Title: LATERALLY LOADED RIGID PILES IN COHESIONLESS SOIL

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ABSTRACT

In this paper, limiting force profile for laterally loaded rigid piles in sand is differentiated from on-pile force profile, from which elastic-plastic solutions are established and presented in explicit expressions. Nonlinear responses of the piles are characterised by slip depths mobilised from mudline and the pile-tip. At the states of tip-yield and rotation point yield, expressions for some critical depths are developed, which allow the on-pile force profiles to be constructed. The solutions and the expressions are developed concerning a constant subgrade modulus \( k \) and a linearly increasing modulus with depth (Gibson \( k \)), respectively. Capitalised on three measurable parameters, the solutions agree well with measured data and numerical predictions. Non-dimensional responses are presented for various eccentricities at the tip-yield state and in some cases at the rotation point yield state. Nonlinear responses are obtained for typical eccentricities from elastic state right up to failure. Comments are made regarding displacement based capacity. A case study is elaborated to illustrate (i) the use of the current solutions; (ii) the impact of the \( k \) distributions; (iii) the evaluation of stresses on pile surface; and (iv) the deduction of soil modulus. The current solutions are easy to be implemented and suitable for general design.

Keywords: piles, lateral loading, rigid piles, nonlinear response, closed-form solutions
1. INTRODUCTION

In-situ full-scale and laboratory model tests (Mayne et al. 1995) were widely conducted on laterally loaded rigid piles (including piers and drilled shaft etc.). They have enabled simple expressions to be established for computing bearing capacity. To assess nonlinear pile-soil interaction, centrifuge and numerical finite element (FE) modelling (Laman et al. 1999) were fulfilled previously. These tests and modelling uphold that the success of any design largely relies on input parameters used for describing in particular, the distributions of soil shear modulus ($G_s$) and limiting force per unit length along the pile ($p_u$ profile, also termed as LFP). Naturally, various techniques have been proposed to gain these parameters, albeit generally for either elastic or ultimate state. The accuracy of these expressions, modelling and techniques is not the motivation for the current study. However, it is not clear about the difference between the $p_u$ profile and the resistance per unit length along a pile (herein referred to as on-pile force profile). Its impact on design needs to be quantified. In addition, a stringent nonlinear model is indeed required to uniquely back-estimate the parameters upon measured nonlinear response, to capture overall pile response at any stage, as the model may then be utilised as a boundary element for simulating more complex soil-structure interaction.

Available expressions are generally deduced in light of force equilibrium on the pile (see Fig. 1), and bending moment equilibrium about the pile-head or tip. The force in turn is calculated utilising a postulated on-pile force profile, which differs from one author to another (Brinch Hansen 1961; Broms 1964; Petrasovits and Award 1972; Meyerhof et al. 1981; Prasad and Chari 1999). The on-pile force profile actually characterises the mobilisation of the resistance along the unique $p_u$ profile (independent of pile displacement). It appears as the solid lines shown in Figs. 2a-2c-left, which is mobilised along the stipulated linearly increasing LFP (Broms 1964) indicated by the two dashed lines. It is elaborated herein for four typical states of yield between pile and soil.

(1) **Pre-tip yield and tip-yield states**: Tip yield refers to that the force per unit length at pile tip just attains the limiting $p_u$ [Fig. 2a-left]. Prior to and upon the tip-yield state, the on-pile force profile follows the positive LFP down to a depth of $z_o$, below which, it is governed by elastic interaction. This generates an on-pile force profile akin to that adopted by Prasad and Chari (1999), as explained later in this paper.
(2) Post-tip yield state: Further increase in load beyond the tip yield state enables the limiting force to be fully mobilised from the tip as well and to a depth of \( z_1 \). This induces a new portion of the on-pile force profile that follows the negative LFP over the depth \( l \) to \( z_1 \). The overall profile is more or less like that illustrated in Fig. 2b.

(3) Continual increase in the load renders the depths \( z_o \) and \( z_1 \) to approach each other. The depths tend to merge eventually with the depth of rotation \( z_r \) (i.e. \( z_o = z_r = z_1 \)), which is practically unachievable. To the depth \( z_r \), the on-pile force profile now follows the positive LFP from the head; and the negative LFP from the tip. This impossible ultimate on-pile force profile is referred to as fully plastic (ultimate) state and is depicted in Fig. 2c. It is adopted previously by some investigators (Brinch Hansen 1961; Petrasovits and Award 1972).

Any solutions underpinned by a single stipulated on-pile force profile unfortunately would not guarantee compatibility with the altering profiles (see Fig. 2-left) recorded at different loading levels for a single pile.

Capacity of a laterally loaded pile is defined as the load at a specified displacement, say, 20% pile diameter upon a measured load-displacement curve (Broms 1964). It is also taken as the load inducing a certain rotation angle upon a measured load-rotation curve (Dickin and Nazir 1999). These two independent definitions would not normally yield identical capacities even for the same test. To resolve this artificial impingement, new displacement-based solutions need to be developed.

In this paper, elastic-plastic solutions are developed for laterally loaded rigid piles. They are endeavoured to be presented in explicit expressions, to facilitate the computation of:

- The distributions of force, displacement, rotation, and bending moment down the pile;
- The slip depths from mudline or pile tip, and the depth of rotation.

The solutions thus allow (1) nonlinear responses of the piles to be predicted; (2) the on-pile force profiles at any loading levels to be constructed; (3) the new yield (critical) states to be determined; and (4) the displacement-based pile capacity to be estimated. By stipulating a linear LFP, the study primarily aims at piles in sand, assuming a uniform modulus with depth or a linearly increasing modulus.

A spreadsheet program will be developed to facilitate numeric calculation of the solutions. Comparison with measured data and FE analysis will be presented to illustrate the accuracy and highlight the characteristics of the new solutions. An example study will
be elaborated to show all the aforementioned facets, apart from the impact of modulus profile, the back-estimation of soil modulus, and the calculation of stress distribution around a pile surface.

2. ELASTIC-PLASTIC SOLUTIONS

2.1 Features of Laterally Loaded Rigid Piles

Under a lateral load, $T_i$ applied at an eccentricity, $e$ above mudline, elastic-plastic solutions for infinitely long, flexible piles have been developed previously (Guo 2006). Capitalised on a generic LFP, these solutions, however, do not allow pile-soil relative slip to be initiated from the pile tip. Thus, theoretically speaking, they are not applicable to a rigid pile discussed herein. With regard to a rigid pile in sand, a linear LFP may be stipulated, upon which the on-pile force (pressure) profile alters as presented previously in Figs. 2a-2c-left.

2.2 Coupled Load Transfer Model

Modelling force (stress) development along the pile and in the surrounding soil in elastic state is critical to development of new solutions. Under lateral loading, stress distribution is unsymmetrical, and shows three-dimensional (3-D) features. Its solution has in general been resorted to complex numerical methods. The essence in the modelling, however, is to capture (1) the non-uniform distribution of force (pressure) on the pile surface in radial (Prasad and Chari 1999) and longitudinal dimensions; and (2) the alteration of modulus of subgrade reaction due to non-uniform soil displacement field around the pile. These two features nowadays can be well captured using a load transfer approach (Guo and Lee 2001) deduced from a simplified soil-displacement model. The approach requires much less computing effort than numerical modelling, it is thus utilised herein to model elastic response. Pertinent expressions/conclusions are recaptured herein:

(1) The stresses change in the soil around a horizontally loaded pile may be approximated by [also owing to Sun and Pires (1993)]:

$$\sigma_r = 2G \mu \frac{\gamma^b}{r_o} \frac{K_1(y^b r/r_o)}{K_0(y^b)} \cos \theta_p \quad \sigma_\theta = \sigma_z = 0$$
\[ \tau_{r\theta} = -G_s \frac{K_1(\gamma_b r/r_o)}{K_0(\gamma_b)} \sin \theta_p \]

\[ \tau_{r\phi} = -G_s \omega \frac{K_0(\gamma_b r/r_o)}{K_0(\gamma_b)} \sin \theta_p \]

\[ \tau_{z\theta} = G_s \omega \frac{K_1(\gamma_b r/r_o)}{K_0(\gamma_b)} \cos \theta_p \]

where \( G_s \) is soil shear modulus; \( \sigma_r \) is radial stress; \( \sigma_\theta \), \( \sigma_z \) are circumferential stress, and vertical stress, which are negligible; \( u \), and \( \omega \) are local lateral displacement, and rotational angle of the pile body at depth \( z \); \( \theta_p \) is an angle between the direction of the loading and the line joining the centre of the pile cross-section to the point of interest; \( r \) is a radial distance from the pile axis; \( K_i(\gamma_b) \) \((i = 0, 1)\) is modified Bessel functions of the second kind, and of order \( i \); \( r_o \) is an outside radius of a cylindrical pile. The factor \( \gamma_b \) in eq. [1] may be estimated by

\[ \gamma_b = k_1(r_o/l) \]

where \( l \) is the pile embedded length; \( k_1 \) is 2.14, and 3.8 for pure lateral loading (free length or eccentricity, \( e = 0 \)), and pure moment loading (\( e = \infty \)), respectively. \( k_1 \) may be stipulated to increase hyperbolically from 2.14 to 3.8 as the free length \( e \) increases from mudline to infinitely large.

(2) Radial stress \( \sigma_r \) and shear stress \( \tau_{r\theta} \) are proportional to the local displacement \( u \) (see eq. [1]). The stress \( \sigma_r \) can be well predicted (see Fig. 3) using eq. [1] compared with measured data, as elaborated later in the section entitled \textit{Case Study}.

(3) A pile is defined as rigid, should the pile-soil relative stiffness, \((E_p/G_s)\) exceed a critical ratio \((E_p/G_s)c\), where \((E_p/G_s)c = 0.052(l/r_o)^4\); and \( E_p \) is Young’s modulus of an equivalent solid pile. Given \( l/r_o = 12 \), for instance, the critical ratio \((E_p/G_s)c\) is 1,078.

In drawing the abovementioned conclusions, the pile-soil interaction is characterised by a series of springs distributed along the pile shaft and within elastic state. In reality, each spring has a limiting force per unit length \( p_u \) at a depth \( z \) \([\text{FL}^{-1}]\). If less than the limiting value \( p_u \), the on-pile force \((\text{per unit length})\), \( p \) at any depth is proportional to the local displacement, \( u \) and to the modulus of subgrade reaction, \( kd \) \([\text{FL}^{-2}]\) (see Fig. 1b):

\[ p = kdu \]

(Elastic state)
where \( d \) is pile outside diameter or width \([\text{L}]\); \( p \) is force per unit depth \([\text{FL}^{-1}]\) for elastic zone. The gradient \( k \) \([\text{FL}^{-3}]\) may be written as \( k_o z^m \) \([k_o, \text{FL}^{-m-3}]\), with \( m = 0 \) and \( 1 \) being referred to as Constant \( k \) and Gibson \( k \) hereafter.

(1) The magnitude of a constant \( k \) may be related to the shear modulus \( G_s \) by

\[
kd = \frac{3\pi G_s}{2} \left\{ 2\gamma_b \frac{K_1(\gamma_b)}{K_o(\gamma_b)} - \gamma_b^2 \left[ \frac{K_1(\gamma_b)}{K_o(\gamma_b)} \right]^2 - 1 \right\} \quad \text{(Constant \( k \))}
\]

(2) The average modulus of subgrade reaction concerning a Gibson \( k \) is \( k_o (l/2) d \), for which the shear modulus \( G_s \) in eq. \([4]\) is replaced with an average \( \overline{G}_s \) over the pile embedment. This allows eq. \([4]\) to be rewritten more generally as

\[
k_o \left( \frac{l}{2} \right)^m d = \frac{3\pi \overline{G}_s}{2} \left\{ 2\gamma_b \frac{K_1(\gamma_b)}{K_o(\gamma_b)} - \gamma_b^2 \left[ \frac{K_1(\gamma_b)}{K_o(\gamma_b)} \right]^2 - 1 \right\} \quad \text{(Gibson \( k \))}
\]

In the use of eqs. \([4]\) and \([5]\), the following points are worthy to be mentioned.

(i) The diameter \( d \) is incorporated into eq. \([3]\), compared to the previous expression by Guo and Lee (2001). This \( d \) is seen on the left-hand side of eqs. \([4]\) and \([5]\). The new introduction greatly facilitates the establishment of the current solutions presented later on.

(ii) The \( G_s \) and \( \overline{G}_s \) are ‘proportional’ to the pile diameter (width). For instance, given \( l = 0.621 \text{m}, r_o = 0.0501 \text{m} \), the factor \( \gamma_b \) was estimated as 0.173–0.307 using eq. \([2]\) and \( k_1 = 2.14\text{–}3.8 \). The \( \gamma_b \) was revised as 0.178 given \( e = 150\text{mm} \). \( K_1(\gamma_b)/K_o(\gamma_b) \) was computed to be 2.898. As a result, the \( kd \) (Constant \( k \)) is evaluated as \( 3.757 G_s \), whereas the \( 0.5k_odl \) (Gibson \( k \)) calculated as \( 3.757 \overline{G}_s \). Conversely, shear modulus may be deduced from \( k \) via \( G_s = kd/3.757 \), or \( \overline{G}_s = 0.5k_odl / 3.757 \), as shown later in Case Study.

(iii) Equation \([5]\) is approximately valid. The average \( k (= 0.5k_od) \) and \( \overline{G}_s \) refer to those at the middle embedment of a pile; whereas the \( k (= k_od) \) and the shear modulus \( G_L \) refer to those at the pile-tip.

(iv) The modulus ratio \( k_m/k_T \) (\( k_m \) and \( k_T \) are due to pure moment loading and lateral loading, respectively) was calculated using eqs. \([2]\) and \([4]\), and it is provided in Table 1. The calculation shows that
The ratio $k_m/k_T$ reduces from 1.56 to 1.27 as the slenderness ratio $l/r_o$ ascends from 1 to 10, with a highest value of 3.153 at $l/r_o = 0$.

The ratio $k/k_T$ reduces from $k_m/k_T (e = \infty)$ to 1 ($e = 0$) as the free length $e$ decreases.

The modulus $k$ may be underestimated by 30–40% for a pile having a $l/r_o$ of 3–8, neglecting the impact of high eccentricity.

Given a pile-head load exerted at $e > 0$, the displacement is overestimated using $k = k_T$ than otherwise. Consequently, the solutions are conservatively deduced using $k = k_T$ in this paper. Influence of the $e$ on the $k$ is catered for by selecting the $k_1$ in determining $\gamma_b$ via eq. [2]. The outmost difference can be obtained by comparing with the upper bound solutions capitalised on $k = k_m$.

### 2.3 Limiting Force Profile (LFP)

Upon reaching plastic state, the interaction force between pile and soil interface attains a maximum. Given a cohesionless soil, the net limiting force per unit length along pile-soil interface (i.e. LFP), $p_u$ varies linearly with depth, $z$, which may be described by (Fleming et al. 1992):

\[ p_u = A_r z d \]

where $A_r z$ is pressure on the pile surface [FL$^2$] that is contributed by radial and shear stresses obtained using eq. [1], which is shown later in Case Study; $A_r$ is given by:

\[ A_r = \gamma_s' K_p^2 \]

where $K_p \equiv \tan^2 (45^\circ + \phi_s'/2)$ is the coefficient of passive earth pressure; $\phi_s'$ is an effective frictional angle of soil; $\gamma_s'$ is an effective unit weight of the soil (dry weight above the water table, buoyant weight below). Equation [7] is consistent with the experimental results (Prasad and Chari 1999), and the pertinent recommendation (Zhang et al. 2002), in contrast to other expressions for the $A_r$ provided in Table 2.

### 2.4 Critical States

Displacement $u$ of a rigid pile varies linearly with depth $z$ (see Fig. 1), and it is expressed as $u = \omega z + u_0$, in which $\omega$ and $u_0$ are rotation and mudline displacement of the pile, respectively. It is rewritten herein in other forms to identify critical states for pile-soil relative slip (yield) developed firstly from the mudline, and later on from the pile tip.
(1) Pile-soil slip developed from mudline to a depth $z_0$: As depicted in Fig. 2a, there exists a depth $z_0$ (called slip depth), above which the pile-soil interface is in plastic state; otherwise in elastic state. At the slip depth, it is noted that

(i) The pile displacement reaches a local threshold $u^*$ of

$$ u^* = \alpha z_0 + u_0 $$

(ii) The limiting force per unit length obtained using eq. [3] is equal to that derived from eq. [6], which permits the threshold, $u^*$ concerning Gibson $k$ to be given by

$$ u^* = A_r/k_o $$

(2) Pile-soil slip developed from pile-tip to a depth $z_1$: As can be seen from Figs. 2a and 2b, pile-soil relative slip (yield) may also initiate from the pile-tip (depth $l$), and expand upwards to a depth $z_1$, at which the local displacement $u$ just touches $-u^*$. This warrants the following relationship, resembling eq. [8].

$$ -u^* = \alpha z_1 + u_0 $$

Upon the pile-tip yield (i.e. $z_1 = l$), the $z_0$ is rewritten as $\bar{z}_o$. The on-pile resistance per unit length is prescribed by eq. [3] regarding the portion bracketed by the depths $z_0$ and $z_1$, and by eq. [6] for the rest portions of $0 \sim z_0$, and $z_1 \sim l$.

(3) Depth of rotation point $z_r$: The pile rotates about a depth $z_r$ ($= -u_0/\alpha$), at which no displacement $u$ ($= \alpha z_r + u_0 = 0$) is anticipated.

Equations [9] and [10] are deduced using Gibson $k$ featured by $p = k_o dz$. Should the $p$ be governed by Constant $k$ with $p = k dz$, eqs. [9g] and [10g] are deduced instead as tabulated in Table 3. Note that the results for Constant $k$ are highlighted using [] brackets, and are placed adjacent to those for Gibson $k$. For instance, assuming a Constant $k$, it is noted in Fig. 2a-right that $u^* = A_r z/k$; and in Fig. 2b-right that the displacement $u$ at the depth $z_1$ is given by $u = -u^* z_1/z_o$, and the limiting displacement at the pile-tip is equal to $u^* l/z_0$ in either figure.

With respect to Constant $k$, solutions for a rigid pile were deduced previously for a uniform $p_u$ profile with depth (Scott 1981), and for a linear $p_u$ profile (Motta 1997). They are presented in new (explicit) form for a linear $p_u$ (Guo 2003), characterised by the slip depths (see Table 3). The latest expressions allow nonlinear responses to be readily
estimated. They are therefore extended in this paper to cater for Gibson $k$, and to examine responses at the newly defined critical states.

### 2.5 Nonlinear Solutions for Pre-tip Yield State

The unknown rotation $\omega$ and displacement $u_0$ in eq. [8] are determined using equilibriums of horizontal force on the pile, and bending moment about the pile-tip, as elaborated in A1, Appendix A. They are expressed primarily as functions of the pile-head load, $T_i$, and the slip depth, $z_o$. As a result, the solutions for a rigid pile prior to tip yield (pre-tip yield state) are simplified to offer the following expressions.

\[ T_i = \frac{1}{A_i d l^2} \left( 1 + 2 \frac{z_o}{l} + 3 \left( \frac{z_o}{l} \right)^2 \right) \]

\[ u_0 k_o = \frac{3 + 2 \left[ 2 + \left( \frac{z_o}{l} \right)^3 \right] e/l + \left( \frac{z_o}{l} \right)^4}{\left( 2 + \frac{z_o}{l} \right) \left( 2 e/l + \frac{z_o}{l} + 3 \right) \left( 1 - \frac{z_o}{l} \right)^2} \]

\[ \omega = \frac{A_i}{k_o l} \left[ \left( 2 + \frac{z_o}{l} \right) \left( 2 e/l + \frac{z_o}{l} + 3 \right) \left( 1 - \frac{z_o}{l} \right)^2 \right] \]

\[ \frac{z_o}{l} = \frac{-u_0}{\omega l} \]

where $T_i/(A_i d l^2)$ is the normalised pile-head load; $u_0 k_o/A_i$ is the normalised mudline displacement; $\omega$ is the rotation angle (in radian) of the pile; and $z_o/l$ is the normalised depth of rotation. Regarding these solutions, the following remarks are worthy to be stressed.

1. Two soil-related parameters $k_o$ and $A_i$ are involved in the expressions. The $k_o$ is related to $G_s$; while $A_i$ calculated using the unit weight $\gamma_s'$, and angle of soil friction, $\phi_s'$. Only the three measurable soil parameters are thus required.

2. Nonlinear responses are characterised by the sole variable $z_o/l$. Assigning a value to $z_o/l$, for instance, a pair of pile-head load $T_i$ and mudline displacement $u_0$ are calculated using eq. [11] and eq. [12], respectively. Note that $e/l$ is a constant.

3. The $T_i$ is proportional to the pile diameter (width) as per eq. [11]. The $u_0$ implicitly involves with the pile dimensions via the $k_o$ that in turn is related to pile slenderness ratio ($l/r_o$) via the $\gamma_b$ (eq. [5]).
(4) The free length \( e \) defined as the height of the loading point \( (T_t) \) above ground level may be regarded as a fictitious eccentricity \( (e = M_0 / T_t) \), to cater for moment loading \( M_0 \) at mudline level.

These remarks are equally applicable to the solutions based on Constant \( k \) that are provided in Table 3, in which eq. [xg] corresponds to eq. [x]. Nevertheless, a plastic (slip) zone for Gibson \( k \) is not initiated (i.e. \( z_o > 0 \)) from mudline until the \( T_t \) exceeds \( A_r d^2 / (24e / l + 18) \); whereas for Constant \( k \), the slip is developed immediately upon loading. In general, elastic-plastic solutions are preferred to elastic solutions (Scott 1981; Sogge 1981).

Features of the current solutions are highlighted for two extreme cases of \( e = 0 \) and \( \infty \).

- At \( e = 0 \), the usage of relevant expressions for \( z_o \leq z_o \) are provided in Table 4.

- Given \( e = \infty \), eqs. [11]-[14] do reduce to those obtained for pure moment loading \( M_0 \) (with \( T_t = 0 \)), for which the normalised ratios for the \( u_0 \), \( \omega \) and \( M_0 \) are provided in Table 5. For instance, the moment per eq. [11] degenerates to \( M_0 (= T_t e) \) given by:

\[
M_0 = \frac{1}{12} \frac{1 + 2z_o + 3(z_o)^2}{2 + z_o} \frac{l}{l} \frac{M_0}{A_r d l^3}
\]

2.6 Solutions for Post-tip Yield State (Elastic-Plastic, and plastic State)

Equations [11] – [14] for pre-tip and tip-yield states are featured by the yield to the depth \( z_o \) (being initiated from mudline only). At a sufficiently high load level, another yield zone to depth \( z_1 \) may be developed from the pile-tip as well. As load increases further, the two yield zones expand gradually towards the practically impossible ultimate state of equal critical depths (i.e. \( z_o = z_1 = z_r \), see Fig. 2c). The advance of the \( z_1 \) from depth \( l \) to \( z_e \) is herein referred to as Post-tip yield state (Figs. 2a-2c). Horizontal force equilibrium of the entire pile, and bending moment equilibrium about the pile-head (rather than the tip) were used to deduce the solutions (see A2 Appendix A). They are featured uniquely by a newly introduced variable \( C (= A_r / (u_0 k_o)) \), which is the reciprocal of the normalised displacement, see eq. [12]). The variable \( C \) must not exceed its value \( C_y \) calculated for the tip-yield state (i.e. \( C < C_y \), and \( u_0 \) being estimated using eq. [12] and \( z_o = z_o \)), to induce the post-tip yield state. The equations/expressions for estimating \( z_e \), \( T_t \) and \( u_0 \) are highlighted in the following:
(1) The rotation depth $z_r$ is governed by the $C$ and the $e/l$

$$\left(\frac{z_r}{l}\right)^3 + \frac{3 + C^2}{2(1 + C^2)} e \left(\frac{z_r}{l}\right)^2 - \frac{1}{4(1 + C^2)} \left(2 + 3 e\right) = 0$$

Solution of eq. [16] may be written as

$$z_r/l = \sqrt[3]{A_0} + \sqrt[3]{A_1} - D_1/6$$

$$A_j = (D_0/8 - D_1^3/216) + (-1)^j [(27D_o^2 - 2D_oD_1^3)/1728]^{1/2} \quad (j = 0, 1)$$

$$D_1 = \frac{3 + C^2 e}{1 + C^2} \quad D_o = \frac{2 + 3 e/l}{1 + C^2}$$

(2) The normalised head-load, $T_t/(A_r d l^2)$ is derived from eqs. [A15] - [A17] as

$$\frac{T_t}{A_r d l^2} = \left(1 + \frac{C^2}{3}\right) \left(\frac{z_r}{l}\right)^2 - 0.5$$

(3) The mudline displacement, $u_0$ is obtained using the definition of the variable $C$.

$$u_0 = A_r/(k_o C)$$

(4) The slip depths from the pile head, $z_o$ and the tip, $z_1$ (see Table 6) are computed using

$$z_o = z_r(1-C) \quad z_1 = z_r(1+C)$$

Equation [17] provides the rotation depth, $z_r$ (thus $z_o$ and $z_1$) for each value of $u_0$ (via the $C$ and $e$). The $z_r$ in turn allows the rotation angle, $\omega$ to be estimated using $\omega = -u_0/z_r$. As for the Constant $k$, the counterparts for eqs. [17] - [21] are provided in Table 6, and those for $z_o$ and $z_1$ in Table 7. The variable $C \equiv A_r z_r/(u_0 k)$, as per eq. [21g] becomes the product of the reciprocal of the normalised displacement $u_0 k/A_r l$ and the normalised rotation depth $z_r/l$.

### 2.7 Load, Displacement, Slope and On-pile Force Profiles

#### 2.7.1 Response at tip yield state

Upon tip yield ($z_o = \overline{z}_o$), eq. [10] is transformed into the following form to resolve the $\overline{z}_o$, by replacing $u_0$ with that given by eq. [12], $\omega$ with that by eq. [13], and $z_1$ with $l$. 

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The $\bar{z}_o/l$ was estimated for $e/l = 0 \text{--} 100$ using eq. [22], and it is illustrated in Fig. 4a. Regarding the tip-yield state, the responses of $u_0k_o l^o/(A_rl)$, and $\omega k_o l^o/A_r$ were calculated and are presented in Figs. 4b and 4c. Two extreme cases of $e = 0$ and $\infty$ are provided in the following.

- Provided that $e = 0$, the $\bar{z}_o/l$ is computed as 0.5437. The normalised $T_t/(A_r l^2)$ is thereby estimated as 0.113, $u_0k_o/A_r$ as 3.383, $\omega k_o l^o/A_r$ as -4.3831, $z_o/l$ as 0.772, and $C_y$ as 0.296, in terms of eqs. [11], [12], [13], [14], and [21], respectively. These values are tabulated in Table 4.

- Given $e = \infty$, the $\bar{z}_o/l$ is 0.366. The normalised values are thus obtained (see Table 5) with $T_t/(A_r l^2) = 0$, $u_0k_o/A_r = 2.155$, $\omega k_o l^o/A_r = -3.154$, $z_o/l = 0.683$, and $C_y = 0.464$. The counterparts for Constant $k$ were obtained, and are provided in the square \{j\] brackets in Tables 4 & 5, as well. For instance, eqs. [14g] and [22g] offer $\bar{z}_o = 0.618 l$ and $z_o = 0.764 l$ (see Table 4) for $e = 0$; whereas $\bar{z}_o = 0.5 l$ and $z_o = 0.667 l$ (see Table 5) for $e = \infty$.

As the $e$ increases from mudline to infinitely large, the $u_0$ reduces by 36% [38%]; and the $\omega$ reduces by 28% [29.2%]. The estimated depths $z_o$ and $\bar{z}_o$ permit the normalised on-pile force profiles of $p/(A_r l^o d)$ at the tip-yield state to be constructed. This is achieved by drawing lines in sequence between adjacent points $(0, 0)$, $(A_r l^o, \bar{z}_o)$, $(0, z_o)$, and $(-A_r l, l)$, which are normalised by $A_r l^o$ and $\bar{z}_o$, respectively for the first and the second coordinates, as exemplified in the following for $e = 0$, and $\infty$.

- Figure 5a provides the profiles constructed using Constant $k$. The $\bar{z}_o/l = 0.618$ and $z_o/l = 0.764$ for $e = 0$ offer a pressure at the pile-tip level of 1.64$A_r l^o$, which raises to 2.73$A_r l^o$, as the $e$ increases to $\infty$ (accompanied by the reduction of $z_o/l$ to 0.5 and $z_o/l$ to 0.667).

- Figure 5b depicts the normalised profiles obtained using a Gibson $k$, in view of the $\bar{z}_o = 0.544 l$ and $z_o = 0.772 l$ (Table 4) upon $e = 0$; whereas $\bar{z}_o = 0.366 l$ and $z_o = 0.683 l$ (Table 5) concerning $e = \infty$. The pile-tip pressure increases from 1.84$A_r l^o$ to 2.73$A_r l^o$, as the $e$ shifts towards infinitely large from 0.

The constructed profiles for $e = 0$ and $\infty$ (see Fig. 5b) well bracket the ‘Test data’ provided by Prasad and Chari (1999). The tri-linear feature of the ‘Test’ profile is also
captured using the Gibson $k$. Likewise, good comparisons with measured $p$ profiles were noted for a few other cases (not shown herein). The profile is governed by the gradient $A_r$, the slip depths $z_o$ (from the head) and $z_1$ (from the tip), and the rotation depth $z_r$. Overall, Fig. 5 indicates the measured $z_o$ (Prasad & Chari, 1999) conforms to pre-tip yield state, as Fig. 4a does (and further confirmed by the reported capacity shown later in Fig. 9). It also reflects the impact of

- The $k$ profile, as a better match is observed using a Gibson $k$;
- Lack of measured points around the critical depth ($z_o$);
- Ignoring the increase in the modulus owing to the free length ($e$) (see Table 1);
- The nonlinear spring behaviour in reality vs the elastic-plastic spring adopted herein.

### 2.7.2 State of yield at rotation point (YRP, Completely plastic state)

The pile-head displaces infinitely large as the $C$ approaches zero, see eq. [21]. This ultimate state featured by $z_r = z_o$ is referred to as yield at rotation point (YRP). Despite practically unachievable, the state provides an upper bound, for which, it is noted that

1. Equation [20] reduces to that proposed by Petrasovits and Award (1972), as the on-pile force profile does (see Fig. 2c).
2. The $z_r/l$ is independent of the modulus $k$. It should be estimated using eq. [17] by substituting $0, 2 + 3e/l$, and $3e/l$ for the $C, D_o$, and $D_1$, respectively in eqs. [18] and [19] regarding $e \neq \infty$; otherwise using eq. [16] directly. This is shown for the following two typical cases.
   - Regarding $e = 0$, eq. [19] allows $D_o = 2$, and $D_1 = 0$ to be deduced. $A_o = 0.5$, and $A_1 = 0$ are thus obtained in terms of eq. [18]. The $z_r/l$ is evaluated as 0.7937.
   - Substituting $\infty$ for the $e$ in eq. [16], the $z_r/l$ is directly computed as 0.7071.
3. On-pile pressure profile can be constructed by drawing lines between adjacent points $(0, 0), (A_rz_o, z_o), (-A_rz_o, z_r), \text{and} (-A_l, l)$ (Fig. 2c-left).

### 2.8 Maximum Bending Moment and Its Depth

#### 2.8.1 Pre-tip yield ($z_o < z_o$) and tip yield ($z_o = z_o$) states

The depth $z_m$ at which maximum bending moment occurs is given by eq. [23] if $z_m \leq z_o$, otherwise by eq. [24], in light of eqs. [A5a] and [A5b] (see Appendix A).

$$\frac{z_m}{l} = \sqrt{\frac{2T}{A_ldl^2}} \quad (z_m \leq z_o)$$
Equation [23] offers \( \frac{T_t}{(A_d l^2)} = 0.5(z_m/l)^2 \). Following determination of \( z_o \) using eq. [22], the \( z_m/l \) was computed for a series of \( e/l \) ratios at tip-yield state. It is plotted in Fig. 4d against \( e/l \). The calculation shows that:

1. The \( z_m \) may in general be computed by using eq. [23], satisfying \( z_m < z_o \) (regardless of \( e/l \)). Nevertheless, a low load level (pre tip-yield state) along with a small eccentricity \( e \) may render \( z_o < z_m \) (not shown herein).

2. The \( z_m \) converges to mudline, as the \( e/l \) approaches infinitely large (see Table 5).

Substituting the \( z_m/l \) from eq. [23] for that in eq. [A7a] allows the maximum bending moment (\( M_m \)) for \( z_m \leq z_o \) to be gained.

\[
M_m = (2z_m/3 + e)T_t \quad (z_m \leq z_o)
\]

Otherwise, substituting the \( z_m/l \) from eq. [24] for that in eq. [A7b] permits the \( M_m \) for \( z_m > z_o \) to be deduced as:

\[
M_m = \left( \frac{3l^4 - 2z_m^3z_o + 2z_o^2z_m^2 - 4l^2z_o^2 - z_m^4}{(l - z_o)^2(3l^2 + 2lz_o + z_o^2)} e + z_m \right)T_t + \frac{dA}{6}z_o^2(2z_o - 3z_m) \quad (z_m > z_o)
\]

Valid to a high load level, eq. [25] is independent of pile-soil relative stiffness. Therefore, it is essentially identical to that for a flexible pile (Guo 2006). Likewise, the counterpart expressions for the \( z_m \) and \( M_m \) were derived regarding a constant \( k \), and they are provided in Table 3. In particular, eqs. [23] and [23g] are identical for either \( k \) profile, so are eqs. [25g] and [25g], although they all are still mentioned later.

With respect to the tip-yield state, as \( z_m < z_o \) (Figs. 4a and 4d), the \( M_m \) was calculated using eq. [25]. It is presented in Figs. 6a and 6b in form of \( M_m/(A_d l^3) \). The \( M_m \) was also evaluated using eq. [25g], and it is plotted in Fig. 6b. The following points are worthy to be stressed for the tip-yield state.

1. The maximum moment \( M_m \) obtained is higher using Constant \( k \) than Gibson \( k \) (see Fig. 6b). With \( e = \infty \), and \( z_m = 0 \) [0], eqs. [25] and [25g] allow the \( M_m \) to be calculated as \( T_t e \) (either \( k \)). More specifically, it is noted that:
o At $e = 0$ (see Table 4), the $M_m/(A_o d^3)$ is estimated as $0.036[0.038]$, as $z_o/l$ was calculated as $0.5473[0.618]$, and the associated $T_i/(A_o d^2)$ is $0.113[0.118]$.

o At $e = \infty$ (see Table 5), the $M_m/(A_o d^3)$ is evaluated as $0.0752[0.0833]$ with the previously calculated values of $z_o/l = 0.366[0.50]$, and $T_i = 0[0]$.

(2) Increase in the $e$ enables $M_m$ to approach $M_o$, e.g. $M_m = 0.947M_o$ given $e/l = 2$ and Gibson $k$ (see Figs. 6a and 6b). Note that a ratio $e/l > 3$ is normally encountered in overhead catenary systems.

(3) The solid circle dotted line of $e = \infty$ in Figs. 6a or 6b denotes the upper limit of the tip-yield state for the corresponding $k$. They were obtained using eq. [15] or $z_o/l/6$ (see Table 5), and overlap with those from eq. [25] [eq. [25g]] as anticipated.

The bending moments are subsequently compared with those induced at ‘yield at rotation point’.

2.8.2 Yield at rotation point (YRP)

At the YRP state, the relationships of $T_i$ versus $z_m$, and $T_i$ versus $M_{max}$ are found identical to those for pre-tip yield state ($z_m \leq z_0$), respectively. Responses at rotation point (see Fig. 2c) may be obtained by substituting $C = 0$ for that in the solutions addressing the post-tip yield state, e.g. eq. [20] is transformed into

\[ \frac{T_i}{A_o d l^2} = 0.5 \left( \frac{z_m}{l} \right)^2 \]

in which the depth $z_m$ is related to the $z_i$ given by the following relationship

\[ \frac{z_m}{l} = [2(\frac{z_i}{l})^2 - 1]^{0.5} \]

where $z_i/l$ is still calculated from eq. [16] or [17], as discussed previously in the section entitled ‘State of yield at rotation point’. Equation [28] was derived utilising the on-pile force profile depicted in Fig. 2c, and shear force $T(z_m) = 0$ at depth $z_m$. The normalised maximum bending moment is derived using eqs. [A7a] and [27] as

\[ \frac{M_m}{A_o d l^3} = \frac{T_i}{A_o d l^2} \frac{e}{l} + \frac{1}{3} \left( \frac{z_m}{l} \right)^3 \]

The normalised $M_m$ was computed using eq. [29]. It is plotted in Fig. 6a as ‘both $k$ at YRP, as eqs. [27] ~ [29] are all independent of the $k$ profiles. The moment of $M_o (= T_e)$ is
also provided in the figure. Equation [29] may be converted into an identical form to eq. [25] using eqs. [27] and [28]. About the YRP state, it is noted that

- The \( z_o/l = z_r/l \) of 0.7937 (\( e = 0 \)) and 0.707 (\( e = \infty \)) allow the \( z_m/l \) to be evaluated as 0.5098 and 0; upon which \( T_t/(A_r d^2) \) is calculated as 0.130 (\( e = 0 \)) and 0 (\( e = \infty \)).

- Equation [29] permits the \( M_m/(A_r d^3) \) to be computed as 0.0442 (\( e = 0 \)) and 0.0976 (\( e = \infty \)). In particular, the \( M_m/(A_r d^3) \) for \( e = \infty \) (see Fig. 2c) is deduced as \( [1-2(z_r/l)^3]/3 \), in light of moment equilibrium about ground line and the on-pile force profile.

Finally, eqs. [23] and [25] are supposedly valid from the tip-yield state through to the ultimate YRP state (with \( z_m < z_o \)). This is not pursued herein, but will be seen later in Case Study.

### 2.8.3 Tip-yield to YRP

Figure 6a demonstrates that as the yield (slip) extends from the pile tip (see Fig. 2a) to the rotation point (see Fig. 2c), \( M_m/(A_r d^3) \) (Gibson \( k \)) increases by 22.8% (from 0.036 to 0.0442) at \( e = 0 \) (see Table 4); or by 29.9% (from 0.0752 to 0.0976) at \( e = \infty \) (see Table 5). (Slightly less increase in percentage is observed using Constant \( k \)). The increase in \( M_{\text{max}} \) consequently would not exceed \( \sim 30\% \) from the tip-yield to the YRP states. In contrast, as the \( e \) increases from 0 to \( \infty \), the \( M_m \) increases by 109% at tip-yield state (from 0.036 to 0.0752), or by 120% at rotation point yield state (from 0.0442 to 0.0976). The eccentricity has more profound impact on the \( M_{\text{max}} \) than the states of yield. These conclusions about the \( M_m \) are valid to other rigid piles of identical non-dimensional parameters.

### 2.9 Calculation of Nonlinear Response

Response of the pile is characterised by two sets of expressions concerning pre- \((z_o < z_o)\), and post- \((z_o > z_o)\) tip yield states. It thus may be obtained pragmatically via two steps:

1. Regarding the pre-tip yield state, a series of slip depth \( z_o (<\bar{z}_o) \) may be specified. Each \( z_o \) allows a set of load \( (T_t) \), displacement \( (u_0) \), and rotation \( (\omega) \) to be evaluated using eqs. [11], [12], and [13], respectively; and furthermore, the moment \( (M_m) \) to be estimated using eq. [25] or eq.[26].
As for the post-tip yield state, a series of \( C (0 \leq C \leq C_y) \) may be assigned. Each \( C \) permits a rotation depth \( z_r \) to be calculated using eq. [17] \((e \neq \infty)\) otherwise eq. [16] \((e = \infty)\). The depth \( z_r \) in turn allows a load and a displacement to be calculated using eqs. [20] and [21] respectively; and a rotation angle to be assessed as \(-u_0/z_r\). The force \( T_t \) estimated is then used to determine \( M_m \) using eq. [25] or [29].

The two steps allow entire responses of the pile-head load, displacement, rotation, and maximum bending moment to be ascertained. For instance, non-dimensional responses of \( u_0 k_0/A_r \) \([u_0 k/A_r]\), \( \omega k_0/A_r \) \([\omega k/A_r]\), and \( M_m/(A_r d l^3) \) were predicted along with \( T_t/(A_r d l^2) \), and those at tip-yield state \((z_o=\bar{z}_o)\), using Gibson \( k \) [also Constant \( k \)] for a pile having \( l/r_o = 12 \), and at six typical ratios of \( e/l \). The ultimate moment at YRP state and \( e/l = 0 \) was also predicted using eq. [29]. All these predictions are shown in Fig. 7. The effect of \( k \) profile on the normalised \( u_0 \) and \( \omega \) is evident, but on the normalised \( M_{\max} \) is noticeable only at low load levels (as anticipated). Conversely, two measured responses \([e.g. T_t \sim u_0 \text{ and } T_t \sim M_{\max} (or \omega) \text{ curves}]\) may be fitted by using the current solutions, to allow values of \( A_r \) and \( k \) to be uniquely back-figured in a principle discussed previously for a flexible pile (Guo 2006), and further illustrated later in Case Study.

### 3. COMPARISON WITH EXISTING SOLUTIONS

The current solutions have been implemented into a spreadsheet program called GASLSPICS operating in Windows EXCEL™. The results presented so far and subsequently were all obtained using this program.

#### 3.1 Comparison with Existing Experiments and Numerical Solutions

Model tests were conducted by Nazir (Laman et al. 1999) at a centrifugal acceleration of 33g \((g = \text{gravity})\) on a pier with a diameter \( d = 30 \text{mm} \) and an embedment length \( l = 60 \text{mm} \) (Test 1); at 50g on a pier with \( d = 20 \text{mm} \) and \( l = 40 \text{mm} \) (Test 2); and at 40g on a pier with \( d = 25 \text{mm} \) and \( l = 50 \text{mm} \) (Test 3), respectively. They are designed to mimic the behaviour of a prototype pier with \( d = 1 \text{m} \) and \( l = 2 \text{m} \). Embedded in dense sand, the prototype pier was gradually loaded to a maximum lateral load of 66.7 kN, generating a moment \( M_0 \) of 400 kNm about mudline \((e = 6 \text{m})\). The sand bulk density \( \gamma_s' \) was 16.4 kN/m³, and frictional angle \( \phi'_s \) was 46.1°. Young’s modulus of the pier was 207 GPa, and Poisson’s ratio was 0.25. In the 40g test (Test 3), lateral loads were applied at a free-length \((e)\) of 120 mm above mudline (Laman et al. 1999) on the model pier. The pier
rotation angle ($\omega$) was measured under various moments ($M_0$) during the test, and it is plotted in Fig. 8a in prototype scale. Tests 1 and 2 were conducted to examine the effect of modelling scale. The test results are plotted in Fig. 8c.

Three-dimensional finite element analysis (FEA 3D) was undertaken (Laman et al. 1999) to simulate the tests as well, adopting a hyperbolic stress-strain model, for which the initial and unloading-reloading Young’s moduli alter with confining stress. The predicted moment ($M$) is plotted against rotation ($\omega$) in Figs. 8a and 8c. It compares well with the median value of the three centrifugal tests, except for the initial stage.

To undertake the current predictions, the two parameters $A_r$ and $k_d$ were calculated.

1. The $A_r$ was estimated as 621.7 kN/m$^3$ using eq. [7], $\gamma'_s = 16.4$ kN/m$^3$, and $\phi'_s = 46.1^\circ$.

2. The $k_d$ was calculated as 3.02$G_s$ (or 1.2$E_s$) in light of eq. [4]. Initial Young’s modulus, $E_s$ was computed as 25.96 MPa, and unloading-reloading $E_s$ as 58.63 MPa, in view of the published data (Laman et al. 1999), and an average confining pressure of 16.4 kPa over the pile embedment. This offers an initial modulus of subgrade reaction $k_d$ of 31.36 MPa, and a unloading-reloading $k_d$ of 70.83 MPa. The $k_d$ was thus chosen (see Table 8) as 34.42 MPa ($d = 1$ m) to simulate Test 3; and as 51.63 MPa ($d = 1$ m) to mimic Test 2.

The $T_t$ and the $\omega$ were estimated via eqs. [11g] and [13g], with the $A_r$ and the selected $k_d$, and assuming a Constant $k$. The $M_0$ was obtained as $T_t* e$, and is plotted against $\omega$ in Fig. 8a and 8c as ‘Current CF’. The figures show the measured responses for Tests 2 and 3 were well modelled. The tip-yield occurred at a rotation angle of 3.8$^\circ$ (see Table 8). Furthermore, a Gibson $k$ was stipulated with $k_0 = 17.5$ kN/m$^3$/m. With the $A_r$, the $M_0$ and $\omega$ were calculated using eqs. [11] and [13], and are plotted in Fig. 8a. The prediction is also reasonable well. Next, the displacement $u_0$ is calculated using eq. [12] and eq. [12g], respectively, and is plotted in Fig. 8b against the respective $T_t$. A softer response is noted given Gibson $k$ than a uniform $k$. The real $k$ profile is not known until the measured $T_t \sim u_0$ becomes available.

The $A_r$ for Test 2 (see Fig. 8c) may be raised slightly from the current value of 621.7 kN/m$^3$, to achieve a better agreement with the measured ($M_0-\omega$) curve. A high value of $A_r$ was used to achieve a good prediction (not shown herein) for the same pier tested in a ‘loose’ sand. The accuracy of any predictions is essentially dominated by the $p_u$ (Guo 2006) mobilised along flexible piles, and perhaps also by the $k$ (or $G_s$) for rigid piles. The
current solutions are sufficiently accurate, in terms of capturing nonlinear response manifested in the tests and the FE analysis (FEA\textsuperscript{3D}).

To validate the current solutions, a continuum-based analysis on a rigid pile ($l = 5 \text{ m}, d = 1 \text{ m}$) was also conducted using the finite-difference program FLAC\textsuperscript{3D} (Itasca 2002). The primary parameters adopted are as follows: $\gamma_s' = 16.4 \text{ kN/m}^3$, $\phi_s' = 46.1^\circ$, $G_s = 5.0 \text{ MPa}$, and $E_p = 200 \text{ GPa}$. The study reveals the $p_u$ is more or less proportional to $z^{1.7}$ (i.e. $p_u \propto z^{1.7}$), as it was noted for majority of flexible piles in sand (Guo 2006). The current solutions (using $p_u \propto z$, eq. [16]) thus predict a lower stiffness (ratio of pile-head load over displacement) at a large displacement than that obtained from FLAC\textsuperscript{3D}. The good comparisons noted among the current solutions, the measured data and the FE analysis vindicate the acknowledged fact that (1) FLAC\textsuperscript{3D} offers 10~20\% higher stiffness than FE analysis; (2) Equation [6] is sufficiently accurate for rigid piles.

### 3.2 Comments on Existing Uncoupled Expressions for $T_o$

Capacity $T_o$ (i.e. $T_i$ at a defined state) of a laterally loaded rigid pile is currently evaluated using such simple expressions as those outlined in Table 2. They were broadly presented in terms of either the normalised rotational depth $z_r/l$ (Methods 1, 4 and 8), or the normalised eccentricity $e/l$ (Methods 2, 3, 5, 6 and 7). Nevertheless, the capacity (alike to $T_i$) relies on the two facets of $e/l$ and $z_r/l$, apart from the critical value $A_r$ and pile dimensions. This may be inferred from eq. [11] or eq. [20]. In eq. [11], for instance, the impact of $z_r/l$ on $T_i$ is represented via $z_o/l$.

Some features of the expressions in Table 2 are highlighted in the following:

1. The relative weight of free-length compared to the constant in the denominator in general varies from $e/l$ (Broms 1964) to $1.5e/l$ (McCorkle 1969; Balfour-Beatty-Construction 1986).

2. Each expression may offer a capacity $T_o$ close to measured data via adjusting the gradient $A_r$. For example, the $A_r$ for McCorkle’s method may be selected as 20–30\% that suggested for Balfour Betty method, as the $T_o$ offered by the former is approximately 3.76 times the latter, given an identical $A_r$.

3. The expressions are not explicitly related to magnitude of displacement or rotation angle. The $A_r$ reported may correspond to different stress states.
The aforementioned ambiguity regarding \( T_0 \) and \( A_r \) prompts us to redefine the capacity \( T_0 \) as the \( T_i \) at (a) Tip-yield state, or (b) Yield at rotation point state (YRP).

(i) The \( T_0 \) for tip-yield state is obtained by substituting \( \bar{z}_o \) for the \( z_o \) in eq. [11]; whereas

(ii) The \( T_0 \) for the YRP state is evaluated using eq. [27], in which the \( z_m \) is calculated from \( z_r \) using eq. [28].

The ratio \( T_0/(A_r d^2) \) was estimated for a series of \( e/l \) ratios, concerning the two states; and is plotted in Fig. 9. The mudline displacement may be accordingly estimated using eq. [12] (or eq. [12g]) concerning the tip-yield state, given \( k_o \) [or \( k \)]; whereas it is infinitely large upon the YRP state. Fig. 9 indicates that

1. The \( T_0 \) at the YRP state exceeds all other predictions, and overlies on the upper bound given by Fleming et al (1992). On the other hand,

2. The \( T_0 \) dictated by the test results of Prasad and Chari (1999) offers the lower bound, which occurs at a pre-tip yield state as mentioned previously.

Especially, \( z_o/l \) at the tip-yield state is obtained as 0.618 if \( e = 0 \) is stipulated. Regardless of the \( e \), substituting 0.618 for the \( z_o/l \) in eq. [11g], a new expression for \( T_0/(A_r d^2) \) is developed, and provided in Table 2 as Mtd 9. Interestingly, it offers a \( T_0 \) (not shown herein) close to the YRP state.

4. CASE STUDY

4.1 Model tests by Prasad and Chari (1999)

A total of 15 steel pipe piles were tested in dry sand. Each model pile was 1,135 mm in length, 102 mm in outside diameter (\( d \)), and 5.6 mm in wall thickness. All piles were embedded to a depth (\( l \)) of 612 mm. The sand had three typical relative densities (\( D_r \)) of 0.25, 0.5, and 0.75; bulk densities (\( \gamma_s' \)) of 16.5 kN/m\(^3\), 17.3 kN/m\(^3\), and 18.3 kN/m\(^3\); and frictional angles (\( \phi_s' \)) of 35\(^\circ\), 41\(^\circ\), and 45.5\(^\circ\), respectively. Lateral loads were imposed at an eccentricity of 150 mm on the piles until failure, which offers the results (Prasad and Chari 1999) of
(1) Distribution of $\sigma_r$ across the pile diameter at a depth of about 0.276 m, shown previously in Fig. 3, with a maximum $\sigma_r = 66.85$ kPa;

(2) Pressure profile ($p$) along the pile that was plotted previously in Fig. 5 as ‘Test data’;

(3) Normalised capacity $T_o/(A_r d l^2)$ versus normalised eccentricity ($e/l$) relationship, signified as ‘Prasad & Chari (1999)’ in Fig. 9;

(4) Lateral pile-head load ($T_t$) ~ mudline displacement ($u_0$) curves, as plotted in Fig. 10a;

(5) Shear moduli at the pile-tip level ($G_{L}$) being 3.78, 6.19 and 9.22 MPa (taking the Poisson’s ratio of the soil $\nu_s$ as 0.3) for $D_r = 0.25$, 0.5 and 0.75, respectively.

Incorporating the effect of diameter, the measured moduli should be revised, for instance for Gibson $k$, as 0.385 MPa ($= 3.78d$), 0.630 MPa ($= 6.19d$) and 0.94 MPa ($= 9.22d$). These values are then quite consistent with 0.22–0.3 MPa deduced from a model pile of similar size (having $l = 700$ mm, $d = 32$ mm or 50 mm) embedded in a dense sand (Guo and Ghee 2005).

Responses (2) and (3) were addressed in previous sections. Other responses are studied herein, along with the $M_m$ and $z_m$, in order to illustrate the use of the current solutions, and the impact of the $k$ profiles.

### 4.1.1 Analysis using Gibson $k$

The measured $T_t$ ~$u_0$ relationships (see Fig. 10a) were fitted using the current solutions for Gibson $k$, following the section entitled ‘Calculation of Nonlinear Response’. This allows the parameters $k_o$ and $A_r$ to be deduced (see Table 9) for each test in the specified $D_r$. (Note: ideally two measured curves are required for the two parameters. The initial elastic gradient and subsequent nonlinear portion of the $T_t$ ~$u_0$ curve may serve this purpose). The $k_o$ in turn permits shear modulus $G_{L}$ to be determined. The $A_r$ allows maximum bending moment $M_m$ to be evaluated, along with its depth of occurrence $z_m$. The $M_m$ is plotted in Fig. 10b against $T_t$, and in 10c against $z_m$, respectively. The study indicates that:

- The $A_r$ was 244.9 kN/m$^3$, 340.0 kN/m$^3$, and 739. kN/m$^3$ for $D_r = 0.25$, 0.5, and 0.75 respectively, which is $\pm \sim 15$ % different from 224.68 kN/m$^3$, 401.1 kN/m$^3$, and 653.3 kN/m$^3$ obtained using eq. [7].

- The $G_{L}$ was 0.31 MPa, 0.801 MPa, and 1.353 MPa, which differ by -19.48%, 27.7% and 43.9% from the revised moduli for $D_r = 0.25$, 0.5 and 0.75, respectively.
The tip-yield for \( D_r = 0.5 \) and 0.75 occurs in close approximation to the displacement \( u_0 \) of 0.2d (= 20.4 mm, see Fig. 10a), whereas the tip-yield for \( D_r = 0.25 \) occurs at a much higher displacement \( u_0 \) of 39.5 mm than 0.2d. The latter yields a higher load \( T_o \) of 0.784 kN than 0.529 kN observed at 0.2d upon the measured \( T_r-u_0 \) curve. The tip-yield, nevertheless, is associated with a rotation angle of 2.1~4.0 degrees (see Table 9), conforming to 2~6 degrees (see Dickin and Nazir, 1999) deduced from model piles tested in centrifuge.

The calculation is elaborated here, for instance, regarding the test with \( D_r = 0.25 \), for which, \( A_r = 244.9 \text{ kN/m}^3 \), and \( k_o = 18.642 \text{ MPa/m}^2 \). It is presented for four typical yielding states (see Tables 10 and 11), shear modulus, and distribution of stress \( \sigma_r \) across the pile diameter.

(1) **Tip-yield** state: The ratio \( \frac{z_o}{l} \) at the tip-yield state is obtained as 0.5007 from eq. [22]. This ratio allows relevant responses to be calculated, as tabulated in Table 10.

(i) The \( T_t/(A_r dl^2) \) is determined, in terms of eq. [11], as:

\[
T_t/(A_r dl^2) = \frac{1}{6} \left( \frac{1+2 \times 0.5 + 3 \times 0.5^2}{(2+0.5)(2 \times 0.245 + 0.5) + 3} \right) = 0.0837
\]

The \( T_t \) is thus estimated as 0.784 kN (= 0.0837\times244.9\times0.102m \times0.612^2 \text{ kN/m}).

(ii) The \( u_0k_o/A_r \) is computed using eq. [12] as 3.005, and \( u_0 \) is evaluated as 39.5 mm (= 3.005\times244.9/18642 m).

(iii) The \( z_m/l \) is estimated as 0.4093 using eq. [23], as \( z_m (= 0.251 \text{ m}) < z_o (= 0.306 \text{ m}) \).

(iv) In view of \( z_m < z_o \), \( M_m/(A_r dl^3) \) is estimated using eq. [25] as 0.0434 [= (2/3\times0.4093 + 0.245) \times 0.0837], and \( M_m \) as 0.248 kNm.

In brief, upon the tip-yield state, it is noted that \( T_t = 0.784 \text{ kN}, u_0 = 39.5 \text{ mm}, z_m = 0.251 \text{ m}, \) and \( M_m = 0.248 \text{ kNm} \), as shown in Figs. 10a-10c. The moment is of similar order to that recorded in similar piles tested under soil movement (Guo and Ghee 2005).

(2) **Pre-tip yield** state (\( z_o/l = 0.3 \), and 0.5): Given, for instance, \( z_o/l = 0.3 \) (< \( \bar{z}_o/l \) ), the \( z_m/l \) is computed as 0.364 using eq. [24]. Thereby, it follows that \( T_t = 0.605 \text{ kN}, u_0 = 22.3 \text{ mm}, \) and \( z_m = 0.223 \text{ m} \). \( M_m \) is estimated as 0.18 kNm using eq. [26] (\( z_m > z_o \)).
As zo/l increases to 0.5, Tt increases to 0.783 kN, Mm to 0.248 kNm, and zm to 0.251 m. The two points, each given by a pair of Mm and zm, are plotted in Fig. 11a.

(3) *Post-tip yield* state (e.g. C < Cy): At the *tip-yield* state, the value of u0 = 39.5 mm allows the Cy to be computed as 0.3326 [= 244.9/(0.0395×18,642)]. Assigning a typical 0.2919 (< Cy) to the C, for instance, the responses were estimated as follows:

(i) The zo/l is evaluated in steps (see Table 12) that follow:

- D1 = 0.6968, and D0 = 2.52049 are gained using eq. [19], which allow A1 = 3.9127×10⁻⁶, and A0 = 0.627 to be obtained using eq. [18].
- zo/l is computed as 0.756 (= 0.627¹/³ + 3.9127¹/³×0.01-0.6968/6) using eq. [17].

(ii) The Tt is calculated as 0.814 kN, as Tt/(Ardl²) = 0.087 using eq. [20].

(iii) The u0 is 45 mm [= 244.9/(18642×0.29193)], as per eq. [21].

(iv) The zo/l is calculated as 0.535 [= (1-0.29193)×0.7555].

(v) The zm/l is calculated as 0.445 [< zo/l] as per eq. [23], and zm = 0.255 m.

(vi) Mm/(Ardl³) is 0.0453 using eq. [25], and Mm = 0.261 kNm.

(4) YRP state: The condition of C = 0 at the YRP state allows D1, D0, Ao, A1 to be estimated (see Table 12), which in turn enable zo/l to be computed as 0.774 (= 0.68¹/³+0.017-0.735/6). Values of zm/l, Tt/(Ardl²), and Mm/(Ardl³) are thus calculated as 0.445 (< zo/l), 0.099, and 0.0536, respectively (see Table 10) using eqs. [28], [27], and [29]. Upon YRP state, it follows Tt = 0.926 kN, zm = 0.272 m, and Mm = 0.307 kNm.

(5) Distributions of the bending moment M(z) and the shear force T(z) with depth were determined in light of eqs. [A7a] and [A7b] and eqs. [A6a] and [A6b] regarding the two ratios of zo/l = 0.3 and 0.5. They are plotted in Figs. 11a and 11b, and agree with the Mmax and zmax predicted before (Figs. 10a-10c). For instance, T(z) at a slip depth of zo = 0.5l was estimated using ω = -0.087, u0 = 39.5 mm (see Table 11), and eqs. [30] and [31].

\[
T(z) = 0.5A_0d\left(z_0^2 - z^2\right) + k_0d\omega\left(l^3 - z_0^3\right)/3 + k_0d\omega\left(l^2 - z_0^2\right)/2 \quad (z \leq z_0)
\]

\[
T(z) = k_0d\omega\left(l^3 - z^3\right)/3 + k_0d\omega\left(l^2 - z^2\right)/2 \quad (z > z_0)
\]
Local shear force $T(z) \sim$ displacement $u(z)$ relationships were predicted using eqs. [A5a] and [A5b], for the normalised depths $z/l$ of 0, 0.3, 0.5, 0.62 and 0.9. They are plotted in Fig. 12

(6) The $\bar{G}_s$ is estimated as 0.1549 MPa ($= 0.5 \times 18.642 \text{ MPa/m}^2 \times 0.612 \text{m} \times 0.102 \text{m} / 3.757$) as per the discussion for eq. [5], in terms of $k_0 = 18.642 \text{ MPa/m}^2$ in Table 9. The shear modulus $G_L$ (at the pile-tip level) is thus inferred as 0.310 MPa ($= 2 \bar{G}_s$).

(7) The measured $\sigma_r$ on the pile surface was mobilised by a local displacement $u$ of 21.3 mm, at a head-displacement $u_0$ of 57 mm (after tip yield, and with $z_o = 0.361 \text{ m}$, and $z_t = 0.470 \text{ m}$).

The discussion about eq. [5] indicates $\gamma_b = 0.178$ and $K_1(\gamma_b)/K_0(\gamma_b) = 2.898$. These values along with $\bar{G}_s = 0.1549 \text{ MPa}$, and $u = 21.3 \text{ mm}$ allow the maximum $\sigma_r$ (with $r = r_o = 0.051 \text{ m}$ and $\theta_p = 0$) to be obtained using eq. [1], with $\sigma_r = 2 \times 154.9 \times 0.0213 \times 0.178 / 0.051 \times 2.898$. The rationale of using the elastic eq. [1] is explained later on. The stress $\sigma_r$ across the diameter is predicted as $\sigma_r = 66.85 \cos \theta_p$ (compared to $\tau_{r\theta} = -33.425 \sin \theta_p$) in light of eq. [1]. It compares well with the measured data (Prasad and Chari 1999), as shown previously in Fig. 3.

The maximum $\sigma_r$ may be cross examined using eq. [6]. In the loading direction, the net force on the pile surface per unit projected area (net pressure) due to the components $\sigma_r$ and $\tau_{r\theta}$ are determined, respectively, as:

$$\int_0^\pi \sigma_r \cos \theta_p d\theta_p = 105.0 \text{ (kPa)}$$
$$\int_0^\pi \tau_{r\theta} \sin \theta_p d\theta_p = -52.5 \text{ (kPa)}$$

Thereby, the total net pressure is 52.5 kPa ($= 105.0 - 52.5$). On the other hand, the ‘net pressure’ at the depth of 0.276 m is estimated as 67.6 kPa ($= 244.9 \times 0.276$) using eq. [6]. The former is less than the latter, as is the measured force compared to the predicted one (see Fig. 10a), showing the effect of $k$ profile (discussed later) and neglecting the $\tau_{rr}$ and $\sigma_0$.

Other features noted herein are: (i) The moment $M_m$ occurs below the slip depth ($z_m > z_o = 0.3l = 0.1836 \text{ m}$) under $T_t = 0.605 \text{ kN}$; or within the depth ($z_m < z_o = 0.5l = 0.306 \text{ m}$) at $T_t = 0.783 \text{ kN}$ (see Fig. 11). (ii) The rotation depth of $z_r$ is largely around 0.62$l$, indicating by a negligible displacement of $u(z_t = 0.62l) \approx 0$ (see Fig. 12). The force below the depth has an opposite direction, as is observed in some field tests. Finally (iii) The non-dimensional responses, e.g. $T_t/(A_r d^2)$, are independent of the parameters $A_r$ and $k_o$, and are thus identical concerning the three piles tested in different $D_r$. 

4.1.2 Analysis using Constant $k$

The solutions for a Constant $k$ (see Tables 3 and 7) were utilised to match each measured pile-head and mudline displacement $T_t \sim u_0$ curve, the $k$ was thus deduced (using the same $A_r$ as that for Gibson $k$). This resulted in the dashed lines in Fig. 10, the shear modulus $G_s$ (= $G_L$) and the angle at tip-yield furnished in Table 9. The predicted curves of $M_{\text{max}} \sim T_t$, and $M_{\text{max}} \sim z_m$ are also plotted in Figs. 10b and 10c, respectively. This analysis indicates:

- The shear moduli deduced for $D_r = 0.25$, 0.5 and 0.75 are 0.105 MPa, 0.327 MPa, and 0.461 MPa, respectively, showing -45.6%, 3.5% and -1.9% difference from the revised measured values of 0.193 MPa, 0.316 MPa, and 0.470 MPa, respectively.
- The tip-yield (thus pile-head force $T_0$), as illustrated in Fig. 10a, occurs at a displacement far greater than $0.2d$ (= 20.4 mm), and at a rotation angle of 10~15 degrees (see Table 9) that are ~5 times those inferred using a Gibson $k$.

In parallel to the Gibson $k$, the calculation for the pile in $D_r = 0.25$ is again presented herein (with $A_r = 244.9$ kN/m$^3$, and $k =3.88$ MN/m$^3$), and is focused on the difference from the Gibson $k$ analysis.

(1) The ratio $z_o/l$ at the tip-yield state was obtained as 0.5885 using eq. [22g]. It allows the following to be gained: $T_t/(A_r d)^2 = 0.0885$, and $u_0 k/(A_r l) = 2.86$ in terms of eqs. [11g] and [12g]; $z_m/l = 0.4208$ via eq. [23g] ($z_o/l>z_m/l$); and $M_m/(A_r d)^3 = 0.0465$ using eq. [25g]. Accordingly, it follows $T_t = 0.8285$ kN, $u_0 = 110.4$ mm, $z_m = 0.257$ m, and $M_m = 0.266$ kNm.

(2) Responses for the pre-tip yield state are tabulated in Table 10 for $z_o/l = 0.3$ and 0.5. Given $z_o/l = 0.3$, $T_t$, $u_0$, and $z_m$ were estimated as 0.462 kN, 21.3 mm, and 0.193 m, based on eqs. [11g], [12g], and [24g], respectively. $M_m$ was calculated as 0.129 kNm via eq. [26g] (with $z_m>z_o$). As the $z_o/l$ increases to 0.5, $T_t$ increases to 0.723 kN; $u_0$ to 65.3 mm; $z_m$ raises to 0.241 m (as per eq. [23g] with $z_m<z_o$); and $M_m$ to 0.224 kNm (according to eq. [25g]). The two points given by the pairs of $M_m$ and $z_m$ agree well with the respective $M(z)$ profiles.

(3)-(4) Calculation for the post-tip yield state is not presented here. Upon the rotation point yield, an identical response to that for a Gibson $k$ (see Table 13) is obtained.
Profiles of bending moment, $M(z)$ and shear force, $T(z)$ at the slip depths of $z_o=0.3l$ and $0.5l$ were determined using expressions given previously (Guo 2003). They are plotted in Figs. 11a and 11b. Local shear force-displacement relationships at five different depths were evaluated and are plotted in Fig. 12.

The $G_s$ was estimated as 0.105 MPa ($= 3.88\times0.102/3.757\text{MPa}$) using eq. [4].

A head-displacement $u_0$ of 92 mm (prior to tip yield) was needed to mobilise the radial pressure $\sigma_r$ of 66.85 MPa at the depth, associating with a local displacement $u$ of 31.3 mm. The displacement occurs at $z_o = 0.342 \text{ m}$, and $z_r = 0.447 \text{ m}$ (note $G_s = 0.105\text{MPa}$). Distribution of the $\sigma_r$ is predicted identical to that for Gibson $k$.

The results shown in Figs. 11 and 12 for Constant $k$ largely support the comments on Gibson $k$ about the $M_m$, $z_m$, $z_r$ and the non-dimensional responses. The measured force (thus $\sigma_r$) at a $u_0$ of 92 mm far exceeds the predicted one (see Fig. 10a), which is opposite to that from Gibson $k$. The actual $k$ should be bracketed by the uniform $k$ and Gibson $k$.

4.1.3 Effect of $k$ profiles

The impact of the $k$ profiles is evident on the predicted $T_t \sim u_0$ curve; whereas it is noticeable on the predicted $M_m$ only at initial stage (see Fig. 10). The latter is owing to the fact that beyond the initial low load levels, the $M_m$ is given by the same value of $A_r$ and the same eqs. [23] and [25]. The deduced $A_r$ is $\pm \sim15\%$ different from eq. [7]. The deduced (constant) $k$ is $\pm \sim3.5\%$ different from the revised measured $k$, except for the pile in $D_r = 0.25$, and those deduced from Gibson $k$, which are explained herein.

Given Gibson $k$, the local displacement $u$ of 21.3 mm ($> u^* = 13 \text{ mm} = 244.9/18640 \text{ m}$) for inducing the measured $\sigma_r$ was associated with plastic response, as the $u = 31.3 \text{ mm}$ for a Constant $k$ was ($u>u^* = 18.6 \text{ mm} = 244.9\times0.276/3880\text{m}$). In contrast, the stress hardening exhibited (see Fig. 10) beyond a mudline displacement $u_0$ of 57 mm implies a higher value of $A_r$ (thus $p_u$) than 244.9 kPa/m adopted herein, and a higher local limit $u^*$ than the currently adopted 13.0$-\sim18.6 \text{ mm}$. The derived $k$ from eq. [1] needs modifications in view of the following:

- Any plastic component of displacement, $u-u^*$, may render the modulus $k$ to be underestimated. The displacement $u$ exceed the elastic limit $u^*$ by 64 % $[= (21.3-13)/13]$ using Gibson $k$; or by 68% $[= (31.3 -18.6)/18.6]$ using Constant $k$. With stress $\sigma_r \propto ku$ (i.e., eq. [1]), the $k$ may be supposedly underestimated by $\sim 68\%$, even if the $\sigma_r$ has been well mimicked using the $k$ profiles. The hardening effect may render less
degree of underestimation, which then seems to be consistent with the 19.5−45.6% underestimation of the measured modulus (Table 9, $D_e = 0.25$), with a predicted $G_L = 0.105−0.31$ MPa.

In contrast, overestimation of the (Gibson) $k$ is noted for $D_e = 0.5−0.75$, although the displacement of $0.2d$ and the angle (slope) for the capacity $T_o$ are close to those used in practice. Real $k$ profile should be bracketed by the constant and Gibson profiles.

4.2 Comments on Current Predictions

(1) The current solutions were developed to cater for net lateral resistance along the shaft only. Longitudinal resistance along the shaft, and transverse shear resistance on the tip are neglected. The shear resistance may become apparent regarding very short, stub piers such as pole foundations (Vallabhan and Alikhanlou 1982). In these circumstances, use of the current solutions will be conservative.

(2) The modulus $k$ was stipulated as a constant or linearly increase with depth (Gibson type), along with a linear $p_u$ profile. The two $k$ profiles should bracket well possible $k$ profiles encountered in practice. Along rigid piles, the linear $p_u$ profile is normally expected, whereas along flexible piles, the $p_u$ may be proportional to $z^{1.7}$ (Guo 2006). Given a pile in a multi-layered sand, the $p_u$ may even become uniform, for which pertinent solutions published previously (Scott 1981) may be utilised.

(3) Equations [1] and [5] are rigorous for the constant $k$, but approximate for the Gibson $k$. The modulus deduced incorporates the effect of pile diameter.

(4) The elastic-plastic load-displacement curve cannot capture the impact of stress hardening as demonstrated in the pile in $D_e = 0.25$.

5. CONCLUSIONS

Elastic-plastic solutions were developed for laterally loaded rigid piles using the load transfer approach. They are presented in explicit form regarding pre-tip and post-tip yield states respectively. Simple expressions are developed for determining the depths $z_o$, $z_1$, and $z_t$ used for constructing on-pile force profiles, and for calculating moment $M_m$ and its depth $z_m$. They are generally plotted against $e/l$, and are elaborated for $e = 0$ and $\infty$. The solutions and expressions are shown to be consistent with available FE analysis and
relevant measured data. They are implemented into a spreadsheet program called GASLSPICS, which was used to conduct a detailed investigation into a well documented case. Comments are made regarding estimation of capacity. In particular, the following features about the current solutions are noted:

(1) The $p_u$ profile is differentiated from on-pile force profile. The former is unique, whereas the latter is mobilised along a specified LFP and may be constructed for any states (e.g. pre-tip yield, tip-yield, post-tip yield, and rotation point yield states).

(2) Characterized by the slip depths $z_0$ and $z_1$, the solutions allow nonlinear response (e.g. load, displacement, rotation and maximum bending moment) to be readily estimated, using the parameters $k$ and $A_r$. Conversely, the two parameters can be deduced using two measured nonlinear responses. The back-estimation is legitimate, as stress distributions along depth and around pile diameter are integrated into the solutions.

(3) For the investigated piles, the deduced $A_r$ is with $\pm15\%$ error from eq. [7]; whereas shear moduli have $\pm3.5\%$ discrepancy from the measured data (except for the $\pm46\%$ underestimation noted for stress hardening case).

(4) Maximum bending moment raises $\sim30\%$ as the tip-yield state moves to the YRP state. It increases 2.1~2.2 times as the $e$ increases from 0 to $3l$ at either state (N. B. $M_{\text{max}} \approx M_0$ given $e/l > 3$).

(5) The impact of $k$ profile is bracketed by the solutions concerning a uniform $k$ and a Gibson $k$. Without catering for the influence of the eccentricity, the $k$ is underestimated by $\sim40\%$ for a rigid pile ($l/r_0 = 3~8$).

The current solutions can accommodate the increase in resistance owing to dilation by modifying $A_r$, while not able to capture the effect of stress-hardening.

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REFERENCES


List of Symbols

The following symbols are used in the paper:

- $A_r = \text{coefficient for the LFP [FL}^{-3}]$;
- $C = A_r/(u_0k_o)$ (Gibson k), or $A_rz_o(u_0k_o)$ (Constant k), used for post-tip yield state;
- $C_y = \text{value of the } C \text{ at the tip yield state};$
- $d(r_o) = \text{outside diameter (radius) of a cylindrical pile [L]};$
- $D_r = \text{sand relative density};$
- $E_p = \text{Young’s modulus of an equivalent solid cylinder pile [FL}^{-2}]$;
- $E_s = \text{Young’s modulus of soil [FL}^{-2}]$;
- $e = \text{eccentricity (free-length) [L] i.e. the height from the loading location to the mudline; or e = } M_o/T_t;$
- $G_s, \bar{G}_s = \text{shear modulus of the soil, and average of the } G_s [FL}^{-2}]$;
- $G_L = \text{shear modulus of the soil at pile tip level [FL}^{-2}]$;
- $k, k_o = \text{modulus of subgrade reaction [FL}^{-3}]$, $k = k_oz_m^m, m = 0 \text{ and } 1 \text{ for Constant and}\$
  \text{Gibson } k, \text{respectively; and } k_o, \text{ a parameter [FL}^{-3-m}]$;
- $K_a = \tan^2(45^\circ - \phi_s'/2)$, the coefficient of active earth pressure;
- $k_n, k_T = \text{modulus } k \text{ due to pure bending moment, and pure lateral loading [FL}^{-3}]$;
- $K_p = \tan^2(45^\circ + \phi_s'/2)$, the coefficient of passive earth pressure;
- $k_1 = \text{parameter for estimating the load transfer factor, } \gamma_b;$
- $K_i(\gamma_b) = \text{modified Bessel function of second kind of } i^{th} \text{ order};$
- $l = \text{embedded pile length [L]};$
- $\text{LFP} = \text{net limiting force profile per unit length [FL}^{-1}]$;
- $M_m = \text{maximum bending moment within a pile [FL]}$;
- $M_o = \text{bending moment at the mudline level [FL]}$;
- $p, p_u = \text{force per unit length, and limiting value of the } p \text{ [FL}^{-1}]$;
- $s = \text{integration variable};$
- $T(z) = \text{lateral force induced in a pile at a depth of } z \text{ [F]}$;
- $T_r, T_o = \text{lateral load applied at an eccentricity of ‘e’ above mudline, and } T_t \text{ at a defined (tip yield or YRP) state [F]};$
- $u, u_0 = \text{lateral displacement, and } u \text{ at mudline level [L]}$;
- $u^* = \text{local threshold } u^* \text{ above which pile-soil relative slip is initiated [L]}$;
- $\text{YRP} = \text{yield at rotation point};$
- $z = \text{depth measured from the mudline [L]}$;
- $z_m = \text{depth of maximum bending moment [L]}$;
- $z_o(z_1) = \text{slip depth initiated from mudline (pile-base)[L]}$;
- $z_t = \text{depth of rotation point [L]}$;
- $\bar{z}_o = \text{slip depth } z_o \text{ at the moment of the tip yield [L]}$;
- $\gamma_b = \text{load transfer factor};$
- $\gamma_s' = \text{effective density of the overburden soil [FL}^{-3}]$;
- $\theta_p = \text{angle between the interesting point and the loading direction};$
- $\nu_s = \text{Poisson’s ratio of soil};$
- $\sigma_r = \text{radial stress in the soil surrounding a lateral pile [FL}^{-2}]$;
- $\sigma_0 = \text{circumferential stress in the soil surrounding a lateral pile [FL}^{-2}]$;
\( \sigma_z \) = vertical stress in the soil surrounding a lateral pile [FL\(^{-2}\)];
\( \tau_{r\theta} \) = shear stress in the \( r-\theta \) plane [FL\(^{-2}\)];
\( \tau_{\theta z} \) = shear stress in the \( \theta-z \) plane [FL\(^{-2}\)];
\( \tau_{rz} \) = shear stress in the \( r-z \) plane [FL\(^{-2}\)];
\( \phi_s' \) = effective frictional angle of soil;
\( \omega \) = rotation angle (in radian) of the pile.
Appendix A: Solutions for Gibson k profile

A.1 Pile Response at Pre-tip Yield State

The force per unit length \( p \) gained from eqs. [3] and [6] allows the horizontal force equilibrium of the rigid pile subjected to a lateral load, \( T_t \) at the pile-head (see Fig. 1) to be written as

\[
T_t - \int_o^{z_o} A_s(s)ds - \int_{z_o}^{l} k_o(s)(o + u_0)ds = 0
\]

The integration is made with respect to ‘s’. The moment equilibrium about the pile-tip offers

\[
T_t(e + l) - \int_o^{z_o} A_s(l - s)ds - \int_{z_o}^{l} k_o(s)(o + u_0)(l - s)ds = 0
\]

Equations [A1] and [A2] along with eq. [8] allow the \( \omega \) and \( u_0 \) to be determined as:

\[
\omega = \frac{12T_t(l + e)}{k_o(l + z_o)(l - z_o)^3} - \frac{2l^3u^*}{(l + z_o)(l - z_o)^3}
\]

\[
u_0 = \frac{-12z_oT_t(l + e)}{k_o(l + z_o)(l - z_o)^3} + \frac{-z_o^4 + 2z_o^3l + l^4}{(l + z_o)(l - z_o)^3}u^*
\]

The solutions prior to tip yield are obtained from eq. [A1] to eqs. [A3a] and [A3b]. They are recast in the normalised form of eq. [11] for the \( T_t \), eq. [12] for the \( \omega \), and eq. [13] for the \( u_0 \).

The shear force at depth \( z \), \( T(z) \) is given by:

\[
T_t - \int_o^{z} A_s(s)ds = T(z) \quad (z \leq z_o)
\]

\[
T_t - \int_o^{z_o} A_s(s)ds - \int_{z_o}^{z} k_o(s)(o + u_0)ds = T(z) \quad (z > z_o)
\]

At \( z = l \), \( T(z) = 0 \), eq. [A5b] is identical to eq. [A1]. Furthermore, \( T(z) \) is rewritten as:
Maximum bending moment, $M_m$ occurs at a depth of $z_m$ at which the shear force, $T(z_m)$ is zero. The moment, $M_m$ is determined from the following expressions:

\[
[A7a] \quad M_m = T_e (e + z_m) - \int_0^{z_m} dA_s (z_m - s) ds \quad (z_m \leq z_o)
\]

\[
[A7b] \quad M_m = T_e (e + z_m) - \int_0^{z_m} dA_s (z_m - s) ds - \int_{z_o}^{z_m} k_o s d(\omega s + u_o)(z_m - s) ds \quad (z_m > z_o)
\]

Equations [A7a] and [A7b] are similar to those given previously (e.g. Scott 1981). They were used to derive the depth $z_m$ and the moment $M_m$.

### A.2 Pile Response Posterior to Tip Yield

As yield expands from the pile-tip, the horizontal force equilibrium of the pile, and the moment equilibrium about the pile-head require:

\[
[A8] \quad T_i - \int_o^{z_o} A_s ddds - \int_{z_o}^{z_i} k_o s d(\omega s + u_o) ds + \int_{z_i}^{z_o} A_s ddds = 0
\]

\[
[A9] \quad \int_o^{z_o} A_s s^2 ddds + \int_{z_o}^{z_i} k_o s d(\omega s + u_o) sdds - \int_{z_i}^{z_o} A_s s^2 ddds + T_i e = 0
\]

These expressions may be integrated to give:

\[
[A10] \quad T_i / d = 0.5 A_r z_o^2 + 0.5 k_o u_o (z_1^2 - z_o^2) + \omega k_o (z_3^2 - z_o^3) / 3 - 0.5 A_r (l_i^2 - z_1^2)
\]

\[
[A11] \quad \frac{1}{3} A_r z_o^3 + \frac{1}{4} k_o \omega (z_4^4 - z_o^4) + \frac{1}{3} k_o u_o (z_3^3 - z_o^3) - \frac{1}{3} A_r (l_1^3 - z_1^3) + T_i e / d = 0
\]

Equation [A10], together with eqs. [8] and [10], gives:

\[
[A12] \quad \omega = -2u^*/(z_1 - z_o)
\]

\[
[A13] \quad u_o = u^* (z_1 + z_o)/(z_1 - z_o)
\]
\[ z_r = -u_0 / \omega \]  
\[ T_t = \frac{1}{6} k_r du^* \left( 2z_1^2 + 2z_2z_o - 3l^2 + 2z_2^2 \right) \]

Given \( C = A_r/(k_0u_0) \), it follows that:

\[ z_o = z_r (1 - C) \]
\[ z_1 = z_r (1 + C) \]
\[ u_0 = A_r / Ck_o \]

Equation [A15] can be simplified to the form of eq. [20], in terms of eqs. [A16]–[A18].
Table 1. \( k_m/k_T \) at various slenderness ratios of \( l/r_0 \)

<table>
<thead>
<tr>
<th>( l/r_0 )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_m/k_T )</td>
<td>1.56</td>
<td>1.47</td>
<td>1.41</td>
<td>1.37</td>
<td>1.35</td>
<td>1.32</td>
<td>1.31</td>
<td>1.29</td>
<td>1.28</td>
<td>1.27</td>
</tr>
</tbody>
</table>

Note **a** \( k_m/k_T = 3.153 \) (\( l/r_0 = 0 \)); 1.27~1.22 (10~20); 1.22~1.19 (20~30); 1.14~1.12 (100~200).

Table 2. Capacity of lateral piles based on limit states

<table>
<thead>
<tr>
<th>Mtds</th>
<th>( T_o/(A_r d l^2) )</th>
<th>( A_r ) (kN/m³)</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Using equilibrium against rotating point</td>
<td>( K_{qa}\gamma_s' ) <strong>a</strong></td>
<td>Brinch Hansen (1961)</td>
</tr>
<tr>
<td>2</td>
<td>([6(1 + e/l)]^{-1})</td>
<td>3 ( K_p\gamma_s' )</td>
<td>Broms (1964)</td>
</tr>
<tr>
<td>3</td>
<td>([2.13(1 + 1.5 e/l)]^{-1} ) <strong>b</strong></td>
<td>(28 ~ 228) kPa <strong>b</strong></td>
<td>McCorkle (1969)</td>
</tr>
<tr>
<td>4</td>
<td>0.5[2(z_r/l)^2]^{-1}</td>
<td>((3.7K_p - K_a)\gamma_s')</td>
<td>Petrasovits and Award (1972)</td>
</tr>
<tr>
<td>5</td>
<td>((1 + 1.4 e/l)^{-1})</td>
<td>(F_bS_{bu}(K_p - K_a)\gamma_s') <strong>c</strong></td>
<td>Meyerhof, et al (1981)</td>
</tr>
<tr>
<td>6</td>
<td>([8(1 + 1.5 e/l)]^{-1} ) <strong>b</strong></td>
<td>(80 ~ 160) kPa <strong>b, d</strong></td>
<td>Balfour Beatty Construction (1986)</td>
</tr>
<tr>
<td>7</td>
<td>(1 - d/l)</td>
<td>4.167(\gamma_s')</td>
<td>Dickin and Wei (1991)</td>
</tr>
<tr>
<td>8</td>
<td>(0.51\frac{z_r}{l}\left[1.59\frac{z_r}{l} - 1\right])</td>
<td>0.8(\gamma_s')10^{1.3}\tan\phi'_{s} +0.3</td>
<td>Prasad &amp; Chari (1999)</td>
</tr>
<tr>
<td>9</td>
<td>(\frac{0.1181}{1 + 1.146e/l})</td>
<td>(K_p\gamma_s')</td>
<td>Derived using (z_ol = 0.618) in eq. [11g]</td>
</tr>
</tbody>
</table>

Note **a** \( K_{qa} \) = also passive pressure coefficient, but it depends on \( l/d \) ratio, etc.  
**b** Dimensional expressions, as a uniform \( p_0 \) is adopted, and \( A_r \) unit is kPa.  
**c** \( F_b \) = lateral resistance factor, 0.12 for uniform soil; \( S_{bu} \) = a shape factor which depends on the depth \( l \) and the angle of \( \phi'_{s} \).  
**d** Smaller values in presence of water.
Table 3. Responses of piles in ‘$k = \text{constant}$’ soil (Pre-tip yield state)

<table>
<thead>
<tr>
<th>Expressions</th>
<th>References</th>
<th>$e/l = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u^* = A_o z_o/k$</td>
<td>eq. [9g]</td>
<td></td>
</tr>
<tr>
<td>$-u^* z_1/z_o = \omega z_1 + u_0$</td>
<td>eq. [10g]</td>
<td></td>
</tr>
<tr>
<td>$T_i/\sqrt{A_i d t^2} = \frac{z_o/l}{2(2 + z_o/l + 3e/l)}$</td>
<td>eq. [11g]</td>
<td>0</td>
</tr>
<tr>
<td>$u_0 k/A_i l = \frac{(2 + 3e/l) z_o/l}{[2 + z_o/l + 3e/l](1 - z_o/l)^2}$</td>
<td>eq. [12g]</td>
<td>$\frac{z_o/l}{(1 - z_o/l)^2}$</td>
</tr>
<tr>
<td>$\omega = \frac{A_o z_o}{k l^2} \left(\frac{z_o/l}{2} + 3(z_o/l - 2) e/l - 3\right)$</td>
<td>eq. [13g]</td>
<td>$\frac{A_o z_o}{k l^2} \frac{z_o/l - 2}{(1 - z_o/l)^2}$</td>
</tr>
<tr>
<td>$z_o/l = -u_i/\omega l$</td>
<td>eq. [14g]</td>
<td></td>
</tr>
<tr>
<td>$\frac{z_o}{l} = \left(1.5 e/l + 0.5\right) + 0.5 \sqrt{5 + 12 \frac{e}{l^2} + 9\left(\frac{e}{l^2}\right)^2}$</td>
<td>eq. [22g]</td>
<td>0.5</td>
</tr>
<tr>
<td>$\frac{z_m}{l} = \sqrt{\frac{2T_i}{A_i d t^2}}$</td>
<td>eq. [23g]</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{z_m}{l} = \frac{1 + (z_o/l + 3e/l)z_o/l}{3(1 + 2e/l) - (z_o/l + 3e/l)z_o/l}$</td>
<td>eq. [24g]</td>
<td>$\frac{z_m}{l} = \frac{z_o/l}{2 - z_o/l}$</td>
</tr>
<tr>
<td>$M_m = (2z_m/3 + e)T_i$</td>
<td>eq. [25g]</td>
<td>$Te$</td>
</tr>
<tr>
<td>$M_m = \left(\frac{z_m^3 - z_o^3}{(l - z_o)^2(2l + z_o)} + \frac{dA_o}{3} z_o^3 T_i e + \frac{dA_o}{6} \frac{z_o}{2} (2z_o - 3z_m)^3 + T_i (e + z_m) + \frac{dA_o}{6} z_o^2 (2z_o - 3z_m) + \frac{dA_o}{2} \left[\frac{z_m z_o - 2l^2 + z_o l}{(l - z_o)(2l + z_o)}\right] \right)$</td>
<td>eq. [26g]</td>
<td>N/A</td>
</tr>
</tbody>
</table>

### Table 4. Response at various states \((e = 0, \text{Gibson } k / [\text{Constant } k])\)

<table>
<thead>
<tr>
<th>Items</th>
<th>(T_t/(A_0 d l^3))</th>
<th>(u_o k_o / A_r) (u_o k_o / A_r)</th>
<th>(o k_o l / A_r) (o k_o l / A_r)</th>
<th>(M_o/(A_0 d l^3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z_o/l) = (z_o/l)</td>
<td>Eq. (11)</td>
<td>Eq. (12)</td>
<td>Eq. (13)</td>
<td>Eqs. (25) or (26)</td>
</tr>
<tr>
<td>(z_o/l)</td>
<td>[Eq. (11g)]</td>
<td>[Eq. (12g)]</td>
<td>[Eq. (13g)]</td>
<td>[Eqs. (25g) or (26g)]</td>
</tr>
<tr>
<td>Tip yield(^a)</td>
<td>0.113</td>
<td>3.383</td>
<td>-4.3831</td>
<td>0.036</td>
</tr>
<tr>
<td></td>
<td>[0.118]</td>
<td>[3.236]</td>
<td>[-4.2352]</td>
<td>[0.038]</td>
</tr>
<tr>
<td>YRP (^b)</td>
<td>0.130</td>
<td>(\infty)</td>
<td>0.5(\pi k_o l / A_r)</td>
<td>0.0442</td>
</tr>
<tr>
<td></td>
<td>[0.130]</td>
<td>[(0.5\pi k_o l / A_r)]</td>
<td>[0.0442]</td>
<td></td>
</tr>
</tbody>
</table>

Note \(^a\) \(z_o/l = 0.5437 / [0.618], z_r/l = 0.772 / [0.764], z_m/l = 0.4756 / [0.4859],\) and \(C_y = 0.296 / [0.236].\)

\(^b\) At the YRP state, all critical values are independent of \(k\) distribution. Thus, \(z_o/l = z_r/l = 0.7937/ [0.7937],\) and \(z_m/l = 0.5098 / [0.5098].\) Also \(M_o = 0.\)

### Table 5. Response at various yield states \((e = \infty, \text{Gibson } k \text{ [Constant } k])\)

<table>
<thead>
<tr>
<th>Items</th>
<th>(u_o k_o / A_r) (u_o k_o / A_r)</th>
<th>(o k_o l / A_r) (o k_o l / A_r)</th>
<th>(M_o/(A_0 d l^3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z_o/l) = (z_o/l)</td>
<td>(2 + (z_o/l)^3) ((1-z_o/l)^2(2+z_o/l))</td>
<td>-3 ((1-z_o/l)^2(2+z_o/l))</td>
<td>(1 + 2z_o/l + 3(z_o/l)^2) (12(2+z_o/l))</td>
</tr>
<tr>
<td>(z_o/l)</td>
<td>[Table 2]</td>
<td>[Table 2]</td>
<td>[(\frac{1}{6}z_o/l)]</td>
</tr>
<tr>
<td>Tip yield(^a)</td>
<td>2.155 / [2.0]</td>
<td>-3.1545 / [-3.0]</td>
<td>0.0752 / [0.0833]</td>
</tr>
<tr>
<td>YRP (^b)</td>
<td>(\infty)</td>
<td>0.5(\pi k_o l / A_r)</td>
<td>0.0976 / [0.0976]</td>
</tr>
</tbody>
</table>

Note \(^a\) \(z_o/l = 0.366 / [0.50], z_r/l = 0.683 / [0.667], z_m/l = 0/[0],\) and \(C_y = 0.464 / [0.333].\)

\(^b\) \(z_o/l = z_r/l = 0.7071 / [0.7071],\) and \(z_m/l = 0 [0].\)

Also \(T_t = 0,\) and \(M_o = M_{in}.\)
Table 6. Responses of piles in ‘k = constant’ soil (Post-tip yield state)

<table>
<thead>
<tr>
<th>Expressions</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{T_t}{A_t \cdot d t^2} = 0.5 \left[ \frac{2}{1 - C^2} \left( \frac{z_r}{l} \right)^2 \right] - 1 ] [ \text{where } C = A_t z_r / (k u_0) ]</td>
<td>eq. [20g]</td>
</tr>
<tr>
<td>[ u_0 = A_t z_r / (k C) ]</td>
<td>eq. [21g]</td>
</tr>
</tbody>
</table>

The ratio \( z_r / l \) is governed by the following expression:

\[
(\frac{z_r}{l})^3 + 1.5(1 - C^2) \frac{e}{l} (\frac{z_r}{l})^2 - (0.5 + 0.75 \frac{e}{l})(1 - C^2)^2 = 0
\]

Thus, the \( z_r / l \) should be obtained using

\[
[-1.5 \frac{e}{l} C^2_g - (0.5 + 0.75 \frac{e}{l}) C^4_g] (\frac{z_r}{l})^4 + (\frac{z_r}{l})^3 + [1.5 \frac{e}{l} + (1 + 1.5 \frac{e}{l}) C^2_g] (\frac{z_r}{l})^2 - (0.5 + 0.75 \frac{e}{l}) = 0
\]

\( z_r / l \) may be approximated by the following solution (Guo 2003)

\[
z_r / l = 0.5(1 - C^2) (\sqrt{A_0} + \sqrt{A_1} + D_0) \quad \text{(Iteration required)}
\]

\[ A_j = (D_0^3 + D_1) + (-1)^j [(D_1(2D_0^3 + D_1))]^{1/2} \quad (j = 0, 1) \]

\[ D_1 = \frac{2 + 3e/l}{1 - C^2} \quad D_0 = -\frac{e}{l} \]

It is generally ~5% less than the exact value of \( z_r / l \). Note that eqs. [27g], [28g] and [29g] are identical to eqs. [27], [28] and [29], respectively.

Table 7. Expressions for depth of rotation

<table>
<thead>
<tr>
<th>( u = \frac{\omega z + u_0}{\omega} )</th>
<th>Depth of rotation</th>
<th>Slip depths</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_r = - \frac{u_0}{\omega} )</td>
<td>( z_o / l ) deduced using eqs. [11]~[14]</td>
<td>( z_i / l = (1 + C)z_r / l ) [ \frac{z_o / l = (1 - C)z_r / l}{[z_i / l = (z_r / l)(1 - C)]} ] [ \frac{z_o / l = (z_r / l)(1 + C)}{z_o / l = z_o / l} ]</td>
<td>2a</td>
</tr>
</tbody>
</table>

Note: \( u_0, \omega, z_r, z_o, z_l \) refer to list of symbols. \( C = A_r / (u_0 k_o) \) (Gibson \( k \)), \( C = A_r z_r / (u_0 k) \) (Constant \( k \))
Table 8. Pile in dense sand reported by Laman, et al. (1999)

<table>
<thead>
<tr>
<th>Input parameters (l = 2 m, d = 1 m)</th>
<th>Output for tip yield state (Test 3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_r$ (kN/m$^3$)</td>
<td>$k$ (MN/m$^3$)</td>
</tr>
<tr>
<td>621.7</td>
<td>34.42$^a$/51.63$^b$</td>
</tr>
</tbody>
</table>

$^a$ Test 3; $^b$ Test 2.

Table 9. Parameters for the model piles (Gibson $k$/ [Constant $k$])

<table>
<thead>
<tr>
<th>$D_r$</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_r$ (kN/m$^3$)</td>
<td>244.9</td>
<td>340.0</td>
<td>739.0</td>
<td>Numerator: Gibson $k$</td>
</tr>
</tbody>
</table>
| $k_o$ (MPa/m$^2$) | \[18.64 \quad 48.2 \quad 81.43 \quad \] | \[3.88 \quad 12.05 \quad 16.96 \quad \] | Denominator: Constant $k$  
| Predicted $G_L$ (MPa) | \[0.31 \quad 0.801 \quad 1.353 \quad \] | \[0.105 \quad 0.327 \quad 0.461 \quad \] | \[a\] Via multiplying the values of 3.78, 6.19 and 9.22 with the diameter $d$ (0.102m), or |
| Measured $G_L$ (MPa)$^a,b$ | \[0.385^a \quad 0.631^a \quad 0.94^a \quad \] | \[0.193^b \quad 0.316^b \quad 0.47^b \quad \] | \[b\] Via multiplying by $0.5d$. |
| Angle at $z_o/l$ (deg.) | \[3.94 \quad 2.11 \quad 2.72 \quad \] | \[14.0 \quad 10.6 \quad 14.4 \quad \] | |

Table 10. States of yield for model piles (valid for any $D_r$, Gibson $k$/ [Constant $k$])

<table>
<thead>
<tr>
<th>Items</th>
<th>$z_o/l$</th>
<th>$u_o k_o/A_r$</th>
<th>$T_i/(A_r d^3)$</th>
<th>$\omega k_o l/A_r$</th>
<th>$z_m/l$</th>
<th>$M_m/(A_r d^3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-tip yield</td>
<td>0.30</td>
<td>[1.695^a \quad [0.5517]^a \quad 0.0647 \quad [0.0494] \quad -2.318 \quad [0.3151] \quad 0.364 \quad [0.0225] \quad 0.031 \quad [0.0031] \quad ]</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>[3.005 \quad [1.691] \quad 0.0838 \quad [0.0773] \quad -4.005 \quad [0.3931] \quad 0.409 \quad [0.0392] \quad 0.0434 \quad [0.0043] \quad ]</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.535$^c$</td>
<td>[3.426 \quad [2.86] \quad 0.087 \quad [0.0885] \quad -4.530 \quad [3.86] \quad 0.4171 \quad [0.4208] \quad 0.0453 \quad [0.0465] \quad ]</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>YRP</td>
<td>[z_o/l = 0.774, z_m/l = 0.445, T_i/(A_r d^3) = 0.099, and M_m/(A_r d^3) = 0.0536. ]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: $^a$Numerator: Gibson $k$; $^b$Denominator: Constant $k$; $^c$Post-tip yield; $^d$Tip yield state.
Table 11. Effect of $k$ profile on predicted responses (Gibson $k$/[Constant $k$])

<table>
<thead>
<tr>
<th>$z_0/l$</th>
<th>$u_0$(mm)</th>
<th>$T_t$(kN)</th>
<th>$\omega$(degree)</th>
<th>$z_m$(m)</th>
<th>$M_m$(kNm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.30</td>
<td>22.3$^a$</td>
<td>0.605</td>
<td>$-0.050(2.85^o)$</td>
<td>0.223</td>
<td>0.181</td>
</tr>
<tr>
<td></td>
<td>[21.3]$^b$</td>
<td>[0.462]</td>
<td>$[-0.053(3.03^o)]$</td>
<td>[0.193]</td>
<td>[0.129]</td>
</tr>
<tr>
<td>0.50</td>
<td>39.5</td>
<td>0.783</td>
<td>$-0.087(4.93^o)$</td>
<td>0.251</td>
<td>0.248</td>
</tr>
<tr>
<td></td>
<td>[65.3]</td>
<td>[0.723]</td>
<td>$[-0.15(8.61^o)]$</td>
<td>[0.241]</td>
<td>[0.224]</td>
</tr>
<tr>
<td>0.535$^c$</td>
<td>45.0</td>
<td>0.814</td>
<td>$-0.097(5.57^o)$</td>
<td>0.255</td>
<td>0.261</td>
</tr>
<tr>
<td>[0.5885]$^d$</td>
<td>[110.4]</td>
<td>[0.8285]</td>
<td>$[-0.244(13.95^o)]$</td>
<td>[0.257]</td>
<td>[0.266]</td>
</tr>
<tr>
<td>YRP</td>
<td>$\propto$</td>
<td>0.926</td>
<td>$\pi/2(90^o)$</td>
<td>0.272</td>
<td>0.307</td>
</tr>
</tbody>
</table>

Note $^a$Numerator: Gibson $k$; $^b$Denominator: Constant $k$; $^c$Post-tip yield; $^d$Tip yield state.

Table 12. Calculation of $z_0/l$ for post-tip and YRP states (Gibson $k$)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(valid for any $D_r$)</td>
<td>$C$</td>
<td>$D_0$</td>
<td>$D_1$</td>
<td>$A_0$</td>
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<tr>
<td>Post-tip yield</td>
<td>0.2919</td>
<td>2.5205</td>
<td>0.6968</td>
<td>0.627</td>
</tr>
<tr>
<td>YRP</td>
<td>0</td>
<td>2.735</td>
<td>0.735</td>
<td>0.680</td>
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</table>

Table 13. Calculation of $z_0/l$ for YRP state (Constant $k$)

<table>
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<th>Eqs. for Constant $k$</th>
<th>Eq. [19g]</th>
<th>Eq. [18g]</th>
<th>Eq. [17g]</th>
<th>$z_0/l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(valid for any $D_r$)</td>
<td>$C$</td>
<td>$D_0$</td>
<td>$D_1$</td>
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<tr>
<td>YRP</td>
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Figure Captions

Fig. 1 Schematic analysis for a rigid pile. (a) Pile - soil system. (b) Load transfer model

Fig. 2 Schematic limiting force profile, on-pile force profile, and pile deformation. (a) Tip yield state. (b) Post-tip yield state. (c) Impossible yield at rotation point (YRP)

Fig. 3 Comparison between the predicted and the measured (Prasad and Chari 1999) radial pressure, $\sigma_r$ on a rigid pile surface

Fig. 4 Effect of free-length on responses of piles at tip-yield state. (a) Normalised slip depth, $z_o/l$. (b) Normalised $u_0k_o l^3/(A_l)$. (c) Normalised $\omega k_o l^3/A_l$. (d) Normalised depth $z_m/l$

Fig. 5 Predicted vs measured (Prasad and Chari 1999) normalised on-pile force profiles upon the tip yield. (a) Constant $k$. (b) Gibson $k$.

Fig. 6 Normalised moment $M_o (= T_o e)$ and $M_m$

Fig. 7 Normalised responses under various ratios of $e/l$. (a) Pile-head load $T_t$ and mudline displacement $u_0$. (b) $T_t$ and rotation $\omega$. (c) $T_t$ and maximum bending moment $M_{max}$

Fig. 8 Comparison among the current predictions, the measured data, and FEA results (Laman, et al. 1999). (a) $M_o$ versus rotation angle $\omega$ (Test 3). (b) $T_t$ versus mudline displacement $u_0$. (c) $M_o$ versus rotation angle $\omega$ (Effect of $k$ profiles, Tests 2 and 1).

Fig. 9 Comparison of the normalised pile capacity at various critical states

Fig. 10 Comparison between the current predictions and the measured (Prasad and Chari 1999) data. (a) Pile-head load $T_t$ and mudline displacement $u_0$. (b) $T_t$ and maximum bending moment $M_{max}$. (c) Maximum bending moment $M_{max}$ and its depth $z_m$

Fig. 11 Effect of $k$ distributions on the profiles of (a) bending moment, and (b) shear force

Fig. 12 Local shear force ~ displacement relationships at five typical depths