Rapid readout of a register of qubits using open loop quantum control

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Measurements are a primitive for characterizing quantum systems. Reducing the time taken to perform a measurement may be beneficial in many areas of quantum information processing. We show that permuting the eigenvalues of the state matrix in the logical basis, using open loop control, provides a $O(n)$ reduction in the measurement time, where n is the number of qubits in the register. This reduction is of the same order as the (previously introduced) locally optimal feedback protocol. The advantage of the open loop protocol is that it is far less difficult experimentally. Because the control commutes with the measured observable at all times, our rapid measurement protocol could be used for characterising a quantum system, by state or process tomography, or to implement measurement-based quantum error correction.

Quantum measurements are typically treated as instantaneous but in many practical situations this is clearly not the case. The noisy signals produced by the detectors need to be integrated over a time interval, called the measurement time, in order to determine the measurement outcome. For example spin qubits made from GaAs double dot systems have measurement times of the order of $\sim 10\mu s$ [1], which is much longer than the ns timescale of the internal dynamics.

Schemes for characterizing quantum systems require a large number of such measurements. The process of characterization can be lengthy for a number of reasons. In quantum state tomography it is necessary to repeatedly produce a state, choose a measurement basis, perform the measurement, and then process all the results into a state estimate. Each of these process can be time consuming. One approach to reduce the time required to characterize a system, provided it is low rank, is to perform compressed sensing [2] or direct fidelity estimataion [3, 4] instead of tomography. Compressed sensing for example, under some reasonable assumptions, drastically reduces: the number of measurements, the post-processing of the data and hence the total characterization time even if the measurements are slow.

A reduced characterization time can also be achieved by speeding up the measurement process. It has been found previously that by applying control throughout the measurement process it is possible to affect the measurement rate [5]. There are two important ingredients in searching for procedures which speed up the measurement process. The first is a description of the measurement process that includes time. This ingredient is the quantum trajectory description of the measurement process. The second ingredient is restricting the allowed controls to a set of operations that make the measurement projective, and allow us to retrodict the basis of the measurement. The relevant restriction on the con-

trol is to operations that commute with the measured observable. In other words, we wish to restrict the control to permutations in the initial measurement (logical) basis [5].
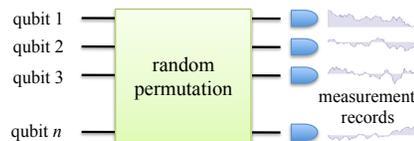

FIG. 1: In our protocol n qubits are independently and continuously monitored in the logical basis. Open loop control consisting of random permutations in the logical basis are applied throughout the measurement to speed up the wavefunction collapse. As the control commutes with the measurements our protocol is comparable with state tomography.

Reference [5] introduced a feedback protocol, optimised locally in time, that reduces the measurement time by a factor that scales linearly in the number of qubits, n, in a register. The ratio of the measurement time without control to that with control is called the speed-up [7]. It was found that the speed-up was $S = 0.7n$. Unfortunately quantum feedback is experimentally demanding. Furthermore the protocol suggested in Ref. [5] requires a rapidly reconfigurable quantum circuit with a large number of CNOT gates. Both of these factors suggest the protocol in Ref. [5] may not be practical to implement.

In this article we present open loop control protocol, inspired by Ref. [6], which achieves the same $O(n)$ improvement in speed-up. The protocol consist of randomly permuting the state in the measurement eigenbasis throughout the measurement; see Fig. 1. The expected asymptotic speed-up of this protocol is $S \sim 0.5n$ and simulations for small n show a $S \sim 0.4n$ scaling. We emphasize that the continuous time description is not necessary

to apply our results in practice. Simply permuting the state in the logical basis throughout the measurement and retrodicting the final result is sufficient. This protocol should be contrasted to rapid purification protocols which can not be used in a tomographic setting [7–9]

This article is organized as follows. In Sec. I, we determine the rate at which information is extracted from a register of qubits using the quantum trajectory description of a continuous measurement. Next, we review the locally optimal protocol feedback protocol, with controls restricted to the permutation group, for rapid measurement in Sec. II. We then reconsider the rapid-measurement problem with the open loop controls restricted to the permutation group in Sec. III. We show analytically that the scaling of the speed-up with the number of qubits register is upper and lower bounded by a quantity that scales with n. Numerical simulations for small n in Sec. IV show the scaling is $S \sim 0.4n$. We conclude in Sec. V with a discussion of open problems.

I.   CONTINUOUS MEASUREMENT

In this section we derive the rate at which a continuous measurement extracts information from a quantum system for a reasonably general measurement model. The starting point for our analysis is the quantum trajectory description of the measurement process.

The physics behind the quantum trajectory description is as follows. Consider using an ancilla, e.g. a field or a current, to probe a quantum system. The interaction between the system and the probe correlates the two systems. Typically the interaction only weakly correlates the system and ancilla. By measuring the ancilla and then using quantum measurement theory we can determine the state of the system conditioned on the results of the measurement. In order to obtain a strong measurement many ancilla must be coupled and then measured. In the limit where there is a continuum of ancilla the measurement process becomes continuous in time. The quantum trajectory approach is a way to describe the conditional state of the system as a functional of the measurement record $R(t)$. One uses a stochastic differential equation for the state of the system $d\rho_{R(t)}$ to describe the change of the system state over an infinitesimal time interval due to any unitary, dissipative and measurement induced dynamics. The system state at the next time interval is given by $\rho(t + dt) = \rho(t) + d\rho_{R(t)}$. Interestingly the form of the differential equation for the system state is similar across many different systems [10–12]. Accessible derivations of this process can be found in Refs. [13–16].

A.   continuous measurement of a single qubit

Consider a finite-dimensional quantum system undergoing a continuous measurement of an observable X. The

change to our state of knowledge of an individual system, ρ, conditioned on the result of the measurement in an infinitesimal interval is described by the stochastic master equation (SME) [13–15]

$$d\rho[t; X] = 2\gamma \, dt \, \mathcal{D}\left[X\right]\rho(t) + \sqrt{2\gamma}\, dW(t)\, \mathcal{H}\left[X\right]\rho(t), \quad (1)$$

where $\mathcal{D}\left[A\right]\rho \equiv A\rho A^\dagger - \frac{1}{2}(A^\dagger A\rho + \rho A^\dagger A)$ and $\mathcal{H}\left[A\right]\rho \equiv A\rho + \rho A^\dagger - \text{Tr}\left[(A^\dagger + A)\rho\right]\rho$. Here we are working in a frame that removes any Hamiltonian evolution. The *measurement strength*, γ, determines the rate at which information is extracted. The measurement result in the interval $[t, t + dt)$ is

$$dR = 2\sqrt{2\gamma}\langle X(t)\rangle dt + dW(t), \quad (2)$$

where dW is a Wiener process and $\langle X(t)\rangle = \text{Tr}\left[X\rho(t)\right]$. Without loss of generality we take X to be traceless. We can do this because Eq. (1) is invariant under $X \to X + \lambda I$ for $\lambda \in \mathbb{R}$ where I is the identity operator.

B.   continuous measurement of a register of qubits

We now generalize the SME in Eq. (1) to a register of n qubits, where each qubit is independently and continuously measured. Instead of one observable X, we now have n, given by

$$Z^r = I^{(1)} \otimes I^{(2)} \otimes \ldots \sigma_z^{(r)} \ldots \otimes I^{(n)}, \quad (3)$$

where r labels the rth qubit. The SME describing such a measurement is

$$d\rho = \sum_{r=1}^n 2\gamma \, dt \, \mathcal{D}\left[Z^r\right]\rho + \sqrt{2\gamma}\, dW^{(r)}\mathcal{H}\left[Z^r\right]\rho, \quad (4)$$

and the n Wiener processes obey $dW^{(i)}dW^{(j)} = \delta_{ij}dt$.

We would like to quantify the amount of information the measurement provides as a function of time. Thus we must choose a measure of information. Previous research has used the logarithm of the infidelity, the log-infidelity [5], which is defined as $\ln \Delta$, where $\Delta = 1 - \lambda_0$, where λ_0 is the largest eigenvalue of ρ. This measure has two advantages: (1) it is often possible to get approximate closed form solutions for the log-infidelity and (2) the time τ at which the average log-infidelity $\langle \ln(\Delta(\tau))\rangle$ reaches a fixed log-infidelity $\ln \epsilon$ is related to the mean time $\langle T \rangle$ to reach that infidelity ϵ [5, 17].

In order to proceed in our analysis of information extraction rates in a register of qubits we need to formulate a solution to Eq. (4). This requires the machinery of linear quantum trajectories [13, 15]. The linear and consequently unnormalised version of Eq. (4) is

$$d\tilde{\rho} = \sum_{r=1}^n 2\gamma \, dt \, \mathcal{D}\left[Z^r\right]\tilde{\rho} + \sqrt{2\gamma}\, dR^{(r)}\tilde{\mathcal{H}}\left[Z^r\right]\tilde{\rho}, \quad (5)$$

where $\tilde{\mathcal{H}}[A]B = AB + BA^\dagger$ and the tilde denotes the lack of normalisation, and $dR^{(r)}$ is related to $dW^{(r)}$ analogously to dR in Eq. (2). The linear trajectory solution for this equation is

$$\tilde{\rho}(\mathbf{R}, t) = e^{-2\gamma It}e^{\sqrt{2\gamma}Z^n R^n}\dots e^{-2\gamma It}e^{\sqrt{2\gamma}Z^1 R^1}$$
$$\rho(0)e^{-2\gamma It}e^{\sqrt{2\gamma}Z^1 R^1}\dots e^{-2\gamma It}e^{\sqrt{2\gamma}Z^n R^n}, \quad (6)$$

where $R^r = \int_0^t dR^{(r)}$, and $\mathbf{R}$ is the vector of records $(R^1, R^2, \dots, R^n)$. We have also taken the initial state to be $\rho(0) = I/2^n$, corresponding to no information about the system. This assumption is not necessary for the asymptotic results below, but makes the derivation more elegant. The linear trajectories expression for the normalization factor of this state is

$$\mathcal{N}(\mathbf{R}, t) = \text{Tr}\left[\tilde{\rho}(\mathbf{R}, t)\right] = \frac{e^{-4\gamma nt}}{2^n}\sum_{q=0}^{2^n-1}e^{2\sqrt{2\gamma}\mathcal{R}^q}. \quad (7)$$

Here the $\mathcal{R}^q$ are linear combinations of the n "bare" records R^r. The records $\mathcal{R}^q$ hide a lot of the complexity of this expression. Thankfully the linear combinations are simply all the 2^n possible combinations of adding and subtracting the n bare records. The index q is actually an n digit binary string which describes how the records $\mathcal{R}^q$ are constructed from the bare records. A '0' in the qth position corresponds to a plus coefficient in front of the rth bare record, for $r = q + 1$ (recall that $q \in \{0, \dots, 2^n - 1\}$). A '1' in the qth position corresponds to a minus coefficient in front of that bare record. For example in a two qubit register the four $\mathcal{R}^q$'s would be $\mathcal{R}^{00} = R^1 + R^2$, $\mathcal{R}^{01} = R^1 - R^2$, $\mathcal{R}^{10} = -R^1 + R^2$, and $\mathcal{R}^{11} = -R^1 - R^2$.

The normalized expression for the conditional state is $\rho(\mathbf{R}, t) = \tilde{\rho}(\mathbf{R}, t)/\mathcal{N}$, which will always be diagonal in the logical basis. Thus, without loss of generality, we can define the local qubit bases (and the sign of the measurement records) such that $\lambda_{\bar{0}}$, the largest eigenvalue of ρ, corresponds to $|\bar{0}\rangle \equiv |00\dots0\rangle$. The expression for the largest eigenvalue is

$$\lambda_{\bar{0}} = \frac{\exp\left(2\sqrt{2\gamma}(R^1 + R^2\dots + R^r\dots + R^n)\right)}{\sum_{q=0}^{2^n-1}\exp\left(2\sqrt{2\gamma}\mathcal{R}^q\right)}, \quad (8)$$

here $\lambda_{\bar{0}}$ does not have the records $\mathbf{R}$ as an argument for notational compactness. It is known that measurement tends to cluster the large eigenvalues around the largest eigenvalue [5] (also see the appendix of [9]). That is, eigenvalues that are close to the largest eigenvalue contain most of the probability. This is graphically depicted in Fig. 2 b. Thus we only consider eigenvalues one Hamming unit away from the largest eigenvalue $\lambda_{\bar{0}}$ and truncate the normalization to a total of $(n + 1)$ terms. With this approximation and in the long-time limit the largest eigenvalue is

$$\lambda_{\bar{0}} = \frac{e^{2\sqrt{2\gamma}(R^1 + \dots R^n)}}{e^{2\sqrt{2\gamma}(R^1 + \dots R^n)} + \sum_{r=1}^n e^{2\sqrt{2\gamma}((R^1 + \dots R^n) - 2R^r)}}. \quad (9)$$

which simplifies to

$$\lambda_{\bar{0}} = \frac{1}{1 + \sum_{r=1}^n e^{-4\sqrt{2\gamma}R^r}}. \quad (10)$$

When $t \gg \gamma^{-1}$ the bare records will, on average, satisfy $R^1 \approx R^2 \approx \dots R^n \approx 2\sqrt{2\gamma}t$ (because $dR^r \approx 2\sqrt{2\gamma}dt + dW$) so that

$$\lambda_{\bar{0}} \approx \frac{1}{1 + ne^{-4\sqrt{2\gamma}(2\sqrt{2\gamma}t)}}. \quad (11)$$

Now that the expression for the largest eigenvalue is suitably simple, we can work out $\ln(1 - \lambda_{\bar{0}})$:

$$\ln(1 - \lambda_{\bar{0}}) \approx \ln\left(ne^{-16\gamma t}\right) = -16\gamma t + \ln n. \quad (12)$$

The long-time expression for $\langle \ln \Delta \rangle$ in the absence of feedback is thus

$$\langle \ln[\Delta(t)]\rangle_{\text{nfb}} \sim -16\gamma t. \quad (13)$$

From this relation we expect that the mean time to attain infidelity $\Delta = \epsilon$ is, for $\ln(\epsilon^{-1}) \gg 1$, [5, 17]

$$\langle T\rangle_{\text{nfb}} = (1/16\gamma)\ln(\epsilon^{-1}). \quad (14)$$

An astute reader will notice that the factor $\ln n$ did not turn up in Eq. (14), this is because it is negligible at long times as it scales as $\ln n/t$.

II. LOCALLY OPTIMAL RAPID MEASUREMENT

In this section we briefly review the rapid measurement protocol developed in Ref. [5]. The basic procedure is to derive a stochastic differential equation for the log-infidelty and then choose a control that maximizes the reduction in the log-infidelty.

Starting with the SME Eq. (4) we transform the observable Z^r to $Z^r - I$. We order the eigenvalues of ρ such that $\lambda_\alpha \geq \lambda_\beta$ when $\alpha < \beta$, where α, β are binary strings, i.e. $\lambda_{00\dots0} \geq \lambda_{00\dots01} \geq \dots \geq \lambda_{11\dots1}$. Again we assume the largest eigenvalue of ρ, i.e. $\lambda_{\bar{0}}$, is placed such that it corresponds to $|\bar{0}\rangle = |00\dots0\rangle$. Then from Eq. (4) the equation of motion for an arbitrary population of ρ, i, in the logical basis is

$$d\lambda_i = 2\sqrt{2\gamma}\sum_{r=1}^n dW^{(r)}(\langle i|Z^r|i\rangle - \text{Tr}[Z^r\rho])\lambda_i, \quad (15)$$

where we have used $\mathcal{D}[Z^r]\rho = 0 \,\forall r$. The equation for the largest eigenvalue $d\lambda_{\bar{0}}$ is

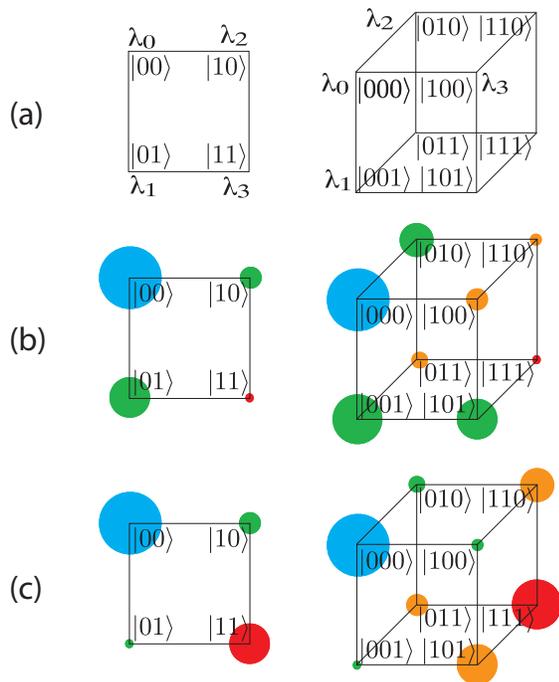$$d\lambda_{\bar{0}} = -2\sqrt{2\gamma}\lambda_{\bar{0}}\sum_{r=1}^n dW^{(r)}\text{Tr}[Z^r\rho]. \quad (16)$$

FIG. 2: To gain some intuition about the measurement process in a register we illustrate the effect of measurement on the eigenvalues of ρ by plotting them on a Hamming cube. A Hamming cube is a way of depicting the distance between n bit binary strings. The distance between two strings is given by the number of bit flips required to get from one string to the other. To plot the state matrix on the Hamming cube we associate the eigenstates with the vertices of the cube and the eigenvalues are placed on the near the vertices as in (a). We used the convention that the largest eigenvalue is always labeled λ_0. By appropriately relabelling the eigenstates one can always think of the largest eigenvalue being associated with the eigenstate $|\bar{0}\rangle$. In figure 1 (b) we plot how a continuous measurement affects the eigenvalues. We now represent the eigenvalues as circles. The magnitude of the eigenvalues is denoted by the size of the circle, while the distance in Hamming space is given by colour. For example blue and red are maximally distant in color space and so are the corresponding eigenvalues. From figure (b) we conclude that measurement tends to clump the large eigenvalues close to the largest eigenvalues. In figure (c) we depict the H-ordered state which is locally optimal for rapid measurement. Our open loop protocol randomly permutes the eigenvalues over all vertices of the Hamming cube.

Using the stochastic change of variables rule (i.e. Ito's lemma) the SDE for the log-infidelty is

$$d\ln(1-\lambda_{\bar{0}}) = d\lambda_{\bar{0}}d\ln(1-\lambda_{\bar{0}}) + \frac{1}{2}(d\lambda_{\bar{0}})^2d^2\ln(1-\lambda_{\bar{0}})$$
$$= \frac{-(d\lambda_{\bar{0}})^2}{2(1-\lambda_{\bar{0}})^2} + \frac{-d\lambda_{\bar{0}}}{(1-\lambda_{\bar{0}})} \quad (17)$$

Thus to complete the change of variables we also need to work out the $(d\lambda_{\bar{0}})^2$ term. Because it is more complicated

we write it in full:

$$(d\lambda_{\bar{0}})^2 = (-2\sqrt{2\gamma}\lambda_{\bar{0}})^2\sum_{r,s}dW^{(r)}dW^{(s)}\langle Z^r\rangle\langle Z^s\rangle$$
$$= 8\gamma\lambda_{\bar{0}}^2dt\sum_r\langle Z^r\rangle^2, \quad (18)$$

where we have used the fact that $dW^{(i)}dW^{(j)} = \delta_{ij}dt$. Finally we find the average rate of change of the log-infidelity to be [5]

$$\mathrm{E}\left[d\ln\Delta\right] = -4\gamma\,dt\sum_r\langle Z^r\rangle^2\frac{(1-\Delta)^2}{\Delta^2}. \quad (19)$$

Now we wish to maximize the average reduction of the log-infidelity for a given ρ using feedback. This is achieved by reordering the elements of ρ, so as to maximize $\sum_r\langle Z^r\rangle^2$, in the following way. By definition (above) the largest eigenvalue is at $|\bar{0}\rangle$. The second largest eigenvalue $\lambda_{00...01}$ is then placed at $|\bar{1}\rangle$ such that it is the maximum Hamming distance [18] away. The next n largest eigenvalues are placed at one Hamming unit away from $|\bar{1}\rangle$, the next $^nC_2$ largest eigenvalues are placed two Hamming units away from $|\bar{1}\rangle$, and so on. This ordering has been called H-ordering [5]. Example H-orderings for a two- and three-qubit register are depicted in the Fig. 2 c. A feedback protocol which H-orders at every time instant is said to be locally optimal (LO) in time. This should be contrasted to globally optimal protocols which require optimal control tools like dynamic programming [19, 20].

We now bound, for a register of qubits, the amount by which the H-ordering algorithm speeds up the measurement process. To do this we must bound

$$\sum_r\langle Z^r\rangle^2. \quad (20)$$

Consider an arbitrary state ρ with an infidelity Δ. The upper bound on Eq. (20) is obtained by considering the minimally mixed state with the same infidelity: $\rho_2 = \mathrm{diag}(1-\Delta,\Delta,0,\cdots,0)$ [5]. H-ordering this state corresponds to $\lambda_{0...0} = 1-\Delta$ placed at $|\bar{0}\rangle$ and the other eigenvalue $\lambda_{0...01} = \Delta$ is placed at $|\bar{1}\rangle$. The lower bound on Eq. (20) is obtained by considering the state: $\rho_F = (1-\Delta,\delta,\cdots,\delta)$ where $\delta = \Delta/(2^n-1)$. H-ordering this state corresponds to placing $\lambda_{0...0} = 1-\Delta$ at $|\bar{0}\rangle$ and distributing the residual probability equally over the remaining 2^n-1 eigenvalues. The reason why ρ_2 and ρ_F provide upper and lower bounds on Eq. (20) can be understood as follows. The feedback (H-ordering) is a permutation of the basis of ρ. Recall that the largest eigenvalue of ρ is fixed at $|\bar{0}\rangle$. The state ρ_2 is very sensitive to permutations, with respect to Eq. (20). Conversely the state ρ_F is invariant under permutations of the remaining 2^n-1 eigenvalues [with respect to Eq. (20)].

The using the states ρ_2 and ρ_F the it is simple to calculate the bounds on Eq. (20). The bounds are

$$[n2^{2n}/(2^n-1)^2]\Delta^2 \le \sum_r\langle Z^r\rangle^2 \le 4n\Delta^2. \quad (21)$$

In the long-time limit (i.e when $t \gg \gamma^{-1}$ or equivalently $\Delta \ll 1$) the bounds on log-infidelity are

$$\mathrm{E}\left[d\ln\Delta\right]_{\mathrm{LO}} = -16\gamma dt S_{\mathrm{LO}}, \qquad (22)$$

where S_{LO} is bounded [from Eq. (21)] as

$$\frac{2^{2n}}{(2^n-1)^2}\frac{n}{4} \leq S_{\mathrm{LO}} \leq n. \qquad (23)$$

The solution to Eq. (22) is bounded as

$$\mathrm{E}\left[\ln\Delta(t)\right]_{\mathrm{LO}} = -16\gamma t S_{\mathrm{LO}}, \qquad (24)$$

In the long-time limit we caculate a speedup factor from Eqs. (13) and (24). Denoting the time taken by a measurement without feedback to reach a given value of $\ln\Delta$ as t_{nfb}, and that for the feedback protocol as t_{fb}, we equate $\ln\Delta(t_{\mathrm{fb}}) = \ln\Delta(t_{\mathrm{nfb}})$ and solve for the ratio $t_{\mathrm{fb}}/t_{\mathrm{nfb}}$. In the long time limit we define the speed up to be $S = (t_{\mathrm{fb}}/t_{\mathrm{nfb}})^{-1}$. Doing so gives $S = S_{\mathrm{LO}}$. That is, we have upper- and lower-bounded the asymptotic speedup for the Locally Optimal closed-loop control strategy.

For $n \gtrsim 7$ the lower bound on S_{LO} is well approximated by $n/4$. Technically are bounds on the speed-up factor with respect to the $\langle\ln\Delta\rangle$ measure. It has been shown that these bounds well-approximate the behavior of the mean time $\langle T\rangle$ to a fixed infidelity and numerical results confirm the $O(n)$ scaling predicted in Eq. (23) [5]. The scaling found in the numerics of Ref. [5] was

$$S_{\mathrm{H}} = 0.718n. \qquad (25)$$

III.   RAPID MEASUREMENT USING OPEN LOOP CONTROL WITH RANDOM PERMUTATIONS

Inspired by Ref. [6] we will now replace the locally optimal feedback control from Sec. II with open loop control and quantum filtering. That is we combine the conditional evolution in Eq. (4) with a control strategy that does not depend on the conditional state. Since the LO feedback is simply a permutation of ρ's basis we will consider the effect of randomly permuting this basis at each instant in time. Working in the Heisenberg picture with respect to the control unitary, this is equivalent to permuting the basis of the observable.

Randomly permuting Z_r's basis at each instant the SME becomes

$$d\rho = \sum_{r=1}^{n} 2\gamma\, dt\, \mathcal{D}\left[PZ^rP\right]\rho + \sqrt{2\gamma}\, dW^{(r)}\mathcal{H}\left[PZ^rP\right]\rho \qquad (26)$$

where the permutation P is independently drawn from the permutation group $\mathfrak{P}(2^n)$ at each time interval. Since P is chosen uniformly from $\mathfrak{P}(2^n)$ for this protocol we explicitly average over all permutations in a single time step to find an analytical approximation to the average

speed-up. We denote the average over permutations symbolically by $\mathbb{E}\left[.\right]$. Averaging Eq. (26) over $\mathfrak{P}(2^n)$ we find

$$\mathbb{E}\left[d\rho\right] = \sum_{s=1}^{2^n}\sum_{r=1}^{n}\left(\frac{2\gamma}{D!}\, dt\, \mathcal{D}\left[P_sZ^rP_s\right]\rho + \right.$$
$$\left. \sqrt{\frac{2\gamma}{D!}}\, dW_{(s)}^{(r)}\mathcal{H}\left[P_sZ^rP_s\right]\rho\right) \qquad (27)$$

where $\mathbb{E}\left[\rho(t+dt)\right] = \mathbb{E}\left[\rho(t)\right] + \mathbb{E}\left[d\rho\right]$. The equation of motion for an arbitrary population $\mathbb{P}_i$ is given by $d\mathbb{P}_i = \mathrm{Tr}\left[\mathbb{E}\left[d\rho\right]\Pi_i\right]$ where $\Pi_i = |i\rangle\langle i|$ and $\mathbb{P}_i(0) = \mathrm{Tr}\left[\rho(0)\Pi_i\right]$. The equation analogous to Eq. (15) is

$$d\mathbb{P}_i = 2\sqrt{\frac{2\gamma}{D!}}\sum_{s=1}^{D!}\sum_{r=1}^{n} dW_{(s)}^{(r)}\left(\langle i|Z_s^r|i\rangle - \mathrm{Tr}\left[Z_s^r\mathbb{E}\left[\rho\right]\right]\right)\mathbb{P}_i, \qquad (28)$$

where $dW_j^\mu dW_i^\nu = \delta_{j,i,\mu,\nu}dt$. The average log-infidelity becomes

$$\mathrm{E}\left[d\ln(1-\mathbb{P}_0)\right] = -(d\mathbb{P}_0)^2/[2(1-\mathbb{P}_0)^2]. \qquad (29)$$

Recall that Eq. (26) is invariant under the transformation $Z^r \to Z^r - I$. Making that transformation we find that $\langle 0|Z_s^r|0\rangle = 0$, which gives

$$\mathrm{E}\left[d\ln(1-\mathbb{P}_0)\right] = -\frac{4\gamma\, dt}{D!}\frac{(1-\Delta)^2}{\Delta^2}\sum_s\sum_r\langle Z_s^r\rangle^2. \qquad (30)$$

Equation 30 is exact, and now we try and derive bounds on this expression. As before we only need to place bounds on $\sum_s\sum_r\langle Z_s^r\rangle^2$ so we substitute the state ρ_2 and ρ_F into Eq. (30), keeping terms $O(\Delta^2)$, to obtain the upper and lower bounds. Further steps are outlined in appendix A. In the long-time limit (i.e when $t \gg \gamma^{-1}$ or equivalently $\Delta \ll 1$) the bounds on log-infidelity for random permutation (RP) control are

$$\mathrm{E}\left[d\ln\Delta\right]_{\mathrm{RP}} = -16\gamma dt S_{\mathrm{RP}}, \qquad (31)$$

where the bounds on S_{RP}

$$\frac{2^{2n}}{(2^n-1)^2}\frac{n}{4} \leq S_{\mathrm{RP}} \leq \frac{2^{n-1}}{2^n-1}n. \qquad (32)$$

Following the argument in the preceding section, we can identify S_{RP} with the speed-up, in terms of the average time for Δ to attain a given value $\epsilon \ll 1$, relative to the no-control case. Notice that the lower bound coincides with the lower bound given in Eq. (23). For $n \gg 1$ the bounds become

$$0.25n \leq S_{\mathrm{RP}} \leq 0.5n. \qquad (33)$$

IV.   NUMERICAL ANALYSIS

While Eq. (33) tells us that the speed up is $\Theta(n)$ [23] it does not tell us if the speed up is closer to the upper

or lower limit. For this reason we perform numerical simulations and calculate the speed up and compare it to the bound.

Our simulation method is to solve equation Eq. (4) while applying a single random permutation at every time step. Then at each time step we calculate $(1-\lambda_{\max})$. By running many simulations we can build up statistics of the mean time to reach a fixed infidelity. The ratio of the mean time without control to the time with control is the speed up.

In Fig. 3 we plot the numerically calculated speed-up for two- and three-qubit registers that are subject to continuous monitoring and random permutation open loop control. In both cases the calculated speed-up is greater than one. The dashed red lines are our numerically estimated speed ups in the limit that $\Delta \to 0$. These lines are calculated as follows. We take the mean time to infidelities between 10^{-4} and 10^{-6}, for the no control and control cases, and perform linear regression to determine the slope. The ratio of these slopes is the expected speed up in the mean time to an infidelity of zero, which we call the *asymptotic speed up*.
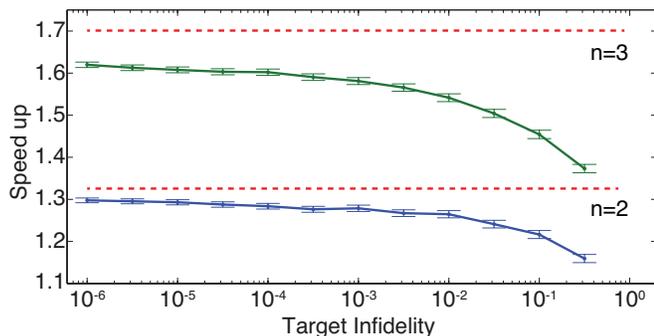
FIG. 3: The speed-up for a random permutation strategy for a register of n qubits with (from top to bottom) $n = 3$ (green) and $n = 2$ (blue) where $\delta t = 6.25 \times 10^{-4} \gamma^{-1}$. The dashed lines are the asymptotically calculated speedups. Averages were taken over 10,000 trajectories.

In order to determine the scaling of the speed up we must look at how the asymptotic speed up scales with n. Due to the difficulty of simulating large quantum systems we are restricted to $n \leq 5$. Time-asymptotic numerical results (see Fig. 4) indicate that random permutations in the logical basis give a similar improvement to the Hamming-ordered feedback scheme in. The fit shown is

$$S = 0.397n + 0.53, \tag{34}$$

as, unlike in the case of the Hamming-ordered feedback scheme, a strictly proportional fit is not adequate.

The fact that the numerically obtained speed-up is closer to the upper bound on Eq. (33) than the lower bound can be understood by the following qualitative analysis. The speed-up in the measurement comes from making the large eigenvalues of ρ more distinguishable
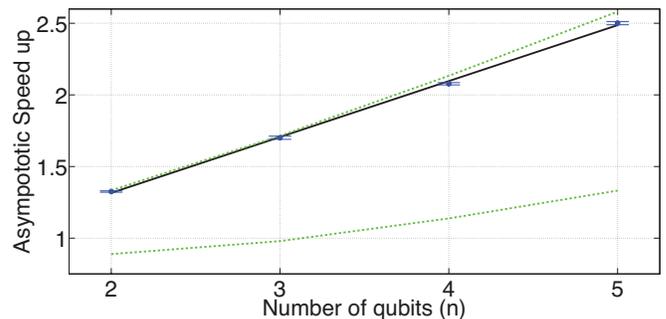
FIG. 4: The asymptotic speed-up in the mean time for a quantum register as a function of the number of qubits in the register. The numerically calculated asymptotic speed up are the data points with one standard deviation error bars. The dashed lines are the upper and lower bounds on the speed up given by Eq. (32). The solid line is the linear fit $S = 0.397n + 0.53$.

[5]. This can be related to the Hamming distance between the two largest eigenvalues of ρ. Now consider the Hamming distance between two binary strings, call it d_{H}, chosen at random with replacement (without replacement is asymptotically the same). The average distance is $\mathrm{E}[d_{\mathrm{H}}] \sim 0.5n$ [21] which is precisely the upper-bound of Eq. (33). Clearly we are not quite achieving that bound for small n.

V. DISCUSSION

We have shown that performing random permutations on a register of qubits increases the effective measurement strength and thus collapses the system into an eigenstate faster.

Because our open loop control procedures are effectively permutations of the eigenstates of the observed quantity, the control commutes with the measurement at all times. So while we have assumed our initial state was initially diagonal in the logical basis (the maximally mixed state) our protocol will work for any state. Specifically, at the end of the protocol one can calculationally retrodict the result of the unpermuted measurement. In some sense we can answer the counterfactual question "had I not performed the control what would my measurement result be?" Consequently our protocol is compatable with state and process estimation protocols such as tomography.

It seems possible that there exists deterministic strategies which may achieve approximate asymptotic speed ups similar to those reported here. For $n = 2$ we have found a single permutation, P_{3124}, which performs as well as the random strategy. The reason is simple, if the initial state is $\rho = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ the permutation maps ρ as follows $\mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \mapsto \mathrm{diag}(\lambda_1, \lambda_2, \lambda_0, \lambda_3) \mapsto \mathrm{diag}(\lambda_2, \lambda_0, \lambda_1, \lambda_3) \mapsto \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$. When visual-

ized on the Hamming cube this permutation makes all possible combinations of pairs of eigenvalues maximally distance at least once in the cycle. For larger registers it may be possible to find sequences with a small number of permutations that produce similar results. This is an open and interesting question for future research and would drastically reduce the requirements of our scheme.

Finally, and perhaps most importantly, we suggest that future work should include imperfections in the control protocols as was done in Ref. [22], such as finite number of permutations in a fixed time interval, to see the advantage of our protocol persists.

Appendix A: bounds on Eq. (30)

1. Upper bound

Recall that the operator $Z^r = I^{(1)} \otimes I^{(2)} \otimes \ldots \sigma_z^{(r)} \ldots \otimes I^{(n)}$ was transformed to $Z_r \mapsto Z_r - I$ and then we defined $Z_s^r = P_s Z^r P_s$ where P_s is an element of permutation group $\mathfrak{P}(2^n)$. Now consider the $D \times D$ ($D = 2^n$) matrix representation of Z_s^r

$$Z_s^r = \begin{pmatrix} Z_{s,11}^r & 0 & 0 & \cdots \\ 0 & Z_{s,22}^r & 0 & \cdots \\ 0 & 0 & Z_{s,33}^r & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (A1)$$

where $Z_{s,ij}^r \in \{0, -2\}$ is the element in the i'th row and j'th column of Z_s^r. Using the state $\rho_2 = \mathrm{diag}(1 - \Delta, \Delta, 0, \cdots, 0)$ we see that

$$\mathrm{E}\left[d\ln(1 - \mathbb{P}_0)\right] = -\frac{4\gamma\, dt}{D!} \frac{(1 - \Delta)^2}{\Delta^2} \sum_s \sum_r \langle Z_s^r \rangle^2 \quad (A2)$$

becomes

$$\mathrm{E}\left[d\ln(1 - \mathbb{P}_0)\right] = -\frac{4\gamma\, dt}{D!} \frac{(1 - \Delta)^2}{\Delta^2}$$
$$\times \sum_s \sum_r [Z_{s,11}^r(1 - \Delta) + Z_{s,22}^r \Delta]^2. \quad (A3)$$

In the expansion of the sum we only keep terms of order Δ^2 because of $(1/\Delta^2)$ out the front

$$\mathrm{E}\left[d\ln(1 - \mathbb{P}_0)\right] \approx -\frac{4\gamma\, dt}{D!} \sum_s \sum_r$$
$$[(Z_{s,11}^r)^2 - 2 Z_{s,11}^r Z_{s,22}^r + (Z_{s,22}^r)^2]. \quad (A4)$$

To make further progress we have two sums to work out. The first sum is

$$\sum_s (Z_{s,ii}^r)^2 = \frac{4}{2} D! = 2D! \quad \forall r \text{ and } i. \quad (A5)$$

The second sum is

$$\sum_s Z_{s,ii}^r Z_{s,jj}^r = 4 \frac{D}{2} \left(\frac{D}{2} - 1\right)(D - 2)!$$
$$= \left[1 - \frac{1}{D - 1}\right] D! \quad \forall r \text{ and } i \neq j. \quad (A6)$$

Using those expressions

$$\mathrm{E}\left[d\ln(1 - \mathbb{P}_0)\right] = -8\gamma\, dt \sum_r \frac{D}{D - 1} \quad (A7)$$

$$= -8\gamma n\, dt \frac{D}{D - 1} \quad (A8)$$

$$= -8\gamma n\, dt \frac{2^n}{2^n - 1} \quad (A9)$$

$$= -16\gamma n\, dt \frac{2^{n-1}}{2^n - 1}. \quad (A10)$$

2. Lower bound

The lower bound on Eq. (30) is obtained by the same procedure as above except we use the state $\rho_F = (1 - \Delta, \delta, \cdots, \delta)$ where $\delta = \Delta/(D - 1)$, instead of ρ_2. Now the important part is

$$\sum_s \sum_r \langle Z_s^r \rangle^2 =$$
$$\sum_s \sum_r \left[Z_{s,11}^r(1 - \Delta) + Z_{s,22}^r \delta + \ldots + Z_{s,nn}^r \delta\right]^2. \quad (A11)$$

When expanding this sum we use Eq. (A5) and Eq. (A6) as well as neglecting terms that do not have order Δ^2 to obtain

$$\sum_s \sum_r \langle Z_s^r \rangle^2 \approx \sum_s \sum_r \left\{ [\Delta^2 + (D - 1)\delta^2](Z_{s,11}^r)^2 + \right.$$
$$\left. [(D - 1)(D - 2)\delta^2 - 2(D - 1)\Delta\delta] Z_{s,11}^r Z_{s,22}^r \right\}. \quad (A12)$$

Summing over permutations and then simplifying we find

$$\sum_r \langle Z_s^r \rangle^2 = \Delta^2 \sum_r \left\{ \frac{D+2}{D-1} - \frac{D-2}{(D-1)^2} \right\} \tag{A13}$$

$$= \Delta^2 n \left\{ \frac{2^n+2}{2^n-1} - \frac{2^n-2}{(2^n-1)^2} \right\} \tag{A14}$$

$$= \Delta^2 n \frac{2^{2n}}{(2^n-1)^2} \tag{A15}$$

Finally we obtain

$$\mathrm{E}\left[d\ln(1-\mathbb{P}_0)\right] = -4\gamma n\, dt \frac{2^{2n}}{(2^n-1)^2} \tag{A16}$$

$$= -16\gamma\, dt \frac{n}{4} \frac{2^{2n}}{(2^n-1)^2}. \tag{A17}$$

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[23] The notation $X = \Theta(Y)$ means X is both upper and lower bounded by quantities scaling as Y.