CAPM Option Pricing

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We thank Nicole Apitz and Wolfgang Schmid for helpful comments.

Abstract

This paper extends the option pricing equations of *Black and Scholes* (1973), *Jarrow and Madan* (1997) and *Husmann and Stephan* (2007). In particular, we show that the length of the individual planning horizon is a determinant of an option's value. The derived pricing equations can be presented in terms of the *Black and Scholes* (1973) option values which ensures an easy application in practice.

Keywords: Capital Asset Pricing Model; Option Pricing; Planning Horizon; Incomplete Markets.

JEL Classification: G13

1. Introduction

Options are redundant securities in a complete market. However, the empirical results of *Vanden* (2004) indicate that options are nonredundant in explaining returns on risky assets. In incomplete lognormal markets, *Jarrow and Madan* (1997) derive a CAPM option price equation. This study extends their results in three respects. First, a pricing equation is derived for an option on an arbitrary non-dividend asset, whereas the option pricing formula by *Jarrow and Madan* (1997) is only relevant for options on the market portfolio. Secondly, *Jarrow and Madan* (1997) assume that the planning horizon of the investor exceeds the time-to-maturity of the option. This study also comprises the case in which the option is sold before maturity. *Husmann and Stephan* (2007) consider a planning horizon equal to time-to-maturity. For a planning horizon longer or shorter than the option's time-to-maturity, we obtain two distinct formulas, making reference to the results presented by *Rubinstein* (1984) and *Husmann and Stephan* (2007). The formulas coincide solely when the planning horizon is exactly equal to the time-to-maturity of the option or when the market is complete. Thirdly, the value of an European option for all considered cases can be presented in terms of the *Black and Scholes* (1973) values of a number of options. This ensures that the results are easy to use in practice.

Sections 2 presents the notation and assumptions used. Section 3 indicates the theoretical results, and Section 4 concludes.

2. Assumptions and Notation

The valuation of options in section 3 is based on the default assumptions of the CAPM. Additionally, we assume that the instantaneous rate of return on any asset and the market portfolio have a joint normal distribution.

The following notation is used throughout the article:

h	Investor's planning horizon
τ	Time-to-maturity of an option
K	Exercise price on an option
S_0	Price of an underlying asset S
C_0	Price of a call on an asset S
$\tilde{S}_t$	Cash flow of the underlying asset
$\tilde{C}_t$	Cash flow of the call
$\tilde{R}_m$	Standardized cash flow of the market portfolio

r	Instantaneous risk-free rate of interest
$\tilde{r}_s$	Instantaneous rate of return on asset S
$\tilde{r}_m$	Instantaneous rate of return on the market index
μ_s	Expected instantaneous rate of return on asset S
μ_m	Expected instantaneous rate of return on the market index
σ_s	Instantaneous variance of the rate of return on asset S
σ_m	Instantaneous variance of the rate of return on the market index

Consider the case of a call option on an arbitrary non-dividend asset S maturing at time s when the planning horizon of the individual investor extends to time t . Our aim is to evaluate this option at time 0. Therefore, the time-to-maturity of the option equals $\tau = s - 0$ and the length of the planning horizon is $h = t - 0$.

Under the given parameters for a bivariate normal distribution of rates of return, the following expressions represent the expected values, variances and covariances of the securities' cash flow and market portfolio's standardized cash flow over a period with length h ¹

$$E[\tilde{S}_t] = S_0 e^{(\mu_m + \frac{1}{2}\sigma_s^2)h}, \quad (1)$$

$$E[\tilde{R}_m] = e^{(\mu_m + \frac{1}{2}\sigma_m^2)h}, \quad (2)$$

$$Var[\tilde{R}_m] = e^{(2\mu_m + \sigma_m^2)h} \left(e^{\sigma_m^2 h} - 1 \right), \quad (3)$$

$$Cov[\tilde{S}_t, \tilde{R}_m] = S_0 e^{(\mu_m + \frac{1}{2}\sigma_m^2 + \mu_s + \frac{1}{2}\sigma_s^2)h} \left(e^{\rho\sigma_m\sigma_s h} - 1 \right). \quad (4)$$

It is assumed that the parameters of the price process remain constant during the planning horizon. Furthermore, for the sake of clarity, we use the following notation for the *Black and Scholes* (1973) price of a call with time-to-maturity y when the price of the current asset is replaced by $S_0 e^{(\mu_s + \frac{1}{2}\sigma_s^2)y+x}$, the strike price is replaced by $K e^{ry}$ and the volatility of the asset amounts to σ_s :

$$C[x, y] = C_0^{BS} \left(S_0 e^{(\mu_s + \frac{1}{2}\sigma_s^2)y+x}, K e^{ry}, y, r, \sigma_s \right). \quad (5)$$

¹To calculate the moments of the lognormal distribution, see Appendix A in *Husmann and Stephan* (2007).

3. CAPM Option Pricing

In an incomplete lognormal market, the CAPM may be used for option pricing. The well-known certainty equivalent valuation formula of the CAPM is

$$C_0 = \frac{E[\tilde{C}_t] - \lambda \cdot Cov[\tilde{C}_t, \tilde{R}_m]}{e^{r \cdot h}} \quad \text{mit} \quad \lambda_h = \frac{E[\tilde{R}_m] - e^{r \cdot h}}{Var[\tilde{R}_m]}. \quad (6)$$

To be able to use this equation to value a call on S, we must first determine the expected cash flow of the call and the covariance between the cash flow of the call and the rate of return for the market portfolio. The expected cash flow of the call is

$$E[\tilde{C}_t] = \int_{-\infty}^{\infty} \tilde{C}_t f(r_s) dr_s, \quad (7)$$

$$\text{where} \quad f(r_s) = \frac{1}{\sigma_s \sqrt{h} \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{r_s - \mu_s h}{\sigma_s \sqrt{h}} \right)^2}.$$

We can simplify the calculation of the covariance using the decomposition theorem,

$$Cov[\tilde{C}_t, \tilde{R}_m] = E[\tilde{C}_t \cdot \tilde{R}_m] - E[\tilde{C}_t] \cdot E[\tilde{R}_m]. \quad (8)$$

$E[\tilde{R}_m]$ and $E[\tilde{C}_t]$ are to be obtained from (2) and (7) and

$$E[\tilde{C}_t \cdot \tilde{R}_m] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{C}_t \cdot e^{r_m} f(r_s, r_m) dr_m dr_s, \quad (9)$$

where the joint density function of the bivariate normal distribution $f(r_s, r_m)$ equals

$$\frac{1}{2\pi \sqrt{\sigma_s^2 \sigma_m^2 h^2 (1 - \rho^2)}} \cdot e^{-\frac{1}{2(1-\rho^2)} \left(\frac{(r_s - \mu_s h)^2}{\sigma_s^2 h} - 2\rho \frac{(r_s - \mu_s h)(r_m - \mu_m h)}{\sigma_s \sigma_m h} + \frac{(r_m - \mu_m h)^2}{\sigma_m^2 h} \right)}.$$

Furthermore, (2) and (3) give the following for the market price of risk

$$\lambda_h = \frac{e^{(\mu_m + \frac{1}{2}\sigma_m^2)h} - e^{r \cdot h}}{e^{(2\mu_m + \sigma_m^2)h} (e^{\sigma_m^2 h} - 1)}. \quad (10)$$

In the following, we discern three cases. First, it is assumed that the investor's planning horizon is longer than to the time-to-maturity of the option ($h > \tau$). Then, the investor's planning horizon coincides with the time-to-maturity of the option ($h = \tau$). Last, we assume that the option is sold before maturity ($h < \tau$) so that the remaining time-to-maturity is $(\tau - h)$. We establish pricing equations for call options. Put prices can be derived in an analogue way.

3.1. Planning horizon longer than the option's time-to-maturity

3.1.1. Expected value

Because any contingent claim paying $\max(S_0 e^{r_s} - K, 0)$ at time s is equivalent to one paying $\max(S_0 e^{r_s} - K, 0) e^{r(h-\tau)}$ at time t , the expected value of the call amounts to²

$$\begin{aligned} E[\tilde{C}_t | \mathcal{F}_0] &= \int_{-\infty}^{\infty} \max(S_0 e^{r_s} - K, 0) e^{r(h-\tau)} f(r_s) dr_s \\ &= e^{r(h-\tau)} \left(S_0 e^{(\mu_s + \frac{1}{2}\sigma_s^2)\tau} N(d_1) - K N(d_2) \right), \quad (11) \end{aligned}$$

where $d_1 = \frac{\ln((S_0 e^{(\mu_s + \frac{1}{2}\sigma_s^2)\tau}) / (K e^{r\tau})) + (r + \frac{1}{2}\sigma_s^2)\tau}{\sigma_s \sqrt{\tau}}$,

$$d_2 = d_1 - \sigma_s \sqrt{\tau}.$$

Therefore, the expected cash flow of a call can be presented as a special case of the compounded option valuation equation of *Black and Scholes* (1973), except that it is evaluated at

$$\begin{aligned} E[\tilde{C}_t | \mathcal{F}_0] &= e^{r(h-\tau)} C_0^{BS} \left(S_0 e^{(\mu_s + \frac{1}{2}\sigma_s^2)\tau}, K e^{r\tau}, \tau, r, \sigma_s \right) \\ &= e^{r(h-\tau)} C[0, \tau]. \quad (12) \end{aligned}$$

3.1.2. Covariance

To establish the covariance between the cash flow of the call and the rate of return on the market portfolio using the decomposition theorem, we first

²For a detailed deviation in the case $h = \tau$, see appendix B in *Husmann and Stephan* (2007). The results can be easily extended to hold for $h > \tau$. To present the results of *Husmann and Stephan* (2007) in terms of *Black and Scholes* (1973) prices, some conversions are necessary.

compute $E[\tilde{C}_t \cdot \tilde{R}_m | \mathcal{F}_0]$. Our task is to integrate

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \max(S_0 e^{r_s} - K, 0) e^{r(h-\tau)} e^{r_m} e^{(\mu_m + \frac{1}{2}\sigma_m^2)(h-\tau)} f(r_s, r_m) dr_m dr_s. \quad (13)$$

Integration yields (see Appendix B in *Husmann and Stephan* (2007), with minor conversions) the following expression for $E[\tilde{C}_t \cdot \tilde{R}_m | \mathcal{F}_0]$:

$$e^{(\mu_m + \frac{1}{2}\sigma_m^2)h + r(h-\tau)} \left(S_0 e^{(\mu_s + \frac{1}{2}\sigma_s^2 + \rho\sigma_s\sigma_m)\tau} N(d_3) - K N(d_4) \right), \quad (14)$$

where $d_3 = \frac{\ln((S_0 e^{(\mu_s + \frac{1}{2}\sigma_s^2 + \rho\sigma_s\sigma_m)\tau}) / (K e^{r\tau})) + (r + \frac{1}{2}\sigma_s^2)\tau}{\sigma_s \sqrt{\tau}},$

$d_4 = d_3 - \sigma_s \sqrt{\tau}.$

It follows that

$$E[\tilde{C}_t \cdot \tilde{R}_m | \mathcal{F}_0] = e^{(\mu_m + \frac{1}{2}\sigma_m^2)h + r(h-\tau)} C[\rho\sigma_m\sigma_s\tau, \tau], \quad (15)$$

and accordingly,

$$Cov[\tilde{C}_t, \tilde{R}_m | \mathcal{F}_0] = e^{(\mu_m + \frac{1}{2}\sigma_m^2)h + r(h-\tau)} (C[\rho\sigma_s\sigma_m\tau, \tau] - C[0, \tau]). \quad (16)$$

3.1.3. Valuation equation

With (11) and (16) employed in (6), the CAPM option price equation is

$$C_0 = \frac{C[0, \tau] - \lambda_h e^{(\mu_m + \frac{1}{2}\sigma_m^2)h} (C[\rho\sigma_m\sigma_s\tau, \tau] - C[0, \tau])}{e^{r\tau}}. \quad (17)$$

3.1.4. Special cases

For calls on the market portfolio, the following applies, $\rho = 1$, $\sigma_m = \sigma_s$ and $\mu_m = \mu_s$. The valuation equation shown in (17) can be easily transformed into

$$C_0 = \frac{C[0, \tau] - \lambda_h e^{(\mu_m + \frac{1}{2}\sigma_m^2)h} (C[\sigma_m^2\tau, \tau] - C[0, \tau])}{e^{r\tau}}. \quad (18)$$

After some conversions, we see that (18) and the valuation equation of *Jarrow and Madan* (1997)³ are identical. However, *Jarrow and Madan* (1997) define

³Equation (20), p. 21.

the parameter μ as the rate of return of the expected value, whereas we use it to identify the expected rate of return. To make our results comparable with theirs, the parameter μ must be replaced with $\mu + \sigma^2/2$ in *Jarrow and Madan* (1997).

In complete markets, a risk-neutral valuation always leads to the correct valuation results (see *Cox and Ross* 1976). As in risk-neutral settings $\mu_m + \frac{1}{2}\sigma_m^2 = \mu_s + \frac{1}{2}\sigma_s^2 = r$, one can use $\lambda_h = 0$ to obtain the *Black and Scholes* (1973) price of the option:

$$C_0 = \frac{C_0^{BS} \left(S_0 e^{(\mu_s + \frac{1}{2}\sigma_s^2)\tau}, K e^{r\tau}, \tau, r, \sigma_s \right)}{e^{r\tau}} = C_0^{BS} (S_0, K, \tau, r, \sigma_s). \quad (19)$$

3.2. Planning horizon equal to the option's time-to-maturity

When the option is held until maturity ($h = \tau$), the valuation equation (17) can be simplified to

$$C_0 = \frac{C[0, \tau] - \lambda_\tau e^{(\mu_m + \frac{1}{2}\sigma_m^2)\tau} (C[\rho \sigma_s \sigma_m \tau, \tau] - C[0, \tau])}{e^{r\tau}}, \quad (20)$$

$$\text{where } \lambda_\tau = \frac{e^{(\mu_m + \frac{1}{2}\sigma_m^2)\tau} - e^{r\tau}}{e^{(2\mu_m + \sigma_m^2)\tau} (e^{\sigma_m^2 \tau} - 1)}. \quad (21)$$

This pricing formula coincides with the one derived by *Husmann and Stephan* (2007), except that the latter is not presented in terms of *Black and Scholes* (1973) prices.

3.3. Planning horizon shorter than the option's time-to-maturity

When the planning horizon is shorter than the time-to-maturity, we use the results presented by *Rubinstein* (1984) who obtained the expected value of the call for $\rho = 0$ in terms of a *Black and Scholes* (1973) value of an European-style call as⁴

$$E[\tilde{C}_t | \mathcal{F}_0] = C_0^{BS} \left(S_0 e^{(\mu_s + \frac{1}{2}\sigma_s^2)h}, K e^{rh}, \tau, r, \sigma_s \right). \quad (22)$$

⁴The deviation is based on a minor extension of a known integral of the normal distribution, see *Owen* (1981), p. 403, integral 10,010.8. The proof is available from the authors on request.

3.3.1. Expected value

We can express the expected value of a call at the end of the holding period using the results obtained using the equation in (20). Our task is to integrate

$$\int_{-\infty}^{\infty} \frac{C_0^* - \lambda_{\tau-h} e^{(\mu_m + \frac{1}{2}\sigma_m^2)(\tau-h)} (C_0^{**} - C_0^*)}{e^{r(\tau-h)}} f(r_s) dr_s, \quad (23)$$

$$\text{where } C_0^* = C[\tilde{r}_s, \tau - h], \quad (24)$$

$$C_0^{**} = C[\tilde{r}_s + \rho \sigma_s \sigma_m (\tau - h), \tau - h], \quad (25)$$

$$\lambda_{\tau-h} = \frac{e^{(\mu_m + \frac{1}{2}\sigma_m^2)(\tau-h)} - e^{r(\tau-h)}}{e^{(2\mu_m + \sigma_m^2)(\tau-h)} (e^{\sigma_m^2(\tau-h)} - 1)}. \quad (26)$$

Splitting (26) in three integrals and utilizing the solution presented by *Rubinstein* (1984) gives us

$$E[\tilde{C}_t | \mathcal{F}_0] = \frac{C_0^1 - \lambda_{\tau-h} e^{(\mu_m + \frac{1}{2}\sigma_m^2)(\tau-h)} (C_0^2 - C_0^1)}{e^{r(\tau-h)}}, \quad (27)$$

$$\text{where } C_0^1 = C[0, \tau], \quad (28)$$

$$C_0^2 = C[\rho \sigma_s \sigma_m (\tau - h), \tau]. \quad (29)$$

3.3.2. Covariance

Again, we compute the covariance between the value of the call and the return on the market portfolio at the end of the holding period using the decomposition theorem. First, to compute $E[\tilde{C}_t \cdot \tilde{R}_m | \mathcal{F}_0]$, we must integrate

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{C_0^* - \lambda_{\tau-h} e^{(\mu_m + \frac{1}{2}\sigma_m^2)(\tau-h)} (C_0^{**} - C_0^*)}{e^{r(\tau-h)}} e^{r_m} f(r_s, r_m) dr_s dr_m. \quad (30)$$

Integration yields the following:⁵

$$E[\tilde{C}_t \cdot \tilde{R}_m | \mathcal{F}_0] = e^{(\mu_m + \frac{1}{2}\sigma_m^2)h} \frac{C_0^3 - \lambda_{\tau-h} e^{(\mu_m + \frac{1}{2}\sigma_m^2)(\tau-h)} (C_0^4 - C_0^3)}{e^{r(\tau-h)}}, \quad (31)$$

$$\text{where } C_0^3 = C[\rho \sigma_m \sigma_s h, \tau], \quad (32)$$

$$C_0^4 = C[\rho \sigma_m \sigma_s \tau, \tau]. \quad (33)$$

⁵The proof is available from the authors on request.

The covariance $Cov [\tilde{C}_t, \tilde{R}_m | \mathcal{F}_0]$ is thus given by

$$e^{(\mu_m + \frac{1}{2}\sigma_m^2)h} \cdot \frac{\left(C_0^3 - C_0^1 \lambda_{\tau-h} e^{(\mu_m + \frac{1}{2}\sigma_m^2)(\tau-h)} (C_0^4 - C_0^2 - C_0^3 + C_0^1)\right)}{e^{r(\tau-h)}}. \quad (34)$$

3.3.3. Valuation equation

Using (27) and (34) in

$$C_0 = e^{-rh} \left(E[\tilde{C}_t | \mathcal{F}_0] - \lambda_h Cov [\tilde{C}_t, \tilde{R}_m | \mathcal{F}_0] \right) \quad (35)$$

with

$$\lambda_h = \frac{e^{(\mu_m + \frac{1}{2}\sigma_m^2)h} - e^{rh}}{e^{(2\mu_m + \sigma_m^2)h} (e^{\sigma_m^2 h} - 1)} \quad (36)$$

results in

$$\begin{aligned} C_0 = & e^{-r\tau} \left(C_0^1 - \lambda_{\tau-h} e^{(\mu_m + \frac{1}{2}\sigma_m^2)(\tau-h)} (C_0^2 - C_0^1) \right. \\ & - \lambda_h e^{(\mu_m + \frac{1}{2}\sigma_m^2)h} (C_0^3 - C_0^1) \\ & \left. + \lambda_{\tau-h} \lambda_h e^{(\mu_m + \frac{1}{2}\sigma_m^2)\tau} (C_0^1 + C_0^4 - C_0^2 - C_0^3) \right), \end{aligned} \quad (37)$$

with C_0^1 - C_0^4 given in (28), (29), (32) and (33) and $\lambda_{\tau-h}$ in (26).

3.3.4. Special cases

When the holding period is equal to the time-to-maturity of the option ($h = \tau$), it follows that $C_0^1 = C_0^2$ and $C_0^3 = C_0^4$. Therefore, (37) becomes

$$C_0 = \frac{C_0^1 - \lambda_h e^{(\mu_m + \frac{1}{2}\sigma_m^2)\tau} (C_0^3 - C_0^1)}{e^{r\tau}}, \quad (38)$$

and is hence identical to (20).

In a risk-neutral world, because $\mu_m + \frac{1}{2}\sigma_m^2 = \mu_s + \frac{1}{2}\sigma_s^2 = r$, one can again obtain the *Black and Scholes* (1973) price using $\lambda_h = \lambda_{\tau-h} = 0$,

$$C_0 = \frac{C_0^{BS} \left(S_0 e^{(\mu_s + \frac{1}{2}\sigma_s^2)\tau}, K e^{r\tau}, \tau, r, \sigma_s \right)}{e^{r\tau}} = C_0^{BS} (S_0, K, \tau, r, \sigma_s). \quad (39)$$

4. Conclusion

This paper extends the valuation results presented by *Jarrow and Madan* (1997), considering the general case of an option on an arbitrary asset in an incomplete lognormal market. Additionally, we consider the case in which the planning horizon of the individual investor is shorter than the time-to-maturity of the option. The case in which the planning horizon equals the option's time-to-maturity is included in *Husmann and Stephan* (2007). The derived pricing equations depend explicitly on the planning horizon of the individual investor and are easy to use due to their simple presentation. In complete markets, the pricing equations for an European call result in the well-known *Black and Scholes* (1973) price.

Black, F., Scholes, M., 1973. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* 81, 637–654.

Cox, J., Ross, S., 1976. The Valuation of Options for Alternative Stochastic Processes. *Journal of Financial Economics* 3, 145–166.

Husmann, S., Stephan, A., 2007. On Estimating an Asset's Implicit Beta. *Journal of Futures Markets* 27, 961–979.

Jarrow, R. A., Madan, D. B., 1997. Is Mean-Variance Analysis Vacuous: Or was Beta Still Born? *European Finance Review* 1, 15–30.

Owen, D. B., 1981. A table of normal integrals. *Communications in Statistics – Simulation and Computation* 9, 389–419.

Rubinstein, M., 1984. A Simple Formula for the Expected Rate of Return of an Option over a Finite Holding Period. *Journal of Finance* 39, 1503–1509.

Vanden, J. M., 2004. Options Trading and the CAPM. *Review of Financial Studies* 17, 207–239.