

THE CONSTRUCTION OF PROPORTIONAL REASONING

Stephen J. Norton

Queensland University of Technology

The development of proportional reasoning has long been recognised as a central but problematic aspect of mathematics learning. In a Year 6 teaching intervention the part/whole notion of fractions was distinguished from the part:part notion of ratio, and the “between” and “within” relationships in ratio were emphasised. Numerous representations of fractions and ratio including LEGO construction activities were used to develop the multiplicative thinking associated with these concepts. The pre-post results indicated this integrated approach helped students to apply proportional reasoning and to enumerate their responses.

BACKGROUND AND RATIONALE

Ratio and proportional thinking and reasoning abilities are seen as a corner stone of middle school mathematics and this observation is reflected in current syllabus documents (e.g., National Council of Teachers of Mathematics, 2004) and by educators such as (e.g., Nabors, 2002). In this article the term “proportional reasoning” is used to describe the concepts and thinking required to understand rate, ratio and proportionality including scale.

A number of authors (e.g., Ilany, Keret & Ben-Chaim, 2004; Lo & Watanabe, 1997) have noted that the essence of such thinking is essentially multiplicative. Ability in such thinking is needed for an understanding of percentages, gradient, trigonometry and algebra. Lamon (1995) noted that proportional reasoning has typically been taught in “a single chapter of the mathematical text book, in which symbols are introduced before sufficient ground work has been laid for students to understand them” (p. 167). It is hardly surprising then, that many adolescent students who can apply numerical approaches meaningfully in addition context, can not apply such approaches to the multiplicative structures associated with proportional reasoning (e.g., Karplus, Pulos, & Stage, 1983). Indeed many of the error patterns that students demonstrate in relation to proportional reasoning problems illustrate that they apply additive or subtractive thinking processes rather than multiplicative processes (Karplus et al. 1983). Unfortunately, exposing students to routine multiplication and division problems alone, has not been effective in helping students to develop deeper understanding of proportional reasoning. This is in part because students need to understand fractions and decimals as well as multiplicative concepts (Lo & Watanabe, 1997).

The teaching and learning of fractions and decimals is problematic (e.g., Pearn & Stephens, 2004). These authors have noted that many misconceptions that students hold are the result of inappropriate use of whole number thinking, including not understanding the relationship between the numerator and the denominator. Pearn

and Stephens (2004) found that a major problem for students because they did not understand the part/whole relationships described in fraction notation, and recommended the use of multiple representations of fractions using discrete and continuous quantities and the number line. Given the challenge in learning fractions, it is not surprising that when the multiplicative thinking associated with proportion is added to the learning cycle, many students struggle with cognitive overload, an observation well noted (e.g., Ilany et al. 2004).

The linkage between fractions and ratio is seen in many mathematics texts books. In particular; “students are shown how to represent the information in proportion word-problems as an equivalent fraction equation, and to solve it by cross multiplying and then dividing” (Karplus, et al. 1983, p. 79). The problem with this approach is that in the context of fractions the numerator represents a part and the denominator the whole, while in the case of ratio both the numerator and the denominator represent parts. Thus, while the use of fraction notation in solving some proportion problems may seem expedient in setting out a multiplication and then division algorithm, it is likely to confuse students as to what really is the whole, in fractions this is the denominator, while in ratio it is the sum of the two parts. Since the mathematics text books generally do not teach fractions and proportional reasoning in an integrated way, and usually this distinction is not made explicit, student confusion is understandable.

The particular issues described are set in a wider agenda of curriculum reform. In particular a curriculum shift towards communication of reasoning, problem bases learning and integration based on authentic tasks that include science and technology (e.g., NCTM, 2004). A second level of integration, which is integration between domains within mathematics subject material has also been recommended (Lamon, 1995). By coincidence, the intervention planning model was remarkably similar to that described by Ilany, Keret and Ben-Chaim, (2004., p. 3-83) in which authentic investigative activities for the teaching of ratio and proportion are described. Thus, the aim of this study was to use an integrated approach, across and within the subject domain of mathematics to the teaching of proportional reasoning and assess the cognitive outcomes.

METHOD

The research approach was one of participatory collaborative action research (Kemmis & McTaggart, 2000). The researcher established a working relationship with the teachers and taught one 90 minute lesson in each class, each week over a 10 week period. The researcher and the two teachers involved in the study planned the unit of work during weekly meetings. The collection of data included observations of students’ interactions with objects, peers and teachers, students planning and construction of artefacts, their explanations of how things worked, and written pre and post-tests.

Subjects

The subjects were 46 Year 6 students in two classes in a private girls' school in metropolitan Brisbane. The two classroom teachers were also part of the study. Annie (all names are pseudonyms) was a very experienced primary school teacher. Louise was a first Year teacher having recently completed her degree in primary teaching and quickly adapted to the concepts and pedagogy.

Procedure and Instruments.

At the beginning and end of the study were tested for knowledge on proportional reasoning. The pencil and paper test had 18 questions. Some questions had simple and familiar contexts with structures as follows:

To make drinks for sports day follow the recipe information given. (a) "Mix 1 litre of juice concentrate with 9 litres of water." What is the ratio of juice to water? (b) How many litres of juice concentrate is needed to make a sports drink that is 20 litres in total?

Such a question can be solved with arithmetic thinking, including the construction of tables which can be done with repeated addition. Other questions required a greater abstraction of the notion of proportion, and are not easily solved without a structural understanding of proportion, e.g.:

My recipe for ANZAC biscuits states that I need two cups of rolled oats to make 35 biscuits. I want to make 140 biscuits, how many cups of rolled oats will I need?

Suppose Challenge College has 800 students and 50 teachers, while Light College has 750 students and 25 teachers. Use mathematics to explain which school is likely to provide better learning opportunities for the students.

The test included questions directly related to the subsequent construction learning contexts such as the inclusion of a diagram, of a bicycle and the following question:

Explain the effect that turning gear A (attached to the peddles) will have upon gear B which has 16 teeth on it (attached to the rear wheel).

Examine the diagram of the pulleys below. If the circumference of pulley A is 20 cm and the circumference of pulley B is 40 cm and the circumference of pulley C is 10 cm, and pulley B is spun twice, describe how pulleys A and C will spin. Explain your answer.

Scoring was on the basis of correctness and completeness of explanations. Simple items such as the first question above, were allocated 1 mark, while more complex questions requiring symbolic manipulation and justification were allocated 2 marks. Over the life of the study student explanations of their understanding of proportional reasoning was recorded in their written and verbal explanations, which on occasions were trapped on audio or video.

During the intervention fractions were taught emphasising on the sharing division and part/whole relationships. Payne and Rathmall's (1975) principles of constructing relationships between concrete materials, language and symbolism were emphasised through out the study. Various representations were used, including area, line set and

volume models. Multiple Attribute Blocks (MAB) was used to link common fractions with fractions with a denominator of 10 or base 10. For example a 10 rod could be viewed as the whole and students were asked to name the fraction shaded if 2 one blocks were shaded ($\frac{2}{10}$ or $\frac{1}{5}$). Similarly the students were asked to view the 100 square as the whole and the 1000 cube as the whole. MAB material was also used to make links to decimal notation, with students having to express parts of the whole as a decimal. While fraction operations were not taught, the multiplicative relationships associated with equivalent fractions were taught using each of the models above (area, set, linear and volume) MAB materials and equivalent fraction strips. Students then represented simple fractions e.g., $\frac{2}{100}$ as decimals (0.02) and percentages (2%). That is, percent was seen as a special way of writing decimal fractions with the whole being 1, or 100 hundredths. Linking to the base 10 number system capitalises on students prior experience with the decimal number system.

Ratio or proportion was introduced by emphasising the part/part relationship of ratio as distinct from the part/whole relationship of fractions. The same models used to teach fractions (area, set, line, volume) were used to teach the part:part relationships involved in ratios including equivalent ratios. The linkages between common fractions, decimals and percentage were taught using the models above including the use of MAB and were consolidated in the first one of three critical learning lessons. In this lesson students were taken to the science laboratory and asked to make up solutions 1 part food dye with 9 parts water (volume model), 1 part food dye with 99 parts water, 1 part food dye with 999 parts water. As the students made up the dilutions they recorded the colour, ratio (1:9); fraction $\frac{1}{10}$; decimal 0.1 and percent 10%. The students repeated the dilution activity, but rather than making up 1 ml of dye with 99 ml of water, they took 1 drop of 10% solution and mixed it with 9 drops of water to make a 1:99 ratio or 1% solution and compared the colours to the solutions they had made earlier. Students repeated the dilution process to make ratios to 1 part per million. Through such activities students were given multiple opportunities to distinguish between the part/whole relationship of fractions and the part:part relationships of ratio. Contextual links were also made.

The second critical learning activity involved proportional reasoning related to body parts. Students designed their methods to test the hypothesis “Is Barbie a Monster?” They compared their own proportions (e.g. leg length to abdomen length (part:part); waist diameter to bust diameter (part:part), foot length to total height (part/whole) with that of the equivalent ratios and fractions on the Barbie dolls they investigated. Subsequently the students make 2 dimensional scale models of Barbie with cardboard, taking measurements of Barbie and multiplying by 5 to make the scale model the height of the average girl in the class. The students concluded that Barbie was indeed a monster.

The third critical lesson involved the students using their knowledge of proportional reasoning, to construct cars with LEGO materials. These materials included axels, blocks and connecting pieces, motors, gears and pullies, that comprised the Simple

and Powered Mechanisms kits (LEGO Educational Division, 2003). The first car was designed to be fast, and the second to win a tug of war competition. Throughout the design and evaluation phases the students were encouraged to make their gearing explanations explicit and formal. In setting out proportion problems, the structural relationships associated with proportion including identifying the “within quantity relationships” and the “between quantity relationships” as described by Lamon (1995. p. 172) was used.

ANALYSIS

The written pre and post-test scores were compared using repeated regression analysis. In assessing the artefacts the associated explanations the description of ratio provided by (Lamon, 1995) and detailed above were used. Emerging assertions were discussed with the teachers and colleagues and tested and refined in the light of further evidence. Triangulation involved the use of multiple data sources identified and this maximised the probability that emergent assertions were consistent with a variety of data.

RESULTS

The results are presented as a number of assertions.

Assertion One: Almost all students improved in their ability to complete questions on the pencil and paper test.

Table 1: Pre and post-test paired results on rate and ratio question, total 24 marks.

Test	N	mean	SD
Maths pre-test	44	10.35	5.24
Maths post-test	44	16.17**	4.62

** significant at $p < 0.01$

Not surprisingly on the pre-test many students were able to give correct responses to questions that could be solved with simple additive projections. For example 35% of students correctly answered the drinks question on the pre-test, while only one student failed to get this question correct on the post-test. On questions where simple arithmetic thinking was less able to assist, such as the ANZAC biscuit question the proportion of students who improved was greater for example on the ANZAC question, pre-test 5% correct, while on the post-test 32% of students answered completely correctly and 33% of the total number of students made computational errors rather than errors related to proportional thinking, that is 65% used proportional reasoning. On the questions related to the construction contexts such as the bicycle gearing; 14% of pre-test answered correctly, while on the post-test 55% answered completely correctly and a further 25% used proportional thinking but made computational errors. Likewise on pre-test pulley question 27% students answered correctly while on the post-test 46% answered completely correctly but

only two students failed to recognise the application of ratio concepts. The most common misconception of those students who erred in this question was to fail to recognise the inverse relationship, for example stating that the smaller pulley would turn fewer times. The question relating to the ratio of students to teachers was revealing. In the pre-test 9% of students provided a complete solution. However, 38% of students provide an answer that indicated the relationship between student and teachers had accounted for (e.g., “Challenge College because there are more teachers to provide better learning in the classes” or “Challenge College because it is easier for the students to ask questions.”). Eight students (18%) provided solutions or reasons which indicated they had added or subtracted. In the post-test 20% provided complete responses, a further 50% provided responses that indicated they had taken account of ratio. While two students provided explanations or solutions indicating additive or subtractive thinking, the responses of the remaining students remained undetermined.

Assertion Two: Most students were able to demonstrate proportional understandings associated with their investigation as to whether Barbie was a monster.

All the groups succeeded in constructing scaled cardboard models of Barbie and all students made comments that indicated an appreciation for Barbie’s proportionality. For example:

S1 Her thighs are normal, but her legs are too long for her body, her waist is tiny, her chest is larger than normal. If she was a human she would die because her organs will not fit in her body.

S2 Barbie’s hands are too small, if she were our height she would have hands the same size as a prep kid, we measured one (5 years old) and that is the age her hands are at.

It is interesting that although students had to use multiplication in the construction of their model, no student provided quantitative explanations in the context of justifying Barbie’s proportionality.

Assertion Three: Most students improved in their abilities to construct and explain the proportional concepts associated with the gearing of their cars and tractors.

Early in the teaching phase students were asked to design, construct and explain a fast car using LEGO materials. All students chose to use pulleys to convey the power from the motor to the rear wheels. Only one group (of 14 groups) used the pulley mechanism appropriately (that is a large pulley attached to the motor and a smaller one attached to the wheels). No students provided explanations that indicated an understanding of the nature of the pulley ratios they constructed. At the end of the intervention the students constructed tractors designed to pull loads, seven groups used pulleys and seven groups used cog gearing. Of the 14 groups only one did not attempt to explain their ratios, but while only one of the groups who used pullies tried to quantify it, all of the groups who used gears did so. Three groups used inappropriate gearing or pulley ratios and had incorrect explanations and 10 groups

used appropriate ratios and had at least partially correct explanations. Samples of student explanations are as follows:

- S3 The gearing is 5:1 which means that the small one goes around 5 times and the one on the wheel, the forty one goes around once.
- S4 The most important factor about our tractor is the pulleys because they make the car go slowly so it gets more force and can pull the other cars better.

DISCUSSION AND CONCLUSIONS

Karlpus, Pulos and Stage (1983) found that when middle school students were asked to compare ratios such as 4:6 and 10:15, or only about 18% of students used proportional strategies. The pre-test data presented in assertion one is consistent with Karlpus et al., (1983) findings, that is a low proportion of students apply proportional reasoning when ratio relationships become less obvious in the data, a finding also supported by Ben-Chaim, et al., (1998). The post-test data indicated that between 50% and 75% of students were using proportional thinking. The data associated with the “Is Barbie a monster?” investigation indicates that these Year 6 students tended to use qualitative rather than quantitative explanations of proportionality. That is they had to be encouraged to “symbolise” proportionality, a finding that confirms that qualitative schemas develop before quantitative schemas. Establishing the relationships between representations has long been recognised (e.g., Payne & Rathmall, 1975). The data under assertion three, associated with the LEGO construction indicated that the medium of sense making was important in the process of enumeration. For example, no students who used pulleys in their construction quantified their ratios. This is not surprising since to do so necessitated the use of the intermediate relationship of diameter and circumference. In contrast, all of those who used gears, which afforded a relatively straight forward opportunity to count the gear teeth, quantified their explanations.

The linkage of fraction thinking, decimals and proportional thinking has not been well explored in the research literature, although Nabors (2002) linked fractional reasoning tasks rate, ratio and proportionality and found that this approach helped one case study student to develop proportional schemas. The relatively rare post-trial confusion as to when to use part/whole and part:part relationships and the relative absence of additive strategies suggest that the linkage of multiplicative structures of the base ten number system with part/whole notions of fractions and part:part notions of proportion was helpful. There may well have been important for two reasons. Firstly, the focus on fractions, especially equivalent fractions may well have helped to emphasise the multiplicative relationships underpinning both fractions and proportion. Secondly, it is likely it helped students distinguish those contextual situations that necessitated the use of use additive or subtractive thinking compared to the use of use multiplicative thinking associated with proportion. The emphasis upon “within quantity relationships” and the “between quantity relationships” as described by Lamon (1995) may well have contributed to the high proportion of students who

presented correct or partially correct solutions to problems that typically perplex students several years older. In summary, this study has indicated, that affording students the opportunity to make links between the fractions, the decimal number system and proportion through the use of common models and authentic contextual problem situations, has assisted them to develop proportional reasoning. Clearly, further research unpacking how these approaches assist students to develop the multiplicative thinking associated with proportional reasoning is needed.

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