Comment on ‘Operator formalism for the Wigner phase distribution’

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The operator associated with the radially integrated Wigner function is found to lack justification as a phase operator.

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1. Introduction

The radially integrated Wigner function has been shown by Garraway and Knight [1, 2] to be negative for important classes of states. As such, it cannot be taken to represent physical properties in the way that probability distributions represent the statistical properties of the things they describe. So it comes as something of a surprise to find the same function being used as a basis for describing the phase observable of a single mode radiation field, as Subeesh and Sudhir have recently done [3]. They derive an operator $\hat{\rho}_W(\theta)$,

$$\hat{\rho}_W(\theta) = \frac{1}{2\pi} \sum_{m,n=0}^{\infty} \sum_{l=0}^n \frac{(-1)^m 2^{m+n/2} \exp[i(n-2l)\theta]}{m!(n-l)!l!} \Gamma\left(\frac{n}{2} + 1\right) \hat{a}^{\dagger m+n-l} \hat{a}^{m+l}, \quad (1)$$

which they call the Wigner phase operator, whose expectation value is the radially integrated Wigner function $P_{\psi}^W(\theta)$, i.e.

$$P_{\psi}^W(\theta) = \text{Tr}[\hat{\rho}_{\psi} \hat{\rho}_W(\theta)] \quad (2)$$

for system density operator $\hat{\rho}_{\psi}$. Calling $\hat{\rho}_W(\theta)$ a *phase operator* is to lay claim that it plays a meaningful role in describing phase. Indeed, the final paragraph of [3] states that “the radially integrated Wigner function captures essentially the same phase information” as the Pegg-Barnett phase formalism. Such extraordinary claims, if left unqualified, may lead the unwary to the conclusion that the operator $\hat{\rho}_W(\theta)$ does indeed faithfully represent quantum phase. The purpose of this comment is to point out that it does not.

It is not difficult to find grounds to counter the claims of [3]. Foremost is the work of Garraway and Knight [1, 2] that shows $P_{\psi}^W(\theta)$ differs significantly from the Pegg-Barnett phase probability density $P_{PB}(\theta)$ [4–6]. In particular, Garraway and Knight have shown that $P_{\psi}^W(\theta)$ can have negative values for wide classes of states whereas $P_{PB}(\theta)$, being a genuine probability distribution, is always nonnegative. The amount of negativity in $P_{\psi}^W(\theta)$ can be significant. For example, Figure 1 (a) illustrates the negativity of $P_{\psi}^W(\theta)$ for the state $(|0\rangle + |2n\rangle)/\sqrt{2}$, for which

$$P_{\psi}^W(\theta) = \frac{1}{2\pi} \left[1 + \frac{2^n n!}{\sqrt{(2n)!}} \cos(2n\theta) \right]$$

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Figure 1. Comparisons of the functions $P_\psi^W(\theta)$ (solid curve) and $P_{PB}(\theta)$ (dashed curve) for various states. In (a) the state is $(|0\rangle + |2n\rangle)/\sqrt{2}$ where, for clarity, only the $\theta < 0$ segment is plotted for $n = 1$ and the $\theta > 0$ segment for $n = 2$. In (b) the state is $c(|\alpha\rangle + |\beta\rangle)$ where c is a normalisation constant and $\alpha = -2$ and $\beta = 8$.

whereas, for comparison,

$$P_{PB}(\theta) = \frac{1}{2\pi}[1 + \cos(2n\theta)]$$

is everywhere nonnegative. While $P_\psi^W(\theta)$ and $P_{PB}(\theta)$ do share some similarities for special classes of states, notably number and coherent states [1–3], their significant differences for other states is sufficient to quash any notion that they capture the same information about phase in any general sense. An example of how different they can be is given by Figure 1 (b) which compares $P_\psi^W(\theta)$ and $P_{PB}(\theta)$ for the superposition of coherent states $c(|\alpha\rangle + |\beta\rangle)$, where c is a normalisation constant and $\alpha = -2$ and $\beta = 8$. Moreover, while $P_{PB}(\theta)$ is a probability density that describes the outcome of an ideal measurement of phase, the same cannot be said for $P_\psi^W(\theta)$ whose negativity *forbids* any interpretation as a probability associated with a measurement, on principle. The claim that $P_\psi^W(\theta)$ represents phase properties *per se* is therefore unjustified.

There are also difficulties associated with the operator $\hat{\rho}_W(\theta)$ defined in [3]. It is immediately apparent from the negativity of $P_\psi^W(\theta)$ that $\hat{\rho}_W(\theta)$ is *not* a positive operator-valued measure. It follows that it is not a projection operator and so it does not project onto states of well defined phase. However, it does have the property that it undergoes phase shifts in the sense that [7]

$$\hat{\rho}_W(\theta) = e^{i\hat{N}\theta} \hat{\rho}_W(0) e^{-i\hat{N}\theta}$$

where $\hat{N} = \hat{a}^\dagger \hat{a}$ is the number operator. This property underlies the relationship in equation (14) of [3],

$$\text{Tr}[\hat{\rho}_W(\theta) \hat{\rho}_{PB}(\phi)] = \text{Tr}[\hat{\rho}_W(\theta') \hat{\rho}_{PB}(\phi')]$$

for $\theta - \phi = \theta' - \phi'$, that the authors call “weak-equivalence” between $\hat{\rho}_W(\theta)$ and the Pegg-Barnett phase operator $\hat{\rho}_{PB}(\theta) = |\theta\rangle\langle\theta|$ [8]. But there are many operators that are weakly equivalent to $\hat{\rho}_{PB}(\theta)$ in the same sense. For example, let

$$\hat{A}(\theta) = e^{i\hat{N}\theta} \hat{A}_0 e^{-i\hat{N}\theta}$$

for an arbitrary operator $\hat{A}_0$. Then

$$\text{Tr}[\hat{A}(\theta)\hat{\rho}_{PB}(\phi)] = \text{Tr}[\hat{A}(\theta')\rho_{PB}(\phi')]$$

for $\theta - \phi = \theta' - \phi'$ and so $\hat{A}(\theta)$ is also weakly equivalent to $\hat{\rho}_{PB}(\theta)$ in the same way. Evidently this weak equivalence is not a stringent condition. It certainly isn't strong enough to justify the definition of $\hat{\rho}_W(\theta)$.

To illustrate the kind of caution that must be used when dealing with $\hat{\rho}_W(\theta)$, consider the angle operator defined by

$$\hat{Q} = \int_{\theta_0}^{\theta_0+2\pi} \theta \hat{\rho}_W(\theta) d\theta$$

for arbitrary phase angle θ_0 . Its expectation value for state $\hat{\rho}_\psi$ is given by

$$\langle \hat{Q} \rangle = \int_{\theta_0}^{\theta_0+2\pi} \theta P_\psi^W(\theta) d\theta$$

in which $P_\psi^W(\theta)$ appears to play the role of a probability but, owing to $\hat{\rho}_W(\theta)$ not being a projection operator, it is straightforward to show that

$$\hat{Q}^n \neq \int_{\theta_0}^{\theta_0+2\pi} \theta^n \hat{\rho}_W(\theta) d\theta$$

for integer $n > 1$ and so it follows that

$$\langle \hat{Q}^n \rangle \neq \int_{\theta_0}^{\theta_0+2\pi} \theta^n P_\psi^W(\theta) d\theta$$

in general. This means that the statistical properties of the operator $\hat{Q}$ are not described by $P_\psi^W(\theta)$. In contrast, it is well-known that the Pegg-Barnett hermitian phase operator $\hat{\phi}_s$,

$$\hat{\phi}_s = \sum_{m=0}^s \theta_m |\theta_m\rangle \langle \theta_m|$$

where $\theta_m = \theta_0 + 2\pi/(s+1)$ and $|\theta_m\rangle$ is a phase eigenstate, has the properties that

$$\begin{aligned} \hat{\phi}_s^n &= \sum_{m=0}^s \theta_m^n |\theta_m\rangle \langle \theta_m| \\ \langle \hat{\phi}^n \rangle &= \lim_{s \rightarrow \infty} \langle \hat{\phi}_s^n \rangle = \int_{\theta_0}^{\theta_0+2\pi} \theta^n P_{PB}(\theta) d\theta \end{aligned}$$

where $\langle \hat{\phi}^n \rangle$ are the moments of the probability density $P_{PB}(\theta)$ [4–6].

In conclusion, the radially integrated Wigner function cannot be interpreted as a probability owing to its negativity, and as such it cannot be used to describe the statistics of an observable

such as phase. Using it as a basis to define the operator $\hat{\rho}_W(\theta)$ to represent phase is therefore without appropriate justification. This leaves the merits of the operator $\hat{\rho}_W(\theta)$ introduced in [3] open to question.

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