Abstract: Watertable waves generated by forcing from oceanic oscillations play an important role in the availability and quality of coastal groundwater resources. Here we present results from laboratory experiments which examine the influence of a boundary slope on the mass transfer from an oscillating clear water reservoir into a homogeneous, unconfined aquifer. The experimental forcing period is sufficiently short \((T = 348\text{ sec})\) to expose limitations of theories based on the simplifying assumptions of a shallow aquifer (hydrostatic pressure) free of any influence from the capillary fringe above the watertable. A new, small amplitude perturbation solution to the linearised Boussinesq equation is derived, taking into account a sloping boundary to an aquifer finite in depth and influenced by capillarity. However, discrepancies between the experimental observations and theory still exist, suggesting other contributing mechanisms. The influence of a truncated capillary fringe is suggested as a possible mechanism and warrants further investigation.

Keywords: watertable waves, sloping boundary, higher harmonics, unconfined aquifer, periodic forcing, capillary fringe, inter-tidal zone.

INTRODUCTION

Mass transfer across the interface between the ocean and coastal aquifer plays a crucial role in determining the quality of coastal water resource. As more and more anthropogenic stresses are applied to this resource, improved knowledge of the details of the transfer process is a necessary requirement for its’ sustainability.

In natural systems, the boundary between an aquifer and an oscillating clear water body will generally be non-vertical. A prime example is a sloping beach face subjected to a complex combination of periodic ocean forcing, the dynamic effects of which on the watertable waves have long been observed in the field (e.g. Lanyon et al., 1982; Nielsen, 1990; Turner et al., 1997; Raubenheimer et al., 1999).

Firstly we present and anlayse a detailed laboratory dataset and estimate wave numbers for the observed pressure wave. The estimated wave numbers are then compared with shallow and finite depth aquifer dispersion relations with and without capillary effects. The existing small amplitude perturbation solutions of Nielsen (1990) and Li et al. (2000a) are then briefly reviewed before deriving a new solution that accounts for a finite depth aquifer and capillarity effects. The predictions of all three of the models are then compared with the observed amplitude and phase profiles.

FLUME EXPERIMENT

Measurements

Figure 1 shows the 9m long by 1.5m high by 14cm wide sand flume used in the present study. The sand used is predominantly quartz sand with \(d_{50} = 0.2\text{ mm}\) and is reasonably well sorted with \(d_{90}/d_{10} = 1.83\). The hydraulic conductivity, \(K\), was determined \(\textit{in situ}\) using a slug test. The average over the flume was calculated to be \(1.32 \times 10^{-4}\text{ m/s}\) with minimum and maximum values
of $1.15 \times 10^{-4}$ m/s and $1.49 \times 10^{-4}$ m/s respectively. The specific yield, $S_y = \theta_s - \theta_r = 0.3$.

The approximately simple harmonic driving head parameters were: $T = 348$ sec, $d = 1.01$ m, $A_\omega = 0.204$ m and $A_{2\omega} = 0.005$ m. The boundary slope, $\beta$ is 0.2 radians and the coordinates, $(x, z)$, of the high, and low water marks were (0.4, 0.8) and (2.46, 1.21) respectively, and (1.57, 1.01) for the mean clear water level.

The piezometric head in the saturated zone was measured with cylindrical piezometers extending 10 cm horizontally into the sand at several locations along the flume (+ in Figure 1). The piezometers are 5 mm stainless steel tubes perforated with numerous 2.5 mm diameter holes screened by stainless steel mesh with 0.1 mm openings. Measurements were taken visually, by reading ID 8 mm manometer tubes connected to the piezometers with a reading accuracy of $\pm$ 1 mm.

**General observations**

A comparison of observed head levels measured near the base ($z = 0.1$ m, solid symbols) with those measured near the watertable ($z = 0.8$ m, open symbols) at several locations along the flume is shown in Figure 2.

Two particular features in the observations stand out and are consistent with observations in the field (e.g. Lanyon *et al.*, 1982; Nielsen, 1990). Firstly, at all locations the non-linear filtering effect of the sloping boundary is clearly apparent with a steep rise as the beach fills and a more gradual fall as it drains, thereby generating higher harmonics at the boundary. Secondly, evidence of seepage face formation is seen when the heads in the active forcing zone, (i.e. between low and high water marks), $0.4 < x < 2.46$ m (squares and circles), become decoupled from the driving head, again contributing to the generation of higher order harmonics.

![Figure 1: Experimental flume setup.](image1)

![Figure 2: Head level observations.](image2)

**Generation of higher harmonics**

The amplitudes and phases shown in Figure 3 and Figure 4 respectively, illustrate the generation of higher harmonics due to the non-linear filtering effect of the sloping boundary. The amplitude of the second harmonic has a maximum around the mid point of the “inter-tidal” zone which is 4 times greater than that which exists in the driving head. Little damping is experienced until landward of the high water mark where the decay is exponential in agreement with small amplitude theory,
\[ \eta_{m\omega}(x,t) = A_{m\omega} e^{-k_{m\omega}x} \cos(m \omega t - k_{m\omega}x) \]  

(1)

where \( A_{m\omega} \) is the harmonic amplitude in the driving head, \( k = k_r + i k_i \) is the wave number, \( \omega \) is the angular frequency, \( m \) is the harmonic component and \( x \) and \( t \) represent space and time respectively. The phase of the second harmonic mirrors the amplitude with a minimum occurring at the midpoint of the “inter-tidal zone”. Again, landward of the high water mark the phase grows linearly, in accordance with the small amplitude theory described by the \( k_{m\omega}x \) term in (1).

Estimation of wave numbers and comparison with theory

Description of watertable waves requires knowledge of their dispersive properties, i.e. their wave number \( k = k_r + i k_i \). As the present data indicates more or less small amplitude behaviour, wave numbers were estimated using least squares fitting to equation (1) with amplitudes and phases extracted from the data (c.f. Figure 3 and Figure 4). The results, shown in Table 1, indicate that \( k_r \neq k_i \), suggesting that the aquifer is either finite depth in nature (Nielsen et al., 1997) and/or influenced by capillarity (Barry et al., 1996; Li et al., 2000b).

Table 1: Wave numbers estimated from data (\( z = 0.8m \)).

<table>
<thead>
<tr>
<th>Value</th>
<th>( k_{1m, r} )</th>
<th>( k_{1m, i} )</th>
<th>( k_{2m, r} )</th>
<th>( k_{2m, i} )</th>
<th>( k_{3m, r} )</th>
<th>( k_{3m, i} )</th>
<th>( k_{4m, r} )</th>
<th>( k_{4m, i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R^2 )</td>
<td>0.984</td>
<td>0.343</td>
<td>0.779</td>
<td>0.311</td>
<td>0.781</td>
<td>0.208</td>
<td>0.713</td>
<td>0.378</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.998</td>
<td>0.995</td>
<td>0.998</td>
<td>0.974</td>
<td>0.988</td>
<td>0.916</td>
<td>0.869</td>
<td>0.837</td>
</tr>
</tbody>
</table>

Figure 5 compares the estimated wave numbers above with those predicted by the shallow aquifer dispersion relation,

\[(kd)^2 = im \frac{n_e \omega d}{K} \]  

(2)

where \( n_e \) is the effective porosity, \( \omega \) is the angular frequency, \( d \) is the mean aquifer thickness, \( K \) is the hydraulic conductivity, \( m \) is the harmonic component and \( i \) is the imaginary unit. Figure 6 provides the comparison with the finite depth aquifer dispersion relation (Nielsen et al., 1997),

\[ kd \tan kd = im \frac{n_e \omega d}{K} \]  

(3)

When capillary effects are neglected, i.e. \( n_e = S_p \) (solid lines), an improved comparison is seen.
when the finite depth dispersion relation is used, however, significant discrepancies still exist suggesting significant capillary influences. Capillary fringe influences can be accounted for in equations (2) and (3) by employing the complex effective porosity concept of Nielsen and Perrochet (2000a,b). Under the Green and Ampt (1911) assumption of a saturated capillary fringe with a constant pressure at the top of the fringe, \( p = \rho g H_\psi \), the corresponding complex effective porosity is (Nielsen and Turner, 2000),

\[
\eta_e = \frac{S_y}{1 + i \frac{S_y \omega H_\psi}{K}} \tag{4}
\]

where \( H_\psi \) is the equivalent saturated, steady capillary fringe height. When inserted into equation (2), equation (4) yields the same shallow aquifer with capillarity dispersion relations obtained by Barry et al. (1996), and when inserted into (3) the Green and Ampt effective porosity yields the same (finite depth with capillarity) dispersion relation as obtained by Li et al. (2000b). Based on the 1D sand column experiments of Nielsen and Perrochet (2000a,b), Nielsen and Turner (2000) showed the form of \( \eta_e \) to be different from that predicted by the Green and Ampt approximation and provided the following empirical curve fit to the column data,

\[
\eta_e = \frac{S_y}{1 + 2.5 \left( i \frac{S_y \omega H_\psi}{K} \right)^{2/5}} \tag{5}
\]

Application of the complex effective porosity formulations above to the theoretical dispersion relations provide improved agreement with the experimental estimates (see —· and --- in Figure 5 and Figure 6). The model providing the best agreement is the finite depth aquifer model with the empirical effective porosity formulation [i.e. equation (3) with \( \eta_e \) by (5), see --- in Figure 6]. Significant discrepancies still persist however. It is speculated that a possible contributing mechanism is that of a truncated capillary fringe. Preliminary experiments have indicated that a truncated fringe will result in a smaller effective porosity which, upon inspection of equations (2) and (3), will lead to smaller wave numbers and improved agreement with the data.
ANALYTICAL MODELS

Under the small amplitude assumption \((A << d)\) the non-linear Boussinesq equation can be linearised to the form,

\[
\frac{\partial h}{\partial t} = Kd \frac{\partial^2 h}{\partial x^2} - n_e \frac{\partial h}{\partial x}
\]

For the case of a vertical boundary, textbook solutions to (6) are readily available. If the boundary is sloping however, solution of (6) is not so straightforward due to the moving shoreline coordinate, \(x_{SL}(t)\), as illustrated in Figure 7.

![Figure 7: Schematic of the sloping boundary condition.](image)

**Existing small amplitude solutions**

Nielsen (1990) and Li et al (2000a) employed a perturbation approach to solve equation (6), accounting for the moving shoreline boundary condition, \(x_{SL}(t)\). Both assumed that the aquifer was shallow and was free of capillary effects, a case where the wave number, \(k = k_r + i k_i\), has equal real and imaginary parts [c.f. equation (2)]. This allows the wave number to be included in the (small) perturbation parameter, \(\varepsilon = k \cot \beta\). Under these assumptions Nielsen (1990) and Li et al. (2000a) derived the following respective solutions to equation (6),

\[
\eta_{Nielsen}(x,t) = A e^{-kz} \cos(\omega t - kx)
+ \varepsilon A \left[ \frac{1}{2} + \frac{\sqrt{2}}{2} e^{-\sqrt{2}k_z} \cos(2\omega t + \frac{\pi}{4} - \sqrt{2}kz) \right]
+ \varepsilon^2 \left[ 1 + \frac{\sqrt{2}}{2} e^{-\sqrt{2}k_z} \cos(2\omega t - \frac{\pi}{4} - \sqrt{2}kz) \right] + O(\varepsilon^3)
\]

\[
\eta_{Li}(x,t) = A e^{-kz} \cos(\omega t - kx)
+ \varepsilon A \left[ 1 + \sqrt{2} e^{-\sqrt{2}k_z} \cos\left(2\omega t - \sqrt{2}kz + \frac{\pi}{4}\right) \right]
+ \varepsilon^2 \left[ \sqrt{2} e^{-kz} \cos\left(2\omega t - kz + \frac{\pi}{4}\right) + \cos\left(kz - \frac{\pi}{4}\right) \right] + O(\varepsilon^2)
\]

The difference between the two solutions is that Li et al. (2000a) introduced the variable, \(z = x - x_{SL}(t)\), and, after applying the appropriate transformation to equation (6), obtained a solution that
exactly matches the boundary condition along the slope as opposed to Nielsen’s (1990) approximate match.

**A new small amplitude solution for \( k_r \neq k_i \)**

Data from the field (e.g. Nielsen, 1990; Aseervatham, 1994) and in the laboratory (e.g. Table 1, Nielsen et al., 1997; Cartwright et al., 2003) indicate that \( k_r \neq k_i \), a consequence of a finite depth aquifer (Nielsen et al., 1997) and/or the influence of capillarity (Barry et al., 1996; Li et al., 2000b). Here we again apply the same perturbation approach, but relax the previous assumptions of a shallow, capillary free aquifer and allow \( k_r \neq k_i \). As a consequence the perturbation parameter can no longer include the wave number and becomes, \( \varepsilon \eta = A \cot \beta \). The resulting small amplitude solution is, \( \eta_{New}(x,t) = \eta_0(x,t) + \eta_1(x,t) + \eta_2(x,t) + \ldots + \eta_n(x,t) \) (9)

where,

\[ \eta_0(x,t) = Ae^{-k_{r,x}} \cos(\omega t - k_{1,r}x) \] (10)

\[ \eta_1(x,t) = \frac{\varepsilon_n^2 A}{2} \left[ k_{r,1} \left( 1 + e^{-k_{s,r}x} \cos(2\omega t - k_{2,s,1}x) \right) - k_{s,1}xe^{-k_{s,r}x} \sin(2\omega t - k_{2,s,1}x) \right] \] (11)

\[ \eta_2(x,t) = \frac{\varepsilon_n^2 A}{4} \left[ \left\{ -k_{s,1}k_{2,s,1} - k_{s,1}k_{1,s} + k_{1,s}k_{1,s} \right\} e^{-k_{s,r}x} \sin(\omega t - k_{1,s}x) + \right. \left. \left\{ -k_{s,1}k_{2,s,1} - k_{s,1}k_{1,s} + k_{1,s}k_{1,s} \right\} e^{-k_{s,r}x} \sin(3\omega t - k_{3,s,1}x) + \right. \left. \left\{ 3k_{s,1}^2 - \frac{3}{2}k_{s,r}^2 + k_{1,s}k_{2,s,1} + k_{1,s}k_{1,s} \right\} e^{-k_{s,r}x} \cos(\omega t - k_{1,s}x) + \right. \left. + O(\varepsilon_n^3) \right\} + \right. \left. \left\{ -k_{s,1}k_{2,s,1} - k_{s,1}k_{1,s} + k_{1,s}k_{1,s} \right\} e^{-k_{s,r}x} \cos(3\omega t - k_{3,s,1}x) \right] \] (12)

**Application of models to flume data**

Attempts to model the flume data with the above models in a purely predictive sense have proven to be unsuccessful to date due to the inability of the corresponding dispersion to accurately predict the wave numbers listed in Table 1 (see section on “Estimation of wave numbers...” on page 3). As a consequence, the modelling efforts described here have been conducted in a quasi-predictive manner only. That is, the wave numbers estimated from the data for \( z = 0.8m \) (see Table 1) have been used as input and the models ability to predict the amplitude decay and phase lag was tested. For the Nielsen and Li et al. models [equations (7) and (8) respectively], which employ a single wave number, the average of \( k_{1,r} \) and \( k_{1,i} \) has been used. The models outlined above only hold true when \( x > x_{SL}(t) \) and as such have been applied as follows,

\[ h(x,t) = d + \eta_{solution}(x,t) \quad ; \quad for \quad x > x_{SL} (t) \] (13)

\[ h(x,t) = h_0(t) = d + A \cos(\omega t) \quad ; \quad for \quad x < x_{SL} (t) \]

The comparison of all three solutions with the experimental amplitude and phase profiles are shown in Figure 8 and Figure 9 respectively. It is immediately clear that allowing for \( k_r \neq k_i \) in the new solution enables it to predict both the amplitude decay rate and phase lag increase for each harmonic. However, for the two existing solutions which allow for a single wave number as input would only ever be able to match one of the profiles. For example, if \( k_{1,r} \) was chosen as
the input wave number they would match the first harmonic amplitude profile but none of the higher harmonics or phase profiles.

Each model performs reasonably well in the “inter-tidal” zone, qualitatively predicting the generation of higher harmonics in this region. A notable discrepancy being observed in the second harmonic amplitude profile where all models predict a maximum amplitude at the high water mark whereas the data indicate it to be reached at the mid point of the “inter-tidal” zone. This could be due to seepage face formation in the experiments (c.f. Figure 2) which is not accounted for in any of the three models.

CONCLUSIONS
A detailed dataset on the vertical and horizontal pressure distribution in a laboratory aquifer subjected to simple harmonic forcing across a sloping boundary has been presented and analysed. The non-linear filtering effect of the sloping boundary is clearly apparent and the resultant generation of higher harmonics was also observed.

Comparison of estimated wave numbers with corresponding theoretical predictions has revealed significant discrepancies, possibly due, in part, to the influence of a truncated capillary fringe.

The dataset has been used to test existing small-amplitude analytical solutions, highlighting their limitations i.e. their simplifying assumptions of a shallow, capillary free aquifer. A new solution is derived taking into account these effects, and is demonstrated to markedly improve agreement with the data for all harmonic components.

Consistent discrepancies are observed for all three models in their ability to generate higher harmonics in the “inter-tidal zone”. Each model predicts a maximum amplitude of the second harmonic at the high water mark whereas the data indicate it to occur at the mid point of the “inter-tidal zone”. The formation of a seepage face in the experiment that is not accounted for in the theory could contribute to the observed discrepancy.

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