An Historical Analysis of the Division Concept and Algorithm Can Provide Insights For Improved Teaching of Division

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The evaluation of historical literature has rarely been used to support the theory that a conceptual understanding of mathematical processes improves learner outcomes. This research paper has been motivated by historical developments in mathematics and their effects on modern teaching strategies. Many of the rotely-taught procedures used in today's classes have been developed out of the methods used from the past but these have been taken out of their cultural contexts and taught to students without reference to their background. This paper explores historical approaches to examine the notion that a conceptual understanding is essential in developing an understanding division.

Introduction

All of society benefits from the technological, systemic, commercial and medical advances created by mathematics. The simplest of mathematical ideas have proven to be a catalyst for ingenious human achievement. Consider the notion of zero and the affects that the full understanding this concept has had on Western civilisation. For example, the essence of digital technology assigns the digits zero and one to electric pulses that in turn create actions. This binary code entwines the networks of the global village. Mathematics education should be designed and delivered to students in a manner that is beneficial to their future requirements and fruitful participation in the global techno-democratic communities in which they live. Otherwise, as Kline (1980, pp. 5–7) argues, there is a travesty in the intellectual quality of mathematics caused by limitations in the current understanding of mathematical concepts.

Solving practical problems that related to the distribution of wealth, resources and food certainly provided an early impetus for the division concept in most societies, it evolved out of a need to share items equally or in some specific ratio. For example, the oldest known example of a division problem (Ifrah, 1998, pp. 121–122) originates from Sumerian civilisation and was discovered at the Iraqi site of Fara (Šuruppak). A tablet thought to date from around 2650 B.C. provides evidence that all the formal elements of the division concept were used for the distribution of grain. There is a dividend, a divisor, a quotient and a precise example of a remainder. The problem seeks to find the division 1 152 000 "granary of barley" between a certain number of people so each person received 7 "granary of barley". It correctly states the quotient as being 164 571 with a remainder of 3.

Errors in division

Rotely acquired methods that have little mathematical relevance constitute many of the reasons for some students' poor understanding of mathematical concepts. Evidence (Booker et al., 2004, p. 302–331) suggests that in some classrooms, the formal teaching of the division concept and algorithm is based on memorised rules. Procedures are described using inappropriate language that has little relationship to mathematics. For example the act of 'putting a number into another', 'the bringing down' of digits or the process of divide, multiply then take-away highlight some of the inappropriate language witnessed in classrooms. The language simply explains the recording process. Research suggests that inappropriate language confuses learners and strongly hinders their understanding of the division concept (Booker et al., 2004, pp. 302–331; Mulligan & Mitchelmore, 1997; Tirosh, Graebe,r & Glover, 1990).

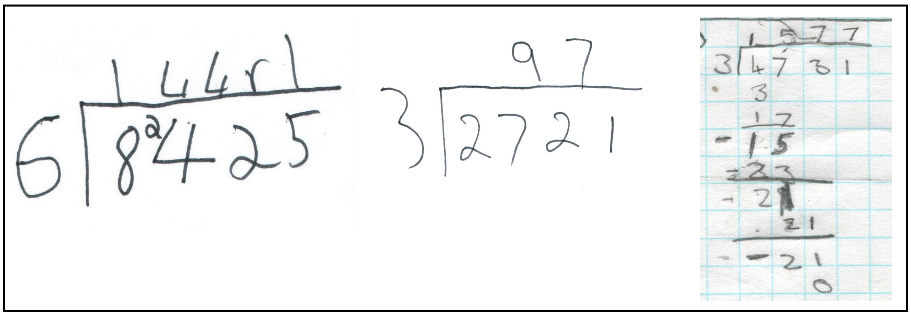


Figure 1.

Three common division errors exhibited by current year six students.

The first example demonstrates difficulties when zero needs to be recorded in the algorithm. Zero poses significant problems for many students and its abstractness of the needs to be appreciated and the concept needs to be carefully taught in a meaningful sequence of lessons. Booker et al., (2004, p. 57) contends that zero causes many difficulties and that the notion of writing something for nothing is rather odd for young children. The second example exhibits difficulties with place value and renaming and shows that a mastery of basic facts will not necessarily lead to students completing the algorithm correctly. In the third example, the student displays poor recording strategies, again exhibiting a lack understanding concerning the significance of place value. All three examples exemplify the necessity of students having secure knowledge of zero, place value and renaming before algorithms can be attempted successfully.

While explicit linking among materials, language and recording for the addition and subtraction processes ensures that students' initial ideas fully develop, teaching sequences for multiplication and division frequently move away from these successful strategies. In particular, there is greater use of the more abstract symbolic representations of division and reference to a language that has little mathematical integrity or respect of the place value system. This example below from a widely used textbook demonstrates the heavy emphasis placed on procedures with the sole objective of obtaining an answer.

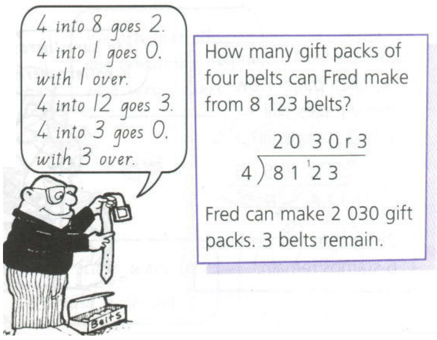


Figure 2.

Current year 6 mathematics text (Parker et al., 2000, p. 39).

Teaching the division concept and subsequent algorithm requires educators to initially explore with students, the relationship between the language of division supported with materials. Only after children have had ample experiences with describing and representing division experiences should the symbolic representations be introduced (Kouba & Franklin, 1993). Madsen, Smith, and Lanier (1995) suggest students' understanding of concepts is greatly enhanced by the use of problem solving, technology and manipulative activities and need to be supported with meaningful language. Children's intrinsic awareness of division can provide the catalysts for them to fully comprehend the division concept. Appropriate teaching sequences focussed on the underlying mathematical ideas can then provide the basis for developing a deep understanding of the division process at the same time fostering improved attitudes and greater confidence in mathematics.

Different approaches to division

The development and advancement of early mathematics was based primarily on practical applications, driven by a need to organise, manage and document the activities of a society. The cornerstone of all early mathematical ideas was arithmetic and measurement (Mankiewicz, 2000, pp. 39–146). Arithmetic emerged from the contact and dealings that people had amongst each other and the importance humans placed on describing and recording the environment that they participated in. Initially the use of whole numbers and the computational method of addition provided the basis and allowed for the development of the other computational methods of subtraction, multiplication and division (Motz & Weaver, 1993, p. 33). For example, Howard Eves (1964, p. 29) states that early mathematics required the computation of a usable calendar, the development of a system of weights and measures to serve in the harvesting, storing and apportioning of food, the creation of surveying methods for canal and reservoir construction and for parcelling. Furthermore the evolution of financial and commercial practices has been crucial in the development of mathematics as the mercantile and trading communities of different civilisations developed mathematical ideas, processes and recording mechanisms appropriate to the needs of their societies.

The Hindu-Arabic system of numeration and computation has flourished because of its ability to adapt to different situations. No other number system has been able to organise, evaluate and manipulate information as successfully (Smith, 1958, pp. 9–18). Nonetheless, two other significant societies, Ancient Egypt and China, also developed powerful number systems and computational procedures for division in response to their political systems and practical economic requirements. Owing to their geographical position, these two societies undoubtedly influenced the development of the dominant Hindu-Arabic numeration and computation system, but there are significant differences in the approaches they took. An examination of the understanding the three number systems provided for the division concepts and procedures points to the somewhat natural origin of some of the ineffective teaching strategies still used in classes today and also highlights the meaning needed for successful division processes.

The additive nature of Egyptian numeration influenced the arithmetic methodology of the Egyptian scholars. Multiplication was performed by a succession of doubling operations based on the fact that any number can be represented as a sum of powers of two (Eves ,1964, p. 39). Calinger (1999, p. 47) states that Egyptian scribes and merchants took duplation for granted and did not question its validity. Division was overtly linked to this multiplication process and the inverse process occurred to solve arithmetic problems. Division in ancient Egypt might be best described as a second kind of multiplication, where the multiplicand and the product were given to find the multiplier (Bunt, Jones, & Bedient, 1976, p. 14). The two operations were so closely linked that it did not require scholars to teach another process. They took the existing process and allowed for the manipulation of data to support both processes effectively.

Gillings (1972, p. 19) articulates that Egyptian scribes considered "What must I multiply by 8 to get 184?" rather than dividing 184 among 8. Using an additive strategy that relied on an understanding of doubling the Egyptian division process required the individual to

(a) Tabulate the factors and multiples of the divisor.

(b) Locate the numbers from the right hand column that would add to or as close to the dividend, which then would be checked marked (/).

(c) Then add the multipliers corresponding the checked numbers that would give the quotient.

If there was a remainder then fractions were introduced. It must be noted that Egyptians only used fractions where the numerator was 1 with the exception of $\frac{2}{3}$ and $\frac{3}{4}$ (Bunt, Jones, & Bedient 1976, p. 14). Figure 3 highlights the four step procedure that the Ancient Egyptians used.

184 divided among 8 = 23	
1	8 /
2	16 /
4	32 /
8	64 /
16	128 /
23	184

Figure 3.

The additive strategy used by the Ancient Egyptians.

The division strategies of the Egyptians met the practical needs of their society. The distribution of bread and beer as pay in given quotas was important in the organisation of resources. It is their practical understanding of the process that allowed the Egyptians to accurately distribute resources in a mercantile system based on bartering. Yet, because mathematics was based on practical notions rather than theoretical conceptualisations the development of Egyptian mathematics remained idle. The systems and processes that were in place solved the problems that were presented to the Egyptian society. Dunham (1994, p. 181) argues that the scholars and merchants of Egypt may have had cumbersome processes but invariably they always provided correct solutions that served their society successfully for over 3000 years.

Prior to the Europeanisation of Chinese Mathematics, astronomers and scholars documented the computational strategies used in Chinese society. These manuals describe the strategies and procedures needed to carry out the operations of addition, subtraction, multiplication and division. The earliest of these is the Ten Computational Canons, a set of 12 textbooks. Under the Sui dynasty (518–617 AD) and Tang Dynasty (618–907 AD) the textbooks were used at the School for the Sons of the State or guozixue (Martzloff, 1988, p. 15). Written in approximately the fifth century by Sun Tsu the Sunzi Suanjing Trilogy provides an insight into the methodology of division and multiplication. Importantly the manuals emphasised that some of the procedures used in China are found in other societies emphasising the interconnectedness of this geographical area. However, there are some computational techniques that are peculiar only to China (Martzloff, 1988, p. 217–221).

Martzloff (1988, pp. 217–218) states the Sunzi texts regarded division as the inverse of multiplication and directly linked the algorithm to multiplication. Similar to other societies the treatise only explained division using small numbers and in case of the Sunzi texts the divisor is never greater than 9. The Sunzi method of division relies on an understanding of place value. There are examples in these historical texts where the remainder is renamed to complete the algorithm (Mikami, 1974, p. 29). With the aid of counting rods and a checkerboard (refer to Figure 5) the division algorithm was completed using methods similar modern short division. The following example translates how the algorithm would be presented on a checkerboard.

Problem 100 divided by 6.	1 hundred divide by 6 → one is not divisible by 6.	10 tens divided by 6 → 1 ten is recorded above the tens place.	40 ones divided by 6 → 6 ones is recorded in the ones place.	The remainder is recorded as a fractional part of the divisor.
Quotient		**1**	**16**	**16**
Remainder	**100**	**100**	**40**	**4**
Divisor	**6**	**6**	**6**	**6**

Figure 4.

Explanation of the Chinese division algorithm.

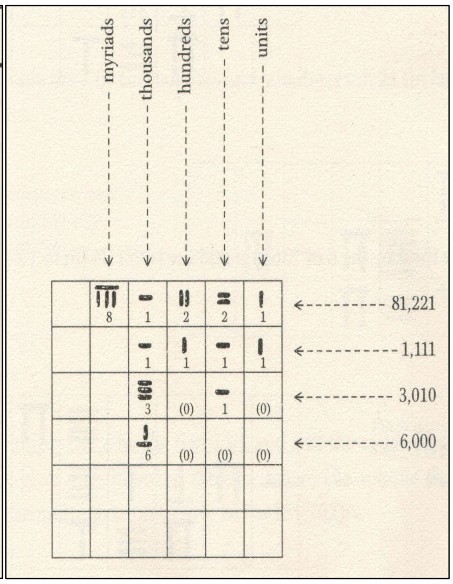


Figure 5.

A Chinese Master teaching the art of calculation.

Figure 6.

An example of how the rods are laid out.

(Ifrah, 1998, p. 284–285)

With the advent of the abacus in the fourteenth century, Chinese mathematics became even more procedurally based. Using short division formulae and rotely learnt procedures, recalled as rhymes or slogans, the Chinese could complete a myriad of division problems. The essence of these procedures and rules ensured that computations would be completed quickly as little or no recording was required. As practitioners became more proficient at bead arithmetic, so did the scope for solving more advanced computational problems. For example the *fei gui jue* rules (formulae for flying division) for divisors between 11 and 99 were developed. The rules stipulated particular procedures that needed to be memorised for every two-digit number. In the journal *The Abacus* (1983, p. 44 cited in Martzloff, 1988, p. 221) the problem 4368 divided by 78 = 56 is calculated in three movements by simply applying the rules for the 78. As Martzloff (1988, p. 220) states, the memorisation of rules and processes is borne out of practical circumstances rather than perambulated reasoning. The Chinese view and practice of division could also be summarised in this manner.

The millennium following the fall of the Roman Empire saw political experimentations and religious reforms initially hinder the growth of mathematical knowledge in Europe. Smith (1951, pp. 361–362) states that mathematics knowledge and activity generally flourish where there is a stable political environment. During this period the Chinese, Indians and Muslims were advancing mathematical thought especially in the area of numeration and arithmetic. Leonardo of Pisa (1180–1250) better known as Fibonacci, was hugely influential in transmitting the Hindu Arabic system of numeration and computation. The son of a merchant, Leonardo of Pisa spent his formative years travelling throughout Northern Africa. It was in Egypt, Syria and Algeria that he came into contact with the teachings of Muslim scholars. With encouragement and support from his father and his Muslim teachers the young Fibonacci immersed himself in the mathematical knowledge of the area. He developed a deep understanding of Hindu Arabic numeration and computation systems. He was rightly convinced that the Hindu Arabic number system was far superior to the Roman numbers used in Europe. This knowledge provided the impetus for the publication of Liber abaci or Book of Calculation in 1202.

Due to the influences of the Hindu Arabic system it is understandable that the primary method of recording the division process was the scratch or galley method. The method can be traced to Eastern societies, with the Hindus employing sand tables to record the process (Boyer, 1989, pp. 200–201). Arab writers from the time of al-Khowarizmi (c.825)—the great mathematician and father of the algorithm—used variations of the scratch method. Numbers were written onto the sand table and once they were used in the algorithm were 'scratch marked' or crossed out. Maximus Planudes (c. 1340 cited in Smith, 1953, p. 136) explained that the scratch method though difficult to perform on paper, would naturally lend itself to the sand abacus. He continued that because of the necessity for erasing certain numbers and the writing of other numbers in their place, it would give rise to much confusion where ink was used, but on a sand table it is easy to write other numbers in their place.

There are different variations of the basic scratch scheme. The algorithm presented below is similar to the example from Tartaglia (cited in Ball, 1960, pp. 192–194). The position of the number in a column is significant not the position of the number within the row. As the method became popular its name was Latinised to galea (galley) or battello (boat) method. Practitioners observed that once the algorithm was completed it seemed to leave an outline of a ship's sail as exemplified by the example below. Interestingly at the time some teachers even encouraged their students to illustrate their recording with the outline of a ship and its sails (Smith, 1953, p. 138; Swetz, 1987, p. 214; Gittleman, 1974, pp. 108–110). This method was efficient and appropriate for the social and economic requirements of the time. It demanded fewer figures than other approaches and this was advantageous as paper and quill pens were still relatively expensive. Mathematicians up to the sixteenth century favoured this method; indeed it had many advocates up to the close of the eighteenth century (Smith, 1953, p. 140).

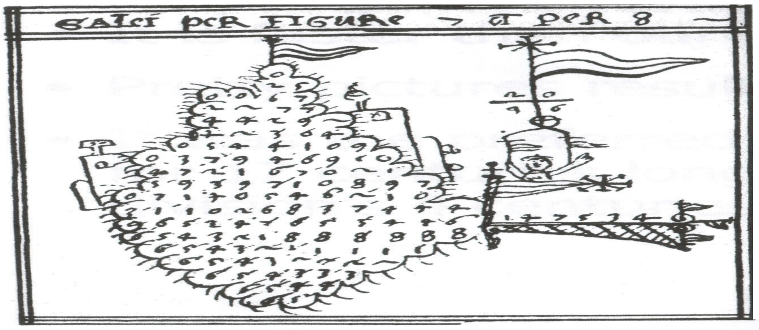


Figure 7.

Galley division 16th century (Smith, 1953, p. 138).

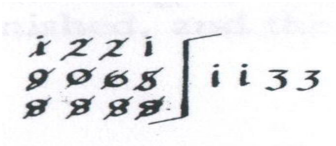


Figure 8.

Treviso arithmetic–galley division algorithm (Swetz, 1987, p. 95).

This algorithm demonstrates the method generally used in Europe especially in the great Italian trading centres of the fifteen and sixteenth century. In the Venetian mathematics textbook, Treviso Arithmetic, it shows 9065 divided by 8 is 1133. The example in the textbook is clarified by a set of instructions that demonstrates, that in Europe some computation processes utilised rotely memorised strategies. The mathematician's intuitive understanding of division was challenged when problems arose that did conform to these strategies. For example division problems where the divisor was greater than the dividend or when new numbers evolved such as negative integers proved problematic. Significantly due to its complexities, difficulties concerning division were been documented. For centuries early textbooks gave only simple examples and scholars such as Gerbert (c.980), Rollandus (1424), and Pacioli (1494) document these difficulties. They would often state that the concept and algorithm could perplex even the most competent of mathematician. Interestingly only simple examples with small numbers that conformed to this limited conceptual framework were used (Smith, 1953, p. 132). No doubt the poor understanding of the Hindu-Arabic numeration system and its nuances hindered a deeper comprehension. Many of the generalisations were based on understanding only the division of whole numbers.

Another issue that hindered the conceptual development of division was the dominant view that division was repeated subtraction. Division treated as repeated subtraction first came to prominence in Europe during the Roman Empire. Due to the additive nature of the Roman number system repeated subtraction was easier to use to solve division problems compared to other computational methods of the time. Suzuki (2002, p. 275) suggests that the popularity of the repeated subtraction method was due in part because practitioners could use mechanical devices like an abacus or counting board. After the Hindu-Arabic system became the prevailing numeration system in Europe repeated subtraction remained popular, as little or no understanding of the new place value system was required. The focus of this procedure was the reliable and accurate attainment of an answer involving natural numbers. Similarly to the Chinese, the memorisation of the division process originates from the intuitive understanding of division rather than a deep mathematical understanding**.**

Example of Repeated Subtraction Method
1128 divide 36 360 10 768 . 360 10 408 . 360 10 48 . 36 1 12 . Answer= 31 remainder 12

Figure 9.

Example of algorithm using repeated subtraction.

> As to the first I say that division is the operation of finding, from two given numbers, a third number, which is contained as many times in the greater third number as unity is contained in the less number. You will find this number when you how many times the less number is contained in the greater…of the third thing which is to be noted, that the number which is to be divided is always greater than or equal to, the divisor. Treviso Arithmetic (Swetz, 1987, p. 85).

It is impossible to fix an exact date to the origins of long division, similar to that used in schools today, due to the fact that it developed gradually (Smith, 1953, p. 140). The earliest printed example appeared in Caldrini's 1491 text though it was not until the end of the seventeenth century that long division, as we know became well established. In Italy where the method originated it was referred to as the D*anda* method, which means 'by giving'. The term originates from the fact that during the division process, after each subtraction of partial products, another figure from the dividend is 'given' to the remainder. A significant and notable aspect of the *danda* method is the placing of the quotient above the dividend as it automatically helps to locate the decimal point (Smith, 1953, p. 143). It could be suggested that the introduction of decimal notation for fractions by Pitiscus in 1608 created the need for a method that would allow for decimal fraction notation to be easily recorded. Importantly as decimal fractions became more widely accepted so did the *Danda* method; the alignment of places ensured that the quotient could be recorded as a decimal fraction.

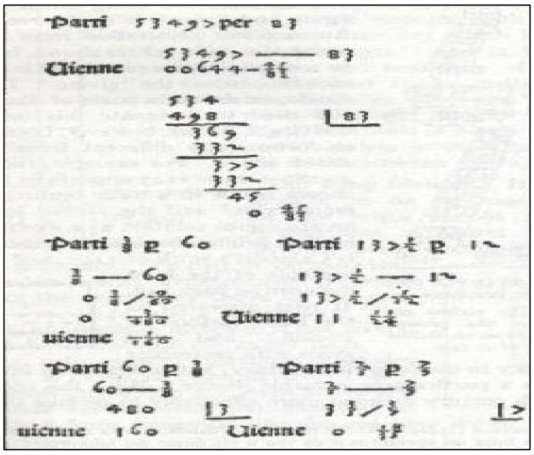
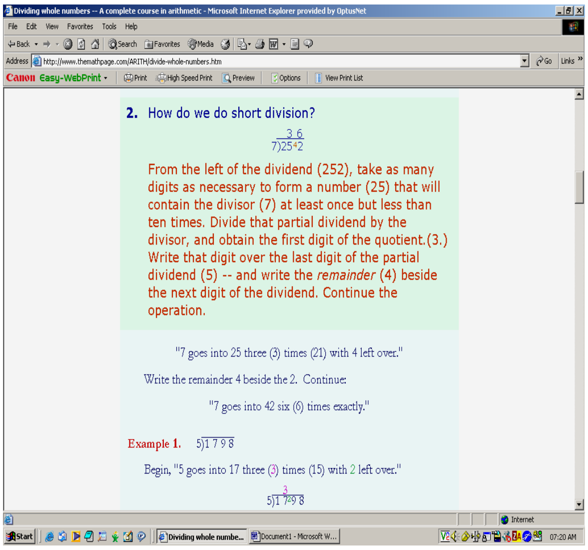


Figure 10.

First printed example of the Danda Method or long division (Caldrini, 1491 cited Smith, 1953, p. 142).

Implications for the teaching of division

Research in mathematics education clearly suggests that a conceptual understanding of the numeration and computational aspects of this discipline improves learner outcomes and teacher practice (Booker, 2004, pp. 10–30; Neuman, 1999, Madsen, Smith, & Lanier, 1995; DeFranco & Curico, 1997). In contrast children at a primary school level are still being taught concepts and algorithms at a procedural and mechanical level. Further, the strategies that some teachers use is based on rotely learnt rules and procedures that encourage a naïve understanding of mathematical concepts (Hart, 1983).



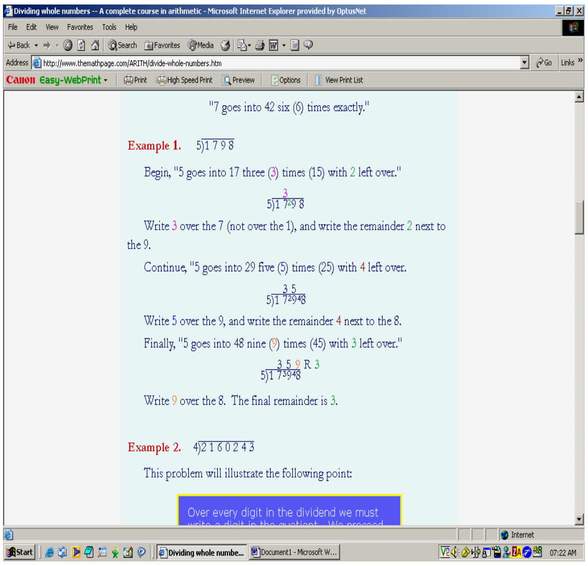


Figure 11.

Examples from the tutorial website themathpage.com that demonstrate the use of a mechanistic approach to the division algorithm using inappropriate language and confusing recording strategies.

This is due in part to poor teacher training and community expectations that require children to be fluent in all computational methods at the expense of a deep and mathematically correct understanding of computation. Twentieth century mathematics teaching and learning is insufficient for the new cohort of students. Booker (2003) states that students need to develop an understanding of the conceptual models that provide meaning to the underlying concepts.

Conclusion

Evidence suggests that the Ancient Egyptians, pre-revolutionary China, and the Europeans of the middle ages have found the concept of division difficult to fully comprehend. For many societies, the need to avoid computational errors forced them to adopt mechanical procedures that were effective for their communities. These procedures were challenged as the need to understand division in alternative settings presented problematic situations. Conceptual issues that arose outside their preconceived frameworks of division were dismissed or not even considered as relevant. Importantly, educators and the institutions responsible for teaching must be aware that unless a clear concept of division is developed then children will dismiss the more abstract issues relating to division as too difficult. Unless a clear conceptual understanding is developed then the ability for children to adapt division to various problem situations will be limited and the basis of their knowledge will be challenged.

Within the context of division, Hedges, Huinker, and Steinmeyer (2005) stated unequivocally that most teachers never really understand the division concept and conversely have difficulty explaining it to their students. No matter how well the procedure is understood, a conceptual understanding is necessary. As students move to the more abstract ideas of fractions, negative numbers, matrices, and algebraic equations, the conceptual understanding is essential for ensuring that students progress from their intuitive ideas, moving onto elementary concepts, then progressing to more advanced mathematics. With a school system that advocates and promotes written computation, surely it is conceivable to expect that the division concept and algorithm be presented to students correctly and effectively. As Booker (2003) argued, an examination of the historical paths that arose from moving from procedures to produce accurate conceptually correct results appears to be a powerful way of providing an insight to both students and teachers.

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