**NEW MEASURES OF FACTOR PRODUCTIVITY IN AUSTRALIA:
A SATO APPROACH**

Anthony J. Makin*
Economics, Griffith Business School
Griffith University, Gold Coast
Australia
t.makin@griffith.edu.au

Sam Strong
Griffith University, Brisbane
Australia

*corresponding author

October, 2011

NEW MEASURES OF FACTOR PRODUCTIVITY IN AUSTRALIA: A SATO APPROACH

Abstract

This paper derives new results of the elasticity of substitution between capital and labour and factor productivity in Australia since the mid-1960s using a Sato production function specification. This specification has unique properties that enable estimation of capital-labour substitution elasticity and changing marginal productivities through time. These estimates reveal that the substitution elasticity and labour productivity rose significantly and remained elevated during a major economic reform period throughout the 1980s and 1990s. A novel contribution of the paper is the depiction of production isoquants to convey how combining labour and capital to produce real GDP has changed over recent decades.

Contents

1. Introduction
2. Aggregate Production Functions and Isoquant Analysis
3. The Sato Specification
 - 3.1 The Error Term
 - 3.2 Data Issues
4. Econometric Estimates
5. Australia's Production Isoquants
 - 5.1 The Trending Data
6. Conclusion

NEW MEASURES OF FACTOR PRODUCTIVITY IN AUSTRALIA: A SATO APPROACH

1. Introduction

Aggregate production functions relating economy-wide output to the capital stock and labour force have long been used to understand the phenomena of economic growth, the productivity of capital and labour, the distribution of national income, and the business cycle. The best-known mathematical specifications of the production process emerged during the mid-twentieth century, although the classical economists Turgot, Malthus and Ricardo, long ago recognised that production was a function of factor inputs (Blaug 1985).¹

Standard production function specifications, such as the linear, Leontief, Cobb-Douglas (and its translog extensions that include materials as an input), and constant elasticity of substitution (CES) forms, have been widely used for analysing both microeconomic and macroeconomic behaviour. Yet another somewhat neglected production function due to Sato (1964) which has unique properties that permit analysis of factor productivity change through time, has, to the authors' knowledge, not previously been used for factor analysis at the macroeconomic level. The main contribution of this paper is to re-examine the properties of the Sato production function and apply the approach to derive new results about factor productivity variation with reference to Australia's experience since the mid-1960s.

Despite earlier controversies about the nature and treatment of capital in the aggregate production models, as surfaced for instance during the 'Cambridge capital controversy' (see Bliss 1975, and Cohen and Harcourt 2003), and criticism of the omission of natural resources

¹ Wicksteed (1894) first proposed production as a mathematical function of inputs, $p = f(x_1, x_2, \dots, x_n)$.

in production (see Daly 1997 and Solow 1997), variants of Cobb-Douglas and CES production functions, in particular, remain highly relevant to contemporary macroeconomic analysis. For instance, variants of aggregate production functions in these forms are central to real business cycle theory, as first proposed by Kydland and Prescott (1982), and to related dynamic stochastic general equilibrium (DSGE) models that seek to explain the causes of short run output fluctuations and what fiscal and monetary policy responses may be appropriate.

In parallel, a literature has grown on the longer run implications of changes in the degree of substitutability between capital and labour, as measured by the elasticity of substitution between these factors. For instance, de La Grandville (1989), Klump and de La Grandville (2000), and Klump and Preissler (2000) focus on the positive relationship between the elasticity of substitution and economic growth. The value of capital-labour elasticity substitution has also been invoked to explain growth convergence (Mankiw 1995) and the effectiveness of employment generation policies (Rowthorn 1999). Nonetheless, there is a paucity of empirical research examining the relationship between the substitutability of capital and labour and its implications for the relative productivity of these two critical factor inputs in the context of ongoing economic growth.

This paper examines this important relationship for Australia, an economy that has grown relatively strongly within the OECD group over recent decades. It provides, for the first time, evidence of a significant rise in the elasticity of substitution between capital and labour at the macroeconomic level, using a Sato production function specification, which allows for non-homothetic patterns of production through time. In preview, it reveals in the case of Australia that the elasticity of substitution and labour productivity relative to capital

productivity rose and remained elevated during a major economic reform period from the 1980s to the 1990s, but has since markedly declined.

The remainder of the paper proceeds as follows. Section 2 briefly surveys alternative macroeconomic production function specifications and introduces key concepts for subsequent analysis, while Section 3 highlights the relevance of the Sato approach for examining the elasticity of substitution between capital and labour and relative factor productivities. Section 4 discusses data issues and the econometric estimation and the production isoquants derived from the data. Section 5 concludes the paper by summarising the results and discussing institutional factors that help explain them over recent decades.

2. Alternative Production Functions and Isoquant Analysis

The CES production function, introduced by Solow (1956) and extended in Arrow, Chenery, Minhas and Solow (1961) assumes constant elasticity of substitution between capital and labor and, as a general functional form, can be shown to encompass linear, Leontief and Cobb-Douglas specifications as special cases, reflecting particular assumptions about the degree of substitutability of the factors.

For example, a two factor macroeconomic production function in linear form assumes labour and capital are perfect substitutes, whereas a Leontief form assumes factor inputs are completely non-substitutable, or are perfect complements, whereas in the popular Cobb-Douglas function, the elasticity of substitution is assumed to have a value of unity. The impossibility theorems of Uzawa (1962) and McFadden (1963) highlighted problems related to elasticities of substitution between factors when extending the CES approach to more than two factors. For related discussion see Sato (1967, 1985). More recently Leon-Ledesma,

McAdam and Willman (2010) have explored the relationship between capital-labour substitution and technical bias in production.

Using a Cobb-Douglas specification and assuming constant returns to scale, diminishing returns to factors, and an elasticity of substitution set at unity, leads to the conclusion that the capital-output ratio and output per capita stabilise at steady state levels. Yet, Antras (2004) concludes the Cobb-Douglas function does not satisfactorily explain aggregate output determination for the United States, its original intention, if the elasticity of factor substitution in production is less than unity. Meanwhile, de La Grandville and Solow (2009) demonstrate that in the standard Solow (1955) and Swan (1956) growth model when the elasticity of substitution exceeds a critical value, an economy can in theory expand perpetually in the absence of technical advance.

Assuming capital and labour are substitutable to some degree, any point on an isoquant depicted in capital-labour space represents a pairing of factor inputs that will produce a given level of aggregate output. By totally differentiating the general functional form,

$$Y = f(K, L) \tag{1}$$

$$dY = f_K dK + f_L dL \tag{1a}$$

where Y is GDP, K is the real capital stock and L is hours worked. Setting

$$dY = 0 \tag{1b}$$

it follows from 1(a) and 1(b) that at any point on an isoquant the marginal rate of technical substitution (MRTS), or rate at which labour can be substituted for capital, is measured by the slope of the isoquant at that point.

From the above it easily follows that the slope of production isoquants in capital-labour space

is $-\frac{dK}{dL} = \frac{f_L}{f_K} = \text{MRTS}$, the ratio of the marginal productivities of labour and capital

respectively. As capital intensity falls moving down an isoquant, the marginal productivity of capital normally rises, whereas a rise in labour intensity implies labour productivity falls.

It also follows that if the slopes of the estimated isoquants steepen (flatten) over time, the productivity of labour relative to capital increases (decreases).

The elasticity of substitution (ES) is the elasticity of the ratio of capital and labour with respect to their marginal products and measures the curvature of an isoquant. ES is defined

as $\frac{d \ln K/L}{d \ln \text{MRTS}}$. Intuitively, ES reflects how the production process changes as the marginal

products and hence relative price of capital and labour change.

When examining the issue of the relative productivity of factors, a criticism of the popular Cobb-Douglas function in particular is that no theoretical rationale exists to explain why the coefficients should be constant over time, implying the ES is constrained to unity value, when the nature and composition of both the capital stock and labour force is ever-changing. The purpose here is to look more directly at the way labour and capital in aggregate combine to produce aggregate output over time. Function (1) above is thus seen to change over time, and technological change is more usefully thought of as being embodied, with specific effects on both capital and labor over time.

3. The Sato Specification

Arising out of a debate in the American Economic Review, Sato (1964) proposed the following aggregate production function

$$Y = L^2K^2/(aL^3+bK^3) \quad (2)$$

where a and b are constant parameters. This function has the desirable economic properties of (i) constant returns to scale, and (ii) the marginal product for each factor (holding the other one constant) can rise and then fall, possibly to less than zero.

These properties are commonly required of microeconomic production functions, and Sato suggested that in aggregate this function is also applicable. These minimal characteristics are all that we require of the production function, as the highly aggregated nature of the data used here precludes a detailed set of technical restrictions for each of the differing sectors of the economy. An artificial example showing the flexibility of this function is shown here, wherein a set of isoquants is drawn.

Foreshadowing results derived later, a graph of an estimated function is shown here as well, with real GDP on the vertical axis:

Other functions have been proposed that have these characteristics, as seen in the articles mentioned in the above debate (the widely used Cobb-Douglas and CES do not have such properties). However, a convenient property of the Sato function is that by inverting each side, a function that is linear in a and b is available.

Before exploring this further, it is observed that if a and b are assumed to be functions of time t then the properties (i) and (ii) above are retained. Allowing a and b to change allows the production function to change, and the marginal product of the factors to vary as well. If the simple linear forms $a(t) = a_0 + a_1.t$ and $b(t) = b_0 + b_1.t$ are used, and both sides of (2) are inverted, then [in the following it is understood that all variables, including Y , L and K , are dated and so will not be shown with an attached subscripted t as is the usual convention]

$$1/Y = a_0.L/K^2 + a_1.t.L/K^2 + b_0.K/L^2 + b_1.t.K/L^2 \quad (3)$$

This is obtained with a little algebra, noting that a and b consist of two terms each.

Quadratic expressions in t could be used for a and b , but later estimate results indicate that these are not necessary.

3.1 The Error Term

If data on the variables Y , L and K are available, then the possibility of using ordinary least squares for estimating the parameters is raised. This is feasible, but for statistical inference purposes care is needed in appending an error term to (3). It has been common practice in econometric work to add an error term as is convenient, with similarly convenient properties. For example, the logarithmic form of the Cobb-Douglas function is often used, with its parameter estimation proceeding with an additive error term. The argument is then made that there is an exponential error in the true production function.

In the model proposed here, if $B(t)$ is the right hand side of (3) and an error term ε with a zero mean and a constant variance σ^2_ε is added to (3), then:

$$1/Y = B(t) + \varepsilon \quad (4)$$

so that
$$Y_t = 1/(B(t) + \varepsilon) \quad (5)$$

Hence the error term enters in a non-linear way, in contrast to the usual setup.

Details of the implications of adding a usual normally distributed error term at this stage is discussed in the Appendix in which the equation numbering follows that in the body of the paper to retain continuity.

As a result, adding a convenient error term ε allows a convenient estimation of the a and b coefficients. As mentioned above, justifications of this kind are often not made: error terms are added at a convenient stage as required.¹

3.2 Data Issues

In the model here, the variables are $1/Y$, L/K^2 and K/L^2 . In detail, Y is an index of real Australian GDP, for 1968 to 2006, L is an index of aggregate hours worked, and K is an index of the real aggregate capital stock. All annual data is sourced from the Australian Bureau of Statistics, Australian System of National Accounts, Catalogue 5204.0, various tables.

In much previous work of this kind, the variable L refers to the size of the work force. In the Australian context, the increase in part-time work combined with the definition of an employed person implies that that previous work cannot be directly compared with that here. Which definition of L is suitable depends on the circumstances, but the functions being estimated here are of a purely technical nature, and so the definition of L used here is appropriate.

¹ Before considering the estimation further, it is noted that the expansion of (5) around $E(\varepsilon_t) = 0$ used the first three term of a Taylor's series for accuracy. But if only the first two terms are used, things are simplified: (7) is replaced with

$$Y = K^2 L^2 / (a(t)L^3 + b(t)K^3) + (1/B(t))^2 \varepsilon \quad (11)$$

The error term is simpler and allows the deterministic part of (12) to retain properties (i) and (ii) of the production function. This strengthens the case for simply adding on the error.

The data have been generated over time, and so care is needed to avoid estimating a spurious relationship. While the standard assumptions of the classical linear model are not necessarily violated if the variables in a model are trending, there is the possibility that there are other trending variables omitted from the set of regressors that could help explain how $1/Y$ varies in (4) (Wooldridge, 2000, pp.334-5).

The omitted variables could lead to biased estimates. What these other omitted variables might be is not be easy to discern: labour and capital are the only aggregate factors of production, and technological change is accounted for by allowing $a(t)$ and $b(t)$ to change over time. Nevertheless, if the variables in a model are trending, it is a safe procedure to try adding a trend (linear or maybe a quadratic term in t). This was done as described later.

Rather than begin with a hypothesis of a unit root in each variable (as in the Dickey-Fuller approach), what is to be tested is the hypothesis that some of the regressor variables L/K^2 , $t.L/K^2$, K/L^2 and $t.K/L^2$ are stationary around a linear or quadratic trend in t . For the first two variables, however, the augmented D-F test (which begins with a null hypothesis of non-stationarity around a linear trend) allows the safe conclusion that non-stationarity around a linear trend can safely be rejected in favour of trend stationarity.²

²The method of Kwiatkowski et al. (1992), which begins with an assumption of stationarity, is applied to all the variables as well. If all of these variables are concluded to be stationary around a trend (either linear or quadratic), then adding a trend (or trends) to the regression model (4) is recommended (Wooldridge, 2000, p.334).

4. Econometric Estimates

Before discussing the data properties in detail, the results of the basic regression are presented:

Table 1. Ordinary least squares estimates

Ordinary Least Squares Estimation			
Dependent variable is 1/Y			
44 observations used for estimation from 1965 to 2008			
Regressor	Coefficient	Standard Error	t-ratio [Prob]
L/K^2	.18313	.004307	41.7080
$t.L/K^2$	.0038381	.0010399	3.69091
K/L^2	2.1995	.21134	10.4074
$t.K/L^2$	-0.037213	.003859	-9.55181
R-Squared	.99703	R-Bar-Squared	.99680
S.E. of Regression	.4573 E-3	F-stat. F(3, 40)	4468.6 [.000]
Mean of Dependent Variable	.020594	S.D. of dependent Variable	.0080865
Residual Sum of Squares	.8365E-5	Equation Log-likelihood	278.0309
Akaike Info. Criterion	274.0309	Schwarz-Bayesian Criterion	270.4626
DW-Statistic	1.5258		

All the estimated coefficients are significant, and the Durbin-Watson statistic does not indicate a problem with serial correlation in the error term. The no-intercept D-W critical

values for testing a lack positive serial correlation are 1.148 and 1.197. No pattern is evident in the estimated residuals.

The high R^2 is expected with this trending data, but it is noted that in this case the usual R^2 is incorrect, as observed by Wooldridge (1991). Furthermore, the lack of an intercept complicates the interpretation of the calculated value of 0.9973. Ramanathan (2002, p.151) discusses this, and suggests that using the correlation coefficient between the dependent variable and its fitted value is a useful alternative as a goodness-of-fit measure. If this is calculated, the value is again high, at 0.997.

If a correction is made for the possible (and expected) existence of heteroscedasticity in the model, done by assuming that the error variance σ^2_ε proportional to the square of the fitted dependent variable, little change in the estimates occurs (the coefficients become .18766, .0053104, 1.9247 and -.032556. All are found to be significant). The unadjusted estimates will therefore be retained for further use. (It is noted, however, that no lesser authorities than Stock and Watson (2007) in their econometrics textbook routinely use OLS results corrected for heteroscedasticity and autocorrelation, virtually as a standard procedure).

A plot of the estimated residuals reveals no trend or other pattern:

Table 2. Residual Plot

Running the regression again with a trend term included produced similar results, and the coefficient on the trend variable is insignificant using the usual levels of significance.

Table 3. Least Squares Estimates with Included Trend

Ordinary Least Squares Estimation			
Dependent variable is 1/Y			
44 observations used for estimation from 1965 to 2008			
Regressor	Coefficient	Standard Error	t-ratio [Prob]
L/K^2	.18230	.004213	42.4814 [.000]
$t.L/K^2$	.0052697	.0012782	4.1228 [.000]
K/L^2	2.1372	.20821	10.2646 [.000]
$t.K/L^2$	-.024069	.0081213	-2.9636 [.000]
TREND[t]	-.1275E-3	.6968E-4	-1.82961 [.075]
R-Squared	.99726	R-Bar-Squared	.99698
S.E. of Regression	.4444 E-3	F-stat. F(3, 40)	3549.0 [.000]
Mean of Dependent Variable	.020594	S.D. of dependent Variable	.0080865
Residual Sum of Squares	.7704E-5	Equation Log-likelihood	279.8425
Akaike Info. Criterion	274.8425	Schwarz-Bayesian Criterion	270.3820
DW-Statistic	1.6253		

Using a squared trend variable similarly did not change the estimates significantly. These results are encouraging, and it therefore useful to examine the shape of the estimated Sato function as it evolves over time. Before this is done, however, it is noted that it is possible to estimate the coefficients directly using the Sato function using non-linear least squares. The estimation results are very similar to the ones above.

Table 4. Non-linear Least Squares Results

Non-Linear Least Squares Estimation			
Parameter	Estimate	Standard Error	t-Ratio[Prob]
a0	.19332	.0090218	21.4283[.000]
a1	.0015323	.0018247	.83997[.406]
b0	2.3727	.35749	6.6371[.000]
b1	-.038438	.0063905	-6.0149[.000]
<hr/>			
R-Squared	.99467	R-Bar-Squared	.99427
<hr/>			
S.E. of Regression	1.6758	F-stat. F(3, 39)	2487.61[.000]
Mean of Dependent Variable	56.4318	S.D. of Dependent Variable	22.1353
Residual Sum of Squares	112.3260	Equation Log-likelihood	-83.0521
Akaike Info. Criterion	-87.0521	Schwarz Bayesian Criterion	-90.6204
DW-statistic	.47802		

Issues to do with the trending nature of the variables are difficult to address in this non-linear estimation method, and by using the inverted form of the Sato function and examining the nature of the data does allow this to be looked at. Park and Phillips (1998) have addressed such issues, but this is not followed up.

5. Australia's Production Isoquants

Estimated production isoquants for selected years show how the economy has changed the way it combines labour and capital to produce real GDP. In the set of estimated isoquants shown below it is seen that the marginal products for both aggregate factors are positive, but certainly not equal. The last two maps are drawn up as contours of the estimated production function. These isoquants depict labour-capital combinations from 1980 up until the global financial crisis. For the years 2005 and 2008 three dimensional diagrams are shown as well, to confirm that the example shown in Section 3 can be successfully implemented.

1968

1970

1978

1988

1998

2008

2005

2008

For all years 1980 – 2008 the marginal products for both factors are almost always positive (from 1971 onwards), a non-trivial result. For while the usual Cobb-Douglas function guarantees that this will be the case, the possibility that over-utilisation of either factor may lead to negative marginal products can be observed, and measured, with the Sato function. That this rarely occurred here suggests that the estimation is proper: businesses and governments are unlikely to routinely employ resources in such a way as to produce negative marginal factor products, but it can be seen that close to the points on the isoquants that have sensible negative slopes, the marginal products do become negative: at times either factor is being used close to its limit of productivity.

The extraction of the elasticities of substitution from the Sato function is simple given the degree one nature of the function. (If $Y = f(L, K)$, then this elasticity is $ES = f_L \cdot f_K / (Y \cdot f_{LK})$, where the subscripts represent derivatives. [Silberberg, 1981, p.316].

The estimated elasticities of substitution (of capital for labour, and vice versa) ES, evaluated at the observed values of L and K, for the above selection of years are:

$$ES(1968) = -.116, ES(1970) = -.007, ES(1978) = .095, ES(1988) = .099, \\ ES(1998) = .077, ES[2005] = .06, ES(2008) = .035.$$

The following plot of the estimated elasticities reveals how they changed over time;

[Approximate confidence intervals using the so-called delta method [Green, 1975] can be obtained. The 95 per cent margin for error for the 2005 and 2008 estimate is .078].

The negative values for a few early years included in the study suggest that there may have been a structural change in the economy that should be addressed in the estimation procedure. However, the sensible results for all of the years after the early seventies, and the plausible economic explanation of the movement of the elasticity coefficients over most of the period as presented in the last section of this paper, allow the estimation results to stand. In any

event, more recent years are of more interest than the early years, and the estimates appear sensible for these latter years.

Furthermore, a plot of the marginal products of labour [L in the first plot below] and capital over the data period shown below can be compared with other estimates, provided by Australian Government's Productivity Commission as shown in the following plot [Australia's line is the one originating at a value of 75 for 1960]. The upward drift in the labour productivity measures in both plots gives added credence to the estimates provided here.

International labour productivity benchmarks

GDP per hour worked, US=100

Mallick (2007) has produced a comprehensive array of estimates for the elasticity of substitution across ninety countries, using the CES function. The estimates vary widely. It is to be noted that the labour variable used in that work is the size of the work force, in contrast to hours worked variable here. Of interest is that Argentina, Brazil, Chile and Mexico have estimates similar to the results found here. However, the elasticity varies greatly around the isoquants for the Sato function, and comparisons with Mallick's estimates are therefore to be made with this in mind.

The regressor variables are highly linearly correlated, as shown in the Appendix, and a common problem in such a case is that the estimated variances of the coefficients are high (often resulting in unexpected signs for the estimated coefficients). In this case, they are not so high as to produce low t-values for the regression coefficients. There appears to be enough year-to-year variation in all variables to produce results in accordance with economic reality.

5.1 The Trending Data

An argument can be made that technical relationships are timeless, and that the sampled data used here is essentially cross-sectional in nature. As a result, matters regarding the temporal characteristics of the data need not be discussed in detail. Lags in behaviour of the kind exhibited in economic relationships must be confronted with time-series techniques, of course, and to the extent that some such temporal phenomena exist here, the following brief remark is in order.

Returning to the nature of the data, the trending nature of the data on L and K produces trends in the four regressors in the model (3). Variables L/K^2 and $t.L/K^2$ decline towards zero from

1965 to 2008, but K/L^2 and hence $t.K/L^2$ trend upwards. If K/L^2 and $t.K/L^2$ are stationary around a linear or quadratic trend then adding trend variables to the regression model allows proper inference to be drawn (Wooldridge, op cit).

6. Concluding Comments

In the broad literature explaining national output determination with reference to capital and labour as key factor inputs, relatively scarce attention is paid to the importance and implications of variation in the substitutability of these factors over time. This is, in part, due to the limitations associated with macroeconomic functions commonly used hitherto to explain dynamic factor productivity. For instance, the constant co-efficients for capital and labour and the unity elasticity of substitution between these factors assumed in the popular and highly tractable, Cobb-Douglas specification appear incongruous for economies experiencing continuous structural adjustment, during which the nature of capital and skill requirements of labour are ever-changing.

The main innovation of this paper is that it invokes a much neglected Sato production function specification which has unique properties that enable us to derive estimates of the elasticity of substitution between capital and labour and changing marginal productivities through time. Application of this approach to Australian real GDP, capital stock and labour force data from the mid-1960s until recently yields new estimates that reveal how the way labour and capital have been combined to produce aggregate output has changed over this period. Specifically, there was a marked increase in the elasticity of substitution of capital

for labour during the 1980s which was maintained until the mid-1990s, although this was reversed from the turn of the century.

The period when the elasticity of substitution reached relatively high levels co-incides with an extensive labour and product market reform era in Australia during the 1980s and 1990s that significantly improved labour and product market flexibility. Over this time the Australian economy also became more open to international trade as a result of the abolition of international trade barriers, a factor Ventura (1997) has identified as important for economic growth, while its financial markets also became more internationally integrated. There was also large scale privatisation of public sector enterprises and liberalisation of foreign investment.

Australia's long term real GDP growth rate as measured by average annual real GDP over the past half century was 3.6 per cent. Annual growth rates tended to exceed this long run rate during the period of productivity-enhancing economic reform. The behaviour of the elasticity of substitution and the production isoquants derived in this paper reflect this pattern, for instance with the isoquants exhibiting a marked steepening from 1980 to 2000, during which time reform of the Australian economy was at its height. This was followed by a subsequent fall in the elasticity of substitution and flattening of the isoquants, particularly between 2005 and 2008, an interval during which the pace of economic reform decelerated.

Appendix

Table A1. Correlation of variables

Sample period	:1965 to 2008				
Variable(s)	: RECIPGDP	LDKK	LDKKT	KDLL	KDLLT
Maximum	: .039063	.15930	.81034	.010078	.44345
Minimum	: .0096339	.0090887	.15930	.0045787	.0047811
Mean	: .020594	.048410	.63878	.0073749	.18659
Std. Deviation	: .0080865	.039739	.14688	.0016858	.13209
Skewness	: .62742	1.3042	-1.2189	-.16523	.36722
Kurtosis - 3	: -.38871	.79868	1.2059	-1.0767	-1.0302
Coef of Variation:	.39267	.82088	.22994	.22859	.70789

The Error Term

Expanding the right hand side of (5) close to the mean of ε_t ($= 0$) yields

$$Y = 1/B(t) - (1/B(t))^2 \varepsilon + (1/B(t))^3 \varepsilon^2 \quad (6)$$

$$\text{or} \quad Y = K^2 L t^2 / (a(t) L^2 + b(t) K^2) + e_t \quad (7)$$

$$\text{for} \quad e = - (1/B(t))^2 \varepsilon + (1/B(t))^3 \varepsilon^2 \quad (8)$$

If it is assumed that $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ then

$$E(e) = (1/B(t))^3 \sigma_\varepsilon^2 \quad (9)$$

$$\text{and } \text{Var}(e) = (1/B)^4 \sigma_\varepsilon^2 + 2(1/B)^6 \sigma_\varepsilon^4$$

(Using a change of variable from ε to e produces a χ^2 -like probability density for e with a minimum of $-1/4.B(t)$).

The variance of e , being dependent on the increasing function of t , $B(t)$, varies over time, in agreement with the need to accommodate heteroscedasticity for the disturbance term in (7). As Y grows over time, so does the error variance, as captured in (7) and (9): such time changing variances are sensible in this context. However, the mean of e not being equal to zero implies that

$$Y_t = K^2 L^2 / (a(t)L^3 + b(t)K^3) + \sigma_e^2 (K^2 L^2 / (a(t)L^3 + b(t)K^3))^3 + \mu \quad (10)$$

where $\mu = e - E(e) = e - (1/B(t))^3 \sigma_e^3$ so that $E(\mu) = 0$.

The question then arises as to how much the non-random part of (10) differs from the original Sato function (2). If σ_e^2 in the middle term of the right hand side of (10) is small, there is no significant problem. Further, if L_t and K_t are multiplied by λ on the R.H.S. of (10) the middle term is then multiplied by λ^3 (not by λ as it is in the first term, implying constant returns to scale). Constant returns to scale requires that λ can be any value, but in practice, the values of λ that are relevant for describing how output expands or contracts, as L_t and K_t , are close to one.

If a whole economy were to halve its stocks of L and K_t it is not envisaged that Y would similarly halve: the structural changes would be immense. Consequently, for λ near one, λ^3 in the middle term of (10) is near to one as well, implying approximately constant returns to scale for (11). If additionally σ_e^2 is small relative to Y then the middle term is of little consequence, so that this second property of the Sato function would be satisfied as well. Estimation of σ_e^2 later does provide evidence that it is indeed small.

References

Antras, P. (2004) "Is the US production function Cobb-Douglas? New estimates of the elasticity of substitution" *BE Journal of Macroeconomics* 4, 1.

Arrow, K., Chenery, H., Minhas, B. and Solow, R. (1961) "Capital-labour substitution and economic efficiency" *Review of Economic Studies* 43, 225-250.

Australian Bureau of Statistics, *Australian System of National Accounts*, Catalogue 5204.0
Available online at: www.abs.gov.au

Blaug, M. (1985) *Economic Theory in Retrospect* (4th ed) Cambridge University Press, Cambridge UK.

Bliss, C. (1975) *Capital Theory and the Distribution of Income*, Elsevier, New York.

Cohen, A. and Harcourt, G. (2003) "Whatever happened to the Cambridge capital controversy?" *Journal of Economic Perspectives* 17 (4), 199-214.

Daly, H. (1997) "Georgescu-Roegen versus Solow-Stiglitz" *Ecological Economics* 22, 261-266.

de La Grandville, O. (1989) "In quest of the Slutsky diamond" *American Economic Review* 79, 468-481.

de La Grandville, O. and Solow, R. (2009) *Capital-labour substitution and economic growth: a unified approach* Cambridge University Press, Cambridge UK.

Green, W.H. 1975 *Econometric Analysis*, Prentice-Hall, Englewood Cliffs, NJ.

Klump, R. and de La Grandville, O. (2000) "Economic growth and the elasticity of substitution: two theorems and some suggestions" *American Economic Review* 90, 282-291.

Klump, R. and Preissler, H. (2000) "CES production functions and economic growth"
Scandinavian Journal of Economics 102, 41-56.

Kydland, F. and Prescott, E. (1982) "Time to build and aggregate fluctuations"
Econometrica, 50(6), pp. 1345-70.

Kwiatkowski, D., Phillips, P.C.B., Schmidt, P. and Shin, Y. (1992), 'Testing the null hypothesis of stationarity against the alternative of a unit root', *Journal of Econometrics*, 54, 159-178.

Leon-Ledesma, M., McAdam, P. and Willman, A. (2010) "Identifying the elasticity of substitution with biased technical change" *American Economic Review* 100 (3), 1330-1357.

Mallick, D. (2008) 'The role of the elasticity of substitution in economic growth: A cross-country test of the de La Grandville hypothesis', Latrobe University (Victoria).

Mankiw, G. (1995) "The growth of nations" *Brookings Papers on Economic Activity* 1, 275-310.

McFadden, D. (1963) "Constant elasticity of substitution production functions" *Review of Economic Studies* 30, 73-83.

Park, J. Y., and Phillips, P.C.B. (1998), 'Nonlinear regressions with integrated time series', *Cowles Foundation Discussion Series*, 1190.

Productivity Commission.....

Ramanathan, R. (2002), *Introductory Econometrics with Applications*, Harcourt College Publishers, Fort Worth, TX.

Rowthorn, R. (1999) "Unemployment, wage bargaining and capital-labour substitution" *Cambridge Journal of Economics*, 23(4), 413-425.

Sato, R. (1985) “CES production function” in J. Eatwell et al (eds.) *The New Palgrave: A Dictionary of Economics* Vol. 1, Macmillan, London, 395-396.

Sato, R. (1975) “The most general class of CES functions” *Econometrica* 43 (6), 999-1003.

Sato, R. (1964), ‘Diminishing returns and linear homogeneity: comment’, *The American Economic Review*, 54, 5, 744-745.

Silberberg, E. [1981] *The Structure of Economics; A Mathematical Analysis* McGraw-Hill, Auckland

Solow, R. (1997) “Reply: Georgescu-Roegen versus Solow/Stiglitz” *Ecological Economics* 22, 267-268.

Solow, R. (1956) “A contribution to the theory of economic growth” *Quarterly Journal of Economics* 70, 65-94.

Swan, T. (1956) “Economic growth and capital accumulation” *Economic Record* 32(2), 343-361.

Schmidt, P. and Phillips, P. (1992), ‘LM tests for a unit root in the presence of deterministic trends’ *Oxford Bulletin of Economics and Statistics*, 54, 3, 257-287.

Stock, J.H. and Watson, M.W. (2007), *Introduction to Econometrics*, Pearson Education, Boston.

Uzawa, H. (1962) “Production functions with constant elasticity of substitution” *Review of Economic Studies* 29, 291-299.

Ventura, J. (1997) “Growth and interdependence” *Quarterly Journal of Economics* 62, 57-84.

Wicksteed, P. (1894) *An Essay on the Co-ordination of the Laws of Distribution* Macmillan and Co, London.

Wooldridge, J.M. (1991), 'A note on computing r-squared and adjusted r-squared for trending and seasonal data', *Economics Letters*, 36, 49-54.

Wooldridge, J.M. (2000), *Introductory Econometrics: A Modern Approach*, South-Western College Publishing, Australia.