CONNECTING ALGEBRAIC REPRESENTATIONS

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A growing number of teachers find themselves teaching middle years
mathematics without a great deal of training to do so. This is
particularly a cause of concern for them in the domain of algebra,
a topic they may remember from their own schooling as one based
upon symbolic manipulation. This paper is for those teachers. It
links various algebraic representations such that students can
make greater sense of them.

Introduction

This paper is designed for teachers of the lower middle school who have had little experience in teaching algebra, or experienced teachers who have tended to focus on teaching symbolic manipulation and wish to add to their repertoire. Thompson and Fleming (2003) noted that about a third of Year 8 mathematics teachers did not have mathematics as their major area of focus. Concerns to improve teachers' repertoires for teaching algebra are echoed in the recent MERGA submission to the National Numeracy Review (2007).

Defining algebraic activity in this article is guided by three ideas; firstly, generational activities, secondly, transformational activities and thirdly, meta-level activity (Stacey & Chick, 2004). Generational activities refer to situations, properties, patterns and relationships and how they are represented and interpreted. Transformational activities refer to changing the form of algebraic descriptions and are frequently linked with symbol manipulation. Finally, meta-level activity is related to using algebra in problem solving contexts, and to prove and notice structure. In the past, algebraic transformation has tended to dominate the teaching of algebra in the early secondary school years, and unfortunately this was frequently not strongly associated with meaning making. Kaput (1995, p. 4) reported that most students saw algebra as "little more than many different types of rules about how to write and rewrite strings of letters and numerals, rules that must be remembered for the next quiz or test."

The goal of this paper is to make the links between some simple generational activities and transformational processes, in particular, transforming relationships into different representations. It is intended to make the features of each representation clear and the links between other representations explicit. The main representations used are: material patterns, tables of ordered pairs, word descriptions of relationships, and symbolic equations. This is not the way the term transformation is usually used. It is usually associated with different ways or representing relationships symbolically.

Generalising from patterns of linear functions

Rich sources of generational activities are the texts books *Access to Algebra* (eg., Lowe, Johnson, Kissane, & Willis, 1993: *Access to Algebra: Book 2*). Suppose toothpicks are used to make the following shapes (Figure 1):

Figure 1. Toothpick pattern.

To add context, students could be informed that each enclosed area (or pen) can be used to contain a small animal (represented by a plastic figurine such as a pig). In the first pattern three toothpicks are used, in the second, five, and in the third, students can observe that seven toothpicks are used. They can be asked to:

- Extend the pattern
- Predict from the pattern how many toothpicks will be needed to make, say, 7 pens
- Predict how many pens can be made from, say, 21 toothpicks

The latter two prompts begin to focus the students' thinking upon the activity of generalisation. This can be further prompted by asking them to develop a "word rule" to enable the students to predict how many toothpicks are needed to make any number of pens designated. For this process I favour group work so students can learn in a social environment. They may develop rules such as:

"Well, you need one (toothpick) to start it off, and two for each pen."

or

"You need three for the first pen and two for each pen after that."

The pattern above (Figure 1) can also be represented in a table of ordered pairs as shown in Figure 2.

Number of pens	Number of toothpicks
1	3
+ 1	+ 2
2	5
+ 1	+ 2
3	7
+ 1	
4	

Figure 2. Table of ordered pairs for toothpick pattern in Figure 1.

There are a number of relationships that the students need to explore and then be able to make explicit. For example: For a given pen number there will be a predicted number of toothpicks needed to make that number of pens. For each new pen made, two toothpicks have to be added.

The same pattern can be represented in a graph. A student discussion can ensue to determine which axis to assign to the pens and which to assign to the toothpicks. In this instance, since we started with pens, and pens determines the number of toothpicks, we call the pens axis the x or independent axis, and the number of toothpicks can be represented on the dependent or y axis. Sometimes students may need to revisit co-ordinate geometry, for example, by playing battle ships, since and understanding of coordinates is a prerequisite of this form of data graphing. Figure 3 shows the graph of the pattern investigated in Figures 1 and 2.
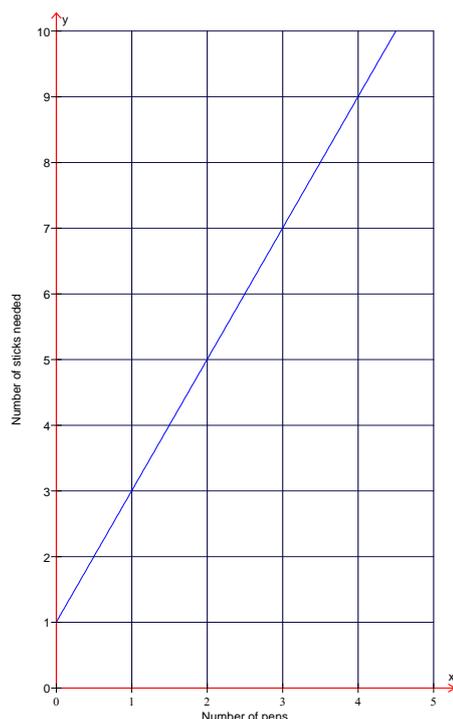
Connective features:

A number of features can be cross-linked between the toothpick pattern, the table of pairs representing the toothpick pattern, the word description, and the graph of the toothpick pattern presented here.

The y coordinate where the graph starts (cuts the y axis) corresponds to the first toothpick that is needed to start the pattern.

The line is straight.

In the pattern, for each increase in pens, two toothpicks are needed. This is the same in the table of pairs (Figure 2), and in this graph is reflected in the slope or gradient of two. Gradient is the change in the number of y (toothpicks) divided by the change in x (pens).

Figure 3. Graph of pens and toothpicks.

Each of these representations of the pattern in Figure 1 can be summarised by students in the short hand we know as algebraic symbolism. For example, students can discuss what letter can be used to represent the number of pens and the number of toothpicks. If the following labels were chosen:

Let p = the number of pens to be made.
Let t = the number of toothpicks needed to make a pattern.
Then the number of toothpicks needed equals twice the number of pens plus one more toothpick. Written symbolically this is:
$t = p \times 2 + 1$
Since there are conventions in algebra, the multiplication sign can be omitted and the numeral written first to make:
$t = 2p + 1$
In the general form this is $y = mx + c$ where m is the gradient and c is the y intercept. We can see from the materials that the y intercept corresponds to the starter toothpick and that the gradient is the extra two toothpicks that must be added to make each new pen.
The pedagogic model is summarised in Figure 4.

Patterns   Tables

                    Natural Language
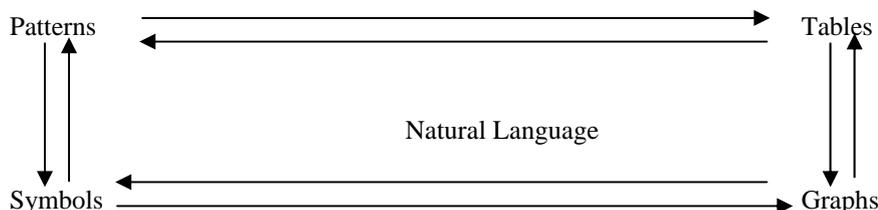

Symbols   Graphs

Figure 4. Pedagogic models for generalising algebra concepts.

It is important that students gain competency in making explicit links between the various representations. For example, they may be asked to represent a symbolic expression of the relationship $y = 3x + 2$ as a toothpicks pattern, a graph, a table of pairs or a word description. Materials such as toothpicks are of less use when representing relationships that have a negative gradient, particularly those with a negative y intercept. While materials are useful in establishing patterns, the tables, graphs and symbols themselves become objects to be considered independently of materials. That is the materials assist the students to develop generalities and abstractions. Materials are an aid to developing thinking, not a replacement for thinking.

Generalising from square patterns

The same processes can be extended to non linear patterns, that is, patterns where the gradient is not constant, where the line is not straight. Incidentally, some may notice that in $y = mx + c$; the power of x is one. That is $y = mx^1 + c$. Suppose we present students with a pattern of dots as shown in Figure 5. Students can readily see that a square pattern is presented.
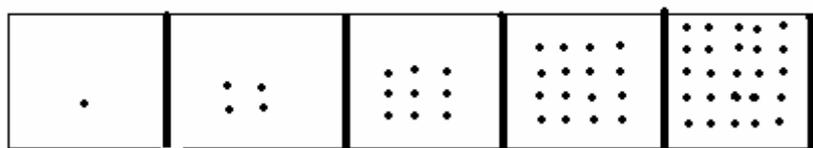

Figure 5. Square numbers dot pattern.

The pattern can be represented in words such as "the base number times itself will give the number of dots" and symbolically as $d = b^2$. Here, d is the total number of dots and b is the number of dots on the base of the square. The pattern can be represented in a table of ordered pairs (see Table 1).

Table 1: Dot pattern 1

Dots on base (b)	Total dots (d)	Change	Change in change
0	0		
1	1	1	
2	4	3	2
3	9	5	2
4	16	7	2
5	25	9	2

Unlike the linear equations, in this case the change in number of dots does not increase by a constant amount. However, students may notice that the change in change is constant and 2.

The index of b is 2, in contrast to the linear equations where the index was 1. These equations are called quadratic equations (why?). Other representations of the relationships, for example, the graphs, are also different. Students can explore these differences through discussion.

The data can be represented graphically, with the base number plotted on the horizontal (b) axis and the total number of dots on the vertical (d) axis. The d intercept can be seen to be zero, with the gradient increasing as the b increases. Students can transfer their learning about the y intercept from linear equations to the square pattern shown in Figure 6, where an extra dot is added to the square of dots. In this instance the d intercept is 1, that is, the equation becomes $d = b^2 + 1$. With the extra dot the ordered pairs become (0, 1); (1, 2); (2, 5); (3, 10); (4, 17); (5, 26), and so on.
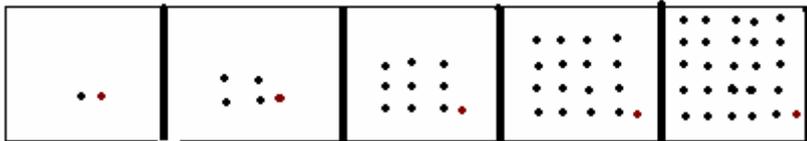

Figure 6. Dot pattern representing $d = b^2 + 1$.

By plotting the graphs, students can see the translation of the curve for $d = b^2 + 1$ compared with the graph of $d = b^2$. The curve $d = b^2 - 1$ can be seen by graphing the ordered pairs for squares in which one dot is removed.

In Figure 7, students can see that the pattern is b^2 plus another row of b, that is, $d = b^2 + b$. This can also be written as $d = b(b + 1)$. Such a rewrite, dependent upon the distributive law, is a transformational activity.
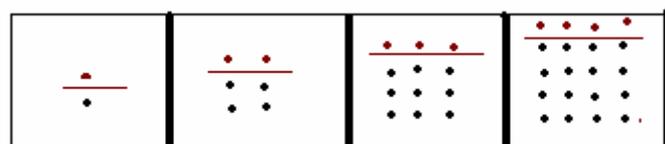

Figure 7. Dot pattern representing $d = b^2 + b$.

As for linear equations, students can explore and develop competency with simple quadratic functions; that is, they can and ought to have the opportunities to connect the various representations of these relationships as expressed in the form of materials (dots in this case), tables of ordered pairs, graphs, word descriptions and, finally symbolic equations. These activities can be supported with the use of technology such as graphing calculators or graphing packages where variables can readily be altered to enable students to gain an intuitive feel for families of functions. Software such as MathsHelper (Vaughan, 1997) can enable students to readily explore the nature of functions by systematically plotting the functions and generating tables of ordered pairs quickly.

The use of materials such as dots or toothpicks have their limitation, not least the representation of negative values. Thus the examples above have tended to focus upon those examples that avoid this limitation. The use of materials is not "the answer," but rather part of a way of enabling students to look at patterns from different perspectives and with different representational tools.

Competency in recognising the different descriptions of linear and quadratic equations can be gained with practice. A fun and social way to practice is via the use of algebra games. For example, Booker (2000) describes how games can readily be constructed and used. In the construction of games designed to consolidate the key mathematical concepts associated with the generational and transformational algebraic processes described above students would be required to analyse, compare and match the representations.

Concluding remarks

MacGregor (2004) indicated that only about 20 per cent of 14 year olds had the literacy and numeracy skills required to cope with algebra study. Stacey and MacGregor (1999) have reported that the major reason for student difficulties with using algebraic methods for problem solving is that they do not understand the underpinning logic of algebra. Students tend to wish to calculate in the first instance, which is a behaviour consistent with their arithmetic learning. However, algebra requires an analysis of the problem and transformation of it into algebraic rules before procedures of symbolic transformation such as simplification, expansion and solving can be attempted. That is, students need to recognise, construct and manipulate algebraic structures before applying their computational skills and transformational processes. Some students require limited opportunities to work with materials; these students need not be forced to engage with activities that do not help them. Other students are likely to need greater opportunities to see the links between representations before they can generalise from the experience and move towards working with purely symbolic models. The pedagogical model and activities described in this paper give students opportunities to perceive algebra as "do-able," "non-mystical," generative, transformational and a problem solving activity.

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