 Preferential Multi-Context Systems

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Abstract

Multi-context systems (MCS) presented by Brewka and Eiter can be considered as a promising way to interlink decentralized and heterogeneous knowledge contexts. In this paper, we propose preferential multi-context systems (PMCS), which provide a framework for incorporating a total preorder relation over contexts in a multi-context system. In a given PMCS, its contexts are divided into several parts according to the total preorder relation over them, moreover, only information flows from a context to ones of the same part or less preferred parts are allowed to occur. As such, the first \( l \) preferred parts of an PMCS always fully capture the information exchange between contexts of these parts, and then compose another meaningful PMCS, termed the \( l \)-section of that PMCS. We generalize the equilibrium semantics for an MCS to the (maximal) \( l \)-equilibrium which represents belief states at least acceptable for the \( l \)-section of an PMCS. We also investigate inconsistency analysis in PMCS and related computational complexity issues.

Keywords : Preferential multi-context systems, equilibrium, inconsistency diagnosis, inconsistency explanation, maximal consistent section

1 Introduction

Many (if not all) real-world applications of sharing and reasoning knowledge are characterized by heterogeneous contexts, especially with the advent of the world wide web. Research in representing contexts and information flow between contexts has gained much attention recently in artificial intelligence [11, 4, 7, 8, 5, 13] as well as in applications such as requirements engineering [10, 15, 14].

Instead of finding a universal knowledge representation for all contexts, it has been increasingly recognized that it may be desirable to allow each context
to choose a suitable representation tool for its own to capture its knowledge precisely. For example, in some frameworks such as Viewpoints for eliciting and analyzing software requirements developers often encourage stakeholders to use their own familiar terms and notations to express their demands so as to elicit requirements as full as possible [10, 15]. Moreover, the heterogeneous nature of contexts representations may allow different monotonic or non-monotonic reasoning mechanisms to occur together in a given system. For example, as stated in [4], there is growing interest in combining ontologies based on description logics with non-monotonic formalisms in semantic web applications. However, the diversity of representations of contexts in such cases brings some important challenges to accessing each individual context as well as to interlinking these contexts [5].

Nonmonotonic multi-context systems presented by Brewka and Eiter [4] can be considered as a promising way to deal with these challenges [5]. Instead of attempting to translate all contexts with different formalisms into a unifying formalism, they leave the logics of contexts untouched and interlink contexts by modeling the inter-contextual information exchange in a uniform way. To be more precise, information flow among contexts is articulated by so-called bridge rules in a declarative way. Similar to logical programming rules, each bridge rule consists of two parts, the head of the rule and the body of the rule (possibly empty). More importantly, each bridge rule allows access to other contexts in its body. This makes it capable of adding information represented by its head to a context by exchanging information with other contexts. In semantics, several equilibria representing acceptable belief states for multi-context systems are also given by Brewka and Eiter [4].

Multi-context systems can be viewed as the first step towards interlinking distributed and heterogeneous contexts effectively. The way they operating contextual knowledge bases is only limited to adding information to a context when the corresponding bridge rules are applicable [5]. To be more applicable to real-world applications, it is advisable to generalize multi-context systems from some perspectives. For example, Brewka et al have considerably generalized multi-context systems to managed multi-context systems (mMCS) by allowing flexible operations on context knowledge bases [5]. Essentially, managed multi-context systems focus on managed contexts, which are contexts together with possible operations on them.

Combining preferences and contexts is still an interesting issue in reasoning about contextual knowledge [3]. In particular, preferences on contexts have an important influence on information exchange between contexts and inter-contextual knowledge integration in many real-world applications. For example, it is intuitive to revise a less reliable knowledge base by accessing more reliable ones. But we cannot use information deriving from less reliable sources to revise more reliable knowledge bases in general case. In legal reasoning, consequences of applying a law to a case can be rebutted by that of applying another law with higher level when there is a conflict, and not vice versa. In such cases, it may be advisable to take into account preferences on contexts in characterizing inter-contextual information exchange in multi-context systems.
Moreover, taking into account the preference relation on contexts makes some subsets of more preferred contexts satisfying some given constraints more significant when the whole set of contexts does not satisfy the constraints. For example, in a multi-party negotiation, an agreement between the most important parties is preferred if it is difficult to achieve an agreement between all parties in many cases. In an incremental software development, only requirements with priorities higher than a given level are concerns of developers at a given stage.

To address these issues, we combine a multi-context system with a total preorder relation on its contexts to develop a preferential multi-context system (PMCS) in this paper. A preferential multi-context system is given in the form of a sequence of sets of contexts such that the location of a set signifies its preference level. Without loss of generality, we assume that the smaller of the location of a set is, the more preferred contexts in that set are. We call each set of contexts in that sequence a stratum. Moreover, we assume that information flow cannot be from less preferred strata to more preferred ones. That is, any bridge rule of a given context does not allow any access to other strictly less preferred contexts in its body. As such, the first several strata also compose a new preferential multi-context system such that all the contexts involved in it are strictly more preferred than ones out of it. We call such a new preferential multi-context system a section of that system. We are interested in all sections as well as the whole preferential multi-context system, and then propose $l$-equilibria to represent belief sets acceptable for at least contexts in the first $l$ strata. In particular, the maximal consistent section describes a maximal section that has an equilibrium. Actually, it plays an important role in inconsistency analysis in a given preferential multi-context system, because it can be considered as maximally reliable part of that preferential multi-context system. We are more interested in finding diagnoses and inconsistency explanations compatible with maximal consistent section instead of all ones. Finally, we discuss computational complexity issues.

The rest of this paper is organized as follows. We give a brief introduction to multi-context systems in Section 2. We propose preferential multi-context systems in Section 3. In section 4, we discuss inconsistency analysis in preferential multi-context systems. We discuss complexity issues in Section 5. In section 6 we compare our work with some closely related work. Finally we conclude this paper in Section 7.

2 Preliminaries

In this section, we review the details of the definitions of multi-context systems presented by Brewka and Eiter [4] and inconsistency analysis in multi-context systems presented in [8]. The material is largely taken from [4] and [8].

The goal of multi-context systems is to combine arbitrary monotonic and nonmonotonic logics. Here a logic $L$ is referred to as a triple $(KB_L, BS_L, ACC_L)$, where $KB_L$ is the set of well-formed knowledge bases of $L$, which characterizes
the syntax of \( L \); \( BS_L \) is the set of belief sets; and \( ACC_L : KB_L \rightarrow 2^{BS_L} \) is a function describing the semantics of the logic by assign to each knowledge base (a set of formulas) a set of acceptable sets of beliefs \([4]\).

**Definition 2.1** \([4]\) Let \( L = \{L_1, L_2, \cdots, L_n\} \) be a set of logics. A \( L_k \)-bridge rule over \( L \), \( 1 \leq k \leq n \), is of the form

\[(k : s) \leftarrow (r_1 : p_1), \cdots, (r_j : p_j), \text{not} \ (r_{j+1} : p_{j+1}), \cdots, \text{not} \ (r_m : p_m)\]

where \( 1 \leq r_i \leq n \), \( p_i \) is an element of some belief set of \( L_{r_i} \), and for each \( kb \in KB_{L_k} \), \( kb \cup \{s\} \in KB_{L_k} \).

Similar to logical programming rules, we call the left (resp. right) part of \( r \) the head (resp. body) of the bridge rule \( r \).

**Definition 2.2** \([4]\) A multi-context system \( M = (C_1, C_2, \cdots, C_n) \) consists of a collection of contexts \( C_i = (L_i, kb_i, br_i) \), where \( L_i = (KB_i, BS_i, ACC_i) \) is a logic, \( kb_i \in KB_i \) is a knowledge base, and \( br_i \) is a set of \( L_i \)-bridge rules over \( \{L_1, L_2, \cdots, L_n\} \).

A multi-context system \( M \) is finite if all knowledge bases \( kb_i \) and sets of bridge rules \( br_i \) are finite \([4]\).

Given a \( L_k \)-bridge rule \( r \), we use \( hd(r) \) to denote the head of \( r \). Further, let \( cnt^+(r) = \{C_i | 1 \leq i \leq j\} \) and \( cnt^-(r) = \{C_i | j + 1 \leq i \leq m\} \). Obviously, \( cnt(r) = cnt^+(r) \cup cnt^-(r) \) is exactly the set of contexts involved in the body of \( r \).

We use \( br_M \) to denote the set of all bridge rules in \( M \), i.e., \( br_M = \cup_{i=1}^n br_i \). For any set \( D \subseteq br_M \), we use \( heads(D) \) to denote the set of all the rules in \( D \) in unconditional form, i.e., \( heads(D) = \{hd(r) \leftarrow | r \in D\} \). Let \( R \) be a set of bridge rules, we use \( M[R] \) to denote the MCS obtained from \( M \) by replacing \( br_M \) with \( R \). For a set \( R \) of sets of bridge rules, we use \( \cup R \) to denote the union of all sets in \( R \).

A belief state for \( M = (C_1, C_2, \cdots, C_n) \) is a sequence \( S = (S_1, S_2, \cdots, S_n) \) such that each \( S_i \in BS_i \). A bridge rule \( r \) is applicable in a belief state \( S = (S_1, S_2, \cdots, S_n) \) iff for \( 1 \leq i \leq j \), \( p_i \in S_{r_i} \) and for \( j + 1 \leq i \leq m \), \( p_k \notin S_{r_i} \). We use \( app(br_i, S) \) to denote the set of all \( L_i \)-bridge rules that are applicable in belief state \( S \).

**Definition 2.3** \([4]\) A belief state \( S = (S_1, S_2, \cdots, S_n) \) of \( M \) is an equilibrium iff, for \( 1 \leq i \leq n \), \( S_i \in ACC_i(kb_i \cup \{head(r) | r \in app(br_i, S)\}) \).

Essentially, an equilibrium is a belief state which contains an acceptable belief set for each context, given the belief sets for other contexts \([4]\).

**Example 2.1** Let \( M_0 = (C_1, C_2, C_3) \) be an MCS, where \( L_1 \) is a propositional logic, whilst both \( L_2 \) and \( L_3 \) are ASP logics. Suppose that

- \( kb_1 = \{a, b\} \), \( br_1 = \{(1 : c) \leftarrow (2 : d), (3 : g)\} \);
Consider \( S = (\{a, b, c\}, \{d, e, p\}, \{f, g, q\}) \). Note that all bridge rules are applicable in \( S \), except \((3 : h) \leftarrow \text{not} (1 : c)\).

Evidently, we can check \( S \) is an equilibrium of \( M_0 \).

Note that it cannot be guaranteed that there exists an equilibrium for a given multi-context system. Inconsistency in an MCS is referred to as the lack of an equilibrium [8]. We use \( M \models \bot \) to denote that \( M \) is inconsistent, i.e., has no equilibrium. In this paper, we assume that every context to be consistent if no bridge rules apply, i.e., \( M[\emptyset] \not\models \bot \).

Example 2.2 Let \( M_1 = (C_1, C_2, C_3) \) be an MCS, where \( L_1 \) is a propositional logic, whilst both \( L_2 \) and \( L_3 \) are ASP logics. Suppose that

- \( kb_1 = \{a, b\} \), \( br_1 = \{r_1 = (1 : c) \leftarrow (2 : e)\} \);
- \( kb_2 = \{d \leftarrow e, e \leftarrow\} \), \( br_2 = \{r_2 = (2 : p) \leftarrow (1 : c)\} \);
- \( kb_3 = \{g \leftarrow, \bot \leftarrow q, \text{not} h\} \), \( br_3 = \{r_3 = (3 : q) \leftarrow (2 : p); r_4 = (3 : h) \leftarrow \text{not} (1 : a)\} \).

Note that all bridge rules are applicable, except \( r_4 \). The three applicable bridge rules in turn adds \( q \) to \( C_3 \), and then activates \( \bot \leftarrow q, \text{not} h \). So, \( M_1 \) has no equilibrium, i.e., \( M_1 \models \bot \).

To analyze inconsistency, inspired by debugging approaches used in the non-monotonic reasoning community, T. Eiter et al have introduced two notions of explaining inconsistency, i.e., diagnoses and inconsistency explanations for multi-context systems [8]. Roughly speaking, diagnoses provide a consistency-based formulation for explaining inconsistency, by finding a part of bridge rules which need to be changed (deactivated or added in unconditional form) to restore consistency in a multi-context system, whilst inconsistency explanations provide an entailment-based formulation for inconsistency, by identifying a part of bridge rules which is needed to cause inconsistency [8].

**Definition 2.4** [8] Given an MCS \( M \), a diagnosis of \( M \) is a pair \((D_1, D_2)\), \( D_1, D_2 \subseteq br_M \), s.t. \( M[br_M \setminus D_1 \cup \text{heads}(D_2)] \not\models \bot \). \( D^\pm(M) \) is the set of all such diagnosis.

Essentially, a diagnosis exactly captures a pair of sets of bridge rules such that inconsistency will disappear if we deactivate the rules in the first set, and add the rules in the second set in unconditional form [8].

**Definition 2.5** [8] \( D^m_m(M) \) is the set of all pointwise subset-minimal diagnoses of an MCS \( M \), where the pointwise subset relation \((D_1, D_2) \subseteq (D_1', D_2')\) holds iff \( D_1 \subseteq D_1' \) and \( D_2 \subseteq D_2' \).
Example 2.3 Consider $M_1$ again. Then
\[ D_{m}^{\pm}(M_1) = \{ (\{r_1\}, \emptyset), (\{r_2\}, \emptyset), (\{r_3\}, \emptyset), (\emptyset, \{r_4\}) \} \].
This means we need only to deactivate one of $r_1$, $r_2$, and $r_3$, or to add $r_4$ unconditionally, in order to restore consistency for $M_1$.

Definition 2.6 Given an MCS $M$, an inconsistency explanation of $M$ is a pair $(E_1, E_2)$ of sets $E_1, E_2 \in {\text{br}}_M$ of bridge rules s.t. for all $(R_1, R_2)$ where $E_1 \subseteq R_1 \subseteq {\text{br}}_M$ and $R_2 \subseteq {\text{br}}_M \setminus E_2$, it holds that $M[R_1 \cup \text{heads}(R_2)] \models \bot$. By $E^{\pm}(M)$ we denote the set of all inconsistency explanations of $M$, and by $E_{m}^{\pm}(M)$ the set of all pointwise subset-minimal ones.

Essentially, an inconsistency explanation captures a pair of sets of bridge rules such that the rules in the first set cause an inconsistency relevant to the MCS, and this inconsistency cannot be resolved by adding bridge rules unconditionally, unless we use at least one bridge rule in the second set.

Example 2.4 Consider $M_1$ again. Then
\[ E_{m}^{\pm}(M_1) = \{ (\{r_1, r_2, r_3\}, \{r_4\}) \} \].
This means that the inconsistency in $M_1$ is caused by $r_1$, $r_2$, and $r_3$ together, moreover, it can be resolved by adding $r_4$ unconditionally.

Note that both addition and removal of knowledge can prevent inconsistency in nonmonotonic reasoning. So, a diagnosis consists of two sets of bridge rules including the set of bridge rules to be removed and that to be added unconditionally. As pointed out in [8], for scenarios where removal of bridge rules is preferred to unconditional addition of rules, we may focus on diagnoses of the form $(D_1, \emptyset)$ only.

Definition 2.7 Given an MCS $M$, an $s$-diagnosis of $M$ is a set $D \subseteq {\text{br}}_M$ s.t. $M[{\text{br}}_M \setminus D] \not\models \bot$. The set of all $s$-diagnoses (resp., $\subseteq$-minimal $s$-diagnoses) is $D^{-}(M)$ (resp., $D_{m}^{-}(M)$).

Similarly, we need only focus on inconsistency explanations in form of $(E_1, {\text{br}}_M)$ if adding rules unconditionally is less preferred.

Definition 2.8 Given an MCS $M$, an $s$-inconsistency explanation of $M$ is a set $E \subseteq {\text{br}}_M$ s.t. each $R$ where $E \subseteq R \subseteq {\text{br}}_M$, satisfies $M[R] \models \bot$. The set of all $s$-inconsistency explanations (resp., $\subseteq$-minimal $s$-inconsistency explanations) is $E^{+}(M)$ (resp., $E_{m}^{+}(M)$).

Example 2.5 Consider $M_1$ again. Then
\[ D_{m}^{-}(M_1) = \{ \{r_1\}, \{r_2\}, \{r_3\} \}, \ E_{m}^{+}(M_1) = \{ \{r_1, r_2, r_3\} \} \].

More interestingly, Eiter et al have obtained the following duality relation between diagnoses and inconsistency explanations:
Theorem 2.1 Given an inconsistent MCS $M$,
\[
\bigcup D_m^\pm(M) = \bigcup E_m^\pm(M),
\]
and
\[
\bigcup D_m^-(M) = \bigcup E_m^+(M).
\]

This duality theorem shows that the unions of all minimal diagnoses and all inconsistency explanations coincide, i.e., diagnoses and inconsistency explanations represent dual aspects of inconsistency in an MCS \[8\].

3 Preferential Multi-context Systems

In this section we formally introduce a class of MCSs that allows us to consider preference information on contexts, called preferential multi-context systems, or simply PMCSs. As explained in the introduction, the motivation for such MCSs is that in many practical applications, it is often the case that some context has higher priority over another context. For example, the ontology SNOWMED CT (a context) will have higher priority over Wikipedia (another context) for medical doctors. In the setting of MCSs, an PMCS $P$ is a pair $(M, \leq_s)$ such that the following conditions are satisfied:

1. $M$ is an MCS that has a splitting $M = \bigcup_{i=1}^m M_i$.
2. $\leq_s$ is a total preorder\(^1\) on the set $\{M_1, \ldots, M_m\}$.

Recall that $M = \bigcup_{i=1}^m M_i$ is a splitting for $M$ if $M_i \neq \emptyset$ for all $i$ and $M_i \cap M_j = \emptyset$ for all $i \neq j$.

Informally, $M_i \leq_s M_j$ means that a context $C$ in $M_i$ is always preferred to a context $C'$ in $M_j$. We assume that the smaller a subscript $i$ is, the more preferred $M_i$ is. Then we use $\langle M_1, \ldots, M_m \rangle$ instead of $\{M_1, \ldots, M_m\}$ from now on.

In an PMCS, preference information controls the information flow from one context to another context. Specifically, a context can be impacted only by more or equally preferred ones. This notion is formally defined as follows.

Definition 3.1 Let $\leq_s$ be a total preorder relation on the set of contexts $M = \langle M_1, M_2, \ldots, M_m \rangle$.

1. The set $br_i$ of bridge rules of $C_i \in M_i$ is compatible with the preorder relation $\leq_s$ on $M$ if for all $r \in br_i$, $cnt(r) \cap M_j = \emptyset$ for all $j > i$.

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\(^1\) A binary relation $\leq$ on some set $A$ is a total preorder relation if it is reflexive, transitive, and total, i.e., for all $a, b, c \in A$, we have that:

1. $a \leq a$ (reflexivity),
2. if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity),
3. $a \leq b$ or $b \leq a$ (totality).
(2) The set \( br_M \) of bridge rules of \( M \) is compatible with the preorder relation \( \leq_s \) on \( M \) if \( br_i \) is compatible with \( \leq_s \) for all \( 1 \leq i \leq n \).

Essentially, the compatibility of \( br_i \) with \( \leq_s \) implies that only information exchange between \( C_i \) with some \( C_k \)'s satisfying \( C_k \leq C_i \) for each \( k \) may activate possible change of \( kb_i \) in \( C_i \).

Given an MCS \( M \) and a total preorder relation \( \leq_s \) on contexts in \( M \), we say that \( M \) is compatible with \( \leq_s \) iff \( br_M \) is compatible with \( \leq_s \).

**Definition 3.2 (Preferential multi-context system)** A preferential multi-context system (PMCS) is a pair \( (M, \leq_s) \), where \( M \) is an MCS, and \( \leq_s \) is a total preorder relation on contexts in \( M \) such that \( M \) is compatible with \( \leq_s \).

An PMCS \( (M, \leq_s) \) is represented in the form of a sequence \( (M_1, M_2, \ldots, M_m) \) such that for \( C_i, C_j \in M, C_i \leq_s C_j \) iff for some \( t_i, t_j : C_i \in M_{t_i}, C_j \in M_{t_j} \) and \( t_i \leq t_j \). In particular, we may consider an MCS \( M \) as a special PMCS \( (M, \emptyset) \), which contains only one stratum, i.e., \( \langle M \rangle \).

Essentially, preferential multi-context systems take into account the impact of preference relation over contexts on inter-contextual information exchange. Only information flow from a context to equally or less preferred ones are allowed to occur in preferential multi-context systems.

Let \( (M, \leq_s) = \langle M_1, M_2, \ldots, M_m \rangle \) be an PMCS. Then the \( i \)-cut of \( (M, \leq_s) \) for each \( 1 \leq i \leq m \), denoted \( M(i) \), is defined as \( M(i) = \bigcup_{k=1}^{i} M_k \). Correspondingly, we call \( M_{i+1} = \langle M_1, M_2, \ldots, M_i \rangle \) the \( i \)-section of \( (M, \leq_s) \). Note that the compatibility of \( M \) and \( \leq_s \) ensures that each \( i \)-section of \( (M, \leq_s) \) is also an PMCS. Correspondingly, each \( i \)-cut of \( (M, \leq_s) \) is an MCS. Informally speaking, given an PMCS, the \( i \)-section is exactly the PMCS consisting of the first \( i \) strata in \( (M, \leq_s) \), in which all the contexts are preferred to ones in \( M_{i+1} \) for each \( 1 \leq i \leq m-1 \). This implies that the \( i \)-section of an PMCS exactly capture the inter-contextual information exchange between contexts preferred to ones in \( M_{i+1} \).

A belief state for \( (M_1, M_2, \ldots, M_m) \) is a sequence \( S = \langle S_1, S_2, \ldots, S_m \rangle \) such that \( S_1 \uplus \ldots \uplus S_i \) is a belief state of \( M(i) \) for all \( 1 \leq i \leq m \), where \( \uplus \) is a concatenation operator. In particular, we use \( \uplus S \) to denote \( S_1 \uplus \ldots \uplus S_m \).

**Definition 3.3** A belief state \( S = \langle S_1, S_2, \ldots, S_m \rangle \) of \( (M, \leq_s) \) is an equilibrium of \( (M, \leq_s) \) iff \( \uplus S \) is an equilibrium of \( M \).

**Example 3.1** Consider an PMCS \( (M_2, \leq_s) = \langle (C_1, C_2), (C_3), (C_4, C_5) \rangle \), where \( L_1 \) and \( L_2 \) are propositional logics, and others are ASP logics. Suppose that

- \( kb_1 = \{ a \}, br_1 = \{ r_{11} = (1 : c) \leftarrow (2 : b) \} \);
- \( kb_2 = \{ b \}, br_2 = \{ r_{21} = (2 : d) \leftarrow (1 : a) \} \);
- \( kb_3 = \{ e \leftarrow f \}, br_3 = \{ r_{31} = (3 : f) \leftarrow (1 : c), r_{32} = (3 : d) \leftarrow \text{not} \ (2 : b) \} \);
Consider $S = \langle \{(a, c), \{b, d\}\}, \{(e, f)\}, \{(h, g), \{p, q\}\} \rangle$. Then all bridge rules are applicable in $S$ except $r_{32}$. Moreover, it is easy to check that $S$ is an equilibrium of $(M_2, \leq_s)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Information flow in $(M_2, \leq_s)$}
\end{figure}

On the other hand, we can use a directed graph $G = (V, E)$ to illustrate the information flow in a (preferential) multi-context system $M = (C_1, \ldots, C_n)$, where $V = \{C_1, \ldots, C_n\}$, and $(C_i, C_j) \in E$ if $\exists r \in br_j$ s.t. $C_i \in cnt(r)$. For example, the information flow in $(M_2, \leq_s)$ is illustrated in Figure 1. Note that in such an information flow graph, there is at most one edge between any two contexts belonging to different strata, moreover, such an edge must be from a preferred context to another context.

As mentioned in [5], inter-contextual information exchange among decentralized and heterogeneous contexts can cause an MCS to be inconsistent. Moreover, inconsistency in an MCS renders the system useless. However, in the case of preferential multi-context systems, inconsistency may not be considered as a totally undesirable. Allowing for preferences on contexts, we are more interested in some consistent sections of an inconsistent PMCS, which are significant in some applications. To address this issue, we generalize the notion of equilibrium to an $l_{\leq}$-equilibrium for an PMCS as follows.

**Definition 3.4 ($l_{\leq}$-equilibrium)** Given an PMCS $(M, \leq_s) = \langle M_1, M_2, \ldots, M_m \rangle$ and a number $l \in \{1, 2, \ldots, m\}$. A belief state $S = \langle S_1, S_2, \ldots, S_m \rangle$ of $(M, \leq_s)$ is an $l_{\leq}$-equilibrium of $(M, \leq_s)$ iff $(S_1, S_2, \ldots, S_l)$ is an equilibrium of the $l$-section $M_{1 \rightarrow l}$ of $(M, \leq_s)$.

Roughly speaking, an $l_{\leq}$-equilibrium of a preferential multi-context system represents belief sets acceptable for at least all the contexts in the first $l$ strata of $(M, \leq_s)$, given the belief sets for other contexts. Note that an $l_{\leq}$-equilibrium
of \((M, \leq_s)\) must be an \(k_\prec\)-equilibrium for all \(k \leq l\). In particular, an equilibrium of \((M, \leq_s)\) is an \(l_\prec\)-equilibrium of \((M, \leq_s)\) for all \(1 \leq l \leq m\). But it does not hold vice versa.

**Definition 3.5** (\(l_\prec\)-equilibrium) \(l\) Given an PMCS \((M, \leq_s) = (M_1, M_2, \cdots, M_m)\) and a number \(l \in \{1, 2, \cdots, m\}\). A belief state \(S = \{S_1, S_2, \cdots, S_m\}\) of \((M, \leq_s)\) is called an \(l_\prec\)-equilibrium of \((M, \leq_s)\) iff

- \(S\) is an \(l_\prec\)-equilibrium of \((M, \leq_s)\),
- but \(S\) is not an \((l + 1)_\prec\)-equilibrium of \((M, \leq_s)\) if \(l + 1 \leq m\).

Essentially, an \(l_\prec\)-equilibrium of a preferential multi-context system \((M, \leq_s)\) represents belief sets acceptable for all the contexts in the first \(l\) strata of \((M, \leq_s)\), but not for at least one context in the \((l + 1)\)-stratum if \(l < m\), given the belief sets for other contexts. Evidently, any equilibrium of \((M, \leq_s)\) is an \(m_\prec\)-equilibrium according to this definition.

**Definition 3.6** (Maximal \(l_\prec\)-equilibrium) \(l\) Given an PMCS \((M, \leq_s) = (M_1, M_2, \cdots, M_m)\) and a number \(l \in \{1, 2, \cdots, m\}\). A belief state \(S = \{S_1, S_2, \cdots, S_m\}\) of \((M, \leq_s)\) is called a maximal \(l_\prec\)-equilibrium of \((M, \leq_s)\) iff

- \(S\) is an \(l_\prec\)-equilibrium of \((M, \leq_s)\),
- For any \(i_\prec\)-equilibrium \(S'\) of \((M, \leq_s)\), \(i \leq l\).

Actually, a maximal \(l_\prec\)-equilibrium of a preferential multi-context system is indeed an equilibrium of that system if that system is consistent, otherwise, it represents belief sets acceptable for contexts in a section which cannot keep consistent if we add the next stratum to it.

**Example 3.2** Consider an PMCS

\[(M_5, \leq_s) = (\langle C_1, C_2 \rangle, \langle C_3, C_4 \rangle, \langle C_5 \rangle, \langle C_6 \rangle),\]

where \(L_1\), \(L_2\), and \(L_6\) are propositional logics, and others are ASP logics. Suppose that

- \(kb_1 = \{a\}, br_1 = \{r_{11} = (1 : c) \leftarrow (2 : b)\}\);
- \(kb_2 = \{b\}, br_2 = \{r_{21} = (2 : d) \leftarrow (1 : a)\}\);
- \(kb_3 = \{e \leftarrow f\}, br_3 = \{r_{31} = (3 : f) \leftarrow (1 : c)\}\);
- \(kb_4 = \{g \leftarrow h, h \leftarrow g\}, br_4 = \{r_{41} = (4 : h) \leftarrow (2 : d), \text{ not } (1 : b)\}\);
- \(kb_5 = \{m \leftarrow, \bot \leftarrow q, \text{ not } p\}, br_5 = \{r_{51} = (5 : g) \leftarrow (3 : f); r_{52} = (5 : p) \leftarrow \text{ not } (2 : b)\}\);
- \(kb_6 = \{\neg r\}, br_6 = \{r_{61} = (6 : r) \leftarrow (4 : h)\}\).
Evidently, all bridge rules are applicable except $r_{52}$. Moreover, applying $r_{13}$, $r_{31}$, and $r_{51}$ in turn adds $q$ to $C_5$, and then activates $\bot \leftarrow q$, not $p$. On the other hand, applying $r_{21}, r_{41}$, and $r_{61}$ in turn adds $r$ to $C_6$, and then results in both $r$ and $\neg r$ occurring in $C_6$. So, $(M_3, \leq_s)$ has no equilibrium, i.e., $M_3 \models \bot$. Moreover, it also implies that its 3-section also has no equilibrium, i.e., $\langle (C_1, C_2), (C_3, C_4), (C_5) \rangle \models \bot$.

However, both the 1-section and 2-section of $(M_3, \leq_s)$ are consistent. Obviously, we can check

- $S_0 = \langle \{(c, e), \{b, d\}\}, \{(m, q)\}, \{(r)\} \rangle$ is an $1_\leq$-equilibrium, but not an $2_\leq$-equilibrium; So, it is an $1_\prec$-equilibrium.

- $S_1 = \langle \{(a, c), \{b, d\}\}, \{(e, f), \{g, h\}\}, \{(m, q)\}, \{(r)\} \rangle$ is an $2_\prec$-equilibrium;

- $S_1$ is a maximal $2_\prec$-equilibrium of $(M_3, \leq_s)$.

An occurrence of inconsistency in a multi-context system makes that system useless. However, considering preferences in preferential multi-context systems makes things better. The section corresponding to a maximal $l_\prec$-equilibrium may be interesting and useful in the presence of inconsistency, because it fully captures the meaningful information exchange among contexts involved in this section.

### 4 Inconsistency Analysis

Now an interesting question arises: how to measure the degree of inconsistency for an PMCS? Note that the value $l + 1$ points out the stratum where we first meet inconsistency if a given inconsistent PMCS $(M, \leq_s)$ has a maximal $l_\leq$-equilibrium. In particular, if we abuse the notation and say that $(M, \leq_s)$ has a maximal $0_\leq$-equilibrium if it has no maximal $l_\leq$-equilibrium for any given $1 \leq l \leq m$. Then $l + 1$ is exactly the inconsistency rank for stratified knowledge bases presented in [1, 2] in essence. To bear this in mind, we present the following inconsistency measure.

**Definition 4.1** Given an PMCS $(M, \leq_s) = (M_1, M_2, \cdots, M_m)$. The degree of inconsistency of $(M, \leq_s)$, denoted $\text{DI}((M, \leq_s))$, is defined as

$$\text{DI}((M, \leq_s)) = 1 - \frac{l}{m},$$

if $(M, \leq_s)$ has the maximal $l_\leq$-equilibrium, where $0 \leq l \leq m$.

Actually, the degree of inconsistency $\text{DI}((M, \leq_s))$ of $(M, \leq_s)$ is a slight adaptation of the inconsistency rank such that

- $0 \leq \text{DI}((M, \leq_s)) \leq 1$;

- $\text{DI}((M, \leq_s)) = 0$ iff $(M, \leq_s)$ is consistent;
Note that the first two properties are called Normalization and Consistency, respectively [12]. The third property says that an PMCS has the upper bound 1 iff there is no consistent section.

**Example 4.1** Consider \((M_3, \leq_s)\) again. Note that
\[
\text{DI}((M_3, \leq_s)) = 1 - \frac{2}{4} = \frac{1}{2},
\]
because it has an maximal \(2_<\)-equilibrium as illustrated above.

The measure \(\text{DI}((M, \leq_s))\) allows us to have a sketchy picture on the inconsistency in \((M, \leq_s)\). In many applications, we need to find more information about the inconsistency. For example, we need to know which contexts and bridge rules of a given PMCS are involved in the inconsistency in order to restore consistency of the PMCS.

Note that any two contexts are considered equally preferred in inconsistency handling in the case of multi-context systems. However, preferences over contexts play an important role in dealing with inconsistency among these contexts, especially in making some tradeoff decisions on resolving inconsistency when we take into account preferences. Generally, the more preferred contexts are considered more reliable when an inconsistency occurs in a preferential multi-context system, moreover, remaining unchanged is preferred to any action of revision for such contexts. For example, in requirements engineering, when two requirements with different priority levels contradict each other, a less preferred requirement will be revised to accommodate itself to another one in most cases.

Given an PMCS, each section actually splits the whole set of contexts into two parts, i.e., itself and a set of other strictly less preferred contexts. Moreover, each consistent section fully captures information exchange among contexts which are strictly preferred to ones not included in that section. Generally, such a section may be considered as one of plausible parts of that PMCS. Allowing for this, we are more interested in a section that contains more preferred strata as much as possible. Moreover, any changes of bridge rules for restoring consistency should not affect information exchange among contexts in such a section. In this sense, identifying a consistent section with the maximal number of strata is central to inconsistency analysis in a preferential multi-context system.

**Definition 4.2 (Maximal consistent section)** Given an PMCS \((M, \leq_s) = \langle M_1, M_2, \cdots, M_m \rangle\), the \(i\)-section \(M_1 \rightarrow_i\) of \((M, \leq_s)\), is called a maximal consistent section of \((M, \leq_s)\), if

- \(M_1 \rightarrow_i \models \perp\);
- \(M_1 \rightarrow_k \models \perp\) for all \(k > i\).
Informally speaking, the maximal consistent section of an PMCS can be considered as a reliable part of that PMCS. We use $M_1 \rightarrow_{kmc}$ to denote the maximal consistent section of $(M, \leq_s)$. Evidently, given an inconsistent PMCS $(M, \leq_s) = \langle M_1, M_2, \ldots, M_m \rangle$, a maximal $l_\leq$-equilibrium of $(M, \leq_s)$ is exactly an equilibrium of the $l$-section $M_1 \rightarrow_l$, because less preferred contexts cannot bring new information to more preferred contexts in an PMCS. This implies that finding the maximal consistent section may be not harder than finding maximal $l_\leq$-equilibrium.

**Example 4.2** Consider $(M_3, \leq_s)$ again. The 2-section $\langle (C_1, C_2), (C_3, C_4) \rangle$ is its maximal consistent section.

As mentioned above, Eiter et al have proposed diagnoses and inconsistency explanations for a multi-context system. We use the following example to demonstrate what will happen when we apply these to a preferential multi-context system.

**Example 4.3** Consider $(M_3, \leq_s)$ again. Note that all of the following sets of rules are $\subseteq$-minimal $s$-diagnoses of $(M_3, \leq_s)$:

- $D_1 = \{r_{51}, r_{61}\}$, $D_2 = \{r_{51}, r_{41}\}$, $D_3 = \{r_{51}, r_{21}\}$;
- $D_4 = \{r_{31}, r_{61}\}$, $D_5 = \{r_{31}, r_{41}\}$, $D_6 = \{r_{31}, r_{21}\}$;
- $D_7 = \{r_{11}, r_{61}\}$, $D_8 = \{r_{11}, r_{41}\}$, $D_9 = \{r_{11}, r_{21}\}$.

Note that all of the $\subseteq$-minimal $s$-diagnoses contains one bridge rule of maximal consistent section except $D_1$. That is, according to $D_i$ for all $i \geq 2$, we need to deactivate some information exchange in maximal consistent section to restore consistency in $(M_3, \leq_s)$. In contrast, $D_1$ leaves information exchange in maximal consistent section unchanged. Allowing for preferences relation over contexts, $D_1$ is more significant for inconsistency handling in $(M_3, \leq_s)$.

The example above illustrates that diagnoses not involving maximal consistent section in inconsistency are more preferred. Allowing for the duality relation between diagnoses and explanations, we have the same opinion on inconsistency explanations. However, the compatibility to more preferred knowledge is considered as one of useful strategies in preferential knowledge revision and integration [1, 2]. Next we adapt diagnoses and inconsistency explanations to accommodate maximal consistent section, respectively.

**Definition 4.3** Given an PMCS $(M, \leq_s)$, a diagnosis $(D_1, D_2)$ of $M$ is compatible to the maximal consistent section of $(M, \leq_s)$ if $(D_1 \cup D_2) \cap \text{br}_M(k_{mc}) = \emptyset$.

Note that if we focus on the maximal consistent section of a preferential multi-context system, then the set $\text{br}_M \setminus \text{br}_M(k_{mc})$ of bridge rules of all contexts out of the section exactly composes a diagnosis $\langle \text{br}_M \setminus \text{br}_M(k_{mc}), \emptyset \rangle$ of inconsistency for that system, because $M[\text{br}_M(k_{mc})] \not\models \bot$. This guarantees that there exists at least one diagnosis compatible with the maximal consistent section.
Example 4.4 Consider $(M_3, \leq_s)$ again. All of $(\{r_{51}, r_{61}, r_{52}\}, \emptyset)$, $(\{r_{51}, r_{61}\}, \emptyset)$ and $(\{r_{61}\}, \{r_{52}\})$ are diagnoses compatible to the maximal consistent section.

Furthermore, we consider minimal diagnoses compatible with the maximal consistent section of a given PMCS.

Definition 4.4 (c-diagnosis) Given an PMCS $(M, \leq_s)$, an $s$-diagnosis $D$ of $M$, is called an $c$-diagnosis of $(M, \leq_s)$, if $D \in D_m^c(M)$ and $D \cap \br(M(k_{mc})) = \emptyset$. The set of all $c$-diagnosis of $(M, \leq_s)$ is $D_c^c((M, \leq_s))$.

Essentially, an $c$-diagnosis $D$ of $(M, \leq_s)$ is an $\subseteq$-minimal $s$-diagnosis that is compatible with the maximal consistent section of $(M, \leq_s)$, i.e., none of bridge rules of the maximal consistent section of $(M, \leq_s)$ is involved in $D$.

Example 4.5 Consider $(M_3, \leq_s)$ again. Then $D_1 = \{r_{51}, r_{61}\}$ is a unique $c$-diagnosis compatible to the maximal consistent section, i.e., $D_c^c((M_3, \leq_s)) = \{D_1\}$.

Note that for all $D \in D_c^c((M, \leq_s))$, $D \in D_m^c((M, \leq_s))$ and $D \cap \br(M(k_{mc})) = \emptyset$. So, $\bigcup D_c^c((M, \leq_s)) \subseteq \bigcup D_m^c((M, \leq_s)) \setminus \br(M(k_{mc}))$, but not vice versa.

Definition 4.5 (c-inconsistency explanation) Given an PMCS $(M, \leq_s)$, an $c$-inconsistency explanation $E$ of $(M, \leq_s)$, is a set $E \subseteq \br_M$ s.t. each $E \subseteq R \subseteq \br_M \setminus \br(M(k_{mc}))$, satisfies $M[\br(M(k_{mc}) \cup R] = \perp$. The set of all $\subseteq$-minimal $c$-inconsistency explanations of $(M, \leq_s)$ is $E_c^+(((M, \leq_s))$.

Essentially, an $c$-inconsistency explanation focuses on the set of other bridges rules need to cause an inconsistency given a set of bridge rules of the maximal consistent section. Both $c$-inconsistency explanations and $c$-diagnoses capture the inconsistency under an assumption that every bridge rule of the maximal consistent section should not be revised or modified to restore consistency.

Example 4.6 Consider $(M_3, \leq_s)$ again. Then both $E_1 = \{r_{51}\}$ and $E_2 = \{r_{61}\}$ are $\subseteq$-minimal $c$-inconsistency explanations compatible to the maximal consistent section, moreover, $E_c^+(((M_3, \leq_s)) = \{E_1, E_2\}$.

More interestingly, we have the following weak duality relation between $c$-diagnoses and $c$-inconsistency explanations.

Proposition 4.1 Given an inconsistent PMCS $(M, \leq_s)$, then

$$\bigcup E_c^+(((M, \leq_s)) = \bigcup D_c^c((M, \leq_s)).$$
Proof: This is a direct consequence of Theorem 2.1 in essence. The main part of this proof is the same as that of Theorem 2.1 provided in [8].

Let \( (M, \leq_s) \) be an PMCS and \( M_1 \rightarrow \text{mc} \) its maximal consistent section. The complement of \( R \) w.r.t. \( br_M \) is denoted as \( \overline{R} = br_M \setminus R \).

We first prove that \( \bigcup E_+^c ((M, \leq_s)) \supseteq \bigcup D_+^- ((M, \leq_s)) \) holds. Let \( D \in D_+^- ((M, \leq_s)) \), then \( br_{M(nc)} \subseteq \overline{D} \). We show that there exists \( E \in E_+^c \) with \( x \in E \), for \( x \in D \).

Consider \( E = \overline{D} \setminus \{ x \} \), then \( br_{M(nc)} \subseteq \overline{E} \) and \( M[\overline{E}] \models \perp \). Let \( E = \overline{E} \setminus \overline{br_{M(nc)}} \). Then for all \( E \subseteq R \subseteq \overline{br_{M(nc)}} \), \( M[\overline{br_{M(nc)}} \cup R] \models \perp \).

Suppose that there exists \( E' \subseteq E \) with \( x \not\in E' \) and \( E' \in E_+^c \). Then \( E' \in E_+^c \), and \( \overline{br_{M(nc)}} \cup E' \subseteq \overline{T} \), then \( M[\overline{br_{M(nc)}} \cup E'] \not\models \perp \). So, \( E' \not\in E_+^c \).

Then we prove that \( \bigcup E_+^c ((M, \leq_s)) \subseteq \bigcup D_+^- ((M, \leq_s)) \) holds. Let \( E \in E_+^c ((M, \leq_s)) \), then \( E \cap \overline{br_{M(nc)}} = \emptyset \). We show that there exists \( D \in D_+^- \) with \( x \in D \), for \( x \in E \).

Consider \( S = \{ R \setminus \{ x \} | E \subseteq R \subseteq \overline{br_{M(nc)}} \} \). Let \( S' = \{ T | E \subseteq R \subseteq \overline{br_{M(nc)}} \} \). Assume that \( S' \not= \emptyset \), then \( M[\overline{br_{M(nc)}} \cup E \setminus \{ x \}] \models \perp \), which contradicts \( E \in E_+^c ((M, \leq_s)) \). So, \( S' \not= \emptyset \).

Let \( T_1 \) be a \( \subseteq \)-minimal set in \( S' \) s.t. \( M[\overline{br_{M(nc)}} \cup T_1] \models \perp \). Then \( \overline{br_{M(nc)}} \cup T_1 \subseteq D_+^- (M) \), since for all \( \overline{br_{M(nc)}} \cup T_1 \subseteq R \), \( M[R] \models \perp \).

Furthermore, \( x \not\in \overline{br_{M(nc)}} \cup T_1 \), then \( x \in \overline{br_{M(nc)}} \cup T_1 \).

This weak duality shows that if we consider bridge rules in the maximal consistent section as reliable ones, then diagnoses and inconsistency explanations compatible with maximal consistent section represent dual aspects of inconsistency caused by bridge rules out of the maximal consistent section.

## 5 Computational Complexity

In this section we are concerned with the complexity aspects of preferential multi-context systems. We assume that the reader is familiar with the classes \( P \), \( NP \), and \( coNP \) as well as polynomial time hierarchy \( (\Delta^p_0 = \Sigma^p_0 = \Pi^p_0 = P) \) and for all \( k \geq 0 \), \( \Delta^p_{i+1} = P^{\Delta^p_i} \), \( \Sigma^p_{i+1} = NP^{\Sigma^p_i} \), \( \Pi^p_{i+1} = coNP^{\Sigma^p_i} \) [10]. We now introduce the following classes:

- \( D^p_i = (\Sigma^p_i, \Pi^p_i) \) is the class of all languages such that \( L = L_1 \cap L_2 \), where \( L_1 \) is in \( \Sigma^p_i \) and \( L_2 \) is in \( \Pi^p_i \) for all \( i \geq 1 \). In particular, \( D^p_1 \) is the class of all languages such that \( L = L_1 \cap L_2 \), where \( L_1 \) is in \( NP \) and \( L_2 \) is in \( coNP \). The well known problem of SAT-UNSAT is one of the canonical \( D^p_1 \)-complete problems.

- More generally, let \( X_1 \) and \( X_2 \) be two complexity classes, then \( \langle X_1, X_2 \rangle \) is the class of all languages such that \( L = L_1 \cap L_2 \), where \( L_1 \) is in \( X_1 \) and \( L_2 \) is in \( X_2 \).

- Let \( X \) be a complexity class, \( P^X \) (resp. \( NP^X \)) is the class of all languages that can be recognized in polynomial time by a (resp. nondeterministic) Turing machine equipped with an \( X \) oracle, where an \( X \) oracle solves
whatever instance of a problem in $X$ class in unit time. In particular, $P^{D^p_k[\log n]}$ is the class of all languages can be recognized in polynomial time by a Turing machine using a number of $D^p_k$ oracles bounded by a logarithmic function of the size of input data.

- $FX$ is the corresponding class of functions of $X$.

At first, we recall the complexity of calculating equilibria by guessing so-called kernels of context belief sets presented in [4], and then we discuss the computational complexity for calculating $l_{\leq}$-equilibria for preferential multi-context systems based on that complexity result. Following this, we discuss complexity aspects for identifying diagnoses and inconsistency explanations compatible with the maximal consistent section.

### 5.1 Complexity for equilibria

We consider the following aspects of computational complexity about finding equilibria for preferential multi-context systems:

- **Consistency checking:** the problem of deciding whether an PMCS $(M, \leq_s)$ has an equilibrium.

- **$l$-consistency checking:** the problem of deciding whether an PMCS $(M, \leq_s) = (M_1, M_2, \ldots, M_m)$ has an $l_{\leq}$-equilibrium for a given $1 \leq l \leq m$.

- **maximal $l$-consistency:** the problem of deciding whether an PMCS $(M, \leq_s) = (M_1, M_2, \ldots, M_m)$ has a maximal $l_{\leq}$-equilibrium for a given $l$.

- **maximal $l_{\leq}$-equilibrium:** the problem of computing $l$ for an PMCS $(M, \leq_s) = (M_1, M_2, \ldots, M_m)$ such that it has a maximal $l_{\leq}$-equilibrium.

Note that the core of these problems is to check consistency of some sections or a whole preferential multi-context systems in essences. However, the complexity aspects of calculating equilibria by guessing so-called kernels of context belief sets has been investigated in [4]. In this paper, we also adopt the following assumption of poly-size kernels about logics used in multi-context systems presented in [4]. A logic $L$ has poly-size kernels, if there is a mapping $\kappa$ which assigns to every $kb \in KB$ and $S \in ACC(KB)$ a set $\kappa(kb, S) \subseteq S$ of size (written as a string) polynomial in the size of $kb$, called the kernel of $S$, such that there is a one-to-one correspondence $f$ between the belief sets in $ACC(kb)$ and their kernels, i.e., $S \equiv f(\kappa(kb, S))$ [4]. Moreover, $L$ has kernel reasoning in $\Delta^P_k$ if given any knowledge base $kb$, an element $b$, and a set of elements $K$, deciding whether (i) $K = \kappa(kb, S)$ for some $S \in ACC(kb)$ and (ii) $b \in S$ is in $\Delta^P_k$ [4].

Brewka et al have pointed out that standard propositional non-monotonic logics such as DL, AEL, and NLP have poly-size kernels, moreover, the standard propositional non-monotonic reasoning formalisms DL and AEL have kernel reasoning in $\Delta^P_k$ [4].

Furthermore, for convenience, we assume that any belief set $S$ in any logic $L$ contains a distinguished element $\text{true}$; then for $b = \text{true}$, (i) and (ii) together
are equivalent to (i), i.e., whether \( K \) is a kernel for some acceptable belief set of \( kb \).

Now we introduce the following theorem about computational complexity about consistency checking for multi-context systems based on assumptions above presented in [4].

**Theorem 5.1** [4] Given a finite MCS \( M = (C_1, C_2, \cdots, C_n) \) where all logics \( L_i \) have poly-size kernels and kernel reasoning in \( \Delta^p_k \), deciding whether \( M \) has an equilibrium is in \( \Sigma^p_k \).

Then we can get the following corollary about consistency checking and proposition about \( l \)-consistency checking for preferential multi-context systems directly from the theorem above, respectively.

**Corollary 5.1** Given a finite PMCS \( (M, \leq) = \langle M_1, M_2, \cdots, M_m \rangle \) where all logics \( L_i \) have poly-size kernels and kernel reasoning in \( \Delta^p_k \), deciding whether \( (M, \leq) \) has an equilibrium is in \( \Sigma^p_k \).

**Proof** Note that \((M, \leq)\) has an equilibrium if and only if \( M \) has an equilibrium. According to Theorem 5.1, the problem of deciding whether \((M, \leq)\) has an equilibrium is in \( \Sigma^p_k \). □

**Proposition 5.1** Given a finite PMCS \( (M, \leq) = \langle M_1, M_2, \cdots, M_m \rangle \) where all logics \( L_i \) have poly-size kernels and kernel reasoning in \( \Delta^p_k \), deciding whether \((M, \leq)\) has an \( l \leq \)-equilibrium for a given \( 1 \leq l \leq m \) is in \( \Sigma^p_k \).

**Proof** Given a number \( 1 \leq l \leq m \), \((M, \leq)\) has an \( l \leq \)-equilibrium if and only if \( M \) has an \( l \leq \)-equilibrium. According to Theorem 5.1, the problem of deciding whether \((M, \leq)\) has an \( l \leq \)-equilibrium is in \( \Sigma^p_k \). □

Next we give the complexity of maximal \( l \)-consistency problem for preferential multi-context systems in the case of given a positive number \( l \).

**Proposition 5.2** Given a finite PMCS \( (M, \leq) = \langle M_1, M_2, \cdots, M_m \rangle \) where all logics \( L_i \) have poly-size kernels and kernel reasoning in \( \Delta^p_k \), the problem of deciding whether \((M, \leq)\) has a maximal \( l \leq \)-equilibrium for a given \( l < m \) is in \( \Delta^p_{k+1} \).

**Proof** Note that \((M, \leq)\) has a maximal \( l \leq \)-equilibrium for a given \( l < m \) if and only if \( M(l) \) has at least one equilibrium and \( M(l+1) \) has no equilibrium. Recall the problem of deciding whether \( M \) has an equilibrium is in \( \Sigma^p_k \), so, the problem of deciding whether \( M \) has no equilibrium is in \( \Pi^p_k \). Then the problem of deciding whether \((M, \leq)\) has a maximal \( l \leq \)-equilibrium for a given \( l < m \) is in \( \langle \Sigma^p_k, \Pi^p_k \rangle \), i.e., \( \Delta^p_{k+1} \). □

Then we are ready to get the following computational complexity for the problem of maximal \( l \leq \)-equilibrium.
**Proposition 5.3** Given a finite PMCS \((M, \leq_s)\) where all logics \(L_i\) have poly-size kernels and kernel reasoning in \(\Delta^p_k\), the problem of computing \(l\) for \((M, \leq_s)\) such that it has maximal \(l_{<}-\)equilibrium is in \(FP^{DP_{k+1}[\log n]}\).

**proof** Consider the following algorithm for computing \(l\):

1. if \(M \not\models \bot\), then \(l = m\);
2. else if \(M(1) \models \bot\), then \(l = 0\);
3. else for \(i\) from 1 to \(n - 1\), if \((M, \leq_s)\) has an \(i_{<}\)-equilibrium, then \(l = i\), break; else \(i = i + 1\).

According to Corollary 5.1 the problem of deciding whether \(M \not\models \bot\) is in \(\Sigma^p_{k+1}\), and checking whether \(M(1) \models \bot\) is in \(\Pi^p_{k+1}\).

From Proposition 5.2 we have obtained that the problem of deciding whether \((M, \leq_s)\) has a maximal \(i_{<}\)-equilibrium for each \(i\) is in \(DP^p_{k+1}\). Therefore, \(l\) can be computed in polynomial time by a Turing machine equipped with an \(DP^p_{k+1}\) oracle. So, the problem of computing \(l\) is in \(FP^{DP_{k+1}}\).

Further, consider two particular singleton multi-contexts systems \(M_1 = (C_1)\) and \(M_2 = (C_2)\), where both \(L_1\) and \(L_2\) are propositional logics with \(kb_1 = kb_2 = \{\}\), \(br_1 = \{(1 : a) \leftarrow \text{not} \ (1 : a)\}\) and \(br_2 = \{(2 : a) \leftarrow\}\). Then

- \(l = m\) if and only if \(\langle M \not\models \bot, M_1 \models \bot\rangle\) holds.
- \(l = 0\) if and only if \(\langle M_2 \not\models \bot, M(1) \models \bot\rangle\) holds.

Moreover, we can use a binary search on \(\{1, 2, \ldots, m - 1\}\) to find \(l\) at step (3). Under such a case, \(l\) can be computed by using \(O(\log_2 m)\) calls to an \(DP^p_{k+1}\) oracle. So, the problem of computing \(l\) is also in \(FP^{DP_{k+1}[\log n]}\). \(\square\)

Note that \(M_1 \rightarrow l\) is the maximal consistent section if and only if \((M, \leq_s)\) has a maximal \(l_{<}\)-equilibrium. Then from Proposition 5.2 we can get the following complexity for the problem of identifying the maximal consistent section of \((M, \leq_s)\) directly.

**Corollary 5.2** Given a finite PMCS \((M, \leq_s)\) where all logics \(L_i\) have poly-size kernels and kernel reasoning in \(\Delta^p_k\), the problem of identifying the maximal consistent section of \((M, \leq_s)\) is in \(FP^{DP_{k+1}[\log n]}\).

We summarize these complexity aspects in Table 1.

### 5.2 Computational complexity for diagnoses and explanations

We focus on diagnoses and inconsistency explanations compatible with the maximal consistent section in a preferential multi-context system, respectively.

At first, we consider the following complexity aspects about diagnoses and inconsistency explanations in the case that the maximal consistent section is
Table 1: Complexity aspects about equilibrium

<table>
<thead>
<tr>
<th>Problem</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>consistency</td>
<td>$\Sigma_{k+1}$</td>
</tr>
<tr>
<td>$l$-consistency</td>
<td>$\Sigma_{k+1}$</td>
</tr>
<tr>
<td>maximal $l$-consistency</td>
<td>$D_{k+1}$</td>
</tr>
<tr>
<td>maximal $l_{&lt;}$-equilibrium</td>
<td>$FP^{D_{k+1}[\log n]}$</td>
</tr>
</tbody>
</table>

given. Note that the complexity aspects about finding diagnoses and inconsistency explanations for multi-context systems have been investigated in [8], respectively. The following proposition shows that problems of finding diagnoses (resp. inconsistency explanations) compatible with the maximal consistent section have the same complexity with that of finding diagnoses (resp. inconsistency explanations) when the maximal consistent section is given.

**Proposition 5.4** Given a finite PMCS $(M, \leq_s)$ and its maximal consistent section $M_{1\rightarrow k_{mc}}$, deciding whether $D \subseteq br_M$ is a diagnosis compatible with $M_{1\rightarrow k_{mc}}$ has the same computational complexity as consistency checking of $(M, \leq_s)$.

**Proof** Note that we only need to check whether $M[br_M \setminus D] \neq \bot$ and $D \cap br_M(k_{mc}) = \emptyset$. □

**Proposition 5.5** Given a finite PMCS $(M, \leq_s)$ and its maximal consistent section $M_{1\rightarrow k_{mc}}$, deciding whether $D \subseteq br_M$ is an $c$-diagnosis has the same computational complexity as minimal diagnosis recognition of $M$.

**Proof** Note that the problem of deciding whether $D$ is an $c$-diagnosis of $(M, \leq_s)$ is equivalent to deciding whether $D$ is a minimal $s$-diagnosis of $(M, \leq_s)$ and $D \cap br_M(k_{mc}) = \emptyset$. □

**Proposition 5.6** Given a finite PMCS $(M, \leq_s)$ and its maximal consistent section $M_{1\rightarrow k_{mc}}$, deciding whether $E \subseteq br_M$ is an $c$-inconsistency explanation has the same computational complexity as inconsistency explanation recognition of $M$.

**Proof** Note that the problem of deciding whether $E$ is an $c$-inconsistency explanation of $(M, \leq_s)$ is equivalent to deciding whether $E$ is an inconsistency explanation of $(M, \leq_s)$ and $E \cap br_M(k_{mc}) = \emptyset$. □

Now we consider the general case of finding $c$-diagnoses and $c$-inconsistency explanations for preferential multi-context systems. Let $C_{ms}(M)$ be the complexity for identifying the maximal consistent section of $(M, \leq_s)$. For example, $C_{ms}(M)$ is $FP^{D_{k+1}[\log n]}$ in the case illustrated in Proposition 5.3. Let $C_{sd}(M, D)$ be the complexity for deciding whether $D \subseteq br_M$ is an $s$-diagnosis of $M$. Let
$C_e(M, E)$ is the complexity for deciding whether $E$ is an inconsistency explanation of $M$. We assume that $C_e(M, E)$ is closed under conjunction, according to discussion about such complexity aspects in [8].

**Proposition 5.7** Given a finite PMCS $(M, \leq_s)$, deciding whether $D \subseteq br_M$ is an $c$-diagnosis is in $\langle C_{ms}(M), C_{sd}(M, D) \rangle$.

**Proof** To decide whether $D \subseteq br_M$ is an $c$-diagnosis, we need to check

1. whether $D \cap br_M(k_{mc}) = \emptyset$ holds;
2. whether $D$ is an $s$-diagnosis.

Note that $D \cap br_M(k_{mc}) = \emptyset$ if and only if $br_M(k_{mc}) \subseteq \overline{D}$. Then (1) is equal to finding the maximal consistent section from sections not involved in $D$. So, the problem of deciding whether $D \subseteq br_M$ is an $c$-diagnosis is in $\langle C_{ms}(M), C_{sd}(M, D) \rangle$. □

**Proposition 5.8** Given a finite PMCS $(M, \leq_s)$, deciding whether $E \subseteq br_M$ is an $c$-inconsistency explanation is in $\langle C_{ms}(M), C_e(M, E) \rangle$, where $C_e(M, E)$ is the complexity for deciding whether $E$ is an inconsistency explanation of $M$.

**Proof** To decide whether $E \subseteq br_M$ is an $c$-inconsistency explanation, we need to check

1. whether $E \cap br_M(k_{mc}) = \emptyset$ holds;
2. whether $E$ is an inconsistency explanation.

Note that (1) is equal to finding the maximal consistent section from sections not involved in $E$. So, the problem of deciding whether $E \subseteq br_M$ is an $c$-inconsistency explanation is in $\langle C_{ms}(M), C_e(M, E) \rangle$. □

**Proposition 5.9** Given a finite PMCS $(M, \leq_s)$, deciding whether $E \subseteq br_M$ is in $E^+_c((M, \leq_s))$ is in $\langle C_{ms}(M), C_e(M, E), coC_e(M, E) \rangle$.

**Proof** Note that $E \in E^+_c((M, \leq_s))$ if and only if $E$ is an $c$-inconsistency explanation and $E$ is minimal w.r.t. $\subseteq$. We have obtained that deciding whether $E \subseteq br_M$ is an $c$-inconsistency explanation is in $\langle C_{ms}(M), C_e(M, E) \rangle$ in Proposition 5.8.

From Lemma 2 in [8], we can check subset-minimality of $E$ by deciding whether none of $E \setminus \{x\}$ is an inconsistency explanation for all $x \in E$. Note that the number of these checks is linear w.r.t. $|E|$, and $C_e(M, E)$ is closed under conjunction. So, deciding whether $E \in E^+_c((M, \leq_s))$ is in $\langle C_{ms}(M), C_e(M, E), coC_e(M, E) \rangle$. □

Note that we consider the general case of complexity. It is not difficult to consider the usual cases discussed in [8].
6 Comparison and Discussion

Preferential multi-context systems provide a framework for incorporating preferences on contexts in multi-context systems. However, the following aspects distinguish preferential multi-context systems from the original multi-context systems presented in [4]. At first, the compatibility of a total preorder relation over contexts with a multi-context system imposes a constraint on bridge rules, i.e., any appearance of less preferred contexts is prohibited in the body of a bridge rule for a given context. Only one-way information flow between any two strata is allowed to occur in a preferential multi-context system. The intuition behind this constraint is that less reliable information cannot be used to revise more reliable knowledge. Second, one-way information flow makes any section of a preferential multi-context system capable of capturing all the information exchange among contexts in that section. This signifies that each section is also a meaningful preferential multi-context system. Third, preferential multi-context systems are concerned with partial equilibria such as \( l_{\leq} \)-equilibria as well as equilibria.

Note that preferential multi-context systems also analyze inconsistency in terms of diagnoses and inconsistency explanations presented in [5]. However, allowing for the role of preferences on contexts in a given preferential multi-context system, we are more interested in diagnoses and inconsistency explanations compatible with the maximal consistent section. More interestingly, diagnoses and inconsistency explanations compatible with the maximal consistent section have duality relation. This implies that the compatibility with the maximal consistent section does not destroy the duality relation between diagnoses and inconsistency explanations [5].

Actually, the compatibility of diagnoses (resp. inconsistency explanations) with the maximal consistent section essentially provides a way to discriminate between all diagnoses (resp. inconsistency explanation) based on preferences on contexts. In this sense, such a compatibility can be considered as some kind of filter to filter some undesirable diagnoses (resp. inconsistency explanations) [6].

Preferential multi-context systems aim to address the total preorder relation over contexts. However, combining preferences with contexts is one of the important issues in integrating and sharing contextual knowledge [3]. Moreover, as stated in [6], there is a multifaceted relationship between nonmonotonic logics and preferences. As a framework for integrating arbitrary monotonic and nonmonotonic logics, it is necessary to incorporating such a relation in multi-context systems. This may be one of directions of our future work.

7 Conclusion

In this paper, we have presented the preferential multi-context system, which provides a promising framework for combining multi-context systems with the total preorder relations on their contexts. Preferential multi-context systems
take into account the impact of preferences among contexts on their inter-contextual information exchange. Only information flow from more preferred contexts to less preferred ones is allowed to occur in preferential multi-context systems. In such a preferential multi-context system, a context may be revised based on only information exchange with more or equally preferred contexts.

This paper presented the following contributions to multi-context systems community:

- We proposed the notion of preferential multi-context system, which consists of a multi-context system with a total preorder relation compatible with that system.

- We extended the equilibrium semantics for multi-context systems and proposed a notion of $l_{\leq}$-equilibrium representing belief states acceptable for at least contexts of the first $l$ strata in a preferential multi-context system. Furthermore, we proposed a notion of maximal $l_{\leq}$-equilibrium describing belief states acceptable for contexts in the maximal consistent section of a preferential multi-context system.

- We proposed inconsistency diagnoses and inconsistency explanations compatible with the maximal consistent section, respectively. Moreover, we discussed their duality relation.

- We investigated the computational complexity aspects for calculating $l_{\leq}$-equilibria and identifying diagnoses and inconsistency explanations compatible with the maximal consistent section, respectively.

References


