Update to "Modeling water erosion due to overland flow using physical principles: 1. Sheet flow"

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[1] We provide corrections to several key formula by Hairsine and Rose (1992) for steady state flow driven sediment transport and at the same time generalize their solution for the case of entrainment limiting flow when the mass of sediment in each size class is arbitrary.

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1. Introduction

[2] *Hairsine and Rose* [1992] presented a theory for the erosion and transport of a multiparticle sized soil when only flow driven erosion processes are operating. When previously eroded sediment settles back on the soil bed it provides a level of protection H ($0 \leq H \leq 1$) to the original soil. *Hairsine and Rose* [1992] provide steady state solutions for the two cases of transport limiting ($H = 1$) and entrainment limiting ($H < 1$) sediment transport. This note is concerned with correcting several slight errors in the original solution for the entrainment-limiting case.

[3] The steady state equations of the Hairsine-Rose model for suspended sediment are given by

$$\frac{d(qc_i)}{dx} = r_i + r_{ri} - d_i \quad i = 1, 2 \ldots I, \tag{1}$$

and

$$\frac{\partial M_i}{\partial t} = d_i - r_{ri} \quad i = 1, 2 \ldots I, \tag{2}$$

where t is time (s), I is the number of particle size classes, i is a counter denoting size class with $i = 1$ being the smallest, c_i (kg m^{-3}) is the sediment concentration in each class and M_i (kg m^{-2}) is the mass of sediment of size class i in the deposited layer. The total suspended sediment concentration, c, and total mass in the deposited layer, M_t, is then found by summing across all size classes as $c = \sum_{i=1}^{I} c_i$ and $M_t = \sum_{i=1}^{I} M_i$. The erosion source and sink terms in (1) and (2) denote the entrainment rate r_i (kg m^{-2} s^{-1}) of original soil in each size class, the deposition rate d_i (kg m^{-2} s^{-1}) of suspended sediment and the reentrainment rate r_{ri} (kg m^{-2} s^{-1}) of deposited sediment. These can be described by

$$r_i = p_i(1 - H)\frac{F}{J}(\Omega - \Omega_0), \tag{3}$$

$$r_{ri} = H\frac{F}{gh}\left(\frac{\rho_s}{\rho_s - \rho}\right)(\Omega - \Omega)\frac{M_i}{M_t}, \tag{4}$$

and

$$d_i = v_i c_i, \tag{5}$$

in which p_i ($0 < p_i \leq 1$ and $\sum_{i=1}^{I}$ pi = 1) is the proportion of sediment in each size class of the original uneroded soil, F is the fraction of excess stream power which is effective in entrainment, J (J kg^{-1}) is the experimentally determined specific energy of entrainment, g is the acceleration due to gravity, ρ and ρ_s are the water and sediment densities respectively, h is the flow depth, v_i are the fall velocities, Ω (W m^{-2}) = $\rho g S_o q$ is the stream power with Ω_0 being the critical threshold stream power and S_o is the bed slope. By allowing for arbitrary proportions of sediment in each size class through (3), we generalize the results of *Hairsine and Rose* [1992], which required equal size class proportions of $p_i = 1/I$.

[4] *Hairsine and Rose* [1992] gave the steady state water flow by the approximate solution of *Rose et al.* [1983]

$$q = Qx, \tag{6}$$

in conjunction with the kinematic approximation

$$q = Kh^m, \tag{7}$$

where Q (m s^{-1}) is the runoff rate per unit area, m is a flow parameter and $K = S_o^{1/2}/n$ with n being Manning's roughness coefficient.

2. Entrainment Limiting Solution ($H < 1$)

[5] Under net eroding conditions the mass of sediment in the deposited layer achieves equilibrium and $\partial M_i/\partial t = 0$, thus from (2) $d_i = r_{ri}$ and from (4) and (5)

$$H = \frac{gh}{F(\Omega - \Omega_0)}\left(\frac{\rho_s - \rho}{\rho_s}\right)\sum_{i=1}^{I} v_i c_i. \tag{8}$$

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Combining (1), (3) and (6) to (8) along with $d_i = r_{ri}$ results in

$$q\frac{dc_i}{dq} + c_i = p_i(a_1 q - a_0) - p_i \eta q^{1/m} \sum_{i=1}^{I} c_i v_i, \tag{9}$$

where

$$\eta = \left(\frac{\rho_s - \rho}{\rho_s}\right)\frac{g}{QJK^{1/m}}, \qquad a_1 = \frac{F\rho g S_0}{QJ}, \qquad a_0 = \frac{F\Omega_0}{QJ}. \tag{10}$$

By multiplying (9) by v_i, defining a new variable

$$w_i = v_i c_i, \tag{11}$$

and summing across all size classes (with $w = \Sigma w_i$), then (9) can be written as

$$\frac{dw}{dq} + \left(1 + \lambda q^{1/m}\right)\frac{w}{q} = \sum_{i=1}^{I} \nu_i p_i (a_1 - a_0/q), \tag{12}$$

where $\lambda = \eta \sum_{i=1}^{I} v_i p_i$. Applying the boundary condition that $c_i = 0$ at the threshold of entrainment $\Omega = \Omega_0$ or $q = q_0$, then the solution to (12) is

$$\frac{w u^m e^{m\lambda u}}{m \sum_{i=1}^{I} v_i p_i} = a_1 \int_{u_0}^{u} z^{2m-1} e^{m\lambda z}\, dz \; - \; a_0 \int_{u_0}^{u} z^{m-1} e^{m\lambda z}\, dz, \tag{13}$$

with u = $q^{1/m}$. Substituting for $w = \sum v_i c_i$ then (13) yields

$$c_i = p_i c, \tag{14}$$

where the total concentration c is given from

$$c\frac{u^m}{m} e^{m\lambda u} = a_1 \int_{u_0}^{u} z^{2m-1} e^{m\lambda z}\, dz \; - \; a_0 \int_{u_0}^{u} z^{m-1} e^{m\lambda z}\, dz. \tag{15}$$

For integer m (15) can be integrated to give

$$\begin{aligned} c = {} & \frac{a_1}{\lambda} u^{m-1}\left[1 - \frac{2m-1}{m\lambda u} + \frac{(2m-1)(2m-2)}{(m\lambda u)^2} + \ldots \frac{(-1)^k (2m-1)(2m-2)(2m-3)..(2m-k)}{(m\lambda u)^k}\right] \\ & - \frac{a_0}{\lambda u}\left[1 - \frac{m-1}{m\lambda u} + \frac{(m-1)(m-2)}{(m\lambda u)^2} + \ldots \frac{(-1)^j (m-1)(m-2)(m-3)..(m-j)}{(m\lambda u)^j}\right] \\ & + \frac{u_0^{m-1} e^{-m\lambda(u-u_0)}}{\lambda u^m}\left\{\begin{array}{l} a_0\left[1 - \frac{m-1}{m\lambda u_0} + \frac{(m-1)(m-2)}{(m\lambda u_0)^2} + \ldots \frac{(-1)^j (m-1)(m-2)(m-3)..(m-j)}{(m\lambda u_0)^j}\right] \\ -a_1 u_0^m \left[1 - \frac{2m-1}{m\lambda u_0} + \frac{(2m-1)(2m-2)}{(m\lambda u_0)^2} + \ldots\ldots \frac{(-1)^k (2m-1)(2m-2)(2m-3)..(2m-k)}{(m\lambda u_0)^k}\right] \end{array}\right\} \end{aligned} \tag{16}$$

where the summation terms in (16) continue until $k = 2m - 1$ and $j = m - 1$. Note that term containing the exponential coefficient in (16) replaces the entire exponential term in (13) of *Hairsine and Rose* [1992]. We note that as the correction occurs in the term with the exponential coefficient, it is only significant near the boundary condition $x = x_0 = q_0/Q$ (from (6)) and decreases rapidly as x or q increases. Consequently comparisons with any data obtained from the end of a flume would not notice the error in the original solution. For noninteger m, (16) becomes a diverging infinite series and the correct solution must be calculated directly from (15), though it can also be written in terms of incomplete gamma functions as

$$\begin{aligned} cu^m e^{m\lambda u} = {} & \frac{ma_1}{(-m\lambda)^{2m}}\left[\Gamma(2m, -m\lambda u_0) - \Gamma(2m, -m\lambda u)\right] \\ & - \frac{ma_0}{(-m\lambda)^m}\left[\Gamma(m, -m\lambda u) - \Gamma(m, -m\lambda u_0)\right]. \end{aligned} \tag{17}$$

Particular values of integer m for which may be of interest are 2 and 3 for which (16) becomes

$$\begin{aligned} & m = 2\left(\xi = \lambda u = \lambda\sqrt{Qx}\right) \\ & c = \frac{a_1}{\lambda^2}\left(\xi - \frac{3}{2} + \frac{3}{2\xi} - \frac{3}{4\xi^2}\right) - \frac{a_0}{\xi}\left(1 - \frac{1}{2\xi}\right) \\ & \quad + \frac{e^{2(\xi_0 - \xi)}}{\xi^2}\left\{a_0\left(\xi_0 - \frac{1}{2}\right) - \frac{a_1}{\lambda^2}\left(\xi_0^3 - \frac{3}{2}\xi_0^2 + \frac{3}{2}\xi_0 - \frac{3}{4}\right)\right\} \end{aligned} \tag{18}$$

$$\begin{aligned} & m = 3\left(\xi = \lambda u = \lambda (Qx)^{1/3}\right) \\ & c = \frac{a_1}{\lambda^3}\left(\xi^2 - \frac{5}{3}\xi + \frac{20}{9} - \frac{20}{9\xi} + \frac{40}{27\xi^2} - \frac{40}{81\xi^3}\right) \\ & \quad - \frac{a_0}{\xi}\left(1 - \frac{2}{3\xi} + \frac{2}{9\xi^2}\right) + \frac{e^{3(\xi_0 - \xi)}}{\xi^3} \\ & \quad \cdot \left\{a_0\left(\xi_0^2 - \frac{2}{3}\xi_0 + \frac{2}{9}\right) - \frac{a_1}{\lambda^3}\left(\xi_0^5 - \frac{5}{3}\xi_0^4 + \frac{20}{9}\xi_0^3 - \frac{20}{9}\xi_0^2 \right.\right. \\ & \quad \left.\left. + \frac{40}{27}\xi_0 - \frac{40}{81}\right)\right\}. \end{aligned} \tag{19}$$

For the special case where the threshold of entrainment is zero $a_0 = u_0 = 0$, (18) and (19) simplify to

$$m = 2\left(\xi = \lambda\sqrt{Qx}\right) \qquad \frac{\lambda^2 c}{a_1} = \xi - \frac{3}{2} + \frac{3}{2\xi} - \frac{3}{4\xi^2}\left(1 - e^{-2\xi}\right), \tag{20}$$

$$m = 3\left(\xi = \lambda (Qx)^{1/3}\right) \qquad \frac{\lambda^3 c}{a_1} = \xi^2 - \frac{5\xi}{3} + \frac{20}{9} - \frac{20}{9\xi} + \frac{40}{27\xi^2} - \frac{40}{81\xi^3}\left(1 - e^{-3\xi}\right). \tag{21}$$

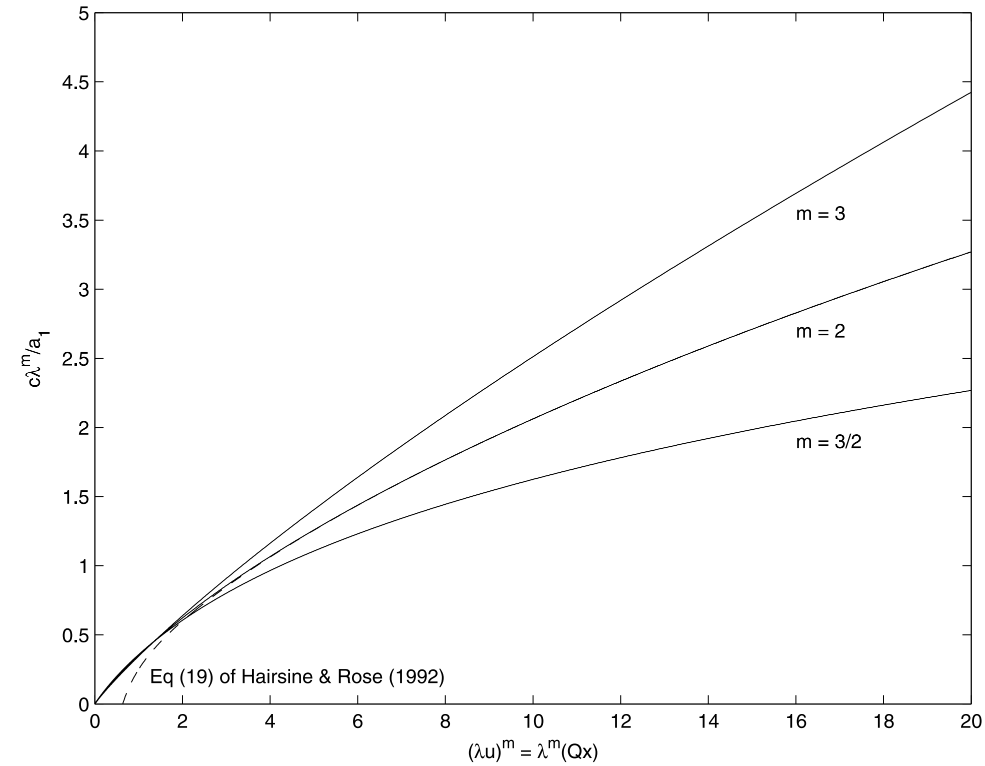

Figure 1. Dimensionless suspended sediment concentration as a function of dimensionless distance for m as indicated.

[6] We note again that there is a mistake in equation (19) of *Hairsine and Rose* [1992] given for m = 2. The correct form is given by (20) which includes the additional term $3e^{-2\xi}/4\xi^2$. For turbulent overland flow m is taken as 5/3 and the integral form of either (15) or (17) must be used. However for $m = 3/2$ and zero threshold of entrainment ($a_0 = u_0 = 0$), the series solution (16) applies out to $k = 3$ since $2m$ is again integer resulting in

$$\frac{\lambda^{\frac{3}{2}}c}{a_1} = \sqrt{\xi}\left(1 - \frac{4}{3\xi} + \frac{8}{9\xi^2}\left(1 - e^{-3\xi/2}\right)\right), \tag{22}$$

where $\xi = \lambda(Qx)^{2/3}$.

3. Discussion

[7] Figure 1 shows a graph of (20), (21), and (22) as well as equation (19) of *Hairsine and Rose* [1992]. Thus it is confirmed that the correction is only significant for small ξ as noted earlier, and in particular for m = 2 when $\xi^2 < 2$.

[8] If we expand each of (20), (21) and (22) for $\xi \to 0$, then c behaves as

$$\frac{c}{a_1} = \frac{Qx}{2}\left[1 - \lambda\left(m - \frac{2m^2}{2m+1}\right)(Qx)^{\frac{1}{m}} + \ldots\right], \tag{23}$$

while from (8), $H = \lambda u^{1-m}c/a_1$, and therefore

$$H = \frac{\lambda}{2}(Qx)^{\frac{1}{m}}\left[1 - \lambda\left(m - \frac{2m^2}{2m+1}\right)(Qx)^{\frac{1}{m}} + \ldots\right]. \tag{24}$$

Thus for ξ small (or $\lambda \ll 1/(Qx)^{1/m}$ for x fixed), c to first order is independent of both the settling velocity distribution and m. This region of the flow is dominated by the suspension of particles and deposition is a second-order effect as shown by (24) which has the deposited layer behaving as $O(\lambda)$ while c behaves like $O(1)$ as $\lambda \to 0$.

[9] If we now consider the limit of (20), (21) and (22) for $\xi \to \infty$ we find

$$\frac{c}{a_1} = \frac{(Qx)^{1-1/m}}{\lambda}\left[1 - \frac{2m-1}{m\lambda}(Qx)^{-1/m} + \ldots\right], \tag{25}$$

and

$$H = 1 - \frac{2m-1}{m\lambda}(Qx)^{-1/m} + \ldots \ . \tag{26}$$

Thus for ξ large (or $\lambda \gg 1/(Qx)^{1/m}$ for x fixed), deposition now dominates the steady state solution as to first order H = 1 (complete coverage) while c behaves like $O(1/\lambda)$ as $\lambda \to \infty$.

[10] It is quite common for entrainment experiments to be carried out for constant flow rate and depth, e.g., *Polyakov and Nearing* [2003]. Under these conditions of q and h constant, the entrainment limiting solution of (1) and (2), is given more simply by [*Rose et al.*, 2006; *Sander et al.*, 2007] as

$$c = \gamma\left(1 - e^{-\beta x/\gamma}\right), \tag{27}$$

where

$$\gamma = \frac{F}{gh}\left(\frac{\rho_s}{\rho_{s-}\rho}\right)\frac{(\Omega - \Omega_0)}{\sum_{i=1}^{I} p_i v_i}, \tag{28}$$

$$\beta = \frac{F}{Jq}(\Omega - \Omega_0), \tag{29}$$

and

$$H = 1 - e^{-\beta x/\gamma}, \tag{30}$$

with the individual size classes, c_i still given through (14).

[11] In summary we have given corrections to equations (13) and (19) of *Hairsine and Rose* [1992] through (16) and (20) above, respectively. In addition we have also generalized their results to account for arbitrary mass in each size class through p_i in (3).

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