A Parallel Interval Computation Model for Global Optimization with Automatic Load Balancing

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Abstract In this paper, we propose a decentralized parallel computation model for global optimization using interval analysis. The model is adaptive to any number of processors and the workload is automatically and evenly distributed among all processors by alternative message passing. The problems received by each processor are processed based on their local dominance properties, which avoids unnecessary interval evaluations. Further, the problem is treated as a whole at the beginning of computation so that no initial decomposition scheme is required. Numerical experiments indicate that the model works well and is stable with different number of parallel processors, distributes the load evenly among the processors, and provides an impressive speedup, especially when the problem is time-consuming to solve.

Keywords parallel processing, automatic load balancing, computation model, interval analysis, global optimization

1 Introduction

In this research, we consider the *global* optimization problem over the search domain $\mathcal{D} \subset \mathbb{R}^n$ which is defined by the lower and upper bounds of all variables:

$$\begin{aligned} \max \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, r \\ & h_j(\mathbf{x}) = 0, \quad j = r + 1, \dots, s \\ & \mathbf{x} \in \mathcal{D}. \end{aligned} \tag{1}$$

where $f : \mathcal{D} \rightarrow \mathbb{R}$ is the objective function, $g_i : \mathcal{D} \rightarrow \mathbb{R}$, $i = 1, \dots, r$ and $h_j : \mathcal{D} \rightarrow \mathbb{R}$, $j = r + 1, \dots, s$ are the inequality and equality constraints, respectively.

Due to the nonlinearity of the objective function and constraints expressed in (1), some effort must be spent if global optimal solutions are wanted.

A large class of global optimization algorithms have adopted divide and conquer (partitioning) approaches. Partitioning approaches divide given domains into smaller sub-spaces whose potential of holding the global optimum is either determined by deterministic approaches, such as [1]; or by stochastic approaches, such as [2].

Deterministic partitioning algorithms can be classified into two major categories. The first category includes algorithms based on interval methods, such as [3] and [4]. These algo-

rithms use interval arithmetics to calculate the range of the objective function and constraints for a specific partition and guarantee reliable box disposal. The second category includes algorithms based on certain *a priori* assumptions on functions, such as Lipschitz methods [1, 5], which generate a partition of the domain based on an assumed rate of change, i.e., the Lipschitz constant. The partition disposal can be reliable if the Lipschitz constant is known with accuracy.

Stochastic partitioning methods assess each partition with certain criteria and subdivide those partitions which are more “promising.” Sub-spaces can be selected based on possible largest expected increase in function values (e.g. [6]); or based on fuzzy assessment (e.g. [2]) where evidence within each partition is collected by using random search techniques.

Interval methods [3, 7], as a deterministic partitioning approach, have the advantage of providing explicit ranges over functions to guarantee assured interval disposal. An interval hypercube, which is commonly referred as ‘box’) can be discarded or removed from consideration when it is verified that there are no better solutions inside it or it is infeasible, or the objective function range over the hypercube is less than some small value, ε_f [7].

Branch-and-bound technique is usually employed for interval methods. It should be noted that in branch-and-bound the number of boxes to be managed usually increases exponentially during execution. This requires both efficient box disposal procedures and powerful box management schemes and computation models. A good candidate for speeding up the computation is parallel processing since each branch can be assessed separately. Some parallel models divide the root problem into a certain number of subproblems before execution to enable each processor has one problem to process. However, due to the fact that no knowledge of the execution behavior is known *a pri-*

ori, it is not suitable to statically balance the load at the very beginning. Furthermore, it is difficult to achieve an optimal initial decomposition [8] even when it is practical. Therefore, parallel computation models must address the load balancing problem.

In this study, we propose a decentralized parallel interval computation model (PICM) for global optimization. The model eliminates the requirement for an initial decomposition scheme to feed every processor at the beginning. It also automatically balances the workload among processors through alternative message passing. These two characteristics make the parallel model easy to implement.

The remainder of the paper is organized as follows: Section 2 briefly introduces interval analysis, constraint handling and some related parallelization approaches; Section 3 presents the PICM model and analyzes the model properties; Section 4 conducts numerical experiments to show the performance of PICM and Section 5 concludes this paper.

2 Interval Analysis and its Parallelization

Interval analysis provides function ranges which cover the objective function and constraints ranges over a hypercube (box) and can be easily embedded into a branch-and-bound computation model. The whole search area can be consecutively subdivided into smaller search areas—branching. Whereas the interval arithmetic provides the function ranges—bounding, which eliminates certain areas that surely do not hold the global optima. A prototype of branch-and-bound for global optimization using interval analysis is presented in Algorithm 1. The whole search space is first treated as a single interval (box), which is subsequently divided into several sub-intervals. The procedure is then recursively applied to those sub-intervals and results in a tree of search nodes

Algorithm 1: Branch-and-bound prototype for global optimization using interval analysis

Data: tree T for managing all the intervals

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1  $T = \emptyset$ ;
2 set the whole search area as an initial box and insert it into  $T$ ;
3 while  $T \neq \emptyset$  do
4   select a box  $B$  from  $T$  according to the selection criteria;
5   subdivide the box  $B$  into child boxes  $B_i$  from selected direction(s);
6   for each  $B_i$  do
7     evaluate objective and constraint functions over the box  $B_i$ ;
8     if better solution is found then
9       | update current best solution and  $T$ ;
10    if  $B_i$  cannot be discarded then
11      | insert  $B_i$  into  $T$ ;
12 report best solution

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(sub-intervals).

Many strategies have been investigated for Steps 4 and 5 in Algorithm 1, which correspond to the two problems that need to be considered when implementing the branch-and-bound prototype: the first is which branch should be selected for processing (branch selection criteria) and the second is how a branch should be partitioned (branching rules). There are some strategies proposed for branch selection: some have long been used in interval methods, such as the strategy to select the box with the best upper bound [9], select the box which is the oldest [10], and select the box which has maximum width [10] and so on. Once the box is selected, it will be partitioned along one or several of its edges [3], such as subdividing along the widest edge [11]. There are other rules that utilize the gradient, such as the rule of Hansen and Walster [3] and the rule of Ratz [12]. Some heuristic selection strategies have also been proposed in [13, 14, 15]. Those strategies have substantially improved the performance of interval analysis. Recent implementations of interval methods can be found in, for example, [16] and [17]. In the rest of this paper, we will not focus on the selection and subdivision strategies since

the main objective of this paper is the parallelization of interval analysis.

In the presence of constraints, the feasibility of a box is checked by examining the range of each constraint. It should be noted that when dealing with equality constraints, it is almost computationally impossible to satisfy them [7]. In practical computation, the equality constraints can be relaxed and replaced by:

$$-\varepsilon \leq h_i(x) \leq \varepsilon \quad i = r + 1, \dots, s,$$

for some positive small valued ε . Similarly, the inequality constraints can also be relaxed:

$$g_i(x) \leq \varepsilon \quad i = 1, \dots, r.$$

Thus, infeasible boxes can be immediately discarded, and feasible boxes can be kept for later processing if their upper bounds are superior to the known best solution. Since the constraints are also evaluated by interval arithmetic, there exist certain boxes which are neither totally feasible nor completely infeasible. Those boxes are called “*pending boxes*.”

Efforts for parallelization of interval computation include centralized and decentralized models. Master/slave architecture (centralized

model), such as the one in [18], has the advantage of efficiently utilizing available information. However, the master processor may easily become a bottleneck for the whole computation model due to data management and communication overheads, especially when the number of processors is large. Decentralized models, on the other hand, do not have such disadvantages. However, they need effective ways to exchange workload so that the total processor idle time can be minimized. Different load balancing mechanisms have been proposed. Such as, Eriksson and Lindstrom [19] investigated dynamic load balance schedulers. Benyoub and Daoudi [20] proposed an approach based on synchronization and uniform redistribution of boxes. Ibraev [21] presented a challenge leadership model where each processor has the chance to become the “leader” when it finds better solution. Gau and Stadtherr [22] provided a stage-wise asynchronous workload information exchange strategy for dynamic load balancing. Berner [23] presented a parallel method using a centralized mediator for the dynamic load balancing. A common observation from these approaches is that the load balancing makes the management of communication a complex task and subsequently complicates the parallel model implementation.

3 The Parallel Model

3.1 Model Topology

Suppose there are p processors in total and each processor is assigned a *rank* ranging from 0 to $p - 1$. The alternative message passing mechanism proposed in this paper sends messages to and receives messages from processors which are a hop away. The hop, h , changes in a periodic way according to a series of $2^0, 2^1, 2^2, \dots, 2^{C-1}$, where $C = \text{round}(\log_2 p)$. The round operation helps the cases where the number of processors does not equal to an integral power of 2. Thus, one particular processor

sends messages to and receives messages from a group of processors alternatively and therefore the mechanism is referred as alternative message passing.

The rank of the processor to which processor P_i should send message can be represented as $(i + h) \bmod p$, and the rank of the processor from which a message should be received can be denoted as $(i - h + p) \bmod p$. Taking a parallel environment with 8 processors as an example, the alternative message passing can be represented in Fig. 1. The operations with $h = 1, 2, 4$ constitute a complete cycle of alternative message passing since $C = 3$ when there are 8 processors.

Fig. 1. The alternative message passing.

3.2 Communications, Message Format and Processing

MPI [24] is adopted for message passing in this work. There are two methods for communicating data in MPI: blocking and non-blocking. In blocking communication, the sending process does not return until the send buffer is available for reuse. In non-blocking communication, the sending process returns immediately, and may only have started the message transfer operation, not necessarily completed it. The application may not safely reuse the message buffer after a non-blocking routine returns. However, the advantage of non-blocking routines is that they enable the communication overlap with the com-

putation [24]. In this research, both blocking and non-blocking message passing operations are employed.

Fig. 2. The message format used by PICM.

The content of the messages passed among processors is as shown in Fig. 2. Each message includes a box, the current best local solution, $\tilde{f}^*$ on the local processor, a status array which records all the processors' status, and a message tag for destination processor to choose corresponding message handling operations. The content of the 'box' section depends on the current status of the message sending processor. If the binary tree T , which stores all the boxes to be processed, is not empty, a box is selected according to the box selection rules; otherwise, the 'box' section will be marked as empty but the message will still be sent out. Sending a message with empty 'box' section keeps the message flow continuous and the order of alternative message passing consistent with other processors. Correspondingly, different message tags are used to instruct the receiving processor to choose the proper message handling operations. There are three tags used: the message tag BOX indicates a box has been sent/received; the message tag IDLE means the message contains no box; and the message tag QUIT instructs the receiving processor to quit the computation.

Sending the current best local solution $\tilde{f}^*$ and the processor status array with each message avoids the time-consuming broadcasting operations. The processor status array is used for termination detection since non-blocking message passing makes the termination detection a tricky task. The status array is set by each processor according to its current status. Algorithm 2 describes the operations for message sending. The best known solution on the

current processor is first packed into the message M_s , followed by the box information. If the binary tree T is not empty, the tag is set to BOX and a box B is retrieved from T for sending. At the same time, the status of the current processor and its destination processor are set to non-idle in the status array; otherwise, the send tag is set to IDLE and the status of current processor is set to idle in the status array.

Algorithm 2: SendMessageProcessing ()

Data: message to be sent M_s

- 1 pack the best known solution into M_s ;
- 2 **if** $T \neq \emptyset$ **then**
- 3 select one box B from the tree T ;
- 4 SENDTAG = BOX;
- 5 set current and destination processors as non-idle in the status array;
- 6 **else**
- 7 $B = \emptyset$;
- 8 SENDTAG = IDLE;
- 9 set current processor as idle in the status array;
- 10 pack B into M_s ;

Algorithm 3: ReceiveMessageProcessing ()

Data: received message M_r , sending message M_s

- 1 copy status array from M_r to M_s ;
- 2 **if** RECVTAG is QUIT and not P_0 **then**
- 3 SEND_MESSAGE ($\emptyset$, destination, QUIT);
- 4 quit;
- 5 **if** a better solution is received **then**
- 6 update best known lower bound and tree T ;
- 7 **if** RECVTAG is BOX **then**
- 8 insert the box into T ;

Algorithm 3 describes the operations for processing received messages. The status array is first copied from received message M_r to the status array of the next sending message M_s .

The messages are then handled by examining the message tag (RECVTAG). If the tag is QUIT, the processor sends a QUIT signal to its destination processor and exits. If a better solution is received, the best known solution on this processor and the binary tree T are updated. If the RECVTAG is BOX, the box is inserted into the tree T for processing.

3.3 The Parallel Algorithm

In PICM, each processor maintains a binary tree T to manage the boxes. Initially, PICM sets the whole search space as a single problem and stores it at processor P_0 , the tree T on other processors are empty at the beginning.

The the parallel model PICM is shown in Algorithm 4. We define *computation cycle* as the cycle from one operation to the next same operation for ease of description (Step 6 to 29). The message sending flag F_s and message receiving flag F_r are used to check the status of non-blocking message passing operations. The status can either be SENT and WAITING for F_s and RECEIVED and WAITING for F_r . Small capital letters are used for description of message passing operations in a simplified way, and “NB” indicates them are non-blocking message passing ones. The simplified operations need to specify content, origin/destination, message tag, and message status flag.

All processors start with a non-blocking message receiving operation except the processor P_0 , which first sends a message out, then starts the non-blocking message receiving operation. This initiates the message flow among all processors. In each computation cycle, first the message receiving flag F_r is checked. If a message has arrived, the message is processed according to Algorithm 3 and another non-blocking receiving operation is started subsequently. Otherwise, if the binary tree T is empty, the processor waits until a message is received and processes the message according to Algorithm 3 and starts another non-blocking

receiving operation. In the processing phase, if the binary tree T is not empty, a box is selected and partitioned; the child boxes are subsequently evaluated and either inserted back into T if they are pending or feasible, or discarded when they are infeasible or inferior to the best known solution. Of course, if a better solution is found during box splitting, the current best solution and binary tree T need to be updated. Subsequently, the processor prepares a message according to Algorithm 2 and sends the message out if a message has been received in this computation cycle. The message sending flag F_s is then checked, in case the binary tree T is empty and F_s is WAITING, the processor waits until this message sending operation finishes. By performing non-blocking sending and receiving, the communication time in each computation cycle can be shortened. When the non-blocking message passing operations in previous computation cycles are not finished, the current computation cycle will only consists of interval processing. Thus, each computation cycle may have message passing operations or not, depending on the status of previous message passing operations.

The termination detection is done by processor P_0 . It should be noted that the status change of a particular processor is not reflected among all processors at the same time. Instead, it is passed through message passing which in turn prevents wrong termination detection. When the status array of all processors in P_0 's sending status array are idle, P_0 sends a QUIT signal out. Consequently, the QUIT signal is passed across all processors to terminate them. The use of non-blocking message passing makes the termination detection complex since even when all processors are currently idle, it is not sufficient to terminate the computation due to the *messages in transit*. These messages in transition will reactivate idle processors.

Algorithm 4: Parallel interval algorithm for global optimization

Data: T, M_r, M_s, F_r, F_s

```

1 if  $P_0$  then
2   include the whole search domain as a box;
3   SendMessageProcessing ();
4   NB_SEND_MESSAGE ( $M_s$ , destination, BOX,  $F_s$ );
5 NB_RECV_MESSAGE ( $M_r$ , origin, RECVTAG,  $F_r$ );
6 repeat
7   check  $F_r$ ;
8   if  $F_r$  is RECEIVED then
9     ReceiveMessageProcessing ();
10    NB_RECV_MESSAGE ( $M_r$ , origin, RECVTAG,  $F_r$ );
11   if  $T = \emptyset$  and  $F_r$  is WAITING then
12     wait until  $F_r$  is RECEIVED;
13     ReceiveMessageProcessing ();
14     NB_RECV_MESSAGE ( $M_r$ , origin, RECVTAG,  $F_r$ );
15   if  $T \neq \emptyset$  then
16     retrieve one box  $B$  from  $T$ ;
17     subdivide  $B$  into subsets  $B_i$ ;
18     for each  $B_i$  do
19       evaluate  $B_i$ ;
20       if better solution is found then
21         update current best solution;
22       if  $B_i$  cannot be discarded then
23         insert  $B_i$  back into  $T$  according to the insertion rule;
24   if  $F_r$  is RECEIVED then
25     SendMessageProcessing ();
26     NB_SEND_MESSAGE ( $M_s$ , destination, SENDTAG,  $F_s$ );
27   check  $F_s$ ;
28   if  $T = \emptyset$  and  $F_s$  is WAITING then
29     wait until  $F_s$  is SENT;
30 until  $M_s$  status array on  $P_0$  are all idle;
31 if  $P_0$  then
32   SEND_MESSAGE ( $\emptyset$ , destination, QUIT);
33   RECV_MESSAGE ( $M_r$ , origin, RECVTAG);

```

3.4 Model Properties

In PICM, although the alternative message passing operations look simple, the mod-

el itself does exhibit some powerful properties. We define the concept of *dominance* for a box as follows:

Definition 1. A **local dominant box** is a box which should be selected according to the selection rules among all boxes on the processor it locates.

Definition 2. A **global dominant box** is a box which should be selected according to the selection rules among all boxes on all processors.

In serial computation, a local dominant box, which at the same time is also a global dominant one, is selected for next subdivision according to the given subproblem selection rules. In PICM, each processor first sends its local dominant box to another processor and processes the local dominant node after the message received has been handled. This makes the dissemination of the globally dominant boxes easily achievable and ensures high qualify (dominant) boxes are processed with priority.

The properties of PICM utilizing p processors can be summarized as follows:

Property 1: If a box is globally dominant and its child boxes are locally dominant on each corresponding processor, then child boxes split from this globally dominant box will dominate all processors after $\text{round}(\log_2 p)$ times of message passing provided no child box will be discarded during the computation.

Bear in mind that PICM sends a box out before processing intervals in a computation cycle, and the alternative message passing enables/forces a processor to send messages to different processors within any given $\text{round}(\log_2 p)$ times of message passing, Property 1 is actually equivalent to the calculation of the depth of a balanced binary tree at the level where at most p nodes can exist.

Without loss of generality, suppose currently the processor P_0 has such a globally dominant node, N_g , and the hop $h = 1$. Since N_g is globally dominant on P_0 , it is selected for subdivision and one child node of N_g is sent to P_1 since it is locally dominant. That is, after

1st message passing, processors P_0 and P_1 have nodes split from N_g . For the second message passing, the hop now becomes $h = 2$, P_2 and P_3 will get such nodes from P_0 and P_1 , respectively. It can be easily verified that the number of processors having such nodes will grow exponentially and without repetition. Eventually, after $\text{round}(\log_2 p)$ message passings, all processors will acquire a node split from N_g once the local dominant property of the child nodes holds and no child box will be discarded during the computation.

Fig. 3. The pervade of nodes among all processors

Property 1 shows that the children of a globally dominant node will be processed by all processors if they cannot be discarded and locally dominant in $\text{round}(\log_2 p)$ times of message passing. Fig. 3 illustrates such a case with 8 processors. It can be easily verified with the help of Fig. 1. This property ensures that more promising nodes are processed as early as possible among all processors. This power is especially important when the number of processors is large. For example, it only takes 10 message passings to disseminate the children of a node in Property 1 in a parallel system with 1024 processors.

Property 2: A processor can only be idle after at most $\text{round}(\log_2 p)$ times of message passing consecutively if there is at least one node left among all processors and at least one of the child nodes split from that node cannot be discarded.

Similar to the proof for Property 1, the proof for Property 2 is equivalent to the problem of calculating the maximum possible depth

for a number i (where $0 \leq i \leq p$ and represents the rank for processor P_i) first appearing in a balanced binary tree where each level is full of nodes and numbered consecutively.

Property 2 indicates that the maximum consecutive idle time a processor might have when there is a node that has Property 1 among all processors. Actually, when there is only one node left, that node is for sure globally dominant and the child nodes will pervade among all processors.

Property 3: If all processors are not idle, computation cycles on each processor will cost almost the same amount of time and the same data managing complexity.

In each computation cycle, the time spent on sending-receiving phase of each processor is the same due to the non-blocking sending and receiving operations. The computation phase costs almost the same amount of time since each processor manipulates a homogeneous problem due to the characteristics of branch-and-bound algorithms.

Property 3 indicates that when the workload is evenly distributed among all processors, the time needed by each processor will be almost identical. Thus, when total effort needed is not increased, the processing time can be reduced correspondingly since each processor will only need to process a portion of the total effort required by a serial implementation and achieve a good performance on parallel efficiency.

4 Numerical Experiments

4.1 Test problems

Sixteen constrained global optimization problems are selected from the literature to evaluate the performance of PICM. Table 1 summarizes the dimensions, type and number of constraints (where E means equality constraints and I means inequality constraints), and the references of the test problems. Detail

description of the test problems is available upon request.

As we mentioned earlier in the paper, the main purpose of this paper is to evaluate the performance of the parallel computation model PICM; therefore we apply rather simple interval strategies from the interval computational perspective. The box with best upper bound is used as the box selection criteria and bisection is applied for box subdivision. The subdivision direction is selected according to the infeasibility index presented in [25]. Some accelerating devices are introduced to speedup the convergence of interval methods, such as the midpoint test which evaluates the middle point of a box. The monotonicity test (see e.g. [3] or [26]) is not used since it is not applicable to the test problems.

Table 1. Summary of test problems.

<i>Prob.</i>	<i>Dim.</i>	<i>Cons.</i>	<i>Reference</i>
1	5	3E	Sect. 8.2.7 [27]
2	6	4E, 1I	Sect. 8.2.8 [27]
3	5	6I	Sect. 3.3 [27]
4	2	3I	Sect. 12.2.6 [27]
5	2	3I	Revised from Spike [28]
6	5	2E, 3I	Sect. 12.2.1 [27]
7	3	3I	Sect. 12.2.2 [27]
8	5	6I	Sect. 7.2.5 [27]
9	4	1E, 1I	Revised from Rastrigin [29]
10	2	3I	Based on Crescent [30]
11	8	6I	Sect. 5.4.2 [27]
12	6	4E, 1I	Sect. 7.2.2 [27]
13	6	6I	Sect. 3.4 [27]
14	8	1E, 1I	Revised from Rastrigin [29]
15	10	1E, 1I	Revised from Rastrigin [29]
16	20	14E	CUTEr [31]

The parallel computation model is tested in a cluster of up to 16 processors (Intel^R XeonTM 3.06 GHz) running on Red Hat Linux 9.0. The MPI version used in the numeri-

cal experiments is MPICH-G2 and the interval package used is PROFIL/BIAS [32].

4.2 Computation Results

Table 2 presents the numbers of objective and constraint function evaluations and the maximum binary tree sizes for each problem using 1, 2, 4, 8 and 16 processors, respectively. Eight problems are selected to further assess the performance of PICM when there are more processors presented (via simulating the runs with 32 and 64 processors by running 2 and 4 processes on a single processor, respectively). Further, we randomly choose two prime numbers 13 and 37, to evaluate PICM's performance with arbitrary number of processors when the number of processors is not integral power of 2. These results are italicized in Table 2.

It can be observed that in most cases PICM uses about the same amount of function and constraint evaluations (with slight fluctuations) for parallel computations compared with the serial one, sometimes even less. However, some problems utilize different numbers of objective and constraint evaluations, such as Problems 7, 11 and 16. The number of interval evaluations changes drastically in some occasions. The reason to this phenomenon might be that PICM processes the boxes in a different order. In serial implementation, only the box with global dominant property can be processed in each iteration. While in parallel implementation, the second-best box at a particular processor has the chance to be processed since the best box might have been packed to be sent out. Such effects are easier to be observed from higher dimensional problems. Overall, Table 2 indicates that PICM does not increase the effort needed to solve the selected test problems. From the results we can conclude that although the problem is treated as a single one at the beginning, the alternative message passing mechanism makes it possible to rapidly dis-

tribute subproblems and the exchange of local dominant nodes effectively controls the total effort required by PICM.

The maximum binary tree size, on the other hand, demonstrates the effect of automatic load balancing of PICM. As more and more processors participate the computation, it would be reasonable to assume that the maximum binary tree size will decrease. The data in Table 2 confirms this assumption. It can be observed that the maximum binary tree sizes change almost inversely proportional to the number of parallel processors, especially for those problems which need big binary trees. There is only one exception, Problem 11, where the binary tree size does not change very much but remain close to the serial case. Compare with the interval evaluations required for Problem 11, it can be concluded that the search space of Problem 11 is explored quite differently when different number of processors are used. It is worthwhile to point out that the maximum binary tree sizes for Problems 7, 14, 15 and 16 are more than halved when the number of processors are doubled in some occasions. Overall, the data indicates that PICM distributes the workload quite evenly among all processors.

The italicized data for selected problems for 13, 32, 37 and 64 processors present similar pattern as we observed above: the numbers of function and constraint evaluations remain stable for the newly tested number of 'processors'. The selection of prime numbers of processors does not exhibit an impact on the performance and the trend on the maximum binary tree size remains the same, i.e., as more processors are used, the tree size decreases.

Fig. 4 presents the CPU time (in seconds, shown on top of the three sub-figures) used for the serial implementation and the parallel efficiency (not speedup) for all test problems. In general, the test problems can be classified into three groups. The first group includes Problems 3, 4, 5, 6, 7 and 10, where sub-linear

Table 2. Objective function and constraint evaluations, maximum binary tree size

<i>Problem</i>	<i>1</i>	<i>2</i>	<i>4</i>	<i>8</i>	<i>13</i>	<i>16</i>	<i>32</i>	<i>37</i>	<i>64</i>
<i>Objective function evaluation</i>									
1	169554	169558	169179	169618	168815	169845	170015	168945	168469
2	181823	181886	177336	178013	181902	177318	173767	169488	182695
3	25293	25235	25101	25240	25244	25104	25569	25405	26110
4	146	154	177	155	—	155	—	—	—
5	263	273	273	273	—	267	—	—	—
6	16229	16224	16062	16207	16314	16279	15106	15008	16650
7	3120	4685	3874	3670	—	2520	—	—	—
8	74239	74152	74293	74512	74742	73833	73262	73977	75105
9	54138	54116	53636	51534	52343	52742	49496	48804	47192
10	9878	9877	9790	9742	9716	9733	9657	9758	9833
11	693668	497564	192572	130421	—	219501	—	—	—
12	365081	361237	322973	326116	—	357013	—	—	—
13	146151	146152	146753	149256	150168	148859	152315	151243	156491
14	309684	358220	335139	282347	—	286554	—	—	—
15	1866677	2144400	1798656	1766403	—	1610199	—	—	—
16	1181529	1850162	2120874	562662	—	879626	—	—	—
<i>Constraint evaluation</i>									
1	815690	815688	813957	815869	812212	816909	817718	812880	810821
2	1394046	1393715	1363809	1368333	1394413	1363054	1339103	1311473	1399550
3	88455	88257	88011	88320	88408	87965	89368	88810	90914
4	566	580	644	583	—	583	—	—	—
5	365	371	371	371	—	371	—	—	—
6	60503	60510	60098	60532	60682	60750	56901	56790	62308
7	51	369	648	408	—	366	—	—	—
8	274353	274121	274907	275214	276358	273372	271730	273712	277331
9	96483	96536	95632	91768	93445	94136	88792	87709	85385
10	216	222	221	233	237	233	233	237	237
11	31805	118169	138600	249322	—	304920	—	—	—
12	2765102	2738602	2484645	2504490	—	2720016	—	—	—
13	542725	542795	544773	552040	554779	550459	560510	554896	562788
14	507397	586906	551975	459382	—	467326	—	—	—
15	3044586	3500170	2977285	2904029	—	2649758	—	—	—
16	23662	22048079	18325953	4096324	—	733156	—	—	—
<i>Maximum binary tree size</i>									
1	13244	6631	3612	1858	1257	1068	516	442	287
2	11486	5934	3302	1689	1073	860	487	410	261
3	3489	1768	897	487	317	232	144	120	87
4	9	6	4	3	—	3	—	—	—
5	26	14	10	4	—	3	—	—	—
6	3964	2001	1005	526	320	273	136	126	81
7	850	344	225	80	—	39	—	—	—
8	10378	5656	2855	1405	978	735	391	359	220
9	7941	3997	2077	1060	657	581	294	276	158
10	505	268	155	92	63	50	27	28	17
11	2,191	1,942	2,131	2,878	—	2,033	—	—	—
12	16,146	9,363	4,125	3,912	—	3,816	—	—	—
13	30939	15530	8171	4301	2651	2233	1199	1074	660
14	87,519	43,380	20,095	4,783	—	3,499	—	—	—
15	544,816	273,087	91,584	41,134	—	13,588	—	—	—
16	20,771	15,482	12,485	7,531	—	802	—	—	—

Fig. 4. Parallel efficiency of PICM

speedup (parallel efficiency < 1.0) is observed, i.e., the speedup deteriorates as the number of processors increases. However, we should point out that for each problem in this group, the time required for the serial implementation is less than 1 second. Of course, in practical computation, there is no need (or not worth) to apply parallel computation to such problems. However, the results are presented here to demonstrate that the performance of PICM is not significantly affected considering the overhead of parallel initialization and message passing. The second group includes Problems 1, 2, 8, 9 and 13, where a linear speedup (efficiency = 1.0) can be observed. The last group includes Problems 11, 12, 14, 15 and 16, where super linear speedup (efficiency > 1.0) can be observed which indicates PICM takes advantage of the parallel implementation and significantly reduces the total effort to solve these problems.

4.3 Rough comparison with other parallel models

It is rather difficult to compare the performance of parallel methods due to the differences in software packages, interval techniques, computer architectures, test problems, programming languages and sometimes lack of data reported. However, the parallel efficien-

cy could be one possible way (since it measures the speedup and scalability) to conduct a *rough* performance comparison between our approach and other methods from the literature. The performance of these models are briefly listed below:

PICM: Tested with up to 16 processors, simulated with up to 64 processors. Maximum parallel efficiency with 16 processors is 53.1. For problems with serial running time over 1–10 seconds, the model scales up quite well.

Ibraev [21]: Tested with up to 16 processors, maximum efficiency with 16 processors reached 9.1. Scales up well when serial running time is over 50–100 seconds.

Eriksson and Lindstrom [19]: Tested with up to 64 processors. For the test results presented in the paper (only three), one exhibits super-linear speedup while the performance for the other two deteriorate quickly when more than 16 processors are employed.

Gau and Stadtherr [22]: Tested with up to 32 processors, maximum efficiency with 16 processors (reproduced from the paper) is about 3.1.

Berner [23]: Tested with up to 32 processors, linear or slightly super-linear speedup for most of the test problems achieved. For problems with serial running time of less than 100 seconds, it was difficult to observe super-linear

speedup.

The above rough comparison indicates that PICM has a high parallel efficiency and is able to provide decent computational performance. The simulated runs with up to 64 processors also illustrates the scalability of the model. When the number of processors gets larger, it might take some time for all processors in the model to be fully loaded. To get rid of this drawback, the processor P_0 can conduct a multi-section split (so more boxes are generated at the beginning) and then send out the boxes to some ‘far away’ processors to speed up the initial load balancing process.

4.4 Discussion

The computational performance of PICM might be contributed by several reasons. First, the workload in PICM could be well-distributed among all processors. The maximum tree sizes shown in Table 2 implicitly suggest this. Smaller tree size not only indicates that each processor handles less boxes, but also suggests less data management complexity since the operations of insertion and deletion of boxes will be quicker. This property is especially useful when the number of boxes to be managed is large. Moreover, the requirement for memory is also reduced, which is meaningful for processors with limited memory and extends the model’s applicability to more difficult problems. Second, each processor sends a local dominant box to its destination processor and handles its local dominant box after considering the received box from its origin processor. This mechanism helps the model process high-quality boxes among all processors. Further, sending local dominant boxes also helps update the best solution in each processor without the time-consuming broadcasting operation. Finally, as discussed in Section 4.2, PICM might be able to explore the search space in a different and more efficient way from that of serial implementation. This effect is more conspicuous

when the problem dimension is high since it normally takes longer (more message passing between processors) to split a high dimensional box into disposable small ones.

5 Conclusion

In this research, a decentralized parallel model for global optimization using interval analysis is presented. The model is adaptive to any number of processors and there is no need to consider initial decomposition of the problem. By performing the alternative message passing with different hop steps, the workload is rapidly and evenly distributed among all processors participating in the computation. Moreover, the communication operations are easy to handle which simplifies the implementation of the parallel model. Numerical experiments indicate that PICM not only speeds up the computation, but also explores the search space in an efficient way, the performance is especially noticeable when the problems are difficult and time-consuming to solve. The model proposed in this paper should have no difficulty to be applied to any other kind of branch-and-bound problems.

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