Relational Processing in Children’s Arithmetic Word problem solving

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Abstract
Solution of arithmetic word problem requires a mental model of task structure that represents variables and relations between them. In arithmetic addition, three variables (augend, addend, sum) are related by the addition operation (English & Halford, 1995). Flexible access to the components of the relation is required especially in non-canonical word problems, in which the augend or addend is missing. Relational processing allows all components of a relation to be accessed, but it is effortful, and is slow to develop during childhood. Therefore, the difference in accuracy between non-canonical problems (augend or addend missing) and canonical problem (sum missing) should be greater in younger than older children. Furthermore, relational processing capacity should predict accuracy on non-canonical problems. In the current research, 132 children aged 6-, 7- and 8-year-olds completed arithmetic word problems in which either the augend, addend or sum was missing and two measures of relational processing ability. Mixed ANOVAs showed significant position effects. Accuracy was lower for problems where the missing sets were in augend and addend positions than in sum position. Hierarchical regression analyses showed that after controlling for accuracy on sum problems, relational processing capacity accounted for more than half of the age-related variance in accuracy on augend/addend problems and also for significant unique age-independent variance. Findings demonstrate the importance of relational processing in development of children’s arithmetic addition, and have implications for designing word problem teaching strategies.

Keywords: Central executive; Relational processing; Arithmetic word problem solving; Position effects.

The relational processing approach to characterising central executive resources focuses on the load imposed by active processing of information. According to this approach, the central executive is specialised to process relational information, which is integral to higher cognitive processes such as planning and goal-directed activities (Halford et al., 2007) including arithmetic word problem solving. Halford, Wilson and Phillips (1998) proposed a metric of relational complexity, in which the number of variables related in a single representation, defines the complexity of that relation.

Unary relations have a single variable as in class membership dog (Fido). Binary relations have two variables as in larger than (5, 4). Ternary relations involve three variables as in addition (5,3,8). A quaternary relation entails four variables as in proportion (a/b = c/d, Halford et al., 1998). Complex relations impose higher processing loads making more complex tasks more difficult. Relational processing capacity increases with age during childhood. Estimated median ages of attainment are 1 year for unary relations, 2 years for binary relations, 5 years for ternary relations, and 11 years for quaternary relations (Halford, 1993).

Arithmetic problem solving requires a mental model of the structure of the task that represents variables and relations between them (i.e., the sets and operators). In arithmetic addition, three variables (augend, addend, sum) are related by the addition operation (English & Halford, 1995). This is a ternary relation.

Relations have properties that distinguish them from associations. Halford and colleagues (1998, section 2.2) describe these properties in detail. One property that is potentially relevant to word problem solving is omnidirectional access (ODA). ODA is a type of flexibility that allows all components of the relation to be accessed. Thus, given the relation, addition (5,3,8), each of the following problems can be solved, 5 + 3 = ? (solve for the sum position); 5 + ? = 3 (solve for the addend position); and ? + 3 = 8 (solve for the augend position).

Arithmetic addition can be investigated using a range of task formats. The current study used word problems which varied in terms of the position of the missing set. Examples are shown in Table 1. Within this missing set paradigm, children need to use their understanding of additive composition of number in a flexible manner (Gilmore & Bryant, 2006).
The relational processing hypothesis predicts that the effect of position will be smaller in older children than in younger children. Although ternary-relational processing is possible from a median age of 5 years, flexible access to relational representations is expected to improve further during mid childhood and this should facilitate performance on problems where the addend and augend are missing.

The relational processing hypothesis also predicts that individual differences in relational processing capacities will account for variance in children’s word problem solving. Relational processing should account for both age-related and age-independent variance in performance on addend and or augend problems, after controlling for accuracy on comparable sum items. Relational processing will be assessed using class inclusion and transitive inferences tasks developed in our earlier research (Andrews & Halford, 2002).

Predictions. Based on prior evidence of position effects (e.g., Gilmore, 2006; Riley et al., 1983), it was expected that problems with the missing set in addend or augend positions would be more difficult than those with the missing set in sum position.

As noted previously, arithmetic addition is ternary-relational. Therefore, children should succeed on tasks based on addition from a median age of 5 years. Andrews and Halford’s (2002, Experiment 2) findings supported this prediction, in that the majority of 5-year-old children performed at above chance level on a task that required a basic understanding of the compositional relations in addition. Furthermore, independent measures of relational processing capacity explained 87.7% of age-related variance in this task.

Children’s performance on arithmetic addition tasks is known to be affected by the position of the missing number (e.g., Gilmore, 2006; Riley, Greeno, & Heller, 1983). For example, Riley and colleagues found that difficulty increased as the missing set was moved closer to the start of a word problem. Participants in Riley et al.’s research were children in Preschool to Grade 2, but the finding holds for older children and numeral format arithmetic tasks also. For example, Gilmore (2006) reported that 6- to 9-year-olds solved numeral and word problems with the solution (sum) missing significantly more accurately than problems with the missing set in the second (addend) position, (e.g., $7 + ? = 13$). In turn, these were easier to solve than problems with the missing set in first (augend) position (e.g., $? - 3 = 8$). These studies demonstrated position effects but did not examine the cognitive factors that might explain the observed differences.

We proposed that success on word problems involving arithmetic addition involves flexible access to the relations between sets and that this is especially the case when position of the missing set varies across problems. To elaborate, the ODA property of relations means that given n -1 elements of a relational instance, the missing element can be computed. When the position of the missing set (position) is varied (e.g., $a + ? = c$; $? + b = c$), flexible access to related elements is required. Consequently, relational processing with the property of ODA would be implicated in accurate processing of such problems. We note that children of the age studied here are unlikely to solve the word problems using direct retrieval of arithmetic facts from long term memory. While such a strategy is possible and likely to be used by for sum missing problems presented in numerical format (Ashcraft & Fierman, 1982; Carpenter & Moser, 1982), direct fact retrieval is much less likely for word problems with missing addends or augends (Gonzalez & Espinel, 2002).

Theoretical frameworks of word problem solving propose a complex interaction of executive functions, including language comprehension, selective attention to relevant information, and mental representation of sets and relations during the encoding process (e.g., Kintsch & Greeno, 1985; Mayer, 1982; Mayer & Hegarty, 1996; Morales, Shute, & Pellegrino, 1985), in addition to arithmetic calculation and response generation processes. Whereas some models of word problem solving describe construction of a mental ‘situation model’ (e.g., Johnson-Laird, 1983), others posit access to a ‘schema’ or knowledge structure held in long term memory (e.g., Briars & Larkin, 1984; Rosenthal & Resnick, 1974). A situation model is a temporary structure constructed in working memory which incorporates qualitative information related to the problem’s context (Thevenot, Devidal, Barrouillet, & Fayol, 2006). A schema is a more formal structure, consisting of a problem frame stored in long term memory which is activated by textual cues, and has empty slots to be filled with information about sets and relations between them (Devidal, Fayol, & Barrouillet, 1997; Kintsch & Greeno, 1985).

Applying a schema model can be interpreted in terms of relational processing, in that it involves identifying relations between the relevant problem elements. Relational processing would be involved in accessing a schema and specifically in analogical mapping of information about sets and their relations from the problem to the schema frame. Analogical mapping is based on relations and so the relational complexity of the problem is a potential source of processing load.

As noted previously, arithmetic addition is ternary-relational. Therefore, children should succeed on tasks based on addition from a median age of 5 years. Andrews and Halford’s (2002, Experiment 2) findings supported this prediction, in that the majority of 5-year-old children performed at above chance level on a task that required a basic understanding of the compositional relations in addition. Furthermore, independent measures of relational processing capacity explained 87.7% of age-related variance in this task.
Method

Participants
The sample consisted of 132 children. There were 50, 6-year-olds, 36, 7-year-olds, and 46, 8-year-olds. In each group 50% were boys. These were normally developing children who were attending two state and one private primary school on the Gold Coast, Queensland during 2007-2008. Schools were located in predominantly middle class locations, with a mean socioeconomic status as measured by the Hollingshead (1975) index of 37.76 (range 9 – 61).

Material and Procedures
All children completed the addition word problems and two relational processing tasks, class inclusion and transitive inference.

Addition word problems. Children attempted nine arithmetic word problems which differed with respect to the position of the missing set. In each problem, either first set augend, second set addend or the final sum was unknown. There were three problems in each position condition. Examples are shown in Table 1. To avoid floor effects among the youngest participants and ceiling effects with the oldest, one problem in each position was of small ‘problem size’ (quantity of the sum value), one was medium and one large, according to previously reported response time data (Adams & Hitch, 1997; Groen & Parkman, 1972; Svenson, 1975). The task was presented on a laptop computer using DMDX software (Forster & Forster, 2003). Children received one of two random presentation orders of the problems. Each problem was read aloud to the child. Children responded verbally, and the experimenter entered their responses using the keyboard.

Relational processing tasks. Two tasks assessed relational processing. Both tasks have established reliability and validity as measures of relational processing and have been used with children of similar age to the current sample (e.g., Andrews, Halford, Murphy, & Knox, 2009).

The class inclusion task adapted by Andrews and Halford (2002) was used. Inferences based on classification hierarchies require recognition of the asymmetric nature of the relations between a superordinate class and two or more subclasses (Markman & Callanan, 1984, as cited in Andrews et al., 2009). Such relations are asymmetric in that all members of a subclass (e.g., cars) are included in the superordinate class (e.g., vehicles), but the reverse is not necessarily true (i.e., not all vehicles are cars). Recognition of this asymmetry requires consideration of the relations among a minimum of three classes (superordinate, subclass 1 and subclass 2), therefore, inferences based on classification hierarchies are ternary-relational (Halford, 1993). In the class inclusion task, each of the six displays of coloured shapes presented on white A4 paper represented an inclusion hierarchy with a superordinate class and two subordinate subclasses. For example, for the display with four green circles and three yellow circles, the superordinate class is ‘circles’, the major (more numerous) subclass is ‘green things’ and the minor (less numerous) subclass is ‘yellow things’. Children responded to questions for each display. Question A required comparison of the two subclasses (Are there more green things or yellow things?) and is binary-relational. The maximum binary score was 6. Question C required comparison of the superordinate and the major subclass (Are there more circles or green things?) and is ternary-relational. Question B required comparison of the superordinate class and the minor subclass (Are there more circles or yellow things?). Responses to question B were used to estimate and correct for guess responses to question C (Hodkin, 1987). The maximum ternary score was 6.

Transitive inference involves inferring A C from premises A R B and B R C, where R is a transitive relation, and A, B and C are the elements related. For example, if we are told that Tom is taller than Mary and John is shorter than Mary, we can conclude that Tom is taller than John. Determining the relation between A and C requires that premises A R B and B R C be integrated to form an ordered triple, A R B R C. Premise integration relates three elements; therefore it is ternary-relational (Halford, 1993).

In the Transitivity task, developed by Andrews and Halford (1998, 2002), children are required to integrate premise information to build vertical towers of colored squares. Each premise display consisted of four vertical pairs of colored squares, in which one color was higher than another. The four pairs together defined a unique vertical ordering of five colored squares in a tower. For example, the premises blue above purple, red above blue, yellow above green, green above red, together define the unique top-down order yellow, green, red, blue, purple. More generally, A>B>C>D>E, where A is the highest position and E is the lowest position. Children used laminated 2-dimensional coloured squares identical to those in the premise displays to build their towers.

Two practice items were administered to familiarize children with the task. Children were instructed on how to use the premises (clues) to determine the correct vertical order of coloured squares. Instructions were given during practice items to ensure children realised that (i) the relation higher than could refer to squares in adjacent and non-adjacent positions (ii) a square could be placed above, below or between the squares already in place and (iii) they might need to look at more than one clue to determine the correct order. Binary- and ternary-relational test items were presented using similar procedures and instructions. A different assignment of colors to ordinal positions was used on each trial, and the left-right order of the premises varied across displays. All items required children to use the premise information to build their towers. Binary-relational items each required one premise to be processed, whereas the ternary-relational items required integration of two premises in a single decision. In binary-relational items children built a 5-square tower starting with an internal pair,
either BC or CD (step 1) followed by D or B (step 2). This concatenation or chaining strategy entails processing one binary relation at a time. One point was awarded for each correctly ordered initial pair and subsequent square. The maximum binary score was 8. In ternary-relational items, children built a 3-square tower, placing B and D on step 1 followed by C on step 2. If the child integrated BC and CD into the ordered array BCD to conclude that B is above D, then the correct placement of C should be obvious. One point was awarded for each item where B, D and C were ordered correctly. The maximum ternary score was 8.

**Results**

**Arithmetic word problems.** Accuracy (% correct) was subjected to a 3(position: addend, augend, sum) × 3(age group: 6, 7, 8) mixed ANOVA with position as the repeated variable. Significant main effects of age, $F(2, 127) = 46.39$, $p < .001$, partial $\eta^2 = .422$, and position, $F(1.86, 236.05) = 83.43$, $p < .001$, partial $\eta^2 = .396$, were qualified by a significant Position × Age interaction, $F(3.72, 236.05) = 3.09$, $p = .019$, partial $\eta^2 = .046$, as shown in Figure 1.

![Figure 1: Mean accuracy on problems with augend, addend or sum missing, by age group. Error bars represent standard error of the mean.](image)

To examine the interaction, simple effects of age in each position were investigated using one-way independent groups ANOVAs. Age effects were significant in augend, $F(2, 127) = 30.99$, $p < .001$, addend, $F(2, 127) = 34.98$, $p < .001$, and sum positions, $F(2, 127) = 24.46$, $p < .001$. Pairwise comparisons revealed that 7-year olds were significantly more accurate than 6-year olds in all positions ($p < .001$), and 8-year olds were significantly more accurate than 7-year olds in augend ($p = .003$) and addend positions ($p = .036$), but not sum ($p > .05$). Thus it seemed age had smaller effect on accuracy when the missing element was in sum position than augend and addend positions. To further examine the interaction, simple effects of position in each age group were investigated with repeated measures ANOVAs. Significant position effects were found in 6-year-olds, $F(2, 94) = 41.22$, $p < .001$, partial $\eta^2 = .467$, 7-year-olds, $F(2, 70) = 28.13$, $p < .001$, partial $\eta^2 = .446$, and 8-year-olds $F(2, 90) = 15.48$, $p < .001$, partial $\eta^2 = .256$. Pairwise comparisons indicated that accuracy did not differ significantly for augend and addend positions in any age group (all $ps > .05$), but both were solved with less accuracy than sum (all $ps < .001$). The predicted position effect was significant in each age group, however, based on comparison of the effect sizes the effect decreased from 6- to 7- to 8-year olds.

**Relational Processing Tasks.** Table 2 shows mean number of correct responses for the binary- and ternary-relational items of the class inclusion and transitive inference tasks by age group. Ceiling performance and zero variance in 8-year olds for binary class inclusion items precluded use of ANOVAs therefore nonparametric analyses were conducted for binary items in that task. As expected, in the class inclusion task a Wilcoxon signed rank test showed that the ternary-relational items were significantly more difficult than the binary-relational items, $Z = 6.89$, $p < .001$. A Kruskall-walls test revealed no significant age effects on the binary-relational class inclusion items, $\chi^2 (2, N = 132) = 3.59$, $p = .166$. A one-way ANOVA on the ternary-relational class inclusion items showed a significant age effect, $F (2, 129) = 28.70$, $p < .001$, such that 7-year olds did better than 6-year olds ($p < .001$), but did not differ significantly from 8-year olds ($p = .123$).

<table>
<thead>
<tr>
<th>Age</th>
<th>Class Inclusion</th>
<th>Transitive Inference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Binary</td>
<td>Ternary</td>
</tr>
<tr>
<td>6-year-olds</td>
<td>5.94</td>
<td>3.29</td>
</tr>
<tr>
<td></td>
<td>(0.35)</td>
<td>(0.33)</td>
</tr>
<tr>
<td>7-year-olds</td>
<td>5.92</td>
<td>5.00</td>
</tr>
<tr>
<td></td>
<td>(0.47)</td>
<td>(0.25)</td>
</tr>
<tr>
<td>8-year-olds</td>
<td>6.00</td>
<td>5.80</td>
</tr>
<tr>
<td></td>
<td>(0.00)</td>
<td>(0.08)</td>
</tr>
</tbody>
</table>

In the transitive inference task, an age (6-, 7-, 8-year olds) × complexity (binary, ternary) mixed ANOVA with repeated measures on the second factor indicated significant main effects of age, $F(2, 127) = 6.43$, $p = .002$, partial $\eta^2 = .092$, complexity, $F(1, 127) = 514.69$, $p < .001$, partial $\eta^2 = .802$, and a significant interaction of Age × Complexity, $F(2, 127) = 5.065$, $p = .008$, partial $\eta^2 = .074$. One-way ANOVAs of age were examined in each level of complexity separately. There was no significant simple effect of age for binary-relational transitive inference items, $F(2, 127) = 1.16$, $p = .315$. However, the simple effect of age was significant for ternary-relational items, $F(2, 127) = 24.11$, $p = .003$, such
that 7-year olds did better than 6-year olds (p = .007), but did not differ significantly from 8-year olds (p > .05).

Scores on binary-relational class inclusion and transitive inference items were not significantly correlated (r = .005, p = .959) due to performance being at ceiling. Scores on the ternary-relational class inclusion and transitive inference items were positively correlated (r = .244, p = .007).

Therefore, a ternary-relational composite score (M = -0.12, SD = 0.79) was computed for each child by averaging standardized ternary-relational scores across tasks. This composite ternary score was used in subsequent analyses.

Relational processing and word problem performance. Accuracy did not differ significantly for problems with missing augends versus addends. Therefore, accuracy for augend and addend position problems was averaged to form a new variable aug/add accuracy. The hypothesis concerning relational processing and position effects was tested using correlational and multiple regression techniques. Correlations between aug/add accuracy, age (in days), composite ternary scores and sum accuracy are shown in Table 3. Notably, composite ternary scores had strong significant linear associations with aug/add accuracy and sum accuracy. As composite ternary scores increased, accuracy increased. SES was not significantly related to accuracy, age or composite ternary scores, therefore it was not included in the analysis.

Table 3: Correlations between aug/add accuracy, age in days, composite ternary scores and sum accuracy, N = 130.

<table>
<thead>
<tr>
<th>Step</th>
<th>Predictors</th>
<th>B</th>
<th>SE B</th>
<th>β</th>
<th>sr²</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Age</td>
<td>0.05</td>
<td>0.04</td>
<td>0.47**</td>
<td>0.153</td>
</tr>
<tr>
<td></td>
<td>Composite ternary</td>
<td>13.30</td>
<td>4.13</td>
<td>0.26**</td>
<td>0.048</td>
</tr>
<tr>
<td></td>
<td>Sum accuracy</td>
<td>0.33</td>
<td>0.10</td>
<td>0.40*</td>
<td>0.023</td>
</tr>
</tbody>
</table>

Note. ** p < .01. * p < .05. ** Age = days between birth and test. Criterion = aug/add accuracy. N = 130.

An hierarchal multiple regression analysis was conducted with aug/add accuracy as the criterion. We examined the amount of variance in aug/add accuracy attributable to age, and the extent to which the unique contribution of age was reduced upon entry of composite ternary scores. It was expected that after controlling for variance explained by age and accuracy on sum position problems, ternary composite scores would explain significant unique variance, independent of both age and sum position. The predictors age and ternary composite scores were entered at Step 1, followed by accuracy on sum position at Step 2. Results are shown in Table 4.

The zero-order correlation indicated that age accounted for 37.2% of the variance in aug/add accuracy. Upon entry of composite ternary scores on step 1 of the regression analysis, the unique contribution of age (squared semi partial correlation) decreased to 15.3%. Thus, composite ternary scores accounted for 58.9% of the age-related variance in aug/add accuracy. Composite ternary scores also accounted for 4.8% variance independently of age. After the entry of sum accuracy on step 2, composite ternary scores continued to account for a small, but significant amount (2.3%, p < .005) of variance independently of both age and sum accuracy. The predictors age, composite ternary scores and sum performance together accounted for a significant 47.3% (46.1% adjusted) of the variance in aug/add accuracy.

Table 4: Hierarchical regression analysis predicting accuracy on aug/add position items.

Discussion

As predicted, word problems missing the augend or addend (i.e., missing set in the first or second position) were significantly more difficult than sum. The present finding is similar to those reported by Riley and colleagues (1983) with preschool to Grade 2 children, and more recently with 6- to 9-year olds by Gilmore (2006) who showed that performance was better when sum was missing versus an addend. We have replicated the typical position effect on word problem solving in 6- to 8-year old children who were in their first, second and third year of formal schooling, respectively. The absence of significant differences in accuracy between augend and addend positions might be due to the limited range of word problem types or semantic classes used in the present study. All of the problems were from the same semantic category describing active change situations (Riley et al.). There are several other variations possible, for example, augend and addend position word problems can be expressed in terms of equalizing, combining or comparing situations, which might make position effects more pronounced. Also, all problems were presented in word format. Further investigation of position effects across a wider range of problem situations, and under different presentation formats such as numeral, would further clarify position effects.

In the current sample, children’s performance on relational processing tasks was comparable to that reported in previous research (Andrews et al., 2009). The relational processing hypothesis predicted that the effect of position would be smaller in older children than in younger children. Although ternary-relational processing is possible from a median age of 5 years, we predicted that during mid
childhood, further improvement in children’s flexible access to relational representations would facilitate their performance on problems where the addend or augend were missing. Age effects on the relational processing tasks support the notion that processing of tasks at ternary levels of complexity continues to develop across mid childhood ages of 6- to 8-years. The current results support the relational processing hypothesis, showing that the magnitude of the position effect decreased with age from 6- to 7- to 8-year olds.

Regression analyses showed that after controlling for accuracy on comparable sum items, individual differences in ternary-relational processing accounted for over half of the age-related variance, and contributed further significant age-independent variance in accuracy for word problems with the unknown set in augend or addend position. As described earlier, word problem solving can be interpreted as involving relational processing in construction of a mental model and analogical mapping of information about sets and their relations from the word problem to this mental representation. Analogical mapping involves relational processing. This study has demonstrated that children’s accuracy in word problem solving can be accounted for, in part, by individual differences in relational processing capacities. Specifically, when position of the unknown set is varied, flexible access to elements of the relation is required, drawing on relational processing resources.

The findings presented here have potential implications for designing effective arithmetic teaching strategies. Based on the position effect observed here and in previous research, it would seem advisable to introduce children first to canonical word problems requiring solving for the sum, and to ensure mastery of these before introducing addend and augend missing problems. Based on the findings involving relational processing and its associations with word problem solution accuracy, activities aimed at improving children’s relational processing capacities might have positive flow-on effects on children’s arithmetic word problem solving. Fostering more flexibility in children’s representations of the part-whole relations would be especially relevant to arithmetic word problems based on addition.

Most previous approaches to children’s word problem solving have focussed on classifying problems, then examining the relative difficulty of the problem types and/or development profiles within each type. The current study goes beyond typologies by presenting a theoretically based explanation for children’s difficulty on augend or addend missing items. Our findings indicate that one reason for the extra difficulty of augend missing and addend missing problems as compared to sum missing problems is that the former depend on explicit mental representations which allow flexible access to the components. The relational processing tasks used here have similar requirements and this explains why relational processing capacities accounted for age-related and age-independent variance in children’s accuracy at solving non-canonical word problems, namely, those where the addend or augend are missing.

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