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6 CALCULATION OF POWER LINE FIELDS 7 BETWEEN CIRCULAR CONDUCTORS

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20 It is important to calculate the electric field at the surface of high voltage direct current
 21 power transmission lines, since it is this field which governs the onset of corona discharge
 22 and the power loss arising therefrom. A method is presented here to calculate the elec-
 23 tric field based on an implementation of the boundary element method for conductors
 24 of strictly circular cross section. Given the circular geometry it is possible to resolve all
 25 integrals involved analytically. A Galerkin approach is adopted, giving the solution in
 26 the spatial frequency domain. That allows a controlled truncation of the system matrix
 27 by choice of which frequency components to keep. It transpires that the low frequency
 28 components are the most important ones. Two test cases are used to quantify the accu-
 29 racy of the solution with respect to truncation and distance from the surface. It is found
 30 that the accuracy increases with distance from the surface, but for all distances can be
 31 controlled by choosing an appropriate level of truncation.

32 *Keywords:* Boundary element method; transmission lines; electric field; Laplace's
 33 equation.

34 1. Introduction

35 The calculation of fields around direct current (DC) power transmission lines arises
 36 in the context of minimizing losses due to corona discharge.¹⁻³ There are two cases
 37 of interest. First is the case before inception of corona, when there is no flow of ionic
 38 current through the region between the conductors. Second is the case after corona
 has been established, and the space between the conductors carries nett charge.

A. Seagar & S. Nilboworn

Both cases are of importance. In the first case the electric field at and nearby the surface of the conductors can be used as a criteria to design electrical components with higher corona inception voltage.^{1,4} In the second case it is of more interest to calculate the current (power) lost and the increase in electric field experienced beneath the power lines.⁵

Many papers are found in the literature describing the calculation of fields around power lines. In general, there are a variety of approaches which differ in five key areas: (1) the intended purpose of the calculation, (2) the geometry, (3) the conditions, (4) the method, and (5) the means of verification.

The intended purpose varies from calculating the field intensity,^{1,4-28} the voltage for onset of corona,^{1,6,7,11,16,17,19} the ionic current density,^{2,5,6,8,9,15,16,18,20,24,27-29} and the power loss.^{6,16,30} Often geometric approximations or simplifications are made. Typically the conductors are taken to occupy either an infinite full space,^{10,14} an infinite half space,^{2,5-9,11,12,16,17,19-21,24,28,30} or a finite circular space.^{2,23} The conductors themselves are assumed as either circular and solid^{2,5,6,20,24} or stranded,^{2,10,11,14,15,21,23,29} arranged either singly^{2,5,6,8-11,14,18,20,24} or in pairs^{12,17,26-29} or in bundles.^{6,12,13,15,16,18,26,27,29} Configurations involving single isolated conductors are more problematic because non-physical results due to a nett charge imbalance or nett current flow to infinity can occur if care is not taken.

The conditions under which calculations are performed are mostly high voltage direct current,^{2,5,6,8,9,12,16-18,20,22,24,26-30} although the corona around conductors carrying high voltage alternating current is also of interest.^{12,13,15,19,21,26,30} Additionally the surface condition of the conductors may be taken as clean, or coated either intentionally⁷ or due to contamination.^{25,31} Environmental processes such as wind^{15,18-20,28} and ionic diffusion²⁰ may or may not be taken into account.

Methods of calculation vary widely, with almost all possible having been tried at one time or another. These include methods of charge simulation,^{4-7,11,13,16,23,28} finite element,^{1,4,5,8,9,18,20,27,28} integral equation,^{25,28} neural networks,³¹ iteration,^{2,5,8,28} flux tracing,^{2,16,17,24} analytic expansions,^{10,23} boundary elements,²² and finally direct measurement.^{15,19,21,29,31} In contrast, verification of calculations involves a more limited repertoire of methods; namely comparison to measurements,^{1,2,6-9,11,17,21,24,27-29,31} other published results,^{5,14,16,20,30} theoretical results,^{4,22} and some other technique.^{2,23} Comparison to measurements is valuable, but the accuracy of the measurements are usually quite limited and do not serve as a very strong means of validating any particular technique. Comparison of one technique to another is fraught with even more problems. If two techniques disagree then that does not indicate which is in the most error, and if they agree then there is no guarantee for either being correct.

Of the techniques employed, the finite element method²⁷ (FEM) and charge simulation method^{23,32} (CSM) are perhaps the most general and well established. They accommodate any number of conductors of effectively any shape.

Although the FEM is widespread and well understood, by nature it requires a discrete approximation to the medium between the conductors. For overhead

Calculation of Power Line Fields between Circular Conductors

conductors the medium, and in principle the approximation, extends to infinity. Artificial boundary conditions can be imposed some finite distance from the conductors in order to overcome that problem. Even when this is done, the solution is essentially a volumetric one. Assuming that axial symmetry in the problem has been exploited to reduce it to two dimensions, halving the size of the finite elements increases the number of unknowns by the square, and the total number of matrix elements in the linear system employed for the solution increases by the fourth power. That can lead to a high computational burden to achieve good accuracy.

The CSM avoids the problems of the FEM. First the field as formulated by the solution tends towards zero at infinity as long as the charges are correctly balanced, so there is no need for any artificial boundary condition away from the conductors. Second, the number of charges typically employed is significantly fewer than the corresponding number of finite elements. Doubling the number of charges increases the number of unknowns in direct proportion, and increases the total number of matrix elements by the square. The computational burden does not increase as fast as for the FEM. However, except in the simplest of situations, the CSM is by construction only approximate. It has been shown by comparison to analytical results to be in error by around 4%.²³

Typically, the geometry of conductors is taken to be either circular or, in the case of cables, a collection of circular strands^{3,11,33} in contact with one another. When the problem is restricted to circular sections it is possible also to entertain analytical or semi-analytical solutions. Parekh *et al.*²³ offer a solution based on an analytical expansion for the case of a multi-stranded cable set at the center of an annular conductor. The advantage of the technique is the speed and accuracy with which it can be calculated. However, the geometry of the method does not apply to all cases. Equally or more common is the case where the conductors are strung above the surface of the earth, which is a flat surface rather than an encircling one.

Since the medium between the conductors is uniform, the problem also admits a solution by the boundary element method²² (BEM). This method has a solid rationale, being directly based on the equation describing volumetric current flow (i.e., Laplace's equation). When implemented much as the FEM in terms of a general algorithm and an effectively arbitrary discrete finite boundary approximation, the BEM offers advantages in speed over the FEM but not the CSM. However, there is no need to take such a general approach, especially for this application where circular conductors are required. Rather than approximating circles by many small straight or curved elements it makes sense to demand the circular geometry *per se* from the outset.

This method has been followed by Nilboworn³⁴ using an analytic integration of the boundary integral equations and then a numerical matching to the known voltages on the surfaces of conductors to construct the matrix elements used in the solution. The advantage of the technique is that it has the same computational

A. Seagar & S. Nilboworn

burden as the CSM but a stronger rationale and a higher accuracy. The disadvantage (in the particular implementation reported) is the use of boundary point matching (collocation). It produces a matrix which is not as well conditioned as often as is desirable.

Here the method of Nilboworn³⁴ has been extended by adopting a Galerkin style approach,³⁵ integrating the equations analytically a second time. This modification delivers a system matrix formulated in the (spatial) frequency domain rather than in the spatial domain itself. The primary advantage is that the matrix can be truncated by choice of frequency limit with the understanding (and experimental evidence) that the higher frequencies do not play as important a role as do the lower frequencies.

The BEM as formulated here is in some ways equivalent to the CSM. The CSM could be called a *multiple-monopole* method, i.e., each conductor is replaced by a collection of distinct monopoles, somewhere beneath the surface of the conductor. The method here is equivalent to a *single-multipole* method, i.e., each conductor is replaced by a collection of multipoles of increasing order (pole, dipole, quadrupole, etc.) all located at the center of the conductor.

This article presents in Sec. 2 and 3 the details of the Galerkin (multipole analytic) BEM for collections of circular conductors. The relationship between this technique and the CSM is explained from a theoretical viewpoint in Sec. 4. Equations developed there which allow the computational time to be calculated are demonstrated later in the discussion. Tests described in Sec. 5 are used to examine the numerical behavior of the technique by comparison in Sec. 6 to theoretical results in terms of the number of spatial frequency components employed (or equivalently the number of multipoles) and in terms of the distance from the surface of the conductors. Particular attention is paid to the accuracy of solution at the surface of the conductors, since it is the field there which governs corona losses.

2. The Model

All highly conducting parts of the model are represented as a collection of individual components each of circular cross section. Cables are modeled as a collection of circular strands in contact with one another, bundles are modeled as a collection of either cables or simply circular conductors which are separated from one another by small distances, and transmission lines are modeled as a collection of bundles separated by larger distances. The region between conductors is taken to be poorly conducting, supporting at least displacement current in unionized air and allowing for conduction current in other situations. The earth is represented as either conducting or nonconducting by the use of image conductors with either opposite or equal polarity.

For a collection of N conductors (including any image conductors) the electric potential $\phi_i(\mathbf{y})$ at point $\mathbf{y}$ on the surface Σ_i of conductor Ω_i and the component of its gradient $\nabla\phi_i(\mathbf{y})$ in the direction of the inward normal $\mathbf{n}_i(\mathbf{y})$ are represented by

Calculation of Power Line Fields between Circular Conductors

Fig. 1. Point $\mathbf{y}$ on surface Σ_i of conductor Ω_i and point $\mathbf{x}$ in region Ω between conductors.

finite Fourier series

$$\phi_i(\mathbf{y}) = \sum_{k=0}^K A_{i,k} \cos k\theta_i + B_{i,k} \sin k\theta_i, \quad (1)$$

$$\mathbf{n}_i(\mathbf{y}) \cdot \nabla \phi_i(\mathbf{y}) = \sum_{k=0}^K I_{i,k} \cos k\theta_i + J_{i,k} \sin k\theta_i \quad (2)$$

1 with geometry as shown in Fig. 1.

The potential over the entirety of each conductor Ω_i for $1 \leq i \leq N$ is a prescribed voltage

$$V_i = \phi_i(\mathbf{y}) = A_{i,0} \quad (3)$$

2 with all other coefficients in the Fourier series in Eq. (1) zero. Over the entirety of
 3 the surface (circle) Σ_0 at infinity the potential $\phi_0(\mathbf{y})$ has unknown value ϕ_0 . It is
 4 important not to presuppose any particular value for ϕ_0 , but rather let it be found
 5 as part of the solution. For an arbitrary choice of V_i , for $1 \leq i \leq N$, the system
 6 may otherwise be inconsistent, since potential can only be defined up to an additive
 7 constant.

Once all of the unknowns ϕ_0 , $I_{i,k}$ and $J_{i,k}$ are determined by solving the equations in Sec. 3, the potential at any point $\mathbf{x}$ in the region Ω away from or approaching the surface of the conductors, as in Fig. 1, is calculated using

$$\begin{aligned} \phi(\mathbf{x}) - \phi_0 = & - \sum_{i=1}^N (\ln r_{\mathbf{x}}) (I_{i,0} r_i) + \sum_{k=1}^K \frac{1}{2k} \sum_{i=1}^N \left(\frac{r_i}{r_{\mathbf{x}}} \right)^k \\ & \times \{ (I_{i,k} r_i) \cos k\theta_{\mathbf{x}} + (J_{i,k} r_i) \sin k\theta_{\mathbf{x}} \} \end{aligned} \quad (4)$$

and the electric field $\mathbf{E}$ is calculated using

$$\begin{aligned} \mathbf{E}(\mathbf{x}) = & \sum_{i=1}^N \frac{1}{r_{\mathbf{x}}} (I_{i,0} r_i) \begin{pmatrix} \cos \theta_{\mathbf{x}} \\ \sin \theta_{\mathbf{x}} \end{pmatrix} + \sum_{k=1}^K \frac{1}{2} \sum_{i=1}^N \frac{1}{r_{\mathbf{x}}} \\ & \times \left(\frac{r_i}{r_{\mathbf{x}}} \right)^k \left\{ (I_{i,k} r_i) \begin{pmatrix} \cos(k+1)\theta_{\mathbf{x}} \\ \sin(k+1)\theta_{\mathbf{x}} \end{pmatrix} + (J_{i,k} r_i) \begin{pmatrix} \sin(k+1)\theta_{\mathbf{x}} \\ -\cos(k+1)\theta_{\mathbf{x}} \end{pmatrix} \right\}. \end{aligned} \quad (5)$$

8 Values of current density ($\mathbf{n} \cdot \mathbf{J} = \mathbf{n} \cdot \sigma \mathbf{E} = -\mathbf{n} \cdot \sigma \nabla \phi$) or voltage gradient ($\mathbf{n} \cdot \nabla \phi =$
 9 $-\mathbf{n} \cdot \mathbf{E}$) on the surface of conductors are calculated more easily from Eq. (2) than from Eq. (5).

A. Seagar & S. Nilboworn

Equations (4) and (5) are in effect multipole expansions for the potential and the field, expanded about the centers of all of the conductors. The coefficients I, J in the expansions are chosen so as to match the potential at the surface of the conductors to known values, simultaneously.

3. Method of Solution

Applying Green's Theorem to Laplace's equation gives the potential $\phi(\mathbf{x})$ on the surface $\Sigma = \bigcup_i \Sigma_i$ of the region Ω between the conductors Ω_i as

$$L_1 = \frac{1}{2}\phi(\mathbf{x}) - \frac{1}{2\pi} \int_{\Sigma} \mathbf{n}(\mathbf{y}) \cdot \phi(\mathbf{y}) \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^2} d\sigma(\mathbf{y}) - \phi_0 \quad (6)$$

$$= L_2 = -\frac{1}{2\pi} \int_{\Sigma} \mathbf{n}(\mathbf{y}) \cdot (\ln |\mathbf{y} - \mathbf{x}|) \nabla \phi(\mathbf{y}) d\sigma(\mathbf{y}), \quad (7)$$

where $d\sigma(\mathbf{y})$ is the infinitesimal measure of surface at point $\mathbf{y}$.

For any point $\mathbf{x}$ on the surface Σ_j of conductor Ω_j as shown in Fig. 2, the integral over the total surface Σ is expressed as a sum of integrals over the surface Σ_i of every conductor Ω_i (including Ω_j). The integral excludes the circle at infinity. That is accommodated in L_1 in terms of potential by ϕ_0 , and is accommodated in L_2 in terms of current by substituting a value of zero for the total current crossing the boundary at infinity. In that case the general solution in Eqs. (6) and (7) reduces to

$$L_1 = \frac{1}{2}V_j - \frac{1}{2\pi} \sum_{i=1}^N V_i \int_{\theta_i=0}^{2\pi} \mathbf{n}_i(\mathbf{y}) \cdot \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^2} (r_i d\theta_i) - \phi_0, \quad (8)$$

$$L_2 = -\frac{1}{2\pi} \sum_{i=1}^N \int_{\theta_i=0}^{2\pi} (\ln |\mathbf{y} - \mathbf{x}|) \sum_{k=0}^K (I_{i,k} \cos k\theta_i + J_{i,k} \sin k\theta_i) (r_i d\theta_i). \quad (9)$$

Fig. 2. Point $\mathbf{x}$ on surface Σ_j of conductor Ω_j expressed using coordinates from center O' of conductor Ω_i .

All of the integrals carry analytically, giving

$$L_1 = V_j - \phi_0, \quad (10)$$

$$\begin{aligned} L_2 = & -(\ln r_j)(I_{j,0}r_j) - \sum_{i=1, i \neq j}^N (\ln r_{\mathbf{x}})(I_{i,0}r_i) \\ & + \sum_{k=1}^K \frac{1}{2k} \{ (I_{j,k}r_j) \cos k\theta_j + (J_{j,k}r_j) \sin k\theta_j \} \\ & + \sum_{k=1}^K \frac{1}{2k} \sum_{i=1, i \neq j}^N \left(\frac{r_i}{r_{\mathbf{x}}} \right)^k \{ (I_{i,k}r_i) \cos k\theta_{\mathbf{x}} + (J_{i,k}r_i) \sin k\theta_{\mathbf{x}} \}. \end{aligned} \quad (11)$$

Kirchhoff's current law dictates that the nett current emerging from any conductor must cancel the sum of all others. This leads to

$$\sum_{i=1}^N (I_{i,0}r_i) = 0. \quad (12)$$

The circle at infinity Σ_0 is excluded from the summation because the current crossing that boundary is identically zero.

Together Eqs. (10)–(12) provide a system of equations in $(2K + 1)N + 1$ unknowns. It is sufficient to apply Eq. (11) at $2K + 1$ equally spaced points on every conductor to balance the number of equations with the number of unknowns and solve for ϕ_0 , $I_{i,k}$ and $J_{i,k}$. Although that works in principle if $K \rightarrow \infty$, it introduces truncation error when K is finite. The error can be reduced by adopting a Galerkin style of solution instead of the pointwise approach. Doing that involves casting Eqs. (10) and (11) in terms of the unknowns' (voltage gradients') basis functions (cosine and sine) rather than in terms of the knowns (voltages) at individual points $\mathbf{x}$.

For any function $G(\theta_j)$ the cosine and sine transforms produce Fourier coefficients (harmonics) H

$$H_{c,0} = \frac{1}{2\pi} \int_{\theta_j=0}^{2\pi} G(\theta_j) d\theta_j, \quad (13)$$

$$H_{c,\ell} = \frac{1}{2\pi} \int_{\theta_j=0}^{2\pi} G(\theta_j) \cos \ell\theta_j d\theta_j, \quad (14)$$

$$H_{s,\ell} = \frac{1}{2\pi} \int_{\theta_j=0}^{2\pi} G(\theta_j) \sin \ell\theta_j d\theta_j. \quad (15)$$

Applying to L_1 in Eq. (10) gives

$$H_{c,0} = V_j - \phi_0, \quad (16)$$

$$H_{c,\ell} = 0, \quad (17)$$

$$H_{s,\ell} = 0, \quad (18)$$

A. Seagar & S. Nilboworn

and applying to L_2 in Eq. (11) gives

$$H_{c,0} = -(\ln r_j)(I_{j,0}r_j) - \sum_{i=1, i \neq j}^N (\ln R_{ij})(I_{i,0}r_i) + \sum_{k=1}^K \frac{1}{2k} (-1)^{-k} \sum_{i=1, i \neq j}^N \left(\frac{r_i}{R_{ij}} \right)^k \times \{ (I_{i,k}r_i) \cos k\psi_{ij} + (J_{i,k}r_i) \sin k\psi_{ij} \}, \quad (19)$$

$$H_{c,\ell} = \frac{1}{2\ell} \sum_{i=1, i \neq j}^N (I_{i,0}r_i) \left(\frac{r_j}{R_{ij}} \right)^\ell \cos \ell\psi_{ij} + \frac{1}{4\ell} (I_{j,\ell}r_j) + \sum_{k=1}^K \frac{1}{4k} \binom{-k}{\ell} (-1)^{\ell-k} \sum_{i=1, i \neq j}^N \left(\frac{r_i}{R_{ij}} \right)^k \left(\frac{r_j}{R_{ij}} \right)^\ell \times \{ (I_{i,k}r_i) \cos[(k+\ell)\psi_{ij}] + (J_{i,k}r_i) \sin[(k+\ell)\psi_{ij}] \}, \quad (20)$$

$$H_{s,\ell} = \frac{1}{2\ell} \sum_{i=1, i \neq j}^N (I_{i,0}r_i) \left(\frac{r_j}{R_{ij}} \right)^\ell \sin \ell\psi_{ij} + \frac{1}{4\ell} (J_{j,\ell}r_j) + \sum_{k=1}^K \frac{1}{4k} \binom{-k}{\ell} (-1)^{\ell-k} \sum_{i=1, i \neq j}^N \left(\frac{r_i}{R_{ij}} \right)^k \left(\frac{r_j}{R_{ij}} \right)^\ell \times \{ (I_{i,k}r_i) \sin[(k+\ell)\psi_{ij}] - (J_{i,k}r_i) \cos[(k+\ell)\psi_{ij}] \}. \quad (21)$$

There is no ambiguity in using the same symbol H for the harmonics of Eqs. (16) to (18) and (19) to (21) because $L_1 = L_2$.

Although the expressions under the summations after cosine and sine transformation are more complicated than before, the advantage is that the truncated linear system as represented by them, for finite K , tends more towards diagonal dominance. Errors introduced by truncation are limited to the high harmonics, near and beyond truncation. The higher harmonics play a limited role in the solution especially when the conductors are far apart in comparison to their size. That makes it possible to adjust the truncation to control the errors in a well-behaved fashion.

Together Eqs. (12) and (16) to (21) are used here to solve ϕ_0 , $I_{i,k}$ and $J_{i,k}$ from the known voltages V_j on all of the conductors.

4. Alternative Sources

The source here of the potential and field is a one-dimensional (curved line) current source. Alternatives include charge sources and sources which are zero-dimensional (discrete points) or two-dimensional (distributed continuously throughout the section of the conductor). Most closely related to the one-dimensional current source as used here is the CSM,^{23,32} which offers a choice of zero, one or two-dimensional sources.

Calculation of Power Line Fields between Circular Conductors

Both current and charge source methods rely on quasi-static limits of Maxwell's equations^{36–40} in two-dimensional space (or a cross section of three-dimensional space with axial symmetry perpendicular to that cross section) written for a point $\mathbf{x}$ on the boundary Σ of a region Ω in terms of the electric potential $\phi(\mathbf{x})$

$$\begin{aligned} \frac{1}{2}\phi(\mathbf{x}) = & -\frac{1}{4\pi} \int_{\Omega \cup \Omega_i} \frac{\rho(\mathbf{y})}{\varepsilon} \ln |\mathbf{y} - \mathbf{x}| d\mathbf{y} \\ & + \frac{1}{2\pi} \int_{\Sigma} \mathbf{n}(\mathbf{y}) \cdot \left(\phi(\mathbf{y}) \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^2} - (\ln |\mathbf{y} - \mathbf{x}|) \nabla \phi(\mathbf{y}) \right) d\sigma(\mathbf{y}). \end{aligned} \quad (22)$$

The sources are the charge density ρ within the region Ω between conductors and also the region Ω_i inside all conductors, and the normal current density $\mathbf{n} \cdot \mathbf{J} = -\mathbf{n} \cdot \sigma \nabla \phi$ crossing the boundary Σ , with $d\mathbf{y}$ and $d\sigma(\mathbf{y})$ respectively as the infinitesimal measures at a point $\mathbf{y}$ of the area within and of the length along the curved surface of the region Ω . The surface function $\sigma(\mathbf{y})$ accommodates whatever parameterization is used to describe the way points $\mathbf{y}$ lie on the boundary. Here the boundary at infinity remains contained as part of Σ , so there is no explicit mention of the term ϕ_0 at infinity, as in Eq. (6).

Setting the charge density to $\rho = 0$ everywhere leaves the equation used here, as in Eqs. (6) and (7). In this case the potential ϕ satisfies Laplace's equation, and $\mathbf{n} \cdot \mathbf{J}$ represents the (largely) displacement current flowing from conductor to conductor through the air-space between them. The current at least in principle can be steady state (DC) or quasi-static (low frequency AC). With this approach the mathematical model is in direct correspondence to the power line application.

Alternatively, setting the current density to $\mathbf{n} \cdot \mathbf{J} = 0$ and $\rho = 0$ only within Ω leaves the equation used in the CSM. In this case the potential ϕ again satisfies Laplace's equation, however there is absolutely no flow of current through the intervening space between conductors. Each conductor is charged to a particular level of completely static voltage, and stays that way without leakage. The model does not correspond well to the real world power line application, but can be used because it does produce a solution which satisfies the same mathematical (Laplace's) equation.

Proper usage for the first term on the right-hand side of Eq. (22) is in the case where corona has already been established and current flows in ionic form through the region Ω between conductors. In that case the charge density is set to $\rho = 0$ only inside the conductors Ω_i and both first and second terms on the right-hand side of Eq. (22) are used in the solution. In this case the potential in the region Ω between conductors no longer satisfies Laplace's equation. We leave the use of this approach on the problem involving ionic current flow to a later paper.

Returning to the case without ionic current flow, there still remains the issue of the nature of the charge and current sources which drive the two alternative models.

The current sources lie on the infinitesimally thin boundary layer Σ which makes the surface of the region Ω . For a given choice of current (supposing we know what it should be) there is only one solution for potential and, most importantly, *vice versa*.

A. Seagar & S. Nilboworn

However, for the CSM the charge sources lie in general as a distribution throughout the entire region Ω_i . Again, for any chosen distribution of charge density there is only one solution for potential, but the converse is no longer true. That is, there are an infinite number of charge distributions which lead to the same potentials in the spaces between the conductors, albeit different potentials within the conductors.

For consistency with the current source approach and uniqueness in terms of charge distribution the charge source ρ can be limited to the infinitesimally thin boundary layer at the surface of the conductors. In that case^{4,25} both methods should in principle be able to yield equivalent results. However, as applied in practice it is usual for the CSM to adopt for the charge source ρ a set of delta functions. These cannot be placed on the surface, because that would lead to infinite potentials, so must be placed away from the surface somewhere in the region Ω_i .

As soon as this is done the two methods in principle and practice are no longer equivalent and cannot yield the same results. Both do still yield solutions which adhere to Laplace's equation (which is indeed necessary). However, for accurate and indeed correct results, particularly at the surface of the conductors as needed here, that is by no means sufficient. Only the current source model or a boundary layer charge source model can do that.

A further complication of the two-dimensional problem for both current and charge sources, is that the fundamental solution to Laplace's equation is the logarithm function, rather than in three dimensions where it is the inverse distance function. In three dimensions the potential naturally decays to zero at infinity, whereas in two dimensions it grows without limit. Extra care must be taken in two dimensions (as here with Eq. (12)) to ensure that no current crosses the boundary at infinity, otherwise the potential at infinity is not well defined and may be nonzero. Indeed, in one particular numerical example when Eq. (12) was not enforced, the potential at infinity as determined numerically turned out to be 19.6 kV; clearly absurd. It appears that others¹⁰ have also fallen into the same trap, later corrected by Parekh *et al.*²³ by imposing an outer boundary at finite distance.

The computational procedure for both the current and charge source methods is effectively the same: (1) set up a linear system of equations, (2) solve the linear system, and (3) calculate the potential at individual points. The computational effort is usually dominated by the numerical solution of the linear system, which is dense rather than sparse, although can be arranged *a priori* to be diagonally dominant if the problem is set up carefully.

For the current source approach the number of coefficients in the linear system is $M \times M$ where $M = (2K + 1)N + 1$. Here K is the highest Fourier component supported on each conductor, meaning physically that there can be no more than $2K$ oscillations (ripples) in current around the surface. N is the number of conductors, which must include both real conductors and image conductors. The value one inside the parenthesis sets the nett current flowing through the surface of each conductor, and the value one outside the parentheses brings the nett current flowing through the surface at infinity to zero. The computational effort to solve the

linear system is of order M^3 . Once the solution has been achieved, the potential at any chosen point can be calculated in the order of M operations. The effort for the latter part of the calculation increases linearly with the number of points chosen.

In comparison, for the charge source method it would be necessary to place $2K$ monopoles (or K dipoles) close to the surface of the N conductors in order to emulate $2K$ oscillations in electric flux density (the equivalent of current density in the purely static case). Another equation would be required to ensure that the nett flux density has the correct value. That could be accommodated by putting one extra monopole in the center of each conductor. Finally to set the potential at infinity to zero, another equation is required. That could be written in terms of a relationship between the monopoles at the center of the conductors. Ultimately there are exactly the same number of equations and same computational effort as for the current source method.

For current sources the computational effort can be quantified by first choosing the error which can be tolerated in the solution and then using the figures in Sec. 6 to deduce the required number of harmonics K . For any particular problem with N conductors it is then possible to calculate the size M of the linear system. Once the speed of the computer is known, that leads directly to the time taken for the calculation.

The accuracy at the surface of the conductors for the same computational effort (time) using discrete charge sources is known to be lower both in principle and in practice.²³ For the application here that accuracy is of crucial importance. It should be possible to use additional monopoles, taking longer than with current sources to obtain the same accuracy. However there has not yet been any study to determine either how many discrete charge sources would be required or where they should be placed.

5. Tests

Tests are based on the conductors shown in Fig. 3. The conductor in Fig. 3(a) is of radius 1 m located 2 m above a conducting earth. The voltages on the conductor and the earth are 1 V and 0 V, respectively. In Fig. 3(b) the same conductor has been

Fig. 3. Test cases (a) solid conductor and (b) stranded conductor occupying same space as solid conductor.

A. Seagar & S. Nilboworn

supplanted by a cable of six strands, together with a maximum radius outer of 1 m. In both cases the conductors are much larger than in practical transmission lines and much closer to the earth in proportion to their size. This is done deliberately to ensure that the fields are quite distorted so that the solution is nontrivially different from that of a single isolated conductor.

In case (a) an analytical solution is known³⁸ in the region outside the conductor. In case (b) there is no known analytic solution for the region outside the conductors, however there is a known solution for the region between conductors. If all of the conductors are at the same potential and touching one another then the region between them should match the potential on them.

Although a uniform potential between conductors seems in one sense like a trivial solution, it is not trivial for the method to produce it since there is no prior knowledge of uniformity built into the method. Indeed, case (b) is more challenging than case (a). The (displacement) current leaving the surface of each conductor in the stranded cable is in principle less uniform than for case (a). That should require higher harmonics to achieve the same accuracy.

6. Results

Table 1 shows the magnitude of values of the harmonics for the gradient of the potential at the surface of the solid conductor; $C_k = (I_{1,k}^2 + J_{1,k}^2)^{1/2}$. The relative size of the individual harmonics decreases monotonically as the frequency k increases. Each harmonic is about 26.5% in size of the adjacent one with lower frequency. If a limited number of harmonics are used to form the solution the ones with the highest values should be chosen to minimize the error. That can be achieved by selecting the ones with the lowest frequencies.

Figure 4 shows the maximum error as the number of harmonics is increased. Figure 4(a) shows the error outside the single solid conductor and Fig. 4(b) shows the error between the strands of the other conductor. For the solid conductor the error reduces at a rate of 12 dB per additional harmonic. A minimum accuracy of

Table 1. Size of individual harmonics $C_k = (I_{1,k}^2 + J_{1,k}^2)^{1/2}$.

k	C_k	k	C_k	k	C_k
0	7.5932572e-01	10	2.8972277e-06	20	5.5272251e-12
1	4.0692143e-01	11	7.7630982e-07	21	1.4810155e-12
2	1.0903427e-01	12	2.0801159e-07	22	3.9683691e-13
3	2.9215644e-02	13	5.5736538e-08	23	1.0633213e-13
4	7.8283082e-03	14	1.4934560e-08	24	2.8491608e-14
5	2.0975889e-03	15	4.0017034e-09	25	7.6343034e-15
6	5.6204724e-04	16	1.0722532e-09	26	2.0456054e-15
7	1.5060010e-04	17	2.8730937e-10	27	5.4811832e-16
8	4.0353176e-05	18	7.6984315e-11	28	1.4686786e-16
9	1.0812601e-05	19	2.0627885e-11	29	3.9353125e-17

Calculation of Power Line Fields between Circular Conductors

Fig. 4. Maximum error for (a) solid and (b) stranded conductors.

Fig. 5. Error for solid conductor with 11 harmonics.

seven decimal places is obtained with 10 harmonics. In the case of the stranded conductor the error reduces at a slower rate of 3 dB per harmonic, and a minimum accuracy of seven decimal places requires 30 harmonics.

Figure 5 shows the error when 11 harmonics are used for the solid conductor, and Fig. 6 shows the error when 11 harmonics are used for the stranded conductor. It is clear from the figures that the maximum error appears at the surface of the

A. Seagar & S. Nilboworn

Fig. 6. Error for stranded conductor with 11 harmonics.

Fig. 7. Error for (a) 7, (b) 11 and (c) 15 harmonics between 1 m and 2 m from the center of the solid conductor.

1 conductor, and reduces quite rapidly with distance from the surface. The angular
 2 distribution of the error follows the first truncated (12th) harmonic, as is seen by
 3 the 12 alternating maxima and minima in Fig. 5.

4 Figure 7 shows the error for 7, 11 and 15 harmonics as the distance from the
 5 center of the solid conductor is increased in a horizontal direction from 1 m to 2 m.

In all cases the error reduces rapidly with distance. The curve for 15 harmonics shows rounding noise below an error of about 10^{-14} , caused by the subtraction of two very close numbers in double precision arithmetic. In the poorest case, with 7 harmonics, the error is reduced by slightly over 47 dB in moving away from the surface by a distance of one radius. When 15 harmonics are used the improvement over the same distance is 95 dB. In terms of decades, the rate at which the error reduces with distance for 7, 11 and 15 harmonics respectively is 237, 355 and 474 dB/decade.

7. Discussion

The method described here trades off the full generality of the BEM with efficiency and accuracy for a particular class of problem involving circular conductors.

Efficiency is achieved first by adopting the BEM itself so as to cast the problem in terms of surfaces rather than volumes, then by adopting the Galerkin method so that the smallest truncated system matrix can be used for any given error, and finally by evaluating all of the integrals involved analytically.

Accuracy is achieved by controlling the level of truncation with the knowledge from Table 1 that the lowest harmonics have the largest value, from Fig. 4 that the error decreases with increasing number of harmonics but is affected by the distance between individual conductors, and knowledge from Figs. 5–7 that the error decreases away from the surface.

For cases where the objective is in calculating the field some distance from the conductors, very few harmonics are required (as indicated by Fig. 7). One such calculation is the field to which workers or the general public are exposed when walking beneath power lines. In this instance it is perfectly feasible to use very few harmonics while representing individually every strand in every cable in every bundle of every circuit, without incurring excessive computation or error. While that is perhaps not entirely necessary, it serves to emphasize the effectiveness of the technique.

Errors near the surface are higher, but can be controlled to within engineering tolerance (never more than six decimal places) by choice of the number of harmonics. As a guideline, for that accuracy 23 harmonics should be used for stranded conductors and six harmonics for solid conductors.

The efficiency and accuracy of the technique makes it suitable for implementation in near real time on the average stand-alone personal microcomputer. The time required for the solution can be calculated directly, in advance. For example, for a high voltage DC power transmission line with bundles of two conductors and cables with 18 strands in the outer layer the model requires $N = 144$. For six decimals of accuracy at the surface of the conductors and significantly higher accuracy away from the conductors, the number of harmonics is $K = 23$. That gives $M = 6769$. The solution requires something less than 301×10^9 floating point operations. For a processor capable of 7 gigaflops, such as the Intel Core i750, the calculation takes

A. Seagar & S. Nilboworn

about 45 s. For 0.1% accuracy (four decimal places): $K = 10$, $M = 3025$, and the calculation time is about 4 s.

With choice between current and charge sources, and between zero-, one- or two-dimensional distributions of source it is possible to formulate a mathematical solution in many ways. The mathematical solution which most closely matches the physical problem uses one-dimensional current sources. A similar solution albeit not strictly related to the physical solution can be formulated in terms of one-dimensional charge sources. The two-dimensional charge source method is problematic because the source is not unique. If accurate results are required at the surface of the conductors the zero-dimensional charge source method should be avoided. If the method presented here is adopted the figures in Sec. 6 can be used to fix the accuracy of the solution in advance.

The method described here provides accurate and efficient solutions in the case without ionic current flow in the region between conductors. Full use of Eq. (22), as noted in Sec. 4, accommodates solutions in the case with ionic current flow.

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Calculation of Power Line Fields between Circular Conductors

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A. Seagar & S. Nilboworn

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