A note on Chow’s description of the weak hydraulic jump

Note sur la description de Chow du ressaut hydraulique faible

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ABSTRACT

Standard analytical and empirical results of hydraulic jumps are reanalyzed. The assumption that momentum but not energy is conserved across a jump is abandoned. The limiting case when energy dissipation is maximum is explicitly considered, which could be useful when hydraulic jumps are used to dissipate energy. Chow’s empirical results and some recent experimental observations are compared to the present predictions.

1 Introduction

The theory of open-channel hydraulics, on which the design of hydraulic structures relies, owes much to Ven Te Chow (1914–1981). By an elegant mixture of theory, basically conservation of mass, momentum and energy, and empiricism, his results still form the basis of much of what we understand today. Of particular interest is the formation of hydraulic jumps, which have many practical applications (Chow, 1959) (see Fig. 1). One of them is its use as an energy dissipator. In the following Chow’s description of hydraulic jumps is re-examined to gain some new theoretical understanding of experimental results and their use as energy dissipators.

2 Theoretical considerations

The standard steady state St. Venant equation can be written based on Darrigol (2002) as

$$\frac{dh}{dx} = gh^3 (S_o - S_f) / (gh^3 - q^2)$$

(1)

where $h$ is the flow depth, $x$ the streamwise distance, $S_o$ the bottom slope, $q$ the discharge, $g$ the acceleration of gravity and $S_f$ the friction slope. The latter can be defined in terms of the normal depth $h^*$ using a Chezy type relation as

$$S_f = S_o h^3 / h^3.$$  

(2)
Equation (1) can be rewritten from conservation of momentum

$$M = (h - q^2/gh^2)dx = hS_0 (1 - h^3/h^3),$$

(3)

or from conservation of energy as

$$(1 - q^2/gh^2)dx = S_0 (1 - h^3/h^3),$$

(4)

which are of course identical to Eq. (1) because of the steady state conditions ignoring correction factors (Liggett, 1994).

Integration of Eq. (3) across the jump yields

$$h_1^2/2 + q^2/gh_1 - h_2^2/2 - q^2/gh_2 + S_0 \int_0^L h \, dx = M,$$

(5)

where $h_1$ is the value of $h$ upstream of the shock and $h_2$ at the end of it, and integration of Eq. (4) gives

$$q^2/2gh_1 - q^2/2gh_2 + h_1 - h_2 + S_0L = E,$$

(6)

where $L$ is the thickness of the shock (it is not taken necessarily as a discontinuity) and $M$ and $E$ are the momentum and energy loss due to the jump. From Eqs (3) and (4)

$$M = S_0 \int_0^L h_{*}^3/h^3 \, dx$$

(7)

and

$$E = S_0 \int_0^L h_{*}^3/h^3 \, dx.$$  

(8)

In the standard treatment, Chow (1959) suggests that $M$ can be neglected in Eq. (5), whereas $E$ in Eq. (6) cannot. This clearly does not have to be the case, e.g. because of non negligible drag on a rough bed (Ead and Rajaratnam, 2002) or three-dimensional effects (Montes and Chanson, 1998).

To continue it is necessary to guess the shape of the jump. We have tried several “reasonable” shapes from a discontinuity to a parabola without affecting our conclusions. Thus we take here the simplest possible case with the jump being discontinuous at $x = 0$ and $L$. This does not mean that $M$ and $E$ in Eqs (7) and (8) are necessarily zero since $h^*$ is then theoretically infinite within the jump, that is, taking $L = 0$ is only for mathematical convenience, in practice $L$ is far from zero, as discussed below. Chow (1959) obtained his Eqs (15–18) when $G = F_1 = V_1/(gh_1)^{1/2}$ as the Froude number of the approach flow as

$$h_2/h_1 = 0.5[(1 + 8F_1^2)^{1/2} - 1]$$

(9)

as obtained from Eq. (5) with $M = L = 0$ with the Froude number also given by

$$F_1 = q^2/gh_1.$$  

(10)

Then, Eq. (6) gives after elimination of $F_1^2$ using Eq. (9) (Chow, 1959)

$$E/h_1 = (h_S/h_1 - 1)^3/[4(h_S/h_1)].$$  

(11)

Outside the jump Chow (1959) made the reasonable assumption that dissipation is negligible. Note that Hogarth et al. (2003) took $h^* = h_1$ at the upstream end of the jump, but it is easy to check that taking either $h_1$ or 0 has essentially no effect on the results; since the primary interest is discussing Chow’s approach, his assumption is kept. Thus in Chow’s case, Eq. (1) with $S_f = 0$ yields

$$S_0 x = h - h_S + q^2/2gh^2 - q^2/2gh_S^2.$$  

(12)

If we do not assume, as Chow does, that $M = 0$ within the discontinuity at $x = 0$, some other condition must be imposed. It is clear that the standard approach, with $M = 0$, corresponds to the minimum energy dissipation. Here then, the other limiting case of maximum energy dissipation is considered which would be a useful approach in practice. Hopefully experimental observations will fall between those two limits, with laboratory experiments conducted on very smooth beds closer to the case of $M = 0$, and those with rough beds closer to “our case”. The value of $h_S$ such that $E$ is maximum is obtained by differentiation of Eq. (6). In this case the jump has a tailwater depth $h = h_2$, at $x = 0$, writing $h_2$ to differentiate from the previous $h_S$, with

$$F_2^2 = q^2/gh_2^2 = 1$$

(13)

and the profile for $x > 0$ is now

$$S_0 x = h + h_2^3/2h^2 - 1.5h_2.$$  

(14)

Note also that a jump to $h_2$ at $x = 0$ is the smallest jump possible and Eq. (1) shows that $dh/dx$ is infinite at $x = 0$, i.e. the profile is continuous in slope with the shock.

3 Discussion

It is noteworthy that the profiles given by Eqs (12) and (14) do not differ significantly, compared to the scatter in data. Figure 2 repeat the 4 cases of Hogarth et al. (2003). The data represent the envelope of the upper waves and thus must be higher than the average surface predictions. The surface of a hydraulic jump was also detailed by Hager (1992).

In all cases the obstruction to the flow is located downstream of the profile. In Fig. 2(a) and 2(b) the predictions of Chow and Eq. (14) are virtually identical and well below the measured upper fluctuations of the turbulent flow. However in Fig. 2(c) and 2(d), Chow’s predictions are too high and Eq. (14) still remains below the recorded profile as in Fig. 2(a) and 2(b). The four examples considered correspond to low slopes and weak “drowned out” hydraulic jumps (Chow, 1955) with $1 < F_1 < 2$. Thus the two limits lead to surface profiles which can barely be distinguished.
Figure 2 Surface profile $h(x)$ ahead of an obstruction located at $x > 1$ m. Chow’s result given by Eq. (12) is shown as a solid line, Eq. (14) as a dashed line for $F_1 = (a) 1.19, (b) 1.27, (c) 1.99$ and (d) 2.25

Now Eq. (6) gives the energy dissipation

$$E_2 = \left( \frac{h_2}{h_1} - 1 \right)^2 \left( 1 + \frac{h_2}{2h_1} \right).$$  \hspace{1cm} (15)

Figure 3 compares the values of $E$ and $E_2$, with $E_2$ roughly twice the value of $E_S$ for $F_1 < 2$. It is interesting that observations of Montes and Chanson (1998) for undular jumps and large energy dissipation give a similar factor of 2 as shown in Fig. 3. Interestingly, the two expressions hardly differ, although it should not be expected that Eq. (15) is reliable for large Froude numbers.

Associated with $h_S$, Chow (1959) also defines $x_S \equiv L$, as the length of the jump. His Figs 15–21 gives experimental values

of $L/h_S$ as a function of $F_1$ for different $S_o$. Equation (14) can be used to estimate $x_S$ for $h = h_S$, since obviously the experimental data were just the values of $x$ when $h = h_S$. In Figs (15–21), $F_1$ close to 1 corresponds to “the parts where the curves are not well defined by the available data” and are shown as dashes in Fig. 4.

Figure 4 gives those results for small $S_o$, and, quite remarkably, the results from Eq. (14)

$$\frac{S_o x_S}{h_S} = \left( 1 - \frac{h_2}{h_1} \right) \left( 1 + \frac{h_2}{2h_1} \right)$$ \hspace{1cm} (16)

are in fairly good agreement with Chow’s experimental results for small $F_1$, when the experimental data are inaccurate.

Figure 3 Energy loss due to hydraulic jump with dashed line, Eq. (15); Solid line, Eq. (11) for (a) large and (b) low values of $F_1$. Stars show experimental results from Fig. 7 of Montes and Chanson (1998) in Fig. 3(b)
4 Conclusions

Replacing the momentum conservation across a hydraulic jump by the maximum energy dissipation leads to several observations for \(F_1 < 2\):

1. The effect on the surface profile is small.
2. The energy dissipation increases by a factor 2. However this effect becomes small as \(F_1\) increases, i.e. when the energy dissipation becomes significant. Thus, if the energy dissipation is the motivation for creating hydraulic jumps, the increase over the standard estimates will often be negligible unless the Froude number is small.
3. Chow’s empirical results for the length of the hydraulic jump can be extended analytically to low values of \(F_1\).

Notation

\[
E = \text{Energy head} \\
E_2 = \text{Energy head associated with } h_2 \\
F = \text{Froude number} \\
g = \text{Acceleration due to gravity} \\
h = \text{Depth of flow} \\
h^* = \text{Normal depth} \\
h_1 = \text{Approach flow depth to shock} \\
h_2 = \text{Downstream flow depth when } E \text{ is maximum} \\
in = \text{Flow depth downstream of shock} \\
L = \text{Thickness of shock} \\
M = \text{Momentum due to hydraulic jump} \\
M_2 = \text{Momentum associated with } h_2 \\
q = \text{Discharge} \\
S_o = \text{Bottom slope} \\
S_f = \text{Friction slope} \\
x = \text{Distance downstream}
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References


