Transient Analytical Solution for the Motion of a Vibrating Cylinder in the Stokes Regime
using Laplace Transforms

Abbreviated title: Analytical Solution - Vibrating Cylinder

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Abstract

A new analytical solution for the motion of an elastic cylinder in a viscous fluid is derived using Laplace transforms. Unlike previously available solutions, full expressions for transient terms are given. The solution is compared with conventional treatments of this problem. It is expected to have particular value for applications related to viscosity measurement using vibrating-wire viscometers applied to higher viscosity fluids.

Highlights

- A new analytical solution for vibration of a cylinder in a viscous fluid
- The effect of the start-up transient terms is shown
- Solution is valuable for ‘ring-down’ vibrating wire viscometers

Keywords

Analytical solution; Transient effects; Vibrating-wire viscometer; Vibrating cylinder; Laplace transforms

1. Introduction

Vibration of cylinders has importance on scales ranging from civil engineering structures (Lu et al., 2013), to viscosity measurement (Tough et al., 1964, Padua et al., 1998, Ciotta and Trusler, 2010, Sullivan et al., 2009, Assael and Mylona, 2013, Caetano et al., 2005, Etchart et al., 2007, Harrison et al., 2007, Correia da Mata et al., 2009), micro-electro-mechanical devices (Shiraishi et al., 2013) and ultimately to studies on the behaviour of nanowires (Yengejeh et al., 2014). For very small Reynolds numbers, the equations of fluid mechanics can be linearized and exact solutions for the transient motion of the cylinder can be derived thanks to the ingenious treatment of the fluid mechanics problem by Stokes (1922). This has lead to the development of a very successful technique for measuring fluid viscosity using an instrument known as a vibrating wire viscometer (Tough et al., 1964). During one mode of operation of this instrument, a fine wire immersed in a fluid sample is set in motion using an electric circuit and the transient decay of the vibration is used to determine the fluid viscosity via an analytical solution to the problem (Retsina et al., 1987). Very accurate viscosity measurements can be

obtained if the instrument is designed in such a way that it meets constraints due to assumptions used in the underlying theoretical basis for the technique (Retsina et al. 1987). Unfortunately, the constraints place limits the range of viscosity that can be measured accurately with any given instrument. Also, it can be difficult to use the technique for analysis of fluids with viscosities higher than the order of 100 mPa·s, although recently significant progress has been made with higher viscosity fluids through careful design choices (Assael and Mylona, 2013, Caetano et al., 2005, Etchart et al., 2007, Harrison et al., 2007, Correia da Mata et al., 2009). One assumption that is of concern for more viscous fluids is the neglect of the initial transient forces associated with the start-up of motion. This is the subject of the present article.

Virtually all theoretical treatments of a vibrating cylinder in an infinite medium use the ‘added mass and damping term’ approach (Stokes, 1922, Retsina et al., 1987, Hussey and Vujacic, 1967, Chen et al., 1976, Mostert et al., 1989, Diller and Van der Gulik, 1991). Analysis of the flow field is considered separately from the solid mechanics in order to yield two drag terms – one proportional to acceleration (added mass term) and another proportional to velocity (damping term). These two terms are then included in the vibration problem for the rod or wire in order to represent drag from the fluid. In this article a different approach is taken where the solid mechanics equation is solved simultaneously with the linearized fluid mechanics equations using Laplace transforms. The merit of this alternative approach is that it yields full expressions for all transient terms which hitherto have been neglected.

It should be expected that transient effects will be largest for fluids of high viscosity where ‘ring-down’ vibrations (i.e. free damped vibrations) are being considered. In the case of steady oscillations (Brushi and Santini 1975) the transient terms are zero by definition. Stokes (1922) argued (rightly) that it is justifiable to neglect the transient start up effects during the transient decay of free oscillations if the amplitude of the oscillation is decreasing slowly. Hussey and Vujacic (1967) and Retsina et al. (1987) showed that with a small modification to the steady oscillation solution, the effect of the log decrement on ‘ideal’ damped harmonic motion could be accounted for. For vibrating wire viscometers, Retsina et al. (1987) went further by proposing an analytical form for the remaining transient terms and then arguing that the terms could be neglected for the purposes of determining fluid viscosity ‘if they are uniformly small’, or ‘if they decay rapidly compared with the decay of the main oscillation’ or alternatively ‘if they decay very slowly compared with the main oscillation’. To date, to the author’s knowledge this assumption has only been justified experimentally – in that acceptable viscosity measurements have been achieved with both ‘ring down’ and forced oscillation viscometers. The transient terms can be reduced further experimentally by exciting only one mode prior to

observing the decay of the free oscillations (Mostert et al., 1989). This case will also be considered.

As mentioned above, one of the remaining challenges for vibrating-wire viscometers where the present investigation may be of particular value is to extend the range of viscosity that can be handled by a single vibrating instrument. The author has an interest in measuring hydrogen gas viscosity (Yusibani et al., 2013) and viscosities of thermosetting resins using vibrating wire techniques (if it is possible). Thermosetting resins have the challenge that the viscosity increases to values well beyond the current limits of this technique as the cure progresses. To the author's knowledge, explicit expressions for the transient terms for damped free vibration in a Stokes fluid have not been published in the literature. This article aims to fill this gap.

Nomenclature

a – Radius of wire

B – Magnetic flux intensity

D_0 – Damping due to internal friction

E – Young's modulus

G – Function defined by Eq. (53)

I – Second moment of area

L – Length of wire

p – Pressure in fluid

q_j – Function defined by Eq. (58)

Q_j – Function of the j th *eigenvalue* defined by Eq. (15)

s – Laplace transform variable

t – time

T – Tension in wire

U_j – Coefficient of *eigenfunction* for defining velocity

V – Output voltage

v_r – Velocity component in radial direction in fluid

v_θ – Velocity component in tangential direction in fluid

x – Coordinate position from one end of wire

y – Displacement of wire

y_j – Coefficient of *eigenfunction* for defining initial condition

Greek

β_1 – Function defined by Eq. (13a)

β_2 – Function defined by Eq. (13b)

θ – Angular coordinate

λ_j – *Eigenvalue* for mode j

μ – Absolute viscosity of fluid

ν – Kinematic viscosity of fluid

ρ – Density of fluid

ρ_w – Density of wire

ψ_j – *Eigenfunction*

χ – Stream function

ω_j – Angular frequency for mode j

2. Mathematical model

Figure 1 shows a geometrical representation of the domain for analysis. The wire is clamped at both ends and deflections (y) are assumed to be small.

Equation (1) is an expression of Newton's second law of motion per unit length of wire.

$$\pi a^2 \rho_w \frac{\partial^2 y}{\partial t^2} + D_0 \frac{\partial y}{\partial t} + EI \frac{\partial^4 y}{\partial x^4} - T \frac{\partial^2 y}{\partial x^2} + 2a \int_0^\pi \left(\left(p - 2\mu \frac{\partial v_r}{\partial r} \right) \cos \theta + \mu \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) \sin \theta \right) d\theta = 0 \quad (1)$$

The term involving the integral in Eq. (1) accounts for the viscous and pressure forces from the fluid acting on the wire surface. As mentioned above, usually this term is treated separately with an assumption about the motion of the wire which then translates into a term proportional to velocity and another proportional to acceleration (Stokes, 1922, Retsina et al., 1987, Hussey and Vujacic, 1967, Chen et al., 1976, Mostert et al., 1989). Here we will take a different approach and solve the equation for the motion of the wire (Eq. (1)) simultaneously with the equations of fluid mechanics. This way we need make no assumptions as to the form of the transient drag forces. The term involving D_0 represents the damping of the motion of the wire due to internal friction or any damping effects other than viscous drag (i.e. damping in a vacuum) (Padua et al., 1998, Woodfield et al., 2012).

Assuming clamped ends of the wire as shown in Fig. 1, the boundary conditions for Eq. (1) are:

$$y|_{x=0} = y|_{x=L} = \frac{\partial y}{\partial x} \Big|_{x=0} = \frac{\partial y}{\partial x} \Big|_{x=L} = 0 \quad (2)$$

Generally, the initial distribution of y can be expressed as a sum of *eigenfunctions* ($\psi_j(x)$):

$$y|_{t=0} = \sum_{j=1}^{\infty} y_j \psi_j(x)$$

To simplify the discussion we will assume that the initial conditions for the wire correspond to the wire being motionless (in a motionless fluid) and elastically deformed into the shape of the fundamental mode (first *eigenfunction* $\psi_1(x)$). This is given by Eq. (3) for $0 \leq x \leq L$.

$$\frac{\partial y}{\partial t} \Big|_{t=0} = 0 \quad (3a)$$

$$y|_{t=0} = y_1 \psi_1(x) \quad (3b)$$

where y_1 is a constant which is proportional to the magnitude of the initial displacement. Extension to the more general initial condition is straight forward.

The governing equations for fluid mechanics are given by continuity and the linearized Navier-Stokes equations for incompressible flow where convective transport of momentum has been neglected. In polar coordinates these are given by:

$$\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_\theta}{\partial \theta} = 0 \quad (4)$$

$$\frac{\partial p}{\partial r} = \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (rv_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right) - \rho \frac{\partial v_r}{\partial t} \quad (5)$$

$$\frac{1}{r} \frac{\partial p}{\partial \theta} = \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (rv_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right) - \rho \frac{\partial v_\theta}{\partial t} \quad (6)$$

The initial conditions are for the entire plane:

$$v_r|_{t=0} = v_\theta|_{t=0} = 0 \quad (7)$$

The boundary conditions far from the wire are for $(0 \leq \theta \leq 2\pi)$:

$$v_r|_{r \rightarrow \infty} = v_\theta|_{r \rightarrow \infty} = 0 \quad (8)$$

The boundary conditions corresponding to the wire surface are:

$$v_r|_{r=a} = U_1(t) \psi_1(x) \cos \theta \quad (9a)$$

$$v_\theta|_{r=a} = -U_1(t) \psi_1(x) \sin \theta \quad (9b)$$

where $U_1(t) \psi_1(x) = \partial y / \partial t$, i.e. the velocity of the wire at position x noting that it is vibrating in its fundamental mode. Notice here that to make the problem manageable, rather than specifying the conditions given by Eq. (9a, 9b) at the actual wire surface (which is moving) we have specified the velocities at the mean position of the surface. This same approach was used by Stokes (1922) and Chen et. al (1976). Stokes (1922), in his paper on oscillating spheres argued that this kind of assumption is justifiable if the actual position of the surface is not far from $r = a$. Retsina et. al (1987) discussed this issue in terms of a Taylor series expansion applied to small deflections. Another point worth noting is that we have neglected any transport of fluid or momentum in the x -direction in Eqs. (4) to (6). This is justifiable because the wire typically has a large aspect ratio.

3. Mode shapes and eigenvalues

For practical implementation it is useful to give explicit forms for the *eigenfunctions*. Retsina et al (1987) gave equations for the symmetric modes. We will give the general result here. The initial shape of the wire $y = y_1 \psi_1(x)$ (i.e. Eq. 3b) satisfies the boundary conditions (Eq. (2)) and the following equation:

$$-EI \frac{d^4 y}{dx^4} + T \frac{d^2 y}{dx^2} = -\lambda_1 y_1 \psi_1(x) \quad (10)$$

The term on the right hand side of Eq. (10) is physically the distribution of the force per unit length needed to hold the wire in its initial mode shape (given by Eq. (3b)). λ_1 is the *eigenvalue* associated with the first mode. In this work for convenience the *eigenfunction* is ‘normalized’ in such a way that

$$\frac{1}{L} \int_0^L \psi_1(x) \psi_1(x) dx = 1 \quad (11)$$

The *eigenvalues* λ_j for boundary conditions in Eq. (2) are given by the roots of:

$$1 - \cosh \beta_1(\lambda) \cos \beta_2(\lambda) + \frac{(\beta_1^2(\lambda) - \beta_2^2(\lambda)) \sinh \beta_1(\lambda) \sin \beta_2(\lambda)}{2\beta_1(\lambda)\beta_2(\lambda)} = 0 \quad (12)$$

where:

$$\beta_1(\lambda) = \sqrt{\sqrt{\left(\frac{TL^2}{2EI}\right)^2} + \frac{L^4\lambda}{EI} + \frac{TL^2}{2EI}} \quad (13a)$$

$$\beta_2(\lambda) = \sqrt{\sqrt{\left(\frac{TL^2}{2EI}\right)^2} + \frac{L^4\lambda}{EI} - \frac{TL^2}{2EI}} \quad (13b)$$

The *eigenfunction* $\psi_j(x)$ is given by

$$\psi_j(x) = \frac{1}{Q_j} \left(C_1 e^{\beta_1(\lambda_j)x/L} + C_2 e^{-\beta_1(\lambda_j)x/L} + C_3 \cos\left(\beta_2(\lambda_j)\frac{x}{L}\right) + C_4 \sin\left(\beta_2(\lambda_j)\frac{x}{L}\right) \right) \quad (14)$$

where

$$C_1 = 1$$

$$C_2 = -1 - C_3$$

$$C_3 = 2 \left(\frac{\beta_1(\lambda_j)}{\beta_2(\lambda_j)} \sin(\beta_2(\lambda_j)) - \sinh(\beta_1(\lambda_j)) \right) \left/ \left(\cos(\beta_2(\lambda_j)) - e^{-\beta_1(\lambda_j)} - \frac{\beta_1(\lambda_j)}{\beta_2(\lambda_j)} \sin(\beta_2(\lambda_j)) \right) \right.$$

$$C_4 = -\frac{\beta_1(\lambda_j)}{\beta_2(\lambda_j)}(C_3 + 2)$$

$$Q_j^2 = \frac{1}{L} \int_0^L \left(C_1 e^{\beta_1(\lambda_j)x/L} + C_2 e^{-\beta_1(\lambda_j)x/L} + C_3 \cos\left(\beta_2(\lambda_j)\frac{x}{L}\right) + C_4 \sin\left(\beta_2(\lambda_j)\frac{x}{L}\right) \right)^2 dx \quad (15)$$

The factor involving Q_j in Eq. (14) has the purpose of ensuring that Eq. (11) is satisfied. Equation (15) is somewhat tedious to evaluate. Using complex numbers, Q_j can be determined a little more compactly in a computer program by recasting Eq. (15) into the form:

$$Q_j^2 = \frac{1}{L} \int_0^L \left(\sum_{k=1}^4 D_k e^{B_k(\lambda_j)x/L} \right)^2 dx \quad (16)$$

From Eq. (16), Q_j is given by:

$$Q_j^2 = \sum_{m=1}^4 \sum_{k=1}^4 D_m D_k E_j \quad (17)$$

where

$$E_j = \begin{cases} 1 & (B_m = -B_k) \\ \frac{e^{(B_m(\lambda_j) + B_k(\lambda_j))} - 1}{B_m(\lambda_j) + B_k(\lambda_j)} & (B_m \neq -B_k) \end{cases}$$

$$B_1(\lambda) = \beta_1(\lambda), \quad B_2(\lambda) = -\beta_1(\lambda), \quad B_3(\lambda) = i\beta_2(\lambda), \quad B_4(\lambda) = -i\beta_2(\lambda)$$

$$D_1 = C_1, \quad D_2 = C_2, \quad D_3 = \frac{1}{2}(C_3 - iC_4), \quad D_4 = \frac{1}{2}(C_3 + iC_4)$$

Solving Eq. (12), Fig. 2 shows the *eigenvalues* for the first four modes as a function of a dimensionless ratio of tension and bending forces. Two regimes are evident. For small tensions, the *eigenvalues* are independent of the tension, while for large tensions the *eigenvalues* increase in proportion with the increase in tension.

Figure 3 shows the *eigenfunctions* for the first two modes evaluated using Eq. (14). It is apparent that as the tension in the wire is increased, the *eigenfunctions* tend to become more sinusoidal in shape (except in the near vicinity to the boundaries).

4. Solution procedures

4.1 Integral and Laplace transforms

Ultimately we are looking for the velocity of the wire ($u_{\text{wire}}(x,t) = U_I(t)\psi_I(x)$) as a function of time. Keeping y as the unknown variable presents a problem when interpreting the boundary conditions for Eq. (9) which we have assumed can be applied at $r = a$. To overcome this ambiguity we can re-express Eq. (1) in terms of the wire velocity. After multiplying each term by the *eigenfunction* $\psi_I(x)$ and integrating from $x = 0$ to L , Eq. (1) becomes:

$$\pi a^2 \rho_w \frac{\partial U_1}{\partial t} + D_0 U_1 + \lambda_1 \left(y_1 + \int_0^t U_1 dt \right) + 2a \int_0^\pi \left(\left(\tilde{p} - 2\mu \frac{\partial \tilde{v}_r}{\partial r} \right) \Big|_a \cos \theta + \mu \left(\frac{\partial \tilde{v}_\theta}{\partial r} \right) \Big|_a \sin \theta \right) d\theta = 0 \quad (18)$$

where for example

$$\tilde{v}_\theta = \frac{1}{L} \int_0^L v_\theta \psi_1(x) dx \quad (19)$$

Note that at $r = a$, we have $\partial v_r / \partial \theta = v_\theta$ from Eq. (9) leading to the simplification of the coefficient of $\sin \theta$ in Eq. (18). Applying the same operation to Eqs (4) to (8) yields equations of identical form with each variable modified in the manner shown in Eq. (19). For example, Eq. (4) becomes:

$$\frac{\partial}{\partial r} (r \tilde{v}_r) + \frac{\partial \tilde{v}_\theta}{\partial \theta} = 0 \quad (20)$$

Equations (9a) and (9b) become:

$$\tilde{v}_r \Big|_{r=a} = U_1(t) \cos \theta \quad (21)$$

$$\tilde{v}_\theta \Big|_{r=a} = -U_1(t) \sin \theta \quad (22)$$

To satisfy the continuity equation (20) it is useful to define the stream function χ in terms of the transformed variables.

$$\tilde{v}_r = \frac{1}{r} \frac{\partial \chi}{\partial \theta} \quad (23)$$

$$\tilde{v}_\theta = -\frac{\partial \chi}{\partial r} \quad (24)$$

Next we apply the operation described by Eq. (19) to Eqs (5) and (6), and substitute (23) and (24) for the velocity components. Through differentiating we can eliminate $\tilde{p}$ to obtain:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\rho}{\mu} \frac{\partial}{\partial t} \right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \chi = 0 \quad (25)$$

The boundary conditions (Eq. (8) and Eq. (9)) become:

$$\frac{1}{a} \frac{\partial \chi}{\partial \theta} \Big|_a = U_1 \cos \theta \quad (26)$$

$$\frac{\partial \chi}{\partial r} \Big|_a = U_1 \sin \theta \quad (27)$$

$$\frac{\partial \chi}{\partial \theta} \Big|_{r \rightarrow \infty} = \frac{\partial \chi}{\partial r} \Big|_{r \rightarrow \infty} = 0 \quad (28)$$

We may set the initial condition for the whole domain:

$$\chi|_{t=0} = 0 \quad (29)$$

We can now proceed in a manner similar to Stokes (1922). By substitution, it is clear that Eq. (25) can be satisfied by

$$\chi = \chi_1 + \chi_2 \quad (30)$$

Where χ_1 and χ_2 satisfy the following equations:

$$\frac{\partial^2 \chi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \chi_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \chi_1}{\partial \theta^2} = 0 \quad (31)$$

$$\frac{\partial^2 \chi_2}{\partial r^2} + \frac{1}{r} \frac{\partial \chi_2}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \chi_2}{\partial \theta^2} - \frac{\rho}{\mu} \frac{\partial \chi_2}{\partial t} = 0 \quad (32)$$

The integral term in Eq. (18) can be re-expressed in terms of the stream function χ using Eq. (23). The individual terms become:

$$\frac{\partial \tilde{v}_r}{\partial r} \Big|_a = \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \chi}{\partial \theta} \right) \Big|_a = \frac{-1}{r^2} \frac{\partial \chi}{\partial \theta} \Big|_a + \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial \chi}{\partial r} \right) \Big|_a = \frac{-1}{a} U_1 \cos \theta + \frac{1}{a} \frac{\partial}{\partial \theta} (U_1 \sin \theta) = 0 \quad (33)$$

Integrating by parts we have:

$$\int_0^\pi \tilde{p} \cos \theta d\theta = [\tilde{p} \sin \theta]_0^\pi - \int_0^\pi \frac{\partial \tilde{p}}{\partial \theta} \sin \theta d\theta = - \int_0^\pi \frac{\partial \tilde{p}}{\partial \theta} \sin \theta d\theta \quad (34)$$

From Eq. (6) we have

$$\frac{1}{r} \frac{\partial \tilde{p}}{\partial \theta} = -\mu \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} - \frac{\rho}{\mu} \frac{\partial \chi}{\partial t} \right) \quad (35)$$

Substituting Eq. (35) into Eq. (30) and making use of Eq. (31) and (32) we have

$$\frac{\partial \tilde{p}}{\partial \theta} = \rho \frac{\partial}{\partial t} \left(r \frac{\partial \chi_1}{\partial r} \right) \quad (36)$$

$$\frac{\partial \tilde{v}_\theta}{\partial r} = -\frac{\partial^2 \chi}{\partial r^2} \quad (37)$$

Adding Eq. (31) and (32) gives

$$-\frac{\partial^2 \chi}{\partial r^2} = \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{\partial \chi}{\partial \theta} \right) - \frac{\rho}{\mu} \frac{\partial \chi_2}{\partial t} \quad (38)$$

Combining Eq. (37) and (38) and evaluating at $r = a$ gives:

$$\left. \frac{\partial \tilde{v}_\theta}{\partial r} \right|_a = -\frac{\rho}{\mu} \frac{\partial \chi_2}{\partial t} \quad (39)$$

Thus, after taking Laplace transforms Eq. (18) can be rewritten as:

$$\pi a^2 \rho_w s \bar{U}_1 + D_0 \bar{U}_1 + \lambda_1 \frac{y_1}{s} + \lambda_1 \frac{\bar{U}_1}{s} - 2as \int_0^\pi \left(a \left. \frac{\partial \bar{\chi}_1}{\partial r} \right|_a + \bar{\chi}_2|_a \right) \sin \theta d\theta = 0 \quad (40)$$

Here we have made use of the initial condition (Eq. (29)) with the added restriction that both χ_1 and χ_2 are zero at $t = 0$.

Let:

$$\int_0^\pi \bar{\chi}_1 \sin \theta d\theta = \bar{\bar{\chi}}_1 \quad (41)$$

Taking Laplace Transforms of Eq. (31) and (32) followed by the operation shown in Eq. (41) gives:

$$\frac{\partial^2 \bar{\bar{\chi}}_1}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\bar{\chi}}_1}{\partial r} - \frac{1}{r^2} \bar{\bar{\chi}}_1 = 0 \quad (42)$$

$$\frac{\partial^2 \bar{\bar{\chi}}_2}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\bar{\chi}}_2}{\partial r} - \frac{1}{r^2} \bar{\bar{\chi}}_2 = \frac{\rho}{\mu} s \bar{\bar{\chi}}_2 \quad (43)$$

The boundary conditions become

$$\frac{1}{a} (\bar{\bar{\chi}}_1 + \bar{\bar{\chi}}_2) \Big|_a = \frac{\pi}{2} \bar{U}_1 \quad (44)$$

$$\frac{\partial}{\partial r} (\bar{\bar{\chi}}_1 + \bar{\bar{\chi}}_2) \Big|_a = \frac{\pi}{2} \bar{U}_1 \quad (45)$$

$$\bar{\bar{\chi}} \Big|_{r \rightarrow \infty} = \frac{\partial \bar{\bar{\chi}}}{\partial r} \Big|_{r \rightarrow \infty} = 0 \quad (46)$$

Equations (42) and (43) have general solutions:

$$\bar{\bar{\chi}}_1 = \frac{A}{r} + Br \quad (47)$$

$$\bar{\bar{\chi}}_2 = CI_1 \left(\sqrt{\frac{\rho s}{\mu}} r \right) + DK_1 \left(\sqrt{\frac{\rho s}{\mu}} r \right) \quad (48)$$

where I_1 and K_1 are modified Bessel functions. Equation (46) can be satisfied if $B = C = 0$.

Applying boundary conditions (44) and (45) gives:

$$D = \frac{-\pi \bar{U}_1}{\sqrt{\frac{\rho s}{\mu}} K_0 \left(\sqrt{\frac{\rho s}{\mu}} a \right)} \quad (49)$$

$$A = \frac{\pi a^2}{2} \bar{U}_1 + \frac{\pi \bar{U}_1 a}{\sqrt{\frac{\rho s}{\mu}} K_0 \left(\sqrt{\frac{\rho s}{\mu}} a \right)} K_1 \left(\sqrt{\frac{\rho s}{\mu}} a \right) \quad (50)$$

Equation (40) can be rewritten as:

$$\left(\pi a^2 \rho_w s + D_0 + \frac{\lambda_1}{s} \right) \bar{U}_1 + \frac{\lambda_1 y_1}{s} = 2 \rho a^2 s \frac{\partial \bar{\bar{\chi}}_1}{\partial r} \Big|_a + 2 \rho a s \bar{\bar{\chi}}_2 \Big|_a \quad (51)$$

Using Equations (47) – (50) to evaluate the terms in Eq. (51) involving the stream function we obtain the transient velocity in Laplace domain:

$$\bar{U}_1 = \frac{-\lambda_1 y_1}{\pi a^2 (\rho_w + \rho) s^2 + \left(D_0 + 4\pi\mu G \left(\sqrt{\frac{\rho s}{\mu}} a \right) \right) s + \lambda_1} \quad (52)$$

where

$$G(z) = z K_1(z) / K_0(z) \quad (53)$$

4.2 Final solution

The inverse Laplace transform of Eq. (52) can be found using the method of residues (e.g. as described by Carslaw and Jaeger (2003)). Finally we obtain:

$$U_1 = 2y_1\lambda_1 \text{Real} \left\{ \frac{-e^{i\omega_1 t}}{b((i - \Delta_1)\omega_1)} \right\} e^{-\Delta_1\omega_1 t} + \frac{y_1\lambda_1}{\pi} \int_0^\infty q_1(R) e^{-Rt} dR \quad (54)$$

where

$$b(z) = 2\pi a^2 (\rho_w + \rho) z + D_0 + 4\pi\mu G \left(\sqrt{\frac{\rho z}{\mu}} a \right) + 2\pi\mu \sqrt{\frac{\rho z}{\mu}} a G' \left(\sqrt{\frac{\rho z}{\mu}} a \right) \quad (55)$$

$$G'(z) = z \left(\frac{K_1(z)}{K_0(z)} \right)^2 - z \quad (56)$$

Δ_1 and ω_1 are positive real numbers for an underdamped system determined by finding the complex root ($z = -(\Delta_1 - i)\omega_1$) of

$$\pi a^2 (\rho_w + \rho) z^2 + \left(D_0 + 4\pi\mu G \left(\sqrt{\frac{\rho z}{\mu}} a \right) \right) z + \lambda_1 = 0 \quad (57)$$

It is worth noting here that the roots of Eq. (57) can be found easily by guessing initial values for Δ and ω in order to estimate $G()$ and then obtaining new estimates for z iteratively by applying the quadratic formula. Once one root is found other root of Eq. (57) is the complex conjugate of z (i.e. $z = -(\Delta \pm i)\omega$). The integral term in Eq. (54) represents the transient start up to the vibration and is one of the main contributions of this study. It can be integrated numerically. $q_j(R)$ is given by Eq. (58):

$$q_j(R) = \text{Imaginary} \left\{ \left(\pi a^2 (\rho_w + \rho) R^2 - \left(D_0 + 4\pi\mu G \left(\sqrt{\frac{\rho R}{\mu}} a i \right) \right) R + \lambda_j \right)^{-1} \right\} \quad (58)$$

4.3 Asymptotic Solutions

It is of interest to investigate the behaviour of Eq. (52) in the limits of large and small s (i.e. small and large time). For large values of s , it is appropriate to rewrite Eq. (52) making use of the Maclaurin series expansion of $1/(1-z)$ and the binomial expansion giving:

$$\bar{U}_1 \approx \frac{-y_1 \lambda_1}{\pi a^2 (\rho_w + \rho) s^2} + \frac{y_1 \lambda_1 D_0}{(\pi a^2 (\rho_w + \rho))^2 s^3} + \frac{y_1 \lambda_1 4\pi\mu G \left(\sqrt{\frac{\rho s}{\mu}} a \right)}{(\pi a^2 (\rho_w + \rho))^2 s^3} + \frac{y_1 \lambda_1^2}{(\pi a^2 (\rho_w + \rho))^2 s^4} + \dots \quad (59)$$

The function $G(z)$ defined by Eq. (53) can be approximated for large z using the asymptotic expansions for the Bessel functions given by Abramowitz and Stegan (1972) to yield after simplification:

$$G(z) \approx z + \frac{1}{2} - \frac{1}{8z} + \frac{237}{1536z^2} + \dots \quad (60)$$

Making use of Eq. (60) in Eq. (59) we obtain for large s :

$$\bar{U}_1 \approx \frac{-y_1 \lambda_1}{\pi a^2 (\rho_w + \rho) s^2} + \frac{y_1 \lambda_1 D_0}{(\pi a^2 (\rho_w + \rho))^2 s^3} + \frac{y_1 \lambda_1 4\pi\mu}{(\pi a^2 (\rho_w + \rho))^2} \left(\frac{\sqrt{\rho a}}{\sqrt{\mu} s^{3/2}} + \frac{1}{2s^3} - \frac{\sqrt{\mu}}{8\sqrt{\rho a} s^{7/2}} + \dots \right) + \dots \quad (61)$$

Applying the inverse Laplace transform to Eq. (61) gives for small values of t :

$$U_1(t) \approx \frac{-y_1 \lambda_1 t}{\pi a^2 (\rho_w + \rho)} + \frac{y_1 \lambda_1 D_0 t^2}{2(\pi a^2 (\rho_w + \rho))^2} + \frac{y_1 \lambda_1 4\pi\mu}{(\pi a^2 (\rho_w + \rho))^2} \left(\frac{4\sqrt{\rho a} t^{3/2}}{3\sqrt{\mu\pi}} + \frac{t^2}{4} - \frac{\sqrt{\mu} t^{5/2}}{15\sqrt{\rho\pi a}} + \dots \right) + \dots \quad (62)$$

It is interesting that the leading term in Eq. (62) does not involve the viscosity, indicating that at very small values of time, inertial effects may dominate the motion of the cylinder.

For small s , the integral to infinity around the branch may be neglected. This gives a result for large values of t as the component of the inverse Laplace transform due to the poles:

$$U_1(t) \approx 2y_1 \lambda_1 \text{Real} \left\{ \frac{-e^{i\omega_1 t}}{b((i - \Delta_1)\omega_1)} \right\} e^{-\Delta_1 \omega_1 t} \quad (63)$$

4.4 Multiple modes

The solution given by Eq. (54) is for one mode but extension to multiple modes is straight forward. The general initial condition is given by:

$$y|_{t=0} = \sum_{j=1}^{\infty} y_j \psi_j(x) \quad (64)$$

In such a case the velocity of the wire would be:

$$u_{wire}(x, t) = \sum_{j=1}^{\infty} U_j(t) \psi_j(x) \quad (65)$$

where

$$U_j(t) = 2y_j \lambda_j \text{Real} \left\{ \frac{-e^{i\omega_j t}}{b((i - \Delta_j)\omega_j)} \right\} e^{-\Delta_j \omega_j t} + \frac{y_j \lambda_j}{\pi} \int_0^{\infty} q_j(R) e^{-Rt} dR \quad (66)$$

A practically realizable example for Eq. (64) would be a constant electrical current I_0 passing through a wire in a uniform magnetic field of intensity B in order to establish the initial condition. For this case, the coefficients y_j in Eq. (64) are given by:

$$y_j = \frac{-BI_0}{\lambda_j} \left(\frac{1}{L} \int_0^L \psi_j(x) dx \right) \quad (67)$$

Where the integral can be determined from Eq. (14) as:

$$\frac{1}{L} \int_0^L \psi_j(x) dx = \frac{1}{Q_j} \left(\frac{1}{\beta_1} (C_1(e^{\beta_1} - 1) - C_2(e^{-\beta_1} - 1)) + \frac{1}{\beta_2} (C_3 \sin \beta_2 + C_4(1 - \cos \beta_2)) \right) \quad (68)$$

After switching off the current through the wire, the measured electromotive force (E.M.F.) during the decay will be:

$$V = \int_0^L B \sum_{j=1}^{\infty} U_j(t) \psi_j(x) dx = BL \sum_{j=1}^{\infty} U_j \left(\frac{1}{L} \int_0^L \psi_j(x) dx \right) \quad (69)$$

4.5 Case of initially forced oscillation

Many practical instruments initially force the wire to oscillate at (or very near) the natural frequency for a particular mode and then switch of the forcing function in order to obtain a decaying signal for analysis (Mostert et al. 1989, Diller and Van der Gulik, 1991, Dehestru et al., 2011). Therefore it is also of interest to investigate the significance of neglected transient

terms for such cases. The driving force per unit length that is responsible for initially maintaining the oscillation in a uniform magnetic field may be defined as:

$$F_{drive}(x,t) = BI_c \sin(\omega_f t) = \sum_{j=0}^{\infty} F_j \psi_j(x) \sin(\omega_f t) \quad (70)$$

where ω_f is the forcing frequency and I_c is the amplitude of the oscillating current. The coefficients of the driving function are given by:

$$F_j = BI_c \left(\frac{1}{L} \int_0^L \psi_j(x) dx \right) \quad (71)$$

The solution for the transient decay can be found using a similar approach to that described above. The final result arises through combining a zero initial velocity solution with the steady oscillation solution via the principle of superposition. In this case the coefficient of the *eigenfunction* for the velocity of the wire is given by:

$$U_j = 2F_j \text{Real} \left\{ g((i - \Delta_j) \omega_j) e^{i \omega_j t} \right\} e^{-\Delta_j \omega_j t} + \frac{F_j}{\pi} \int_0^{\infty} w_j(R) e^{-Rt} dR \quad (72)$$

where

$$g(z) = \frac{-z \omega_f / (z^2 + \omega_f^2)}{2\pi a^2 (\rho_w + \rho) z + D_0 + 4\pi \mu G \left(\sqrt{\frac{\rho z}{\mu}} a \right) + 2\pi \mu \sqrt{\frac{\rho z}{\mu}} a G' \left(\sqrt{\frac{\rho z}{\mu}} a \right)} \quad (73)$$

$$w_j(R) = \text{Imaginary} \left\{ \frac{-R \omega_f / (R^2 + \omega_f^2)}{\pi a^2 (\rho_w + \rho) R^2 - \left(D_0 + 4\pi \mu G \left(\sqrt{\frac{\rho R}{\mu}} a i \right) \right) R + \lambda_j} \right\} \quad (74)$$

As before, Δ_j and ω_j are determined using Eq. (57) and the measured E.M.F. is given by Eq. (69). In this case $t = 0$ corresponds to switching off the driving electrical current at the start of an oscillation period where the wire velocity is instantaneously zero.

5. Simulation results

To illustrate the characteristics of the present solution it is helpful to select some practical examples. Mostert et al. (1989) recommend that practical viscometers use wires of diameters in the range from 10 to 500 μm with lengths of a few centimetres. Fig. 4 shows simulation results

using Eq. (69) with Eq. (66) or (72). In addition to the simulation results, a non-linear least squares fitting procedure (Woodfield et al., 2008) is employed to fit Eq. (75) to the first 15 complete oscillations. The fit is also shown for the simulations in Fig. 4 using a dash-dot line.

$$V = V_0 e^{-\Delta \omega t} \sin(\omega t + \phi) \quad (75)$$

The log decrement Δ and frequency ω obtained from fitting Eq. (75) to the simulated results were then used to determine the viscosity through the model derived by Retsina et al. (1987) (Note: there is a misprint in one of the equations in (Retsina et al., 1987). See, for example, (Assael and Mylona, 2013) for the corrected version). The viscosity was also determined using the classical Stokes model (e.g. (Tough et al., (1964)) which assumes the coefficients of the added mass and damping terms are identical to those of the steady oscillation terms (Stokes, 1922). Comparing the determined viscosities with the actual viscosity used for the present simulation gives an indication of the differences between the present model and the other more conventional treatments given in the literature. In Figs 4(a), 4(b) and 4(c) the initial condition was established using a simulated direct current (DC) electrical current of 1 mA. For the purposes of simulation the magnitude of the initial current only affects the magnitude of the initial displacement and not the determined viscosity. For the cases in Fig. 4 the initial displacement (with a current of 1 mA) was the order of 1 % of the diameter of the wire (In Eq. (64) y_1 is 1.3% of $2a$).

In Fig. 4(a) Eqs. (69) and (63) were used for the simulation with only one mode (and the transient term involving the integral to infinity in Eq. (54) neglected). Without the transient term, the assumptions used in the present derivation correspond to those of Retsina et al. (1987). Therefore, as expected, the viscosity obtained using Retsina's model and Eq. (75) was identical to that used in the simulation. The steady-oscillation Stokes model gave an overestimate of the viscosity by 3.5 % in this case. It should be noted here that if the log decrement was smaller, the accuracy of the steady Stokes model would improve.

Figure 4(b) shows the same simulation as that in Fig. 4(a) except that the transient term in Eq. (54) is included. In this case the model by Retsina et al (1987) underestimates the viscosity by about 0.5%. The largest deviation of the fitted equation (Eq. (75)) to the simulated results is at $t = 0$ where it is of the order of 1% of the maximum output voltage in Fig. 4(b). It should be mentioned here that the fit using Retsina's model would be improved if the first few oscillations were not included in the analysis.

Figure 4(c) shows the effect of including the first ten modes in the simulation instead of only the fundamental mode. The fit itself (which assumes a single oscillation mode) was somewhat poorer than in Fig. 4(b) with deviations of up to about 4%. The accuracy of the viscosity

prediction however is similar for both Fig. 4(b) and 4(c). This indicates that the single mode fitting procedure is effective in eliminating higher frequency modes. Figure 4(d) shows that the commonly-used procedure of starting with an initial oscillation is effective both in reducing the effects of other modes and the transient term. For this simulation the oscillating current had an amplitude $I_c = 0.1$ mA. The simulation in Fig. 4(d) was done using Eqs. (69) and (72).

All of the simulations in Fig. 4 were done assuming the tension in the wire corresponded to 10% of the yield strength for tungsten. Figure 5 shows the effect of changing the tension in the wire while keeping all other parameters the same as those used for Fig. 4(b). In Fig. 5(a) the wire has no tension while in Fig. 5(c) the tension is at 50% of the yield strength of the wire. For the same initial current (1 mA) the amplitude of the measured signal is smaller for the case of greater tension. The important observation that can be made from Fig. 5 is the improved reliability of Retsina's model for the case of higher frequency oscillations.

Figures 4 and 5 illustrate cases where a 50 μ m diameter could quite successfully be used to measure the viscosity of water. If an attempt was made to use the same instrument to analyse a fluid of significantly higher viscosity problems would be encountered. This is illustrated in Fig. 6(a) where the simulation is applied to a fluid with a viscosity of 200 mPa·s. In this figure the effect of the transient term is very significant as is evident by the large differences between the simulated output voltages for the cases with and without the transient term. One of the constraints proposed by Retsina et al. (1987) was that the log decrement Δ must be much smaller than one. Obviously this is not satisfied for Fig. 6(a) (where in fact $\Delta = 2.0$). For the case of a 200 μ m wire shown in Fig. 6(b) the relative importance of the transient term is much smaller than for the case of a 50 μ m wire. This is consistent with the observation that viscometers designed for higher viscosity fluids tend to use heavier wires (Assael and Mylona, 2013, Caetano et al., 2005, Etchart et al., 2007, Harrison et al., 2007). In principle, the present analytical solution could be used to determine the viscosity from data (such as is illustrated in Fig. 6(a)) which would be unusable based on other available analytical solutions.

Figure 7 gives an example of the behaviour of the asymptotic solutions (Eqs. (62) and (63)) relative to full solution for a single mode (Eq. (54)). For this simulation, the wire properties and geometry were the same as those used in the simulations for Fig. 4. In order to clearly show the asymptotes for small and large values of time on the same graph, the viscosity of the fluid was increased to 20 mPa·s. For this particular set of conditions it is evident that asymptotic solution for small values of time (Eq. (62)) gives a reliable prediction only during the very early stages of the motion prior to the first oscillation peak. The leading term in Eq. (62) may be suitable for determining the initial acceleration of the wire. For large values of time, Eq. (63) captures the

oscillatory motion of the wire well but loses accuracy near time zero, particularly if the viscosity of the fluid is large (Fig. 6).

6. Conclusions

An exact analytical solution for the transient decay of the motion of a vibrating cylinder in a stationary Newtonian fluid was derived using Laplace transforms. The solution has the merit that full expressions for the transient start-up terms are given. These terms become increasingly important and may dominate the motion of a wire in a high viscosity fluid. The solution has immediate practical applications in analytical treatments of 'ring down' type vibrating-wire viscometers.

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Fig. 1. Geometrical description of vibrating wire

Fig. 2. Dimensionless *eigenvalues* as a function of ratio of tension to bending forces

Fig. 3. Mode shapes for the fundamental and first harmonics

Fig. 4. Simulation results showing expected output voltage from a 50 μm diameter tungsten wire immersed in water for different model configurations

Fig. 5. Effect of wire tension

Fig. 6. Expected response in a more viscous fluid ($\mu = 200$ mPa·s)

Fig. 7. Asymptotic solutions ($\mu = 20 \text{ mPa}\cdot\text{s}$, single mode)