Mathematics education and manipulatives

Which, when, how?

Kevin Larkin
Griffith University
<k.larkin@griffith.edu.au>

Kevin Larkin suggests a framework for teachers to use in their selection and use of concrete and virtual materials, to support student learning at different experiential stages.

Introduction

This article proposes a framework for classroom teachers to use in making pedagogical decisions regarding which mathematical materials (concrete and digital) to use, when they might be most appropriately used, and why. Two iPad apps will also be evaluated to demonstrate the usefulness of the framework in assisting teachers to evaluate digital resources in terms of their pedagogical, cognitive and mathematical fidelity (Bos, 2009).

Using materials to support mathematical learning

There is a wealth of literature on the benefits of using concrete materials to support mathematical learning. I present briefly here the findings of Carbonneau, Marley and Selig (2013), who conducted a meta-analysis of 55 studies investigating the benefits of the use of mathematical materials. Each of the studies involved groups of students using concrete objects compared with control groups of students using only mathematics symbols. The major finding was a small to medium positive effect on student learning for students using materials compared with those using mathematical symbols only. There were some provisos on these findings in relation to the strengths of the effects:

- **Developmental level of the user.**
  The materials were most effective in the 7–11 years age group. They were still effective, but less so, in the 12 years and older age group. Least positive effects were noted in students aged 3–6 years; perhaps due to the difficulty that younger students have in discerning between the materials and their mathematical representations.
- **Perceptual richness of the object**.
  In the studies, perceptually rich objects were considered to be those closely related to actual objects (e.g. toy bears) and when used resulted in a larger positive effect. Although perceptual richness was important in encouraging conceptual development, care is needed in their use with pre-operational students who may become distracted from the mathematics understandings intended by their use.
- **Level of guidance during manipulative use.**
  Students whose use of manipulatives was scaffolded, were better able to establish connections between objects and mathematical representations.

In relation to virtual (digital) manipulatives, a meta-analysis conducted by Moyer-Packenham and Westenskow (2013), found that virtual manipulatives:

- Allow exploration in a different manner to concrete materials or pen and paper;
- Support the development of individual representations as the learner is in control; and
- Have a moderate effect on student achievement.

What these findings confirm is that with timely teacher support, and given the selection of appropriate mathematics materials, student mathematical achievement is enhanced by their use.

A framework for choosing materials to support student learning

Given that the use of materials to teach mathematics is endorsed both by the tacit knowledge of teachers and also the research, questions remain as to the which, when, and how of materials usage. Bruner (1966) proposed that students learn through three experiential stages:

- **Enactive (direct sensory) experience** where students take an active part in their learning through the manipulation of their learning environment;
- **Iconic representation of experience** where enacted experiences are represented via diagrams, film clips, charts etc.; and
- **Symbolic representation** including written language symbols such as words and mathematical symbols.

More recently, the terms 'concrete', 'representational' and 'abstract' (Cooper, 2012) are used to describe these three stages. The intent of this article is to explain the use of a framework (Figure 1) that primary school teachers can use to direct their decision-making regarding the use of mathematical materials, both concrete and digital, to facilitate student learning at the 'Enactive' and 'Iconic' stages in particular. In order to assist with this decision-making process, I have adapted Dale's *Cone of Experience* (1969), which explored the use of materials to support student learning in any domain, to create a framework that is useful in determining the which, when and how of mathematical materials use at each of Bruner's experiential stages of learning.

In using the framework, teachers should note a number of important points. Firstly, that the framework tracks the use of materials from concrete to the abstract and, depending on where students are in their conceptual development, materials may be helpful or harmful to their learning (See Carbonneau et al., 2013). For example, it would be inappropriate to only use square tiles when developing area concepts as this may detract from the development of an understanding of area as a 'covering' and may also promote the misconception that only regular figures have area. Secondly, it would normally be expected that students spend a significant amount of time at the Enactive stage of learning to ensure robust conceptual development of mathematics content.

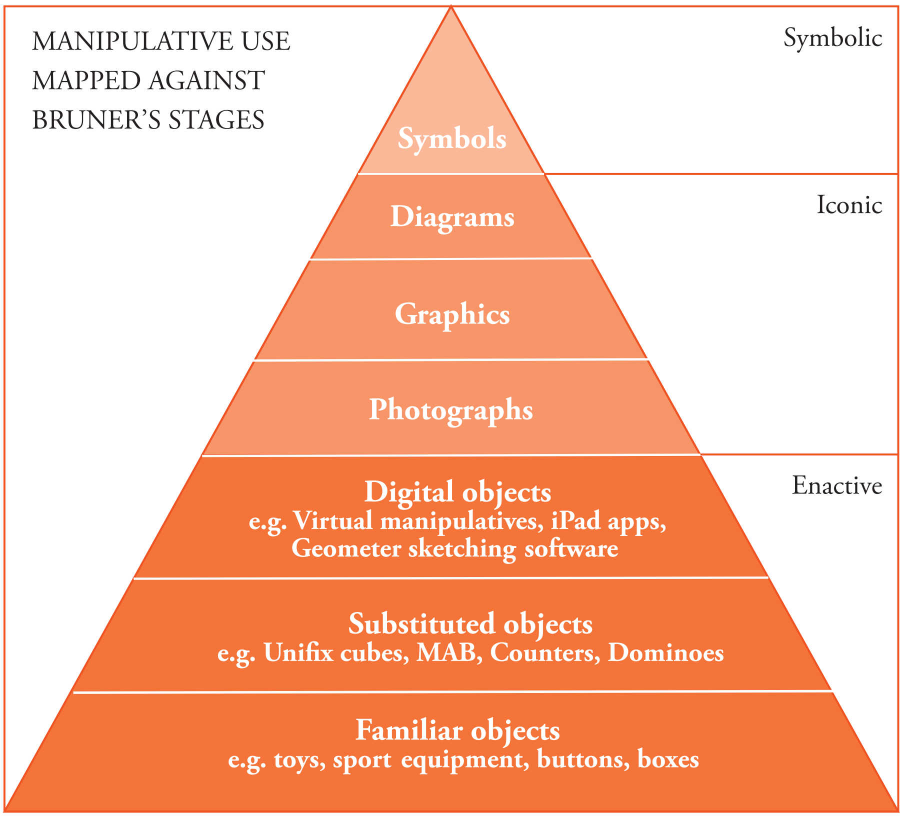

Figure 1. Mapping mathematics materials to Bruner's experiential stages of learning.

Finally, the framework depicts a separation of concrete (familiar and substituted) and digital objects as digital objects add an additional level of abstractness to the use of materials at the Enactive stage.

Using the framework for developing algebraic thinking

In order to demonstrate the application of the framework as a teaching scaffold, I have indicated briefly below its enactment in relation to the teaching of a mathematics concept; namely, the notions of balance or equivalence which underpins early algebraic thinking. This concept is included at differing levels of complexity across the primary year levels; however, the focus here is on initial exposure to the concept. Students are not required to use number sentences until Year 2 and unknowns in number sentences are only introduced in Year 4 so it is likely that many students will operate at the Enactive and Iconic stages in the very early years of schooling. The framework indicates that materials used at the Enactive stage should initially be familiar to students from their real world contexts and that these materials are then substituted with materials found in mathematics classrooms or digitally via the web or mobile devices. In each of the three cases, the major consideration is that the students can develop deeper mathematical representations of the concepts via their engagement with materials.

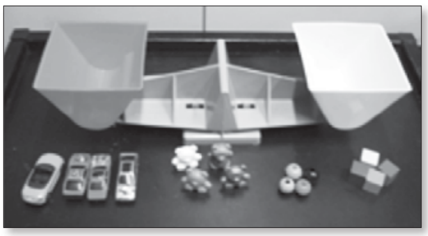

Figure 2. Scales to develop the concept of balance or equivalence.

Initially, a simple balance scale can be used to develop the notion of balance or equivalence in algebraic expressions (Figure 2). Objects such as cars, toy animals, beads or blocks can be used. Students will begin to develop an understanding that changes can be made to elements in each of the pans such that balance is either maintained or lost.

Further in the development of the concepts, the balance scale can be replaced with mathematics materials such as the *Number Balance Equaliser* (Figures 3 and 4). Depending on the understanding of the students, the Number Balance can be used in various ways. In Figure 3 the students can see the numbers on the Number Balance as the teacher demonstrates that (2 × 3 + 2 = 6 + 2) are balanced.

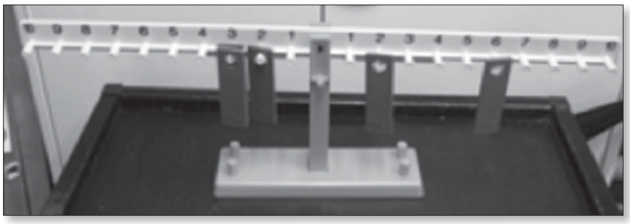

Figure 3. Mathematics manipulatives used to represent simple number sentences.

At a later stage, by hanging the blue weights on the far side of the Number Balance, elements of an algebraic expression can be hidden from student view.

Finally, digital materials that allow students to manipulate objects via a mouse or touch screen, can be used. A number of these resources are available in either web based or iPad format.

Figure 4. Screen shot of *Hands on Equations.*

The screen shot depicted in Figure 4 is from an iPad app titled *Hands On Equations* and illustrates a partially worked example to solve an equation with one unknown.

At the Iconic stage, students use visual representations to assist in their learning. Examples of two different 'balance' representations are included in Figure 5. These materials encourage students to recall their earlier enacted activities and provide a scaffold between these activities and the symbolic representations of relationships to come. Short instructional videos, such as the one demonstrated here (https://www.youtube.com/watch?v=r9lmoySahVs) are also appropriate at this stage.

Figure 5. Iconic representations of the notion of balance or equivalence.

Finally, at the Symbolic stage of the experiential learning sequence, students solve simple number sentences or algebraic equations (Figure 6), without the use of concrete or digital materials.

$$7 + 4 = \square + 2$$
$$2p + 5 = 13$$

Figure 6. Symbolic representations of the notion of balance or equivalence.

Using the framework to evaluate digital materials including apps

As well as providing a mechanism for suggesting the types of materials likely to be useful in supporting student learning at the Enactive and Iconic stages, the framework can also be used by teachers evaluating the quality of digital materials. This is particularly pertinent as concrete mathematics materials are increasingly becoming digitised. When concrete materials are digitised, their usefulness may diminish as they become distanced from the concrete nature of the resource; thus limiting their ability to be enacted upon by students. Bos (2009) discussed this increased distance from the initial intent of the materials in terms of three levels of fidelity:

- **Pedagogic** where the resource allows the student to do mathematics without being distracted or limited by technical features;
- **Cognitive** which refers to whether a concept is better understood when an action is performed on or with the object; and
- **Mathematical** where the object conforms to mathematical accuracy and embodies accurate representations of the concept

In addition to the three issues of fidelity, there are more mundane, but equally problematic, issues of teachers finding the time to properly review digitised resources and also the lack of quality information about apps that is provided by the iTunes store (Larkin, 2014). The remainder of this article uses the framework to evaluate two notionally similar apps: *Area of Shapes (Parallelogram)* and *Area of Parallelogram.*

Area of Shapes (Parallelogram) app

This app consists of four components; an interactive lesson, a virtual geoboard, a multiple-choice test, and a challenge component.

Component 1

This lesson consists of 21 interactive 'slides' with voice and diagrammatic support. Students have the option to complete activities within the lesson that incorporate the use of manipulatives. This component supports experiences in the Enactive and Iconic stages.

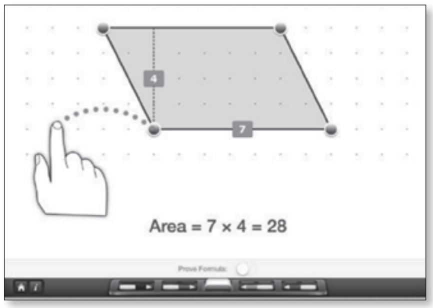

Figure 7. Screen shot of *Area of Shapes* (Parallelogram).

Component 2

Component 2 is a virtual geoboard where students can draw, and then manipulate, their own parallelogram. They can fill the parallelogram and change its base and height to see how this changes the area. They can also translate the triangular area from one end of the shape to the other to develop the relationship between parallelograms and rectangles as a specific subset of parallelograms. This component supports student experiences in the Enactive stage.

Component 3

Component 3 is a ten question multiple-choice test. This component is not as useful as the others but does include Iconic and Symbolic experiences.

Component 4

Component 4 is a free challenge area where users can manipulate various parallelograms to assist them in determining their area. They are also required to manipulate the geoboard to create parallelograms of various sizes. In addition, there are drawing tools available for students to write on the screen. This challenge component supports learning at all three experiential stages.

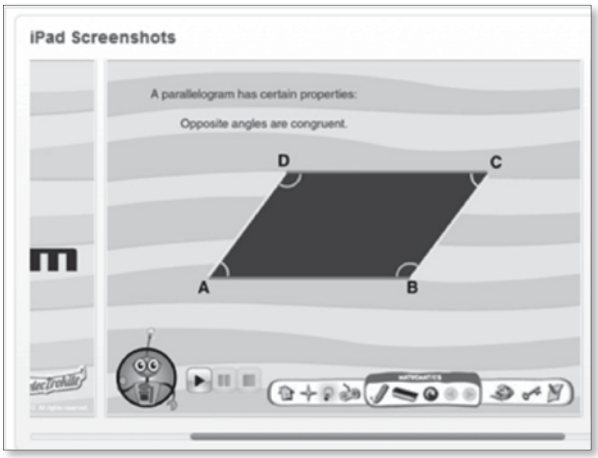

Figure 8. Screen shot of *Area of Parallelogram.*

Area of Parallelogram

This app consists only of a lesson with voice-overs and diagrams explaining to students how to determine the area of a parallelogram. It is one in a series of apps for various shapes from this developer. Students have no control over the creation of the parallelograms. Once the lesson is complete, the students are prompted to complete worksheets which are only available when you email the creators of the software. There is no opportunity for students to manipulate the parallelogram to establish the relationship between its area and the area of a rectangle with the same dimensions nor to translate the triangle from one end of the parallelogram to the other. This app only supports learning at the Iconic (in a minimal way) and Symbolic stages.

The use of the framework to determine levels of experiential learning supported by the two apps indicates that the first app is more useful for supporting initial student conceptual development of area at the Enactive stage, and then scaffolds this learning further across the Iconic and Symbolic stages; whereas the second app is only useful to reinforce this conceptual understanding at the Iconic (minimally) and Symbolic stages. According to the framework used in this article, the first app is therefore much more appropriate to use with students across a range of stages than the second which would only be used to support learning during the later stages of the learning process.

Conclusion

This article has suggested a framework for teachers to use in their selection and use of materials to support student learning at various experiential stages. As indicated, the iTunes store is not an appropriate source of advice on the quality of apps; however, some of the teacher decision making can be outsourced to reputable providers of digital materials. I encourage classroom teachers to visit the following sites:

- **The National Library of Virtual Manipulatives website** http://nlvm.usu.edu/;
- **Illuminations website** http://illuminations.nctm.org/; and
- **Shodor** http://www.shodor.org/interactivate/

In addition, if teachers are looking for appropriate apps to use with their students, the author has reviewed 142 mathematics apps that, to varying degrees, are useful in supporting student learning at either the Enactive or Iconic stages. These reviews are available at http://tinyurl.com/ACARA-Apps. A brief summary of the process followed in evaluating their use is available in an earlier edition of APMC (Larkin, 2014).

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Manipulatives