] Javanbakht Z, Hall W, Öchsner A. Effective thermal conductivity of fiber reinforced composites under orientation clustering. In: Öchsner A, Altenbach H, editors. *Engineering design applications. (Advanced structured materials, 1869-8433; Vol. 92)*. Cham, Switzerland: Springer; 2019. p. 507–519.
- [4] Advani SG, Tucker CL. The use of tensors to describe and predict fiber orientation in short fiber composites. *Journal of Rheology*. 1987;31(8):751–784.
- [5] Müller V, Böhlke T. Prediction of effective elastic properties of fiber reinforced composites using fiber orientation tensors. *Composites Science and Technology*. 2016;130:36–45.

- [6] Millet O, Gu S, Kondo D. A 2d micromechanical modelling of anisotropy in granular media. *Comptes Rendus Mécanique*. 2007;335(4):231–237.
- [7] Srivastava PK, Mahajan P, Satyawali PK, et al. Observation of temperature gradient metamorphism in snow by x-ray computed microtomography: measurement of microstructure parameters and simulation of linear elastic properties. *Annals of Glaciology*. 2010; 51(54):73–82.
- [8] Odgaard A. Three-dimensional methods for quantification of cancellous bone architecture. *Bone*. 1997;20(4):315–328.
- [9] Krajcinovic D, van Mier JGM. Damage and fracture of disordered materials. (CISM courses and lectures; Vol. no.410). Wien: Springer-Verlag; 2000.
- [10] Tomic A, Grillo A, Federico S. Poroelastic materials reinforced by statistically oriented fibres—numerical implementation and application to articular cartilage. *IMA Journal of Applied Mathematics*. 2014;79(5):1027–1059.
- [11] Kanatani KI. Distribution of directional data and fabric tensors. *International Journal of Engineering Science*. 1984;22(2):149–164.
- [12] Jones MN. Spherical harmonics and tensors for classical field theory. (Electronic & electrical engineering research studies. Applied and engineering mathematics series; Vol. 2). Letchworth: Research Studies; 1985.
- [13] Cowin SC. The relationship between the elasticity tensor and the fabric tensor. *Mechanics of Materials*. 1985;4(2):137–147.
- [14] Oda M, Konishi J, Nemat-Nasser S. Some experimentally based fundamental results on the mechanical behaviour of granular materials. *Géotechnique*. 1980;30(4):479–495.
- [15] Shertzer RH. Fabric tensors and effective properties of granular materials with application to snow Phd. Department of Civil Engineering; 2011.
- [16] Zysset PK, Curnier A. An alternative model for anisotropic elasticity based on fabric tensors. *Mechanics of Materials*. 1995;21(4):243–250.
- [17] Cowin SC. Fabric dependence of an anisotropic strength criterion. *Mechanics of Materials*. 1986;5(3):251–260.
- [18] Javanbakht Z, Hall W, Öchsner A. The effect of partitioning on the clustering index of randomly-oriented fiber composites: A parametric study. *Defect and Diffusion Forum*. 2017;380:232–241.
- [19] Javanbakht Z, Hall W, Öchsner A. Finite element evaluation of effective thermal conductivity of short carbon nano tubes: A comparative study. *Defect and Diffusion Forum*. 2016;372:208–214.
- [20] Javanbakht Z, Hall W, Öchsner A. Automatized estimation of the effective thermal conductivity of carbon fiber reinforced composite materials. *Defect and Diffusion Forum*. 2016;370:177–183.
- [21] Böhm HJ. A short introduction to basic aspects of continuum micromechanics. 2018;.
- [22] Zohdi TI, Wriggers P, Huet C. A method of substructuring large-scale computational micromechanical problems. *Computer Methods in Applied Mechanics and Engineering*. 2001;190(43):5639–5656.
- [23] Gasser TC, Ogden RW, Holzapfel GA. Hyperelastic modelling of arterial layers with distributed collagen fibre orientations. *Journal of the Royal Society, Interface*. 2006;3(6):15–35.
- [24] Müller V, Brylka B, Dillenberger F, et al. Homogenization of elastic properties of short-fiber reinforced composites based on measured microstructure data. *Journal of Composite Materials*. 2016;50(3):297–312.
- [25] Eduljee RF, McCullough RL. Elastic properties of composites. In: Cahn RW, Haasen P, Kramer EJ, editors. *Materials science and technology*. Vol. 19. Weinheim, Germany: Wiley-VCH Verlag GmbH & Co. KGaA; 2006. p. 693.
- [26] Spencer AJM. Continuum theory of the mechanics of fibre-reinforced composites. (Courses and lectures / International Centre for Mechanical Sciences; Vol. no.282). Wien: Springer; 1984.
- [27] Virk AS, Hall W, Summerscales J. Modulus and strength prediction for natural fibre

- composites. *Materials Science and Technology*. 2013;28(7):864–871.
- [28] Virk AS, Hall W, Summerscales J. Multiple data set (mds) weak-link scaling analysis of jute fibres. *Composites Part A: Applied Science and Manufacturing*. 2009;40(11):1764–1771.
 - [29] Javanbakht Z, Hall W, Öchsner A. Computational evaluation of transverse thermal conductivity of natural fiber composites. In: Öchsner A, Altenbach H, editors. *Improved performance of materials*. (Advanced structured materials; Vol. 72). Cham: Springer; 2017. p. 197–206.
 - [30] Hashlamoun K, Federico S. Transversely isotropic higher-order averaged structure tensors. *Zeitschrift für angewandte Mathematik und Physik*. 2017;68(4):1.
 - [31] Comon P, Golub G, Lim LH, et al. Symmetric tensors and symmetric tensor rank. *SIAM Journal on Matrix Analysis and Applications*. 2008;30(3):1254–1279.
 - [32] Vasta M, Gizzi A, Pandolfi A. On three- and two-dimensional fiber distributed models of biological tissues. *Probabilistic Engineering Mechanics*. 2014;37:170–179.
 - [33] Summerscales J, Virk AS, Hall W. Enhanced rules-of-mixture for natural fibre reinforced polymer matrix (nfrp) composites (comment on lau et al. in volume 136). *Composites Part B: Engineering*. 2019;160:167–169.
 - [34] Krenchel H. *Fibre reinforcement; theoretical and practical investigations of the elasticity and strength of fibre-reinforced materials*. Akademisk forlag; 1964.
 - [35] Breuer K, Stommel M, Korte W. Analysis and evaluation of fiber orientation reconstruction methods. *Journal of Composites Science*. 2019;3(3):67.
 - [36] Advani SG, Tucker CL. Closure approximations for three-dimensional structure tensors. *Journal of Rheology*. 1990;34(3):367–386.
 - [37] Herakovich CT. *Mechanics of fibrous composites*. New York: Wiley; 1998.
 - [38] Altenbach H, Altenbach J, Kissing W. *Mechanics of composite structural elements*. Berlin and London: Springer; 2010.
 - [39] Virk AS, Hall W, Summerscales J. Modelling tensile properties of jute fibres. *Materials Science and Technology*. 2011;27(1):458–460.
 - [40] Virk AS, Summerscales J, Hall W. A finite element analysis to validate the rule-of-mixtures for the prediction of the young’s modulus of composites with non-circular anisotropic fibres. In: Figueiro RMEdS, Rana S, editors. *Natural fibres*. (RILEM book-series, 2211-0844; Vol. 12). Dordrecht: Springer; 2016. p. 173–182.
 - [41] Öchsner A. *Computational statics and dynamics: An introduction based on the finite element method*. Singapore: Springer; 2016.
 - [42] Javanbakht Z, Öchsner A. *Advanced finite element simulation with msc marc: Application of user subroutines*. Cham: Springer; 2017.
 - [43] Virk AS, Hall W, Summerscales J. Physical characterization of jute technical fibers: Fiber dimensions. *Journal of Natural Fibers*. 2010;7(3):216–228.