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Evaluating Renewable Resource Assets Under Uncertainty: Analytical and Numerical Methods with Case-Study Applications

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ABSTRACT

This study derives an innovative working asset valuation model, termed the RRM (Renewable Resource Model), within the real options methodology to evaluate renewable natural resource assets. The RRM calculates the value of an investment project as well as computing the critical strike prices at which it becomes optimal to exercise various options over the asset, including when to invest (commence or recommission operations), disinvest (temporarily decommission or delay operations) or abandon the asset altogether. An implementation of the RRM including a user-friendly interface is presented. A case study, in which agricultural investments are evaluated, demonstrates the applicability of the model to a real-world setting.

This study is expected to make several contributions to the capital budgeting literature, particularly the growing body of research on real options. First, it provides an innovative extension to the seminal academic work of Brennan and Schwartz (1985b) by developing a real options model, the RRM, which is generally applicable to evaluating renewable resource investments. Second, it develops a practical solution and implementation of this model with a view to making it accessible to practitioners. Third, some theoretical work is also presented which equates the RRM with a traditional valuation framework to calculate an exact risk-adjusted discount rate applicable to traditional discounted cash flow valuations for a whole equity firm. Fourth, it demonstrates how the RRM can be applied generally to renewable resource assets using a real-world example.

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STATEMENT OF ORIGINALITY

This work has not previously been submitted for a degree or diploma in any university. To the best of my knowledge and belief, the thesis contains no material previously published or written by another person except where due reference is made in the thesis itself.

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Marcus Smith

TABLE OF CONTENTS

CHAPTER 1 INTRODUCTION 1

1.1 Background and Motivations for the Research 1

1.2 Research Questions and Objectives 4

1.2.1 Research Question 1 5

1.2.2 Research Question 2 6

1.3 Overview of Methods of Analysis 7

1.4 Contributions to Knowledge 8

1.5 Structure of the Study 9

CHAPTER 2 CONCEPTUAL FRAMEWORK TO THE EVALUATION OF NATURAL RESOURCE ASSETS UNDER UNCERTAINTY 11

2.1 Overview 11

2.2 Traditional Methods for Valuing Investment Opportunities 13

2.2.1 The DCF method 13

2.2.2 Methods for specifying the risk-adjusted discount rate 16

2.3 The Real Options Approach 20

2.4 Key Contributions to the Real Options Literature 23

2.4.1 Real options theory 23

2.4.2 Empirical studies of real options 26

2.5 Real Options Applications Involving Renewable Resources 28

2.6 Review of Option Pricing Techniques 31

2.6.1 A worked example of pricing by replication using a one-period binomial lattice approach 33

2.6.2 A stylised example of the binomial lattice approach applied to the optimal investment timing decision 41

2.6.3 Option pricing using continuous-time models 44

2.7 Key Features of the Brennan and Schwartz (1985b) Model 48

2.7.1 The deterministic relationship between spot and futures prices 49

2.7.2 Specifications of the Brennan and Schwartz (1985b) Model 54

2.8 Specific Applications Using the Brennan and Schwartz Model 60

2.9 Chapter Summary 62

CHAPTER 3 EVALUATING RENEWABLE RESOURCE INVESTMENTS UNDER UNCERTAINTY USING THE REAL OPTIONS APPROACH 64

3.1 Introduction 64

3.2 Foundations of the RRM 65

3.3 The RRM: A New Model for Evaluating Renewable Resource Assets 67

3.3.1 Redefining the boundary conditions 68

3.3.2 The RRM 69

3.4 Solution to the RRM for the Untapped Asset 70

3.4.1 The homogeneous solution, $w_H(s)$ 71

3.4.2 The particular solution, $w_p(s)$ 73

3.4.3 The solution in closed form 73

3.5 Solution to the RRM for an Operational Asset 74

3.5.1 The homogeneous solution, $v_H(s)$ 75

3.5.2 The particular solution, $v_p(s)$ 77

3.5.3 The solution in closed form 78

3.6 General Solution to the RRM 79

3.7 Chapter Summary 82

CHAPTER 4 SOLUTION IMPLEMENTATION AND VALIDATION OF THE RRM 84

4.1 Numerical Methods Utilized 84

4.1.1 Gauss-Seidel iteration 84

4.1.2 Newton-Raphson iteration 86

4.1.3 Convergence of the entire system 88

4.2 Computer Implementation of Model Solution 88

4.3 Results and Discussion 92

4.4 The Exact DCF Risk-Adjusted Discount Rate for Renewable Resource Assets 93

4.4.1 Calculating the analytic value for the risk-adjusted discount rate as a function of the spot price for an unlevered firm 95

4.5 Chapter Summary 99

CHAPTER 5 EVALUATING CORN CROP INVESTMENTS UNDER UNCERTAINTY IN THE U.S. CORN BELT: A CASE STUDY APPLICATION OF THE RRM 101

5.1 Introduction and Background 101

5.2 The Setting for the Case Study 104

5.3 Estimating the Data Parameters for the Case Study 106

5.3.1 Estimating the historical commodity price variance of average U.S. corn prices 106

5.3.2 Annual production rates and operating costs for Corn Belt states in the Heartland Region 107

5.3.3 Estimating the real interest rate and convenience yield for corn 108

5.3.4 Estimates for federal income taxes, state property taxes and the royalty rate 113

5.3.5 Maintenance costs, costs of closure and costs of reinstating operations 113

5.4 Results for Cropland Evaluation 116

5.4.1 Results for RRM Cropland Valuations 120

5.4.2 Results for the Risk-Adjusted Rates of Return 126

5.5 Results for Exercise Prices of Operational Options 130

5.6 Conclusion 133

CHAPTER 6 CONCLUSION AND FUTURE RESEARCH 135

6.1 Overview of Research 135

6.2 Main Findings and Contributions of Research 137

6.3 Implications of the Research 140

6.3.1 Theoretical Implications 140

6.3.2 Practical Implications 141

6.4 Directions for Future Research 143

APPENDIX A DERIVATION OF THE ORIGINAL BRENNAN AND SCHWARTZ (1985B) MODEL 145

A.1 The Mathematics of Random Processes 145

A.2 Ito's Lemma 148

A.3 The Brennan and Schwartz (1985b) Model 151

APPENDIX B SOURCE CODE FOR THE RENEWABLE RESOURCE MODEL IMPLEMENTATION 162

APPENDIX C DATA FOR THE CASE STUDY 186

APPENDIX D OUTPUT SCREENS FOR THE CASE STUDY 190

REFERENCES 199

LIST OF FIGURES

Figure 4.1 – The RRM Solution Screen for a Renewable Resource Asset 89

Figure 4.2 – Graphical Display of the Results for the Copper Mine Example of Brennan and Schwartz Solved Analytically via the RRM 91

Figure 5.1: U.S. Farm Resource Regions 105

Figure 5.2: Evaluation for an Average Corn Cropland Asset in Illinois, 250 Acres 118

Figure 5.3: U.S. Average Monthly Corn Prices, January 2010 – June 2011................ 124

Figure C.1: U.S. Average Monthly Corn Prices, January 1975 – June 2011 186

Figure C.2: Statistical Tests for Monthly U.S. Corn Price Series 187

Figure D.1: Evaluation for an Average Corn Cropland Asset in Illinois, 1000 Acres . 190

Figure D.2: Evaluation for an Average Corn Cropland Asset in Indiana, 250 Acres .. 191

Figure D.3: Evaluation for an Average Corn Cropland Asset in Indiana, 1000 Acres 192

Figure D.4: Evaluation for an Average Corn Cropland Asset in Iowa, 250 Acres 193

Figure D.5: Evaluation for an Average Corn Cropland Asset in Iowa, 1000 Acres 194

Figure D.6: Evaluation for an Average Corn Cropland Asset in Missouri, 250 Acres 195

Figure D.7: Evaluation for an Average Corn Cropland Asset in Missouri, 1000 Acres 196

Figure D.8: Evaluation for an Average Corn Cropland Asset in Ohio, 250 Acres 197

Figure D.9: Evaluation for an Average Corn Cropland Asset in Ohio, 1000 Acres 198

LIST OF TABLES

Table 2.1 – Net Present Value Analysis for a Hypothetical Asset Over a 10 Year Horizon 14

Table 2.2: One-Period Binomial Example – Data and Notation 42

Table 4.1 – Results of Implementation and Comparison with Results Published in Brennan and Schwartz (1985b) Example 92

Table 5.01: US Corn Belt Average Annual Harvest Yields 108

Table 5.02: Calculation for Average Real Interest Rate 2000-2009. 110

Table 5.03: Calculation for Average Convenience Yield (Corn) 2000-2009........ 112

Table 5.04: Data Parameters Summary 115

Table 5.05: RRM Evaluation of Corn Cropland and Unlevered Risk-adjusted Discount Rates at Various Spot Prices........ 121

Table 5.06: USDA Average Cropland Values by State........ 122

Table 5.07: Results for the Appropriate Corn Prices at which the RRM Matches the USDA Estimates for Cropland Values 123

Table 5.08: Discrepancies (in percentage terms) for Average Monthly Corn Prices and Appropriate Prices at which the RRM Matches the USDA Estimates for Cropland Values 125

Table 5.09: A Comparison of the Unlevered Returns on Corn Cropland to Returns on Other Asset Classes Over the Period January 2010 to May 2011 129

Table 5.10: Real Options Analysis of Critical Option Prices........ 131

Chapter 1

Introduction

This study investigates the evaluation of renewable natural resource assets using a real options approach. This research has implications for financial practitioners concerned with the well-known shortfalls of using traditional methods to evaluate investment under uncertainty. In addition, the research lends itself to practical applications which could assist policy-makers in gaining a better understanding of the effects of uncertainty on investments in renewable resources as well as the effectiveness of policies relating to taxes and subsidies at a firm level.

The remainder of this chapter outlines the research program to be undertaken for this study and its significance within the real options research field. The background and motivations for the research are first presented. A statement of the research questions and the key objectives of this study is subsequently discussed. An overview of the methods of analysis used in the study is then presented along with a brief description of the expected contributions to knowledge. The chapter ends with an outline of the structure of this study.

1.1 Background and Motivations for the Research

Real options approaches to evaluating investment opportunities are heralded as significant theoretical developments in financial economics (Schwartz & Trigeorgis, 2004). Within these approaches, the managerial flexibility to react to new information and the management of uncertainty in future conditions confer significant value to an investment (Trigeorgis & Mason, 1987; McNamara, 2003). Over recent decades these approaches have been applied to evaluate investment decisions across a wide range of assets albeit mainly theoretically-based. While much of the early literature has focussed on natural resource assets which have finite reserves there has been a growing research interest in applications involving renewable resource assets. This is unsurprising given that renewable resources such as water, food and renewable energy, are of vital importance to the world's population.

The economic foundation of the real options approach is based on the condition that desired payoff patterns can be replicated and evaluated via a portfolio of transactions involving market-traded assets (Trigeorgis & Mason, 1987). Where the portfolio replicates the payoffs of an investment opportunity in all states of the world the portfolio and the opportunity must have the same market price in the absence of arbitrage opportunities (McNamara, 2003). A seminal work in the area of evaluating natural resource investments is Brennan and Schwartz (1985b) who developed a continuous-time model that evaluates investment opportunities involving natural resource assets as real options over the underlying commodity.

There are a number of advantages in using the Brennan and Schwartz evaluation framework when valuing risky investments (such as natural resource assets) instead of the traditional and widely-used discounted cash flow (DCF) method. First, unlike the DCF method[1], the Brennan and Schwartz model explicitly accounts for risk to the future

[1] The DCF method requires forecasting future output prices to determine the expected cash flows of an asset and subsequently formulating an appropriate risk-adjusted discount rate which may involve both objective and subjective aspects.

cash flow stream using a systematic, market-based approach to deal with output price uncertainty. Second, using this approach obviates the need to assign a risk-adjusted discount rate generated from a market equilibrium model (such as the Capital Asset Pricing Model). Third, the model allows one to determine the critical output prices at which it becomes optimal to exercise various operational policies over the life of the asset.

Whilst there are a number of related publications that are essentially theoretically-based there is a paradoxical dearth of empirical studies to support the validity, or otherwise, of the basic theory[2]. In addition, despite the conceptual advantages that the Brennan and Schwartz model offers over traditional valuation methods, practitioners in the finance industry have generally not adopted the model for practical use. A possible reason for this is the complexity in developing implementations of their model. It is acknowledged that, in stark contrast to the DCF method, it is exceedingly difficult to develop (and implement) an algorithm that can solve the Brennan and Schwartz model numerically under general conditions (see, for example, Kelly, 1998; McNamara, 2003).

Another possible reason is that the typically private and confidential nature of a firm's financial and operational record makes it difficult to compile relevant and complete sets of data needed to conduct this type of analysis (Colwell, Henker & Ho, 2003). Furthermore, as McNamara (2003) points out, when markets for an underlying asset are

[2] As published by Brennan and Schwartz (1985b).

essentially incomplete[3] the lack of market information will lead to an inaccurate options analysis. Given these points it seems reasonable why, despite the seminal nature of this work, there has been sparse follow-on research and thus very few documented examples involving real-world applications using the model. Moreover, an extensive examination of the literature shows that there has not been a single, well-documented, published example describing (in sufficient detail to adequately reproduce) how the model can be solved numerically.

1.2 Research Questions and Objectives

Two main research questions are addressed in this study. The primary objective is to develop an innovative working asset valuation model based on the Brennan and Schwartz approach to evaluate a natural resource asset whose underlying resource is renewable. The principal research question is thus whether a model (specifically developed for valuing a renewable resource based asset and ideally with an analytically tractable solution) can be derived within a generalized framework based on the seminal Brennan and Schwartz (1985b) Real Options theory. Whilst this first question has a theoretical focus, the second research question takes a more practical orientation involving the application of the model produced in response to the first research question.

Hence, the research questions (RQs) are as follows:

[3] For instance, consider a water asset that functions as a state-owned monopoly and environmental technologies which are still in the early stages of development.

RQ1 : *What is a feasible and implementable approach to a generalised solution of a model, derived within the Real Options framework, which is applicable to renewable resource investments?*

RQ2 : *What case-study examples would demonstrate the feasibility and suitability of the above approach for the valuation of renewable resource investments?*

The basis and justification for each research question is now discussed.

1.2.1 Research Question 1

RQ1 : *What is a feasible and implementable approach to a generalised solution of a model, derived within the Real Options framework, which is applicable to renewable resource investments?*

The application of the Brennan and Schwartz model has been significantly hampered by the fact that no analytic solution exists to the generalised problem. Furthermore there are only vaguely stated guidelines of how to implement a numerical solution to the model equations.

The work involved for this first research question will focus initially on the derivation of a real options model, called the RRM (the Renewable Resource Model), which evaluates an asset with a continuously renewable underlying resource. Once such a model has been derived the focus will then switch to developing a closed form solution and the implementation of that solution in a computerised system.

The outcomes of the first research question will yield both the correct format and viable solution techniques for applying a model to value renewable resource assets which is derived from the seminal work of Brennan and Schwartz (1985b). Ideally the new model will retain the key features embedded in the original work of Brennan and

Schwartz such as the operationally sensitive prices at which various options can be exercised.

1.2.2 Research Question 2

RQ2 : *What case-study examples would demonstrate the feasibility and suitability of the above approach for the valuation of renewable resource investments?*

The second research question is predicated upon a successful outcome from the first. Once a generalised solution to a model for renewable resource assets is developed, the second question involves demonstrating the implementation of that model using an appropriate case-study application.

To illustrate the applicability of the RRM an investment in agricultural resources is selected as a suitable focus for the study. Investments in perennial agricultural resources involve decision-making on various operational levels in order to optimise the value of those investments. An example of such operational decision making is where a farmer reserves the option to decide whether current prices are sufficient to warrant planting for this season and, if so, when to harvest and deliver their crop. The farmer must also decide whether it is appropriate to rotate their crops or abandon growing any particular agricultural commodity.

Under this setting, a generalized implementation of the RRM is applied to evaluate perennial corn crop investments across the United States Corn Belt which includes the Heartland States of Illinois, Indiana, Iowa, Missouri and Ohio. Real-world data is sourced to calculate the required input parameters used in the model. In addition to calculating the present value of the cash flow streams for hypothetical corn crops in each state, computations are made for the critical spot prices at which it is appropriate for farmers to consider exercising various operational options over the crop to optimise the value of their investments. These prices include the minimum spot price at which it is optimal to delay planting and, if this decision has been previously made, the price at which it becomes optimal to resume normal operations over the crop if market prices improve. Furthermore, the price at which it becomes optimal to abandon the crop and let the land lie fallow should prices continue to fall is also calculated. The latter price is interpreted also as the price at which other types of land usage should be considered.

1.3 Overview of Methods of Analysis

The first research question requires intensive use of formal, mathematical analysis as well as systems analysis and design, and computer programming. In particular, theoretical techniques of model design and development will be required initially. Logical analysis of the fundamental aspects of the Brennan and Schwartz (1985b) real options modelling will be undertaken and then extended to the derivation of a new model applicable to renewable resource assets.

Next it will be necessary to apply mathematical analytic techniques to determine a closed form solution for the newly derived model. A complete numerical analysis will then be necessary to ensure an accurate, convergent implementation of the model. The correctness of the implementation can be validated by a rough comparison with the published data of the finite-inventory copper mine example in Brennan and Schwartz (1985b) paper as direct comparison is not possible.

In addressing the second research question involving a case-study application of the model to evaluate perennial corn crop investments across the United States Corn Belt this will require real-world data to be sourced. Parameters and parameter estimates will need to be computed from historical data and fed through the implementation developed from the results of the first research question. Statistical tests will be conducted to assess the extent to which key assumptions of the model are consistent with actual data. The results obtained from the case-study application of the model are then validated by direct comparison to official estimates of market values reported by the United States Department of Agriculture (USDA).

1.4 Contributions to Knowledge

This research is expected to make a number of key contributions to the capital budgeting literature, particularly the growing body of research on real options. First, this research develops a new real options model that is generally applicable to evaluating renewable resource investments, while at the same time, it provides an innovative extension to the seminal work of Brennan and Schwartz (1985b) in evaluating natural resource assets. A case study example is also presented which demonstrates how to apply the RRM generally, using a real-world example, to evaluate renewable resource investments in agriculture.

Second, it develops a practical solution and implementation of the new model with a view to making it accessible to practitioners. In particular, the computer program used to run the model includes a user-friendly interface which allows for ready adjustment of all parameters and spot price ranges. While this is useful for practitioners to perform tests on the sensitivity of asset value to changes in the market parameter estimates such as interest rates and output price volatility, it also lends itself to further uses such as readily determining the affect that changes in the tax or regulatory environment has on asset value and optimal operational decision-making of firms in a particular industry. This is ideal for policy-makers considering reforms to current tax and industry subsidy policies.

Third, while the DCF method remains the standard approach to evaluating investment opportunities despite its well-known shortcomings, this research includes some original theoretical work which incorporates the real options approach into a traditional DCF valuation framework. By deriving the discount rate as a function of the spot price, an objective means by which to calculate an exact risk-adjusted discount rate applicable to a DCF valuation for an unlevered firm is provided.

1.5 Structure of the Study

The structure of this study is as follows. Chapter 2 presents the background to the research area in general including an examination of approaches to evaluating investment under uncertainty. The real options approach is presented from both a theoretical and empirical perspective. A review of the research literature of relevance to the research problem itself is then presented. Significant emphasis is placed on the specification of the Brennan and Schwartz (1985b) continuous-time model in the specific context of natural resource assets.

A renewable resource evaluation model (the RRM, complete with an analytical closed form solution based on the original Brennan and Schwartz (1985b) framework) is derived in Chapter 3. The requirements for natural resource investments to be analysed in a renewable resource context are explicitly laid out and discussed. An implementation of the RRM is developed in Chapter 4. It is noted that the numerical procedure required to enable the successful convergence of the closed form analytic solution derived in Chapter 3 is a non-trivial matter.

Chapter 4 also examines a theoretical approach to objectively calculate the risk-adjusted discount rate for a DCF valuation of renewable resource assets using the RRM framework. Chapter 5 then presents a case study application involving 5 States located in the United States Corn Belt using the RRM implementation developed in Chapter 4. Results obtained via the evaluation process are documented and discussed including cropland values, calculations for the unlevered rate of return at various spot price intervals, as well as the appropriate prices at which to exercise various options over this cropland. Chapter 6 then concludes the study and discusses potential future research directions.

Chapter 2

Conceptual Framework to the Evaluation of Natural Resource Assets Under Uncertainty

This chapter examines several approaches to evaluating investment under uncertainty. Following a brief overview of these approaches in Section 2.1, a critique of the traditional DCF approach to evaluating investment opportunities is presented in Section 2.2. The real options approach is introduced as an alternative to the traditional approach in Section 2.3 while Section 2.4 reviews in some depth key contributions to this growing literature on real options, from both a theoretical and empirical view. Section 2.5 focuses more narrowly on contributions dealing with real options involving renewable resources.

In Section 2.6 several worked examples of conventional option pricing techniques are provided and discussed. The seminal work of Brennan and Schwartz (1985b) and the original continuous-time model they developed to value natural resource assets is treated by itself in Section 2.7. A discussion on several applications involving the Brennan and Schwartz (1985b) model is the subject of Section 2.8. The chapter concludes with a summary in Section 2.9 which also reiterates the research objectives of the study in a view of identified gaps in the literature.

2.1 Overview

Evaluating investment under uncertainty is a well-known problem of central importance in financial economics (Cortazar & Schwartz, 1993). In particular, specific issues arise when the valuation of such investments is contingent on the realisation of one or more stochastic variables (Cortazar & Schwartz, 1993; Colwell et al., 2003). While these issues are not unique to any particular industry, they are especially relevant to valuing natural resource investments. This is due to the value of natural resource assets being dependent on the volatile nature of market prices for their underlying commodities.

The complexity in properly accounting for this volatility highlights the many difficulties inherent in using traditional methods to value risky assets. These difficulties are particularly salient when considering that accounting standards warrant cash flow information and discount rates that appear reasonable in relation to historical cash flows, market information and future expectations. The research effort to properly evaluate investment under uncertainty and, in particular, the implications for contingent managerial decision-making over the life of real assets, have resulted in significant levels of development in financial theory.

The traditional DCF method provides an accurate valuation of the investment when the risk is zero or relatively minimal and the future cash flows of the asset are assumed to be known with certainty, such is the case with government bonds and discount securities. However, these ideal conditions are typically not synonymous with the riskier business conditions faced by real firms. On this basis, the traditional DCF method is widely criticised for neglecting the stochastic nature of output prices and its inability to capture value created by the strategic capacity of management to respond operationally to output price volatility over the life of an investment (Brennan and Schwartz, 1985b). The shortcomings of the traditional methods, which essentially ignore these strategic concerns, led to the development of alternative methods to value real investments. The best-known of these approaches is the real options pricing approach by arbitrage methods.

The real options approach has its foundations in methods originally developed to price financial options. A financial option is a derivative security[4] which gives the option holder the right, without obligation, to purchase (a call option) or sell (a put option) a financial instrument at a predetermined price to be exercised at a future date[5]. Real options are, in a practical sense, analogous to financial options but include various strategic and operational flexibilities that impact on the market value of the asset.

2.2 Traditional Methods for Valuing Investment Opportunities

Finding profitable investment projects is the key objective of capital budgeting (Brennan and Schwartz, 1985a). The traditional and most widely applied technique for valuing investment projects is by conducting a net present value (NPV) analysis using the DCF method (Schwartz & Trigeorgis, 2004). The DCF method essentially exploits the fundamental principle of the NPV criterion that an investment in capital is viable and worthwhile if the sum of its expected cash flow stream, discounted back to present values, is worth more than the cost of the investment.

2.2.1 The DCF method

The DCF method is modelled as follows (Brennan & Schwartz, 1985a):

---

4 A derivative security is defined as “a financial instrument whose value depends on, or is derived from, some underlying security” (Kidwell, Brimble, Beal & Willis, 2007).

5 A European style option can only be exercised at maturity. An American style option can be exercised at any point in time up to and including the exercise date.

$$PV = \frac{C_1}{1+i} + \frac{C_2}{(1+i)^2} + \ldots + \frac{C_n}{(1+i)^n} \qquad (2.01)$$

where

PV = present (market) value,

C_t = expected incremental net cash flows in period t, and,

i = is the discount rate of return.

For example, let us consider a simple case of an investment in a hypothetical asset that produces Widgets, has a 10 year life and has no salvage value. The cost of the machine is $10,000 and its expected cash flow stream is estimated to be $2,000 per year. The discount rate of return is 10% and it is assumed that this rate remains constant over the 10 years[6]. Applying the DCF method using equation (2.01) to calculate the present value of the asset's cash flows results in a NPV of the investment as summarised in the following Table 2.1.

Table 2.1 – Net Present Value Analysis for a Hypothetical Asset Over a 10 Year Horizon

Time (Year)	0	1	2	3	4	5	6	7	8	9	10
Cash Flows	-$10,000.00	$2,000.00	$2,000.00	$2,000.00	$2,000.00	$2,000.00	$2,000.00	$2,000.00	$2,000.00	$2,000.00	$2,000.00
Discount Factor		1.10	1.21	1.33	1.46	1.61	1.77	1.95	2.14	2.36	2.59
Present Value		$1,818.18	$1,652.89	$1,502.63	$1,366.03	$1,241.84	$1,128.95	$1,026.32	$ 933.01	$ 848.20	$ 771.09
NPV	$ 2,289.13										

[6] Treating the discount rate as constant is considered the standard procedure when using the traditional present value method (Geltner & Mei, 1995).

By summing the discounted expected cash flows this results in a present value of \$12,289.13. The NPV of the income stream of this investment is equal to \$2,289.13 after subtracting the initial cost of \$10,000 at time 0. Given that the NPV of this example is positive, it is therefore considered a worthwhile investment.

Note that this simple example considers a relatively small capital investment in machinery for which we have assumed constant expected future cash flows and also a constant discount rate which accounts for the risks to those cash flows over the life of the investment. However, the known difficulties associated with using the DCF approach to valuing investment opportunities under uncertainty are well documented (see, for example, Brennan & Schwartz, 1985a; Kelly, 1998; Cortazar, Schwartz & Salinas, 1998; Duku-Kaakyire & Nanang, 2002; Colwell et al., 2003; McNamara, 2003; Schwartz & Trigeorgis, 2004). One of the oft-cited difficulties is estimating the expected future cash flows based on the naïve assumption that future output prices can be forecasted with sufficient accuracy. To illustrate this point, consider that it is common to observe annual variations of between 25 to 50 percent in output prices for natural resource industries (Brennan and Schwartz, 1985a) which amounts to significant uncertainty regarding any DCF-based cash flow calculations.

Such cases highlight why this assumption is challenged on the basis that the DCF method fails to adequately account for the stochastic nature of market prices. Further, once the expected future cash flow stream has been estimated, the DCF approach requires that the discount rate be inclusive of all risks associated with the investment project over the life of the cash flows. Hence the fundamental issue with DCF valuations is the need to specify the exact risk-adjusted rate of return. It follows that the present value of any set of future cash flows can only be determined accurately if the assigned discounted rate of return captures, in its entirety, all risk associated with the future cash flow stream.

2.2.2 Methods for specifying the risk-adjusted discount rate

The discount rate used to determine the present value of the future cash flows of a project represents the opportunity cost of investing in a particular project rather than in another with similar risk characteristics (Pindyck & Rubinfeld, 2009). However, the inherent risks associated with investing in various projects differ across firms and industries. A common practice of accounting for the uncertainty relating to future cash flows is not to make particular adjustments to the cash flows themselves, but rather make adjustments to the discount rate. That is, in common practice, the DCF method discounts the expected cash flows of a project using a risk-adjusted rate of return.

Whereas diversifiable risk associated with a single project can be mitigated by investing in many other projects, nondiversifiable risk cannot be diversified away due to the returns on an investment being dependent on the more general uncertainty associated with future economic conditions. For instance, when economic growth is positive it is an expectation that profits will rise. Conversely, in times of recession profits of firms are expected to be relatively less. Since investors bear nondiversifiable risk, the opportunity cost of investing in risky projects is higher than in a project where this risk can be diversified away. It is therefore appropriate to incorporate into the discount rate a premium over and above the risk-free rate of return.

2.2.2.i The Capital Asset Pricing Model

The Capital Asset Pricing Model (CAPM) is the standard procedure used to measure the appropriate risk premium which accounts for nondiversifiable risk in a project. The CAPM formula is generally presented as follows[7]:

$$r_i - r_f = \beta_j (r_m - r_f) \qquad (2.02)$$

where

r_i is the expected rate of return on the asset,

r_m is the expected rate of return on the market portfolio, which consists of all assets in the market,

r_f is the risk-free rate of return, and,

β is the beta of the asset, where $\beta = \dfrac{Cov(r_i, r_m)}{Var(r_m)}$.

The CAPM posits that the risk premium for a capital investment depends on the correlation of the return on the investment to the return on the entire stock market. The constant beta defines the relative sensitivity of the investment's returns to market movements (Kidwell et al., 2007). Once the beta is determined, the expected return of the investment is computed as follows:

$$r_i = r_f + \beta (r_m - r_f) \qquad (2.03)$$

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[7] See Kidwell et al., 2007, p.283.

If the project is a commonly listed stock, the beta can be readily determined statistically from historical data. Alternatively, in considering whether to make an investment in capital for a new project the CAPM can be used to estimate the relative risk adjustment for the rate of return using the risk premium from the beta of projects with similar risk profiles (often termed twin securities, see for example, Brennan & Schwartz, 1985a). However, significant issues arise when importing the beta values from twin securities and incorporating them into the DCF procedure. Brennan and Schwartz (1985a) argue that transferring the betas of established firms to a new firm is problematic because firms often consist of portfolios of unrelated industries which typically have specific, unrelated risk characteristics.

2.2.2.ii The Weighted Average Cost of Capital

Levered firms often use the company cost of capital (also known as the weighted average cost of capital (WACC)) as a nominal discount rate which is a weighted average of the expected return on the company's stock[8] and the interest rate that it pays for debt. Formally, the after-tax WACC can be written as follows[9]:

$$WACC = (E/V)R_E + (D/V)R_D(1 - T_c) \qquad (2.04)$$

where

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[8]The expected return on a stock is the cost of equity and may be calculated using the CAPM framework or the dividend growth model approach (see Ross, Christensen, Drew, Thompson, Westerfield & Jordon, 2011).

[9] See Ross et al., 2011, p. 597.

V is the total combined market value represented by debt and equity,

E is the total market value of a firm's equity,

D is the total market value of a firm's debt,

R_E is the cost (required return) of a firm's equity,

R_D is the cost of a firm's debt, and,

T_C is the corporate tax rate.

The WACC is interpreted as the rate of return a firm is required to earn on its current assets to maintain the value of its shares (Ross et al., 2011). However, when considering investments in new projects, Pindyck and Rubinfeld (2009, p. 566) note:

> "This approach is correct as long as the capital investment in question is typical for the company as a whole. It can be misleading, however, if the capital investment has much more or much less nondiversifiable risk than the company as a whole…it may be better to make a *reasoned guess*[10] as to how much the revenues from the investment are likely to depend on the overall economy".

In particular, a latent degree of managerial discretion to respond operationally to evolving market conditions, which is clearly also not objective, will impact on these risk characteristics over the life of the project. This managerial flexibility introduces a

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[10] My emphasis.

dynamic complexity surrounding the investment decision that cannot be resolved when applying traditional capital budgeting approaches. By adopting a static approach to investment decision-making, whereby all contingencies relating to the reversibility of an investment are decided at the outset, the DCF analysis neglects the flexible nature of real investment opportunities in addition to the compounding interdependencies of each and every subsequent investment decision (Trigeorgis & Mason, 1987; Colwell et al., 2003).

Finally, from a practical perspective, assigning a discount rate is difficult at best when analysing assets traded in markets that are essentially incomplete (thus not exchange-traded). This difficulty arises because important information that is reflected in market prices is not readily observable. For instance, consider an investment in commercial property assets which derive their value from office rents (the underlying asset). A study by Cowley (2007) found that commercial property valuations are not systematically standardised and that valuers adopt several subjective and objective processes to assign a discount rate[11]. Cowley (2007) noted that due to their typically private nature discount rates are rarely published and their publication is mainly limited to, for instance, independent valuation reports produced by assessors for their clients.

2.3 The Real Options Approach

As a consequence of its relative simplicity the DCF valuation method is particularly difficult to apply to assets that operate in markets which exhibit high levels of uncertainty (Cortazar & Schwartz, 1993). Despite its limitations, the DCF method

[11] This study was focussed on the Brisbane central business district office market.

remains widely used in practice. Schwartz & Trigeorgis (2004) acknowledge that this is generally the case because it is normally thought easier to account for risk to a project by adjusting the discount rate rather than to make adjustments for risk to the expected cash flows. The alternative methods that were developed to incorporate risk into the expected cash flow stream are collectively known as the real options approach[12].

The real options approach to value assets (which treats the investment decision as an option) has its foundations in the parallel field of financial options pricing. A characteristic feature of basic financial options is that they give the option holder the right, without obligation, to purchase (referred to as a call option) or to sell (a put option) an underlying security at an agreed price at a future date (Boyle & Boyle, 2001). In an analogous sense, a real option exists where a firm or investor reserves various strategic and operational alternatives pertaining to a particular asset which can be activated (or deactivated) under different sets of business conditions.

For example McDonald and Siegel (1986) describe how the investment opportunity is similar to a perpetual call option whereby it gives the investor the right to invest (the exercise price of the option) and receive an asset (the share of a stock). Similarly, Kester (1984) views “growth options” as the discretionary opportunity to invest in productive assets like plant, equipment and brand names with the cost outlay equivalent to the exercise price of a call option. In a broader sense real options may be categorised as being either strategic flexibilities or operating flexibilities (McNamara, 2003).

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[12] The first use of this term is generally ascribed to Myers (1977, p. 22) who described *real options* as “opportunities to purchase real assets on possibly favourable terms” (see, for example, McNamara, 2003).

Strategic flexibilities can in many cases be thought of as compound options whereby initial investment opportunities, upon being exercised, are prerequisites for further discretionary investment opportunities (Trigeorgis, 1988). These options include contingent decisions relating to the timing of investment as well as potential future growth options[13] (Merton, 1997). For example, a lease purchase for the purpose of exploration for potential resource reserves may lead to a subsequent option to develop the site. Further examples of this type of analysis include investment in advertising and acquiring unrelated companies to pursue opportunities that may become apparent through diversification and impending synergies.

Operational flexibilities, on the other hand, are those options that imply the ability to dynamically adjust project operations in response to transitory changes in the business cycle that will, in time, be resolved and/or prices become optimal. These options apply to production and operating parameters such as altering production rates, temporarily closing, expanding or contracting operations, abandoning production, and re-optimising production cut-off grades (McNamara, 2003). An ideal example is the case of a mine that has been temporarily mothballed due to adverse market conditions. While mining operations remain suspended while commodity prices are low the owners retain the option to recommission operations when the commodity price returns to an appropriate threshold.

Real options theory posits that the markets value the rights of firms (or investors) to exercise specific strategic and/or operational options during the course of the business

[13] Such as the ability to undertake research and development and creating excess capacity.

cycle. Furthermore, the more volatile the business conditions, the more valuable are the options that enable the owners of assets to react under uncertainty in order to maximise gains and limit potential losses. This is consistent with a number of empirical studies which have found that options inherent to investment opportunities tend to be valued positively (as measured by a premium) in markets involving real, physical assets (see, for example, Quigg, 1993; Berger, Ofek & Swary, 1996; McNamara, 2003). Accordingly, the real options approach has important practical implications for investment decisions in the sense that it is useful to augment the traditional NPV measure to effectively enhance the decision-making capabilities of firms (Kelly, 1998).

2.4 Key Contributions to the Real Options Literature

This section presents a review of the Real Options literature. This review focuses on the key contributions which are relevant to this research beginning with the theoretical literature. Following from this some well-known empirical studies are discussed.

2.4.1 Real options theory

The published research that is generally attributed to have first established a continuous-time model for pricing financial options was the Black and Scholes (1973) option pricing model. This work is attributed as presenting the first completely satisfactory equilibrium option pricing model (see Cox, Ross & Rubinstein, 1979). At the time this breakthrough in the financial options pricing research area essentially paved the way to address well-known problems associated with conventional valuation methods. In addition, by applying its fundamental principles to real assets, it provided a way to value the very real managerial options which come with owning an asset but for which conventional capital budgeting techniques could not capture.

Whilst the Black and Scholes (1973) model is considered the seminal paper for pricing financial options, the first, and generally also considered seminal, paper that significantly defined the area for evaluating real options using continuous-time modelling is Brennan and Schwartz (1985b). In this paper they derived a real options pricing model and further illustrated its use via a manufactured copper mine example and a contrived infinite resource mine. In the spirit of the work of Black and Scholes, Brennan and Schwartz (1985b) develop a system of partial differential equations whereby the value of the mine's cash flows is dependent on the volatility of output prices, the respective operational policy and the output production rate[14].

During the same period, other theoretical developments in real options valuation models include the work of McDonald and Siegel (1985) which developed a similar methodology for valuing risky investment projects which reserve the option to shut down production temporarily and without cost whenever variable costs exceed operating revenues. McDonald and Siegel (1986) and Madj and Pindyck (1987) consider the use of the investment rate as the control variable instead of the production rate, the former investigating the value of the option to wait to invest, while the latter applies option pricing methods to derive optimal decision rules for investment outlays over the entire construction program.

Key literature spanning the 1990s includes works by He and Pindyck (1992) which considers two production level controls for two different products and Cortazar and Schwartz (1993) who analyse a two-stage production system. Other such contributions

[14] The equations involve an optimisation over that production rate.

were by Trigeorgis (1993) who investigates a variety of both real and financial options and their implied interactions, emphasising the value in flexibility to adjust the terms of the financial deal to better reflect the evolution of risk in project operations. Kogut and Kulatilaka (1994) treated uncertainty affecting multinational facility locations while Kulatilaka and Trigeorgis (1994) examined the option to switch between alternative technologies and operational modes. Meanwhile the decade engendered some of the limited number of empirical studies which found evidence that would seem to validate the real options pricing theory (see, for example, Quigg, 1993; Berger et al., 1996).

Works toward the end of that decade include Cortazar, Schwartz and Salinas (1998) which presented a model that determines the optimum price level at which to invest in environmental technologies, discussing various parameters that influence this decision. Perlitz, Peske and Schrank (1999) investigated specific issues involving real option applications to research and development projects (see also, Faulkner, 1996). Cortazar and Casassus (2000) developed a compound model for evaluating multi-stage natural resource investments that explicitly considers multi-stage production decisions viewing investments as compound options.

While the theoretical literature generally applies real options analysis in the context of a particular production technology or ownership of land or natural resources, implicit features of firms have also been studied as real options. For example, Pindyck (1988) acknowledges a variety of intrinsic factors that effectively differentiate firms and create real options: “a firm’s managerial resources and expertise, reputation, market position, and possible scale, all of which have been built up over time, enable it to productively undertake investments that individuals or other firms cannot undertake”. Moreover, Majd and Pindyck (1989) treat the ability of a firm to learn over time as a valuable option affecting the profitability of a firm. By focusing on the implications of the learning curve on the cost structure of a firm, they derive an optimal production rule under uncertain demand that maximises the firm's market value based on the assumption that accumulated production reduces unit costs (see also, Seta, 2009).

2.4.2 Empirical studies of real options

It is well-known that empirical research has indeed lagged the development of conceptual frameworks of real options theory (Colwell et al., 2003; McNamara, 2003). That is, although a good deal of work has been developed with respect to conceptual models that treat various aspects of real options from a theoretical perspective, attempts to empirically test the validity of the value of optionality in a real-world setting remain sparse. In the attempts that have been made, a significant portion of this research has been applied to specific cases involving natural resource assets. Commodity markets, in particular, are suitable case studies due to the fact that they are typically highly irreversible investments which exhibit significant underlying spot price volatility (thus the problem is especially acute for valuing these particular assets) and have well developed futures markets (Cortazar, Gravet & Urzua, 2005). However, the typical private and confidential nature of a firm's financial and operational records does indeed make it difficult to compile a relevant and complete set of data to conduct this type of analysis (Colwell et al., 2003).

One of the best-known early empirical studies on the use and accuracy of real options valuation methods is Paddock, Siegel and Smith (1988). This paper develops a model for valuing options associated with offshore petroleum leases and juxtaposes these model prices with government estimates and actual market bid prices for 21 tracts in the Gulf of Mexico. Although the real options approach did not predict the bids as closely as was hoped[15], it produced the first empirical evidence of comparisons between option pricing valuations and actual DCF-based bids. In particular, they noted that their real option approach required less data, less computational cost and was less subject to error than currently used DCF techniques (Paddock et al., 1988).

Another classic empirical study by Quigg (1993) focussed on option-based land valuations. The study searched for empirical evidence for the value of the option to wait to develop land parcels by examining market prices of 2700 land transactions in Seattle during the period 1976-1979. In particular, the study found that market prices reflect a premium for the option to wait to be on average 6% of the land value. This work suggests that investors appear either to directly apply a real option valuation framework, or otherwise, they behave in a manner consistent with predictions of real options theory.

A few years later, Berger et al. (1996) researched how firms value the option to abandon an asset. Their approach was to investigate whether investors use balance sheet information to value their option to abandon the continuing business at the exit value of the asset. They treated the option to abandon as analogous to owning an insurance policy that pays off if the firm performs below expectations. Their method was to analyse the relationship between book value and exit value for major asset classes by examining the discontinued operations footnotes of 157 firms. They found a positive, highly significant relationship between market and exit values and that, if all other

[15] McNamara (2003) notes that the projects being deep in-the-money (which implies that the options to close or abandon the projects were less valuable) may help to explain the findings of low correlations between options-based valuations and industry bids.

factors are equal, investors prefer firms with greater exit values over those with lower values.

More recently, Moel and Tufano (2002) searched for empirical evidence on how well the real options theory can predict entry and exit behaviour of firms using the Brennan and Schwartz (1985b) model. They were specifically interested in the factors affecting the decision of firms to exercise real options such as the option to open or close a mine. The study compiled data from a proprietary database consisting of 285 North American gold mines over the period 1988 to 1997 finding that a mine was more than 90% likely to remain open if it was open in the prior period. Similarly, a mine was more likely to remain closed if it was closed in the prior period. Moel and Tufano (2002) suggested that hysteresis[16] was a feature of entry and exit decisions in this particular case. Furthermore, they noted that one reason for this is that the decision to close a mine is often made at the firm-level rather than the plant-level.

2.5 Real Options Applications Involving Renewable Resources

Renewable resources are those natural resources useful to human economies that exhibit growth, maintenance, and recovery from exploitation over an economic horizon (Erickson, 2002). While some renewable resources involve a rotational period subject to growth and maintenance (such as forestry resources, seasonal orchids and live fish stocks harvesting) others (for example, energy derived from solar and wind farms as

[16] Dixit (1989, p. 622) defines hysteresis as “the failure of an effect to reverse itself as its underlying cause is reversed”.

well as water catchments and desalination utilities) are subject to regular, ongoing recovery to inventories. Literature involving applications of real options approaches to evaluate renewable resource investments tends to be growing, particularly with respect to such areas as agriculture, forestry and renewable energy.

Examples of real options applications to agricultural investments include soybean processing (Plato, 2001), tart cherry production (Nyambane & Black, 2004), organic wheat and barley production (Ehmke, Golub, Harbor & Boehlje, 2004), hog production (Odening, Mußhoff, & Balmann, 2005), dairy operations (Engel & Hyde, 2003; Tauer, 2006), coffee planting (Luong & Tauer, 2006), cotton farming (Seo, Segarra, Mitchell & Leatham, 2008), citrus operations (Iwai, Emerson and Roka, 2009) and livestock farms (Bartolini, Gallerani & Viaggi, 2010). Among these applications, various types of real options are analysed such as the decision to replace existing manual operating systems with automatic systems (Engel & Hyde, 2003), evaluating optimal orchid replacement policies (Nyambane & Black, 2004), adoption of more efficient irrigation technology (Seo et al., 2008) and deciding when it becomes optimal to change from conventional production to organic production (Ehmke et al., 2004).

In particular, Tauer (2006) and Luong & Tauer (2006) use the analytic approach of Dixit (1989), which is close in spirit to the Brennan and Schwartz (1985b) model, to calculate the entry and exit prices for New York dairy farming and coffee growing in the Central Highlands of Vietnam, respectively. Tauer's (2006) study found there was a wide range between exit and entry prices for individual farmers which was affected by the relative size of the farm and that smaller farms required higher prices to induce entry and would exit at higher prices also. Luong & Tauer's (2006) study found similar results where efficiency was a key determinant. Efficient farmers enter production at lower prices and less efficient farmers enter at higher prices, while less efficient farmers start to exit at higher prices than efficient farmers.

Real options applications to forestry investment, which spans several decades, has predominantly examined the investment timing options involving the underlying timber resource. These include early works by Clarke and Reed (1987) who derive an analytical model which determines economically optimal harvesting policies under both single and rotational conditions, and Morck, Schwartz and Strangeland (1989) who develop a model to evaluate a white pine forestry asset in Canada which holds a ten-year logging lease. Morck et al. (1989) use the cutting rate as the optimal control whereby both the timber price and level of inventories are governed by stochastic processes that follow geometric Brownian motion. The authors acknowledge that their model is equally applicable to fisheries resources and any other similar renewable resource management problem.

Investment in forestry assets is also examined by Duku-Kaakyire and Nanang (2002) who evaluate a set of managerial options using the binomial option pricing method. These options include the timing option to delay reforestation, the option to expand the size of the wood processing plant, the option to abandon the processing plant if timber prices fall below a certain threshold or due to corporate takeover, and multiple options in which all three options are evaluated together. While others such as Kerr, Martin, Kimura, Perera and Lima (2009) analyse the optimal timing decision of when to harvest a stand of eucalyptus trees in Brazil.

Recent studies involving renewable energy investments include Schmit, Luo and Tauer (2008) who evaluate the optimal trigger prices for exercising the options of entering, suspending, reactivating and exiting a corn-based, dry-grind ethanol facility. They note that while research into ethanol firm operations have been analysed from the perspective of NPV, return on investment and break-even analysis, however, little attention has been paid to evaluating these optimal investment decisions from a real options perspective. By analysing the gross margin of ethanol price over corn price they find that, compared to a standard NPV analysis, option values increase entry prices and lower exit prices.

The option to switch between alternative sources of production using renewable energy is analysed by Kjærland and Larsen (2009). They apply the switching model developed by Kulatilaka (1988) to evaluate the operational flexibility relating to a hydro-based operator who holds the option to add thermal power to augment production. This switching option creates value for a hydro operator as it allows the operator to reschedule more production during peak price periods. The study finds that significant option values arise when thermal power plants are controlled by a hydro operator.

2.6 Review of Option Pricing Techniques

The conceptual advantage of using the options approach is that it recognises the ability of firms to react to uncertainty as it unfolds over the life of an asset (Trigeorgis & Mason, 1987). By treating the investment as an option to invest (or to undertake a number of subsequent investments) this approach is able to capture the implicit value of managerial flexibility in the valuation procedure. Consequently, a number of methodologies have arisen in the literature resulting in real options models ranging in complexity from the commonly applied discrete time analyses using multi-period decision tree lattices, to various simulation techniques such as the Monte Carlo method, and the complex analytic methods which employ the tools of stochastic calculus to develop continuous-time models.

The most well-known of the options pricing approaches is the binomial lattice method. A significant operational advantage of using this approach, and the real options approach more generally, is that it does not require an estimated value of the expected cash flows based on forecasted output price movements in the underlying asset. In applying the binomial method, the expected future price movements are calculated using the current output price and the volatility of output prices which is inferred from an historical distribution. The binomial approach calculates the value of the resulting set of unconditional expected cash flows using their respective conditional probabilities (Trigeorgis & Mason, 1987).

Calculating these probabilities explicitly involves equating the expected rate of change in the price of the asset to the risk-free rate less any net convenience yield of the asset[17]. This enables the set of risk-neutral probabilities of future price movements to be determined. The probabilities are calculated solely from the current market price, the estimated price volatility, the risk-free rate, and any appropriate payout rates (such as dividends and convenience yields). By weighting the expected payoff of each possible outcome of the option (the investment) using these unique probabilities results in the risk-neutral measure of the value of the expected cash flow stream.

Significant operational advantages arise as a consequence of using this risk-neutral approach to calculate the expectation. For instance, due to the risk to the cash flows being explicitly accounted for in the expectation this obviates the need to arbitrarily

[17] The convenience yield is an adjustment made to the cost of carry and is derived from the relationship between the spot and futures prices of an asset (see Section 2.7.1).

adjust the discount rate. This implies that the present value can be determined by discounting the future cash flow stream using the risk-free rate.

Other option pricing techniques, such as the continuous-time models, are the most operationally complex of the real options models. The complexity in using these techniques is due to the need to implement various numerical methods to solve the resultant set of differential equations that these models are cast as (see, for example, Brennan & Schwartz, 1985b). The method generally arrives at these differential equations after careful consideration and application of the mathematics of a standard Gauss-Wiener process in conjunction with Ito's Lemma[18] to derive the total differential of a function of stochastic variables (see Appendix A). The resulting set of differential equations establish a continuous-time arbitrage condition which describes the value of an investment based upon the current spot price, the resource inventory and a number of other model parameters. However these equations are rarely amenable to analytic, or closed-form, solution and hence they require unique implementations of numerical methods tuned specifically to cover the applications in real options cases.

2.6.1 A worked example of pricing by replication using a one-period binomial lattice approach

Pricing by replication involves valuing the option to invest in a risky asset[19] by way of synthetic replication. This strategy can be used to calculate the price of the option to

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[18] Ito's Lemma is a technique for differentiating and integrating functions of continuous-time stochastic processes.

[19] The asset is regarded as risky if the spot price of the underlying asset behaves stochastically leading to future spot prices being uncertain.

invest in a risky asset by equating the value of it's expected future cash flows to the payoff of a replicating portfolio[20] consisting of a position in the risky asset and a position in an asset which is free from risk. An example is presented, similar to the one in Boyle and Boyle (2001), which uses a government security as the risk-free asset[21] (which implies that cash can be borrowed or lent over the period at the risk-free rate). It is assumed that the underlying asset does not pay any dividends and that positions in the replicating portfolio are completely divisible.

For this example consider that the exercise price of a one year call option on a stock is $1000 which is equal to the current market price of this asset. The price of a one year government security (zero coupon) is presently $909.09 with an interest rate of 10% per annum for the purpose of this exercise. This results in a value of $1000 (909.09*1.10) for the risk-free asset at maturity in one year.

Suppose, from historical data, that we have a distribution of *changes* in stock prices that have a annualised standard deviation (volatility, σ) of 40%. Thus, in the absence of any other information, if the price moves in this period, then the expected price movement is $S^*(1+\sigma)$ for a price increase and $S^*(1- \sigma)$ for a price decrease assuming that the price movement is normally distributed. An upward movement in the market price of the stock would thus result in the expected future value in one year of $1400

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[20] A portfolio constructed in such a way as to replicate the expected cash flows of the investment is known as a replicating portfolio.

[21] Government securities are assumed to carry no risk and their yield is often referred to as the risk-free rate (Kidwell et al., 2007). The example presented in Boyle and Boyle (2001) assume interest rates are zero.

(=$1000(1+0.4)). If the call option is exercised the payoff will be equal to the future value of the stock less the exercise price of the option. The payoff of the call option in this case will be $1400-$1000 = $400.

Alternatively, if the market value of the stock declines the value of the asset would be $600 (=$1000(1-0.4)). The payoff will be equal to the value of the stock less the exercise price which is $600-$1000 = -$400. In this case the option would not be exercised (as the investor would incur a loss) which implies the option would expire worthless (i.e. payoff = $0). Consequently the option payoff will be either $400 (if exercised) or $0 (if it is allowed to expire).

We now have the data necessary to calculate the respective positions to be taken to create an appropriate hedge that will replicate the option payoff in one year and subsequently price the call option. This is done by constructing a portfolio that replicates the option payoff calculated in the previous paragraph which is expected to be $400 or $0. In this case consider a long position in the stock and a short position in the risk-free security[22]. The appropriate portfolio can be easily shown to be 0.5 units in the underlying asset and -0.3 units in the government security.

[22] Taking a short position in the risk-free asset (i.e. selling bonds) is the same as borrowing cash at the risk-free rate while taking a long position in the risk-free asset (i.e. buying bonds) is the same as lending cash at the risk-free rate (see Cox et al. (1979)).

A position of 0.5 units in the asset and -0.3 units of the risk-free security will yield a payoff in one year of, should the market move upwards, 0.5(1400) – 0.3(909.09*1.1) [23] = $400. Should the market move downwards, however, then the payoff will be 0.5(600) – 0.3(909.09*1.1) = $0. Thus, by using trial and error, we now have a portfolio that mimics the expected future value of the asset.

The selection of the appropriate positions that are required to hedge the portfolio need not be derived from inspired guesswork. As shown by Cox et al. (1979) the positions in the hedge can be determined systematically by equating the call option payoffs to the values of the portfolio for each expected outcome at expiry of the option.

Consider a portfolio consisting of Δ positions in the stock where the current market price is *S* (= $1000), and λ positions in the dollar amount of a risk-free security in one year, *B* (= $909.09*1.1). Denote *u* as the expected upward price movement of *S* in one year (1+σ = 1.4), and *d* as the expected downward price movement in one year (1-σ = 0.6). C_u is the payoff of the call if the stock price increases over the period (= $400), and C_d is the payoff of the call if the stock price decreases over the period (= $0). Formally, this is expressed in equations (2.05) and (2.06) as follows:

$$\Delta uS + \lambda B = C_u \qquad (2.05)$$

$$\Delta dS + \lambda B = C_d \qquad (2.06)$$

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[23] The dollar amount of bonds (cash) includes the current price 909.09 and interest (factor 1.1) over the period.

The positions required to hedge the portfolio in such a way as to replicate the payoffs of the call option under both outcomes can thus be determined by solving equations (2.05) and (2.06) simultaneously.

Thus, solving equations (2.05) and (2.06) [24] simultaneously to find Δ gives,

$$\Delta uS + \lambda B = C_u$$

$$-\Delta dS - \lambda B = -C_d$$

$$\Rightarrow \Delta uS - \Delta dS = C_u - C_d$$

$$\Rightarrow \Delta(uS - dS) = C_u - C_d$$

$$\Rightarrow \Delta = \frac{(C_u - C_d)}{(uS - dS)}$$

$$\Rightarrow \Delta = \frac{(C_u - C_d)}{(u-d)S} \qquad (2.07)$$

Substituting Δ from equation (2.07) into equation (2.06) to find λ gives

$$\left(\frac{(C_u - C_d)}{(u-d)S}\right) dS + \lambda B = C_d$$

$$\Rightarrow \lambda B = C_d - \left(\frac{(C_u - C_d)}{(u-d)S}\right) dS$$

---

[24] Rearranging equation (2.02) (or indeed multiplying each term by -1) gives $-\Delta dS - \lambda B = -C_d$.

$$\Rightarrow \lambda B = C_d - \left(\frac{(C_u - C_d)}{(u-d)} \right) d$$

$$\Rightarrow \lambda B = C_d - \left(\frac{(dC_u - dC_d)}{(u-d)} \right)$$

$$\Rightarrow \lambda B = C_d - \left(\frac{(dC_u - dC_d)}{(u-d)} \right)$$

$$\Rightarrow \lambda B(u-d) = C_d(u-d) - (dC_u - dC_d)$$

$$\Rightarrow \lambda B(u-d) = uC_d - dC_d - dC_u + dC_d$$

$$\Rightarrow \lambda B(u-d) = uC_d - dC_u$$

$$\Rightarrow \lambda = \frac{uC_d - dC_u}{(u-d)B} \qquad (2.08)$$

Substituting the numerical values for each variable in equation (2.07) gives the appropriate position(s) in the stock required to hedge the portfolio in such a way as to replicate the call option payoffs.

$$\Delta = \frac{(C_u - C_d)}{(u-d)S}$$

Δ = 400- 0 / (1.4– 0.6)1000

Δ = 400 / 800

Δ = 0.5

Similarly, substituting the numerical values into equation (2.08) gives the appropriate position(s) to be taken in the risk-free security.

$$\lambda = \frac{uC_d - dC_u}{(u-d)B}$$

$$\lambda = [1.4(0) – 0.6(400)] / (1.4 – 0.6)(909.09*1.1)$$

$$\lambda = - 240 / 800$$

$$\lambda = - 0.3$$

As before, the appropriate hedging strategy requires the portfolio to be constructed using 0.5 *long* positions in the stock and 0.3 *short* positions in the risk-free security. Having replicated the expected option payoff using this portfolio, we substitute these positions into the prices at inception to arrive at the price of the call option: 0.5(1000) – 0.3(909.09) = $227.27. Thus the price of this call option must be $227.27.

The no-arbitrage principle asserts that the market price of the call option must be equal to the value of the replicating portfolio. Accordingly, a call option that is priced more (or less) than $227.27 will result in an opportunity for arbitrage. For example[25], if the market price of the call was $250 then an investment bank would immediately sell the option to receive the proceeds and create the replicating portfolio. The portfolio would involve shorting 0.3 bonds to receive $272.73 (0.3*909.09) and buying 0.5 stocks at a cost of $500 resulting in a total cost of $227.27 for the bank. This leaves the bank with $22.73 (250-227.73) from this transaction.

[25] This is following the reasoning of Boyle and Boyle, 2001. For a detailed clarification of the no arbitrage opportunities argument see Cox, Ross and Rubenstein, 1979.

If the price rises the bank will pay the investor $400 ($1400 - $400 = $400), but the portfolio will also earn $400 over the period (0.5 positions in the asset will be worth $700 and 0.3 short positions in the bond will cost -0.3*(909.09*1.1) = -$300). If the price falls and the option expires worthless, the bank owes nothing to the investor while the value of the portfolio is 0 (the positions in the stock will be worth $600*0.5 = $300 and the short position will still be worth -$300). In either case the bank will profit $22.73.

Similarly, the situation whereby the market price is lower than the value of the replicating portfolio would result in an arbitrage opportunity whereby the bank would purchase the call option and short the portfolio. For instance, consider a call option with a price of $220. Creating the portfolio would involve short-selling 0.5 stocks to receive $500 and purchasing 0.3 bonds for $272.73 leaving a balance of 227.27. The bank would then purchase the call option for $220 resulting in the bank having $7.27 (227.27 - 220) left over.

If the price moves up in one year the bank would exercise the call option to receive (1400-1000) = $400. The bonds in the portfolio would earn 0.3*(909.09*1.1) = $300, and the 0.5 stocks could be purchased to settle the short position for 0.5*1400 = $700. Since the loss on the portfolio (-$400) is offset by the payoff of the call option ($400), the bank profits by $7.27. If the price falls over the period the option will expire worthless, and the stock can be purchased to cover the short position for 0.5*600 = $300 which is offset by the $300 earned on the bonds. In each case the bank profits $7.27.

The next section progresses to a worked example of a stylised application of the binomial lattice approach in the context of a one-period binomial model to price the option for a natural resource mine to develop its reserves.

2.6.2 A stylised example of the binomial lattice approach applied to the optimal investment timing decision

Following from the previous introduction to pricing through replication, the binomial lattice option approach is a technique that can be used to value various option-like opportunities faced by firms. Its appeal lies in its simplicity in implementation typically only requiring an excel spreadsheet to perform the algebraic equations that result from this approach. To demonstrate the basic concepts of this approach in a practical sense this section will analyse the optimal investment timing decision using a stylised example. This example will use a one-period binomial model to price the option for a natural resource mine to delay investment given that that the spot price is currently in-the-money and assuming that the volatility in the output price of the underlying commodity and the risk-free rate are both known.

This example shows how the investment timing option is analogous to a European call option[26] where the costs are equivalent to the strike price of the financial option. For simplicity we make a further assumption that the costs of storing the physical commodity are zero (in addition there is no convenience yield). This implies that the procedure to value the real option is consistent with the method used to value a financial option[27]. Here we note that the option payoff at maturity has two possible states: the market price is greater than the contract price and the option is exercised; the market

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[26] A European call option can only be exercised upon maturity.

[27] Financial assets, unlike real assets, can be stored without physical cost. The additional complications that arise from the existence of inventories and the convenience yield of a commodity are discussed in Section 2.7.1.

price is less than the contract price and the option expires worthless (zero)[28]. Table 2.2 presents the data and notation to be used in this example.

Table 2.2: One-Period Binomial Example – Data and Notation

Variable	Notation	Value
Current price of commodity	S	$500
Cost of production (exercise price)	K	$450
Price volatility	σ	20%
Riskless rate	r	5%
Risk-neutral probability of price increase	p	
Risk-neutral probability of price decrease	1-p	

From this data the value of immediate exercise of the option is the current market price less any costs incurred from production and is calculated as follows.

$$S - K = \$500 - \$450$$

$$= \$50$$

To compare this value against the value of the option to wait using the binomial lattice method we derive the price movements of the underlying commodity using estimated prices (based on the volatility), the payout rate (which we have excluded from this example for simplicity) and the risk-free rate (Kelly, 1998). This approach requires us to treat the expected price in a year (one period), S*(1+r), as a function of the expected price movements (which are estimated from the current price and the price volatility) weighted by their respective risk-neutral probabilities. Formally, this is written as[29]:

[28] It would be counterintuitive to exercise any option when the commodity can be purchased directly for a market price which is less than the exercise price of the option (McNamara, 2003).

[29] See Kelly (1998).

$$S(1+r) = S(1+\sigma)p + S(1-\sigma)(1-p) \qquad (2.09)$$

Substituting the numerical values from Table 2.2 into equation (2.09) gives

525 = (500*(1+0.2)*p) + (500*(1-0.2)*(1-p))

1.05 = (1.2*p) + (0.8*(1-p))

1.05 = 1.2p + 0.8 – 0.8p

0.4p = 0.25

p = 0.625

1-p = 0.375

Thus using a simple one-period binomial lattice, the risk-neutral probabilities of future price movements in this example have been calculated. For a risk-neutral position, given the volatility in the spot price of the commodity, calculating the probability of an upward move in prices to be 62.50% implies that the corresponding probability of a downward move will be 37.50%. The next step is to compute the expected payoff in one year.

Using the current price, the price volatility and the costs of production, the expected payoff of the investment in one year can be shown to be

S*(1+ σ) – k = $150 in the up-state and

S* (1-σ) – k = -$50 in the down-state.

If the price moves down the option will not be exercised and will expire worthless. By substituting the risk-neutral probabilities into the expected payoff we find the expected value of the option in 1 year to be

$$\$150(0.625) + 0(0.375) = \$93.75$$

Discounting this back to obtain the present value using the risk-free rate results in the option value of $89.29. Compare this to the value of immediate exercise which was valued in this case to be $50.

This example illustrates that although the value of immediate exercise is currently positive, the volatility in the commodity price over time results in there being greater value in delaying the current investment opportunity. This example has shown how the option pricing method can enhance the decision-making capabilities of firms by accounting for value implied by the element of uncertainty in future market conditions. The binomial lattice approach can be further expanded to include a multi-period analysis incorporating continuous compounding. The specific modelling of this approach is detailed in Cox et al. (1979). A practical example of this approach is the subject of Kelly (1998) which applies the binomial lattice approach to value a mining property initial public offering.

2.6.3 Option pricing using continuous-time models

The Black and Scholes continuous-time model for pricing financial options is widely acknowledged to be the first complete option pricing model. The Brennan and Schwartz (1985b) model applies a similar approach to evaluate real options over natural resource assets by creating a theoretical portfolio involving the underlying commodity. Due to their significance within the options pricing literature, these models are discussed in detail below.

2.6.3.i The Black and Scholes Model

A significant advance in the area of pricing financial options using continuous-time modelling was achieved by the pioneering efforts of Black and Scholes (1973) and Merton (1973). These seminal papers specified the first successful financial options pricing formula and also provided a framework for further pricing of derivatives and ultimately real options. This work culminated from the cumulative contribution of many researchers over the 20th Century. (For a detailed chronological annotated bibliography and summary see, for example, Boyle and Boyle (2001).)

Using advanced algebraic techniques, the resulting formula specifically prices European call or put options via the following set of equations. The original equations of their model are given by:

$$c = s\Phi(d_1) - xe^{-rt}\Phi(d_2) \qquad (2.10)$$

$$p = xe^{-rt}\Phi(-d_2) - s\Phi(-d_1) \qquad (2.11)$$

where

$$d_1 = \frac{\log(s/x) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

and

$c =$ is the current price of a European call option

$p =$ is the current price of a European put option

$s =$ price of underlying stock

$x =$ strike price

$r =$ continuously compounded risk-free rate

$t =$ time in years until expiration

$\sigma =$ implied volatility of the stock

$\Phi =$ standard normal cumulative distribution function

The Black and Scholes model is based on the creation of a theoretical portfolio involving positions in the stock and its underlying security (option) which share the same element of risk, in this case, the random movements of stock prices. By dynamically adjusting opposite positions in the portfolio together in time this effectively eliminates uncertainty by keeping the value of the portfolio in equilibrium (i.e. changes in the value of these positions effectively cancel each other out). Merton's contribution was to employ the use of Ito calculus to effectively divide time into infinitely small parcels to formulate the model in continuous time. The key insight of this work is that to price an option on a stock one only needs to know the current price of the stock, its volatility, the current risk-free rate and the time to maturity of the option.

The Black and Scholes model specifically addresses the obvious problems that arise due to the inherent volatility of stock prices where, otherwise, the expected future value of a stock cannot be known with certainty. In what is the standard approach to modelling the stochastic price process in continuous time[30], the stock price is assumed to follow a random walk and has a log-normal distribution. Thus the model assumes a standard normal distribution curve for the change in the stock price where the volatility of the stock price (which is estimated using historical stock price data) determines the height and width of the curve.

2.6.3.ii The Brennan and Schwartz Model

Adopting a similar approach, a continuous-time model to evaluate real options involving natural resource assets was published by Brennan and Schwartz (1985b). This model calculates the present value of the cash flows of an asset based on a no arbitrage argument between the investment opportunity itself and a self-financing portfolio of risk-free bonds and future contracts over the underlying commodity (McNamara, 2003). This seminal paper demonstrated a stylised application of this approach to a copper mine example where the only source of uncertainty is the output price of the underlying commodity.

The Brennan and Schwartz (1985b) model represents an insightful way of capturing salient features of the situation facing the owner of a real asset, like a mine, whose value depends on the fluctuating price of the commodity that the mine produces as well as the key decisions regarding the opening, closing and potential abandonment of the mine. Natural resource assets are aptly suited to this valuation approach because the markets for the underlying commodities often have well-developed spot and futures markets and

[30] For an extensive examination of stochastic price modelling see Schwartz (1997).

thus observable market prices can be used to quantify the relationship between risk and return of the asset (McNamara, 2003).

Similar to the Black-Scholes (1973) financial option pricing model, the real asset is valued by creating a theoretical portfolio consisting of a long position in the mine and a short position in the underlying commodity. The mathematics of the Wiener process is used in conjunction with Ito's Lemma (a mathematical technique for differentiating and integrating functions of continuous-time stochastic processes) to derive the total differential of a function of stochastic variables[31]. Ito's Lemma is analogous to taking the limit of a Taylor series style expansion, where in the limit of the infinitesimal process, the higher than second order terms disappear. The resulting set of partial differential equations value the asset based on the current spot price, the volatility of the spot price, the real risk-free rate, the convenience yield of the commodity and various operational cost parameters. (The derivation of the model is contained in Appendix A.)

2.7 Key Features of the Brennan and Schwartz (1985b) Model

The Brennan and Schwartz (1985b) model assumes that the value of the asset is a function of the commodity price and the quantity of reserves. This model assumes that the only source of uncertainty is the future volatility in the price of the underlying commodity (McNamara, 2003). The spot price of an underlying commodity is modelled as a continuous stochastic process in which the logarithm of the randomly changing quantity is assumed to follow standard geometric Brownian motion. Thus the

[31] For an extensive summary of Wiener Processes and Ito's Lemma see Pindyck (1991) and Hull (2009).

spot price is assumed to be determined competitively and is given by the exogenous stochastic process (Brennan and Schwartz, 1985b):

$$\frac{dS}{S} = \mu dt + \sigma dz \qquad (2.12)$$

where

μ is the local trend (drift) in the price,

dz is an increment to a standard Gauss-Wiener process, and,

σ is the instantaneous standard deviation of the spot price.

The first term in the stochastic process described by equation (2.12) captures the expected value (drift) of the change in price. The second term captures the deviation away from this expected value at any point in time due to the random walk that the price is assumed to follow.

As discussed previously in Section 2.6.3.ii, the present value of the mine's cash flows are replicated using a portfolio consisting of a long position in the investment opportunity itself (the mine) and a short position in futures contracts over the underlying commodity. Within this framework the returns to the replicating portfolio depend on the deterministic relationship that exists between spot and future prices of the underlying commodity.

2.7.1 The deterministic relationship between spot and futures prices

There are several classical theories that attempt to explain the relationship between spot and futures prices. One of the classic approaches is the normal backwardation theory which was developed by John Maynard Keynes in 1923[32] (see Caumon & Bower, 2004). This theory posits that the normal condition of the futures market is backwardation[33] which is necessary as an incentive to encourage speculators to become counter-parties to shorted futures contracts. (In this sense it is analogous to an insurance premium.) However, the obvious shortfall in this theory is that it fails to explain the basis when the futures market is in contango[34].

An alternative theory to normal backwardation is the theory of storage. Physical commodities, unlike financial securities, are able to be stored in inventory by both producers and end-users. On the one hand there are advantages in holding physical inventories on a speculative basis. For instance, holding inventories provides benefits in times of temporary shortages which help avoid interruptions to production. However, on the other hand, acquiring these benefits accrues opportunity costs such as physical storage costs as well the opportunity costs associated with the interest on the funds tied up in the inventory. Rational holders of commodities will optimise their current holdings of inventory based on the opportunity costs of investing in other assets, such as bonds for example.

[32] See Keynes, 1923.

[33] Backwardation is the market condition whereby the current futures price is below the expected (contract) futures price.

[34] Contango is the market condition whereby the current futures price is above the expected (contract) futures price.

Accordingly, the theory of storage explains the basis between spot and futures prices by way of the arbitrage approach[35]. This theory focuses on the term structure of prices relative to the costs associated with holding physical inventories of stock. The classical theory of storage equation is presented as follows (Caumon & Bower, 2004):

$$F_{t,T} = S_t e^{(r+c)(T-t)} \qquad (2.13)$$

where

$F_{t,T}$ is the forward price of the commodity

S_t is the current spot price

e^i is the exponential function of the compounding interest rate

c is the cost of storage

r is the risk-free rate of return

$T-t$ is an increment of time to maturity

The theory of storage equation (2.13) implies that the futures price must always be greater than or equal to the spot price. This is due to the fact that it is practically impossible for any of the underlying components that make up the cost of carry[36] of

[35] The arbitrage approach implies that the equilibrium price is maintained due to cash and carry and reverse cash and carry trading operations (Caumon & Bower, 2004).

[36] This is assuming the cost of carry of physical commodities includes such costs as market interest rates, insurance, shrinkage and transit and warehousing fees (Caumon & Bower, 2004).

commodities to realistically assume negative values. Thus, in contrast to the normal backwardation theory, a significant concession is that the theory of storage cannot explain the basis when futures prices are in backwardation. Intuitively, there are physical limitations on borrowing physical stocks from the future for present use. Therefore the relationship between spot and futures prices described by the theory of storage formula becomes one of inequality and not one of equilibrium as the model implies (Caumon & Bower, 2004).

The notion of a commodity convenience yield was first proposed by Kaldor (1939). The convenience yield is defined as a measure of the benefits or flow of services that accrue to the holder of physical inventories of a commodity which do not accrue to the owner of a futures contract over those inventories (Brennan and Schwartz, 1985b; Grafstrom & Lundquist, 2002; McNamara, 2003). Because a futures contract over the commodity cannot be used for productive purposes, it is unlikely to be regarded the same way by, for example, a manufacturer or a refiner who uses the commodity as an input within the production process (Hull, 2009).

The *net* convenience yield consists of four components that make up the cost of carry including storage costs, insurance, transportation and financing costs (McNamara, 2003). In a practical sense the convenience yield can be considered as analogous to the dividend yield on a stock. Where the price of the stock can be regarded as the present value of the expected future flow of dividends, the spot price of a commodity can be regarded as the present value of the expected future flow of convenience yields (Borak, Hardle, Truck & Weron, 2006).

Brennan and Schwartz (1985b) assume a competitive storage argument which implies the net convenience yield of the marginal unit of inventory will be equal across inventory holders. Rationally, commercial holders of commodities will continue to build up their inventories until the point where the marginal convenience yield is equal to the interest on the funds tied up in the inventory (Brennan and Schwartz, 1985a). As noted by Grafstrom and Lundquist (2002), so far as the convenience yield can be viewed as the opportunity cost of not producing the commodity, the financial benefits of postponing investment decreases as the benefit of holding inventories increases.

The convenience yield is often viewed as a reflection of investor's, or indeed the market's, forecasts regarding future supplies and availability of the commodity (Grafstrom & Lundquist, 2002; McNamara, 2003; Hull, 2009). A high convenience yield implies that the benefits of holding inventory are greater when there is low availability in the market. Holding inventory provides economic benefits during periods when demand increases and stocks become relatively low, thus the value of the option to delay investment decreases. For example, if inventories are held by manufacturers of copper products then holding a substantial stockpile of inventory negates the problems of delays in production and supply (Brennan and Schwartz, 1985a). Furthermore, holding adequate supplies enables the firm to take advantage of unexpected spikes in demand (McNamara, 2003).

On the contrary, high levels of inventory result in a low convenience yield. As the convenience yield decreases or becomes negative the value of the option to delay investment in inventories increases. By incorporating the convenience yield into the theory of storage the relationship between spot and futures prices is conventionally modelled as follows (Brennan and Schwartz, 1985b; Caumon & Bower, 2004):

$$F_{t,T} = S_t e^{(r-\kappa)(T-t)} \qquad (2.14)$$

where

$F_{t,T}$ is the forward price of the commodity

S_t is the current spot price

e^i is the exponential function of the compounding interest rate

κ is the net convenience yield

r is the risk-free rate of return

$T-t$ is an increment of time to maturity

Thus for all practical purposes, using the spot price, futures price and risk-free rate allows us to determine the convenience yield as follows (Gibson and Schwartz, 1990):

From the relationship between the spot and futures prices defined previously in equation (2.14) we have:

$$F(T) = S(t)e^{(r-\kappa)(T-t)}$$

rearranging gives the convenience yield as

$$\kappa = r - \frac{1}{\Delta t} * \ln\left[\frac{F(T)}{S(t)}\right] \qquad (2.15)$$

2.7.2 Specifications of the Brennan and Schwartz (1985b) Model

The Brennan and Schwartz (1985b) model values the mine as a function of the commodity price and quantity of reserves and the only source of uncertainty is the future volatility in the returns on the underlying asset. (The derivation of the original

Brennan and Schwartz (1985b) model is contained in Appendix A.3) The equations specifying this model allows the value of an open and operating mine, $v(s,Q)$, and the value of a closed (temporarily shut down) mine, $w(s,Q)$, to be calculated. Under these conditions, Brennan and Schwartz (1985b) specify the deflated[37] value of a mine as follows:

Let $v(s,Q)$ be the value of the open mine and $w(s,Q)$ be the value of the closed mine. The Brennan and Schwartz (1985b) equations that must be solved to find v and w are given by:

$$\max_{q \in [\underline{q}, \bar{q}]} [\frac{1}{2}\sigma^2 s^2 v_{ss} + (r - \kappa)s v_s - q v_Q + q(s - a) - \tau - (r + \lambda_1)v] = 0 \quad (2.16)$$

$$\frac{1}{2}\sigma^2 s^2 w_{ss} + (r - \kappa)s w_s - f - (r + \lambda_0)w = 0 \quad (2.17)$$

where

$\tau = t_1 qs + \max\, [t_2 q[s(1-t_1) - a], 0]$

$r = \rho - \pi$ is the real interest rate

subject to the boundary conditions:

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[37] Brennan and Schwartz make a simplifying assumption that all prices increase uniformly at the same rate of inflation to exclude time as a determining variable so that the value of the mine is a function of the current spot price and quantity of reserves only (see Appendix A.3). This implies that the deflated value of each parameter is constant.

$$w(s_0^*, Q) = 0 \tag{2.18}$$

$$v(s_1^*, Q) = \max[\, w(s_1^*, Q) - k_1(Q), 0] \tag{2.19}$$

$$w(s_2^*, Q) = v(s_2^*, Q) - k_2(Q) \tag{2.20}$$

$$w(s,0) = v(s,0) = 0 \tag{2.21}$$

$$w_s(s_0^*, Q) = 0; \tag{2.22}$$

$$v_s(s_1^*, Q) = \begin{cases} w_s(s_1^*, Q) & if \quad w(s_1^*, Q) - k_1(Q) \geq 0, \\ 0 & if \quad w(s_1^*, Q) - k_1(Q) < 0; \end{cases} \tag{2.23}$$

$$w_s(s_2^*, Q) = v_s(s_2^*, Q). \tag{2.24}$$

A solution to the Brennan and Schwartz (1985b) model equations (2.16 to 2.24) determines the optimal policy for opening, closing and abandoning the mine and sets the optimal output rate. Note that, referring to the equation (2.16) for the open mine $v(s, Q)$, the optimisation involving the rate of extraction, q, is not intended to drive the system of partial differential equations. Rather, it can be interpreted as saying that for any one spot price, there is an optimal rate of production, bounded by the minimum and maximum production/capacity constraints, that maximises the value of the mine $v(s, Q)$.

The Brennan and Schwartz (1985b) model makes use of many variables in the mathematical specification of the model. These variables, along with their interpretation and use within the model, are now explained.

2.7.2.i The Spot Price, s

The spot price, s, is the current market price of a single unit of a homogenous commodity. The spot price is determined competitively and is assumed to follow an exogenous stochastic process known as a Markov process. As noted by Hull (2009, p. 261) the underlying feature of a Markov process is that “probability distributions of the price at any particular future time is not dependent on the particular path followed by the price in the past”. In other words, as profitable trading opportunities arise and are subsequently arbitraged away this ensures that past market information is embedded into the current commodity price.

When stocks of a physical commodity are low relative to market demand then the spot price will be driven upwards. Conversely when stocks are relatively high the spot price will drop. Because the underlying commodity spot price is a determining factor in the value of the asset, it is a significant factor supporting managerial discretion regarding the optimal operational policy for that asset. For instance, when the price is high enough, it is economically rational to exercise the option to begin or reopen operations. Also, if the price drops to a sufficiently low level, in order to minimize losses it will then be rational to temporarily close or ultimately abandon the asset. The Brennan and Schwartz (1985b) model defines these critical operational policy points as follows:

$s_2(Q,t)$ denotes the price at which a closed operation is opened,

$s_1(Q,t)$ denotes the commodity output price at which operations are ceased, and

$s_0(Q,t)$ denotes the price at the asset is abandoned.

2.7.2.ii Variance in the spot price, σ^2

Over a short period of time, Δt, the return on holding a unit of an asset is normally distributed, $\frac{\Delta S}{S} \approx \phi(\mu \Delta t, \sigma^2 \Delta t)$. In the absence of uncertainty (i.e. the variability of the return is zero), the return in a short period is independent of the price and is equal to its expected rate of return. It follows from this assumption that the standard deviation of returns in Δt is thus $\sigma\sqrt{\Delta t}$ meaning that the standard deviation is proportional to the square root of time. This implies that, for any given price, investors are equally uncertain about future movements in the price and uncertainty about future prices increases the further one looks ahead in time (Hull, 2009).

The volatility of the spot price is calculated as the standard deviation of the continuously compounded annual rate of return from holding a unit of the commodity. The annualised volatility[38] used in the model is calculated as follows (assuming daily data is used):

$$\sigma = \sqrt{\text{var}\,iance(\ln(\frac{S_{t+1}}{S_t})) * n_{TD}}$$

where n_{TD} = the number of trading days in a year.

2.7.2.iii Interest and inflation rates: r, ρ, and π

ρ is a nominal annual interest rate and π is the inflation rate. The real rate of interest r is the nominal rate of interest ρ less the rate of inflation π, thus $r = \rho - \pi$. The

[38] See Appendix A.1.

model requires the real risk-free interest rate to be specified (and thus known). The conventional approach when using real option pricing is to use the rate of return on government securities as a proxy for the risk-free rate.

2.7.2.iv Inventory Stock and Depletion, Q and q

Q is the mine's resource inventory (reserves) whilst q is the annual extraction rate. The extraction rate is assumed to be constant at the optimal rate of production between the upper and lower bounds of extraction rates, $\bar{q}$ and $\underline{q}$.

2.7.2.v Convenience Yield, κ

The symbol κ is used to denote the net convenience yield of the commodity. The instantaneous (log) change in the future price is determined by the difference between the risk-free rate of return and the net convenience yield of the underlying commodity. Since Brennan and Schwartz assume the interest rate to be constant, the relationship is simplified to the convenience yield being proportional to the spot price, κS.

2.7.2.vi Maintenance Costs, f

The annual periodic maintenance cost for a closed mine which is not abandoned are denoted by f. This includes any outlay of funds necessary to maintain a mine which is currently in a closed state. Costs associated with maintaining the closed mine are assumed to increase at the inflation rate.

2.7.2.vii Property Tax Rates, λ_1 and λ_0

The tax rates on the property value of the mine land are simply given by:

λ_1 is the proportional rate of property tax on the value of the open mine.

λ_0 is the proportional rate of property tax on the closed mine value.

Brennan and Schwartz note that in some contexts this may also be interpreted as representing the risk premium associated with the possibility of uncompensated expropriation of the mine.

2.7.2.viii The Tax Function, τ

The tax function $\tau = t_1 qs + \max\,[t_2 q[s(1-t_1) - a], 0]$ consists of income taxes levied on the net cash flow of the mine and the appropriate royalty rate. These are denoted as follows:

t_1 is the royalty rate, and

t_2 is the income tax rate.

2.7.2.ix Operational Costs a, k_1 and k_2

a is the average cost of production per unit of commodity. This cost is assumed to increase at the inflation rate.

k_1 represents the cost of closing the mine which includes relevant redundancies, outstanding general liabilities and termination payments on current contracts. k_2 represents the costs of opening a previously closed mine and may include hiring and training costs, miscellaneous costs such as uniforms as well as other general recommissioning costs. These costs are also assumed to increase at the rate of inflation.

2.8 Specific Applications Using the Brennan and Schwartz Model

What is interesting to note is that during the period from 1985 to 2003 there were few implementations or empirical studies based on the seminal model of Brennan and Schwartz. For example, a number of papers applied the model, or indeed tested their model, in the specific setting of gold mines including Brennan and Schwartz (1985a) themselves as well as Moel and Tufano (2002), Colwell et al. (2003) and McNamara (2003). Typical of most reports that reference the Brennan and Schwartz (1985b) paper, Moel and Tufano (2002) do not actually solve their equations. Rather they compare general expectations of the model against empirical data they had collated over a 10 year period.

The Moel and Tufano work though is not without consideration. In light of their observation that North American mines are typically parts of mining companies with portfolios consisting of various divisions including both mining and non-mining assets, they make the point that “Brennan and Schwartz’s work takes the mine as the unit of analysis, as if it were a stand alone unit” (Moel & Tufano, 2002, p. 55). An important implication of this, especially in the case where divisions in the firm share a common destiny, is the notion that the profitability at other locations may be taken into consideration when deciding a particular operation policy such as to close down or abandon a particular mine. As a result, the assessment of a firm’s operational policies may be influenced by spill-over effects in the case where firms have available the capacity for cross-subsidisation and loss-offsetting practices.

McNamara’s (2003) work, on the other hand, directly compares the valuation of a mine as given by a numerical solution to the partial differential equations in Brennan and Schwartz (1985b). This study looked at Australian gold mining ventures and compared mine valuation techniques using the Hotelling Valuation Principle (HVP) (see also Miller & Upton, 1985) which is a special case of DCF analysis (specially tailored for the valuation of mineral assets) and the Brennan and Schwartz (1985b) option pricing model. Findings of this study showed that the HVP method significantly undervalued the market value of those firms surveyed. Furthermore, the option to close an open mine to avoid future loses if commodity prices fall sufficiently below extraction costs accounted for a substantial portion of the variation between the market values of sample firms and their valuations using the HVP.

Similarly, Cortazar et al. (2005) also solved the Brennan and Schwartz model numerically using a computer-based simulation. They then proceeded to compare this numeric solution to an expansion of the one-factor approach of Brennan and Schwartz which they developed to include a multifactor specification for uncertainty. They acknowledge that much research has been done to in an attempt to better capture the stochastic behaviour of commodity prices in a more sophisticated way. However, they further conceded the added complexity to obtain numerical solutions to these multi-factor settings. By increasing the number of stochastic variables this renders analytical solutions to these models intractable.

2.9 Chapter Summary

The first section of this chapter presented the traditional DCF method to determine an asset's present economic value. The well-known difficulties associated with using the traditional DCF method to evaluate investment under uncertainty were discussed and the real options approach was then presented as an alternative method to evaluate such investment opportunities. Subsequently, the chapter reviewed key theoretical and empirical works within the real options literature with a particular emphasis on the Brennan and Schwartz (1985b) model which was developed specifically for evaluating natural resource investments under uncertainty. A survey of the related literature showed that whilst there have been a number of publications that are essentially theoretically based there are few empirical studies to support the validity, or otherwise, of the basic theory[39] or of any extensions of this work.

Brennan and Schwartz briefly treat, with a number of unrealistic assumptions, the case of a natural resource asset with infinite inventory. This led to, under those constraining assumptions, an analytic solution to their model albeit to a class of mines which could never realistically exist. Furthermore, no solution of the Brennan and Schwartz (1985b) model for the general case of an asset which derives its value from an infinite or renewable resource has ever been published. Thus the following chapter will address the first research question of this study and develop a continuous-time real options model based on the Brennan and Schwartz framework which can be generally applied to evaluate renewable resource investments.

Given the above gaps in the literature, the next chapter will address the first research question of this study and will, accordingly, develop a new, continuous-time real options model which, although based on the Brennan and Schwartz framework, can be generally solved, implemented and applied to evaluate renewable resource investments.

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[39] As published by Brennan and Schwartz (1985b).

Chapter 3

Evaluating Renewable Resource Investments Under Uncertainty Using the Real Options Approach

3.1 Introduction

This chapter directly addresses the first research question of the study:

RQ1 – *What is a feasible and implementable approach to a generalised solution of a model, derived within the Real Options framework, which is applicable to renewable resource investments?*

More specifically, a new model entitled the RRM for evaluating renewable resource investments is derived. This model is based upon the seminal Brennan and Schwartz (1985b) framework, but involves recasting that framework to suit assets that have a renewable inventory, under the assumption that the rate of extraction does not exceed the rate at which the resource is replenished. An integral part of the derivation of the RRM is the specification of boundary conditions similar to, albeit mathematically distinct from, the boundary conditions of the original framework. More generally, the new model is derived and analysed through the application of the concept of analytic continuation (Harris, 1992) which involves extending the basic functions and equations of the model, continuously to first derivatives, over the entire phase space of the parameters. It will be shown that the resultant analytically continued model's equations can be solved to produce closed form solutions. Specific solutions can only be calculated via numerical techniques and consequently computational implementations of the general solution for the final model are developed.

The chapter is organized as follows. The next section presents the Brennan and Schwartz (1985b) model as the framework from which the RRM is to be developed. This framework is the basis from which the model (specifically developed under assumptions that are consistent with the characteristics of a renewable, non-exhausted, resource asset) is derived in the section after that. In turn, a new set of boundary conditions are presented. Finally, the model equations are then solved with these boundary conditions to produce a closed form solution for the RRM. The chapter is closed with a summary.

3.2 Foundations of the RRM

The Brennan and Schwartz (1985b) model consists of two differential equations which describe the value of a mine based upon the current spot price and a number of other model parameters. (The derivation of these equations is contained in Appendix A for the sake of completeness.) The original model (equations 2.16 to 2.24) for a finite life, diminishing inventory (when being operated) asset is:

$$\max_{q\in(0,\bar{q})}[\frac{1}{2}\sigma^2 s^2 v_{ss} + (r-\kappa)s v_s - q v_Q + q(s-a) - \tau - (r+\lambda_1)v] = 0 \quad \text{(open mine)}^{40}$$

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[40] We recall from Section 2.7.2 that the optimisation involving the rate of extraction, q, is not intended to drive the system of partial differential equations. Rather, it can be interpreted as saying that for any one spot price, there is an optimal rate of production, bounded by the minimum and maximum production/capacity constraints, that maximises the value of the mine $v(s,Q)$.

$$\frac{1}{2}\sigma^2 s^2 w_{ss} + (r-\kappa)s w_s - f - (\rho+\lambda_0)w = 0 \quad \text{(closed mine)}$$

where

$$\tau = t_1 q s + \max\,[t_2 q[s(1-t_1)-a],0]$$

$r = \rho - \pi$ is the real interest rate

subject to the boundary conditions:

$$w(s_0^*, Q) = 0$$

$$v(s_1^*, Q) = \max[\,w(s_1^*, Q) - k_1(Q), 0]$$

$$w(s_2^*, Q) = v(s_2^*, Q) - k_2(Q)$$

$$w(s,0) = v(s,0) = 0$$

$$w_s(s_0^*, Q) = 0;$$

$$v_s(s_1^*, Q) = \begin{cases} w_s(s_1^*, Q) & if \quad w(s_1^*, Q) - k_1(Q) \geq 0, \\ 0 & if \quad w(s_1^*, Q) - k_1(Q) < 0; \end{cases}$$

$$w_s(s_2^*, Q) = v_s(s_2^*, Q).$$

The development of the RRM, applicable to evaluating renewable resource assets, based on the Brennan and Schwartz theory, involves the use of analytic continuation. Analytic continuation is a well-known technique in the area of complex function analysis mathematics (Harris, 1992). The technique involves extending the domain of an analytic function beyond its original (or defined) range in a manner which keeps that function continuous to first derivative into the extended domain. The practical implications of the use of the method of analytic continuation is that the analytically-continued function can then be used to extend the original function into a new domain where the original function initially defined becomes divergent, is undefined, has no physical meaning, or has non-analytic behaviour or form (see, for example, Harris, 1992).

In this study analytic continuation is used to cover values of the spot price lower than the abandonment option, a physically unreasonable range of the spot price domain. In their original model, Brennan and Schwartz imposed the reasonable constraint that the value of the mine is zero when the spot price is lower than the abandonment option. This causes a non-analyticity in their model (in the first derivative) and is a contributory reason to the generally accepted belief that their model is insoluble analytically.

3.3 The RRM: A New Model for Evaluating Renewable Resource Assets

In this section we produce an analytically continued version of the real options Brennan and Schwartz (1985b) functions $v(s,Q)$ and $w(s,Q)$ by allowing the functions to continue smoothly beyond their boundary endpoints and into regions of the spot price phase space where they are not defined. That is, the real options functions $v(s,Q)$ and $w(s,Q)$ are smoothly continued into the lower spot price region $s < s_0^*$ and the upper spot price region of $s > s_2^*$. Achieving an analytic solution to a version of this model provides a more complete coverage of phase space thus furthering conventional insight into the behaviour of the model. Necessarily this requires a reworking of the currently discontinuous boundary conditions specified by Brennan and Schwartz (1985b).

3.3.1 *Redefining the boundary conditions*

The RRM is further defined on the basis of a number of observations and assumptions relevant to a renewable resource asset. The RRM equations are smooth and continuous over their entire range of spot prices and are not modified, even beyond the physical limits imposed by the boundary conditions. This is in stark contrast to the boundary conditions originally specified which contain embedded discontinuities and discontinuous behavior.

In the case of an asset with a renewable inventory, $Q \to \infty$, resulting in the Q dependence in the boundary conditions disappearing. Consequently, the set of physical boundary conditions simplify to:

$$w(s_0) = 0 \tag{3.01}$$

$$v(s_1) = w(s_1) - k_1 \tag{3.02}$$

$$v(s_2) = w(s_2) + k_2 \tag{3.03}$$

$$v_s(s_1) = w_s(s_1) \tag{3.04}$$

$$v_s(s_2) = w_s(s_2) \tag{3.05}$$

Similarly, the discontinuity in the tax function τ is replaced by a smooth tax function thus allowing for a full loss tax off-set (see Brennan and Schwartz, 1985b). Such a tax function becomes

$$\tau = t_1 qs + t_2 q(s(1-t_1) - a) \tag{3.06}$$

Under the above assumptions the (deflated) value of the operational asset $v(s,Q)$ (when it is operating at a production rate q which does not exceed the rate of replenishment) satisfies the ordinary differential equation (ODE)

$$\frac{1}{2}\sigma^2 s^2 \frac{d^2v}{ds^2} + (r-\kappa)s\frac{dv}{ds} + q(s-a) - t_1 qs - t_2 qs(1-t_1) + t_2 qa - (r+\lambda_1)v = 0 \quad (3.07)$$

It is of interest to note that the ODE which values the decommissioned renewable resource asset in the RRM remains unchanged from the original model equation for a closed mine.

3.3.2 The RRM

Recasting the original boundary conditions, as developed above, with the equations reworked to remove their Q dependence, results in the new model, the RRM, for an renewable resource asset. The RRM can then be written as the following system of equations:

$$\frac{1}{2}\sigma^2 s^2 \frac{d^2v}{ds^2} + (r-\kappa)s\frac{dv}{ds} + q(s-a) - \tau - (r+\lambda_1)v = 0 \qquad (3.08)$$

$$\frac{1}{2}\sigma^2 s^2 \frac{d^2w}{ds^2} + (r-\kappa)s\frac{dw}{ds} - f - (r+\lambda_0)w = 0 \qquad (3.09)$$

where

$$\tau = t_1 qs + t_2 q(s(1-t_1) - a)$$

$$r = \rho - \pi$$

subject to the boundary conditions:

$$w(s_0) = 0$$

$$v(s_1) = w(s_1) - k_1$$

$$v(s_2) = w(s_2) + k_2$$

$$v_s(s_1) = w_s(s_1)$$

$$v_s(s_2) = w_s(s_2)$$

This model has the advantage of being smooth and twice-differentially continuous over the entire spot price phase space thus implying that conventional solution techniques can now be applied to this system of equations. In the next section of the chapter such techniques are applied to derive an analytic solution to the system in closed form. Numerical methods are then applied to provide specific values for the solution.

3.4 Solution to the RRM for the Untapped Asset

The equation presented in the previous section for the *untapped asset,* equation (3.09), can be solved analytically. Solution is readily recognized when the equation is reformulated as a second order ordinary differential Euler equation. That is, equation (3.09),

$$\frac{1}{2}\sigma^2 s^2 \frac{d^2 w}{dw^2} + \left(-\kappa \right) s \frac{dw}{ds} - f - \left(+ \lambda_0 \right) w = 0$$

is recast as an Euler Equation:

$$\frac{1}{2}\sigma^2 s^2 \frac{d^2 w}{dw^2} + \left(-\kappa \right) s \frac{dw}{ds} - \left(+ \lambda_0 \right) w = f$$

Thus the value of the closed asset, $w(s)$, is the sum of the homogeneous and particular solutions to this Euler equation. At this point it is interesting to note that the actual form of the ODE for the closed asset is identical in both the original Brennan and Schwartz (1985b) description and in the description provided in this chapter. That is, there is no real difference between the two approaches in providing an analytic solution to the equation describing the non-operational asset in isolation: an analytic solution is available for both renewable and non-renewable, non-operational resource assets.

3.4.1 The homogeneous solution, $w_H(s)$

The homogeneous equation (same equation with a zero RHS) is:

$$\frac{1}{2}\sigma^2 s^2 \frac{d^2 w}{ds^2} + (r - \kappa)s\frac{dw}{ds} - (r + \lambda_0)w = 0$$

Following standard techniques, (see, for example, Spencer, Parker, Berry, England, Faulkner, Green, Holden, Middleton & Rogers, 1977a), it is well known that the form of the homogeneous solution will be

$$w(s) = cs^n$$

where

s is the spot price,

$w(s)$ is the value of the closed asset at a particular spot price, s

c is a constant to be determined by boundary conditions

n is the exponential change in value of the asset with respect to s.

The derivatives of w are:

$$\frac{dw}{ds} = ncs^{n-1}$$

$$\frac{d^2w}{ds^2} = n(n-1)cs^{n-2}$$

Substituting back into the homogeneous equation gives

$$\frac{1}{2}\sigma^2 s^2 n(n-1)s^{n-2} + (r-\kappa)sns^{n-1} - (r+\lambda_0)s^n = 0$$

$$\Rightarrow \frac{\sigma^2}{2} n(n-1)s^n + (r-\kappa)ns^n - (r+\lambda_0)s^n = 0$$

$$\Rightarrow \left[\frac{\sigma^2}{2} n(n-1) + (r-\kappa)n - (r+\lambda_0)\right]s^n = 0$$

$$\Rightarrow \frac{\sigma^2}{2} n^2 - \frac{\sigma^2}{2} n + (r-\kappa)n - r - \lambda_0 = 0$$

$$\Rightarrow \frac{\sigma^2}{2} n^2 + (r-\kappa-\frac{\sigma^2}{2})n + (-r-\lambda_0) = 0$$

Consequently n is the solution of a quadratic equation and thus determined by the standard quadratic solution formula:

$$n_{\pm} = \frac{-(r-\kappa-\frac{\sigma^2}{2}) \pm \sqrt{(r-\kappa-\frac{\sigma^2}{2})^2 + 2\sigma^2(r+\lambda_0)}}{\sigma^2}$$

where it is noted that $n_+ > 0$ and $n_- < 0$

Subsequently the full homogeneous solution is given in closed form by

$$w_h(s) = c_+ s^{n_+} + c_- s^{n_-}$$

Imposing the physical conditions that $w(s)$ is a monotonically increasing function of s implies $c_- \equiv 0$, and hence the homogeneous solution to the Euler ODE becomes:

$$w_h(s) = c_+ s^{n_+}$$

3.4.2 The particular solution, $w_p(s)$

By inspection $w_p(s)$ must equal some constant independent of spot price s. That is $\frac{d^2w}{ds^2} = 0$ and $\frac{dw}{ds} = 0$, and hence:

$$\frac{1}{2}\sigma^2 s^2 \frac{d^2w}{ds^2} + \left(- \kappa s \right)\frac{dw}{ds} - \left(+ \lambda_0 \right) w = f$$

$$\Rightarrow -\left(+ \lambda_0 \right) w = f$$

$$\Rightarrow w = -\frac{f}{r + \lambda_0}$$

$$\Rightarrow w_p(s) = -\frac{f}{r + \lambda_0}$$

In practical terms, the particular solution $w_p(s)$ is that component of the closed asset value that is independent of the spot price. This component is thus the maintenance costs of the asset discounted by the appropriate interest and tax rates.

3.4.3 The solution in closed form

Combining the above, including a notational change of + to 0 in the constants for clarity with the overall model, results in the general, analytic solution to the closed asset to be:

$$w(s) = c_0 s^{n_0} - \frac{f}{r + \lambda_0}$$

where c_0 is an integration constant, and

$$n_0 = \frac{-(r - \kappa - \frac{\sigma^2}{2}) + \sqrt{(r - \kappa - \frac{\sigma^2}{2})^2 + 2\sigma^2 (r + \lambda_0)}}{\sigma^2}$$

The value of c is determined by appropriate boundary conditions. At this point it is noted that isolating the closed asset invalidates those conditions relating the values between the operational and non-operational states. That is, the only valid boundary condition (that does not depend on the value of the closed asset) is

$$w_s(s_0) = 0\,.$$

We see that this boundary condition requires the value of the spot price at which the value is zero which would imply $c = 0$, a non-physical case, and thus discarded.

3.5 Solution to the RRM for an Operational Asset

We begin by rearranging equation (3.08) for the operational asset (operating in the presence of a continuously renewed resource) into an Euler equation:

$$\frac{1}{2}\sigma^2 s^2 \frac{d^2 v}{ds^2} + (r - \kappa) s \frac{dv}{ds} + q(s - a) - t_1 qs - t_2 qs(1 - t_1) + t_2 qa - (r + \lambda_1) v = 0$$

$$\Rightarrow \frac{1}{2}\sigma^2 s^2 \frac{d^2 v}{ds^2} + (r - \kappa) s \frac{dv}{ds} + qs - qa - t_1 qs - t_2 qs(1 - t_1) + t_2 qa - (r + \lambda_1) v = 0$$

$$\Rightarrow \frac{1}{2}\sigma^2 s^2 \frac{d^2 v}{ds^2} + (r - \kappa) s \frac{dv}{ds} + qs(1 - t_1) - t_2 qs(1 - t_1) - qa + t_2 qa - (r + \lambda_1) v = 0$$

$$\Rightarrow \frac{1}{2}\sigma^2 s^2 \frac{d^2v}{ds^2} + (r-\kappa)s\frac{dv}{ds} + qs(1-t_1)(1-t_2) - qa(1-t_2) - (r+\lambda_1)v = 0$$

$$\Rightarrow \frac{1}{2}\sigma^2 s^2 \frac{d^2v}{ds^2} + (r-\kappa)s\frac{dv}{ds} - (r+\lambda_1)v = -qs(1-t_1)(1-t_2) + qa(1-t_2) \qquad (3.10)$$

This Euler equation can be solved using standard techniques to find the solution to the operational state of the asset. The solution is simply the sum of the homogeneous and particular solutions.

3.5.1 The homogeneous solution, $v_H(s)$

The homogeneous solution is the solution of the following ODE:

$$\frac{1}{2}\sigma^2 s^2 \frac{d^2v}{ds^2} + (r-\kappa)s\frac{dv}{ds} - (r+\lambda_1)v = 0 \qquad (3.11)$$

Following standard techniques, (see, for example, Spencer et al., 1977a), the form of the homogeneous solution will be

$$v_H(s) = cs^n$$

where

c is a constant to be determined by boundary conditions

n is the exponential change in value of the asset with respect to s.

The values of c and n are found by substitution. The derivatives of v are:

$$\frac{dv}{ds} = ncs^{n-1}$$

$$\frac{d^2v}{ds^2} = n(n-1)cs^{n-2}$$

Substituting back into the homogeneous equation (3.11) and removing c yields

$$\frac{1}{2}\sigma^2 s^2 n(n-1)s^{n-2} + (r-\kappa)sns^{n-1} - (r+\lambda_1)s^n = 0$$

$$\Rightarrow \frac{\sigma^2}{2}n(n-1)s^n + (r-\kappa)ns^n - (r+\lambda_1)s^n = 0$$

$$\Rightarrow \left[\frac{\sigma^2}{2}n(n-1) + (r-\kappa)n - (r+\lambda_1)\right]s^n = 0$$

Therefore either $s^n = 0$ or $\left[\frac{\sigma^2}{2}n(n-1) + (r-\kappa)n - (r+\lambda_1)\right] = 0$

$$\Rightarrow \frac{\sigma^2}{2}n^2 - \frac{\sigma^2}{2}n + (r-\kappa)n - r - \lambda_1 = 0$$

$$\Rightarrow \frac{\sigma^2}{2}n^2 + (r-\kappa-\frac{\sigma^2}{2})n + (-r-\lambda_1) = 0$$

Thus, the solutions for n are given by substitution into the standard quadratic solution formula, leading to:

$$n_{\pm} = \frac{-(r-\kappa-\frac{\sigma^2}{2}) \pm \sqrt{(r-\kappa-\frac{\sigma^2}{2})^2 + 2\sigma^2(r+\lambda_1)}}{\sigma^2}$$

where it is noted that $n_+ > 0$ and $n_- < 0$ (as was the case for the closed asset).

The homogeneous solution is given by

$$v_H(s) = c_0 s^{n_+} + c_1 s^{n_-}$$

Imposing the physical condition that $v(s)$ remains bound to being no greater than linear in *s* as $s \to \infty$ implies that $c_0 \equiv 0$, and hence

$$v_H(s) = c_1 s^{n_1} \qquad (3.12)$$

where

$$n_1 = \frac{-(r-\kappa-\frac{\sigma^2}{2}) - \sqrt{(r-\kappa-\frac{\sigma^2}{2})^2 + 2\sigma^2(r+\lambda_1)}}{\sigma^2}$$

3.5.2 The particular solution, $v_p(s)$

By inspection of (3.10), $v_p(s)$ must equal some function that is linearly dependent on spot price s. That is, $v_p(s)$ can be written as:

$$v_P(s) = As + B$$

where A and B are constants. Thus, the derivatives with respect to spot price, s, will be

$$\frac{dv}{ds} = A$$

and

$$\frac{d^2v}{ds^2} = 0$$

Substituting this back into the Euler equation (3.10) yields

$$\to (r-\kappa)sA - (r+\lambda_1)(As+B) = -qs(1-t_1)(1-t_2) + qa(1-t_2)$$

$$\Rightarrow \left(r - \kappa\right)sA - As\left(r + \lambda_1\right) - B(r + \lambda_1) = -qs(1 - t_1)(1 - t_2) + qa(1 - t_2)$$

$$\Rightarrow \left(r - \kappa - r - \lambda_1\right)sA - B(r + \lambda_1) = -qs(1 - t_1)(1 - t_2) + qa(1 - t_2)$$

$$\Rightarrow -A\left(\kappa + \lambda_1\right)s - B(r + \lambda_1) = -qs(1 - t_1)(1 - t_2) + qa(1 - t_2)$$

Grouping according to common terms in s yields:

$$-A\left(\kappa + \lambda_1\right) = -q(1 - t_1)(1 - t_2)$$

$$\Rightarrow A = \frac{q(1 - t_1)(1 - t_2)}{\left(\kappa + \lambda_1\right)}$$

and

$$-B(r + \lambda_1) = qa(1 - t_2)$$

$$\Rightarrow B = -\frac{qa(1 - t_2)}{(r + \lambda_1)}$$

Consequently, the particular solution to the operational state of the renewable resource asset is:

$$v_P(s) = \frac{q(1 - t_1)(1 - t_2)}{\left(\kappa + \lambda_1\right)} s - \frac{qa(1 - t_2)}{(r + \lambda_1)} \qquad (3.13)$$

3.5.3 The solution in closed form

Combining the above two solutions (3.12) and (3.13) yields the full general solution to the operational state of the renewable resource asset within the RRM:

$$v(s) = c_1 s^{n_1} + \frac{q(1 - t_1)(1 - t_2)}{\left(\kappa + \lambda_1\right)} s - \frac{qa(1 - t_2)}{(r + \lambda_1)} \qquad (3.14)$$

where the constant of integration, c_1, is found from the boundary conditions and

$$n_1 = \frac{-(r-\kappa-\frac{\sigma^2}{2}) - \sqrt{(r-\kappa-\frac{\sigma^2}{2})^2 + 2\sigma^2(r+\lambda_1)}}{\sigma^2}$$

3.6 General Solution to the RRM

The general solution to the RRM is thus the combination of the above 2 solutions, one case being for when the asset is decommissioned and the other for when it is operational. That is, the general solution is:

$$w(s) = c_0 s^{n_0} - \frac{f}{r+\lambda_0}$$

$$v(s) = c_1 s^{n_1} + \frac{q(1-t_1)(1-t_2)}{(\kappa+\lambda_1)} s - \frac{qa(1-t_2)}{(r+\lambda_1)}$$

To determine the various constants it is necessary to apply the boundary conditions listed (3.01) to (3.05). The first of these boundary conditions (3.01) uniquely specifies the value of s_0 only. This then leaves 4 equations in 4 unknowns, being c_0, c_1, s_1 and s_2. Equations $v_s(s_1) = w_s(s_1)$, and $v_s(s_2) = w_s(s_2)$ require first derivatives as follows:

$$\frac{dv}{ds} = c_1 n_1 s^{n_1-1} + \frac{q(1-t_1)(1-t_2)}{(\kappa+\lambda_1)} \qquad (3.15)$$

and

$$\frac{dw}{ds} = c_0 n_0 s^{n_0-1} \qquad (3.16)$$

Substituting equations (3.07), (3.14), (3.15) and (3.16) into the boundary conditions (3.02) – (3.05) yields the standard problem of solving the following 4 simultaneous equations to determine the unknown constants c_0, c_1, s_1 and s_2:

$$c_1 s_1^{n_1} + \frac{q(1-t_1)(1-t_2)}{(r+\lambda_1)} s_1 - \frac{qa(1-t_2)}{(r+\lambda_1)} = c_0 s_1^{n_0} - \frac{f}{r+\lambda_0} - k_1$$

$$c_1 s_2^{n_1} + \frac{q(1-t_1)(1-t_2)}{(r+\lambda_1)} s_2 - \frac{qa(1-t_2)}{(r+\lambda_1)} = c_0 s_2^{n_0} - \frac{f}{r+\lambda_0} + k_2$$

$$c_1 n_1 s_1^{n_1-1} + \frac{q(1-t_1)(1-t_2)}{(r+\lambda_1)} = c_0 n_0 s_1^{n_0-1}$$

$$c_1 n_1 s_2^{n_1-1} + \frac{q(1-t_1)(1-t_2)}{(r+\lambda_1)} = c_0 n_0 s_2^{n_0-1}$$

The solution of these 4 equations is made difficult due to their explicit non-linearity. For brevity define the following constants:

$$\frac{q(1-t_1)(1-t_2)}{(r+\lambda_1)} \equiv p$$

and

$$\frac{qa(1-t_2)}{(r+\lambda_1)} \equiv m$$

Thus the system of non-linear equations to be solved simultaneously becomes:

$$c_1 s_1^{n_1} + p s_1 - m = c_0 s_1^{n_0} - \frac{f}{r+\lambda_0} - k_1 \qquad (3.17)$$

$$c_1 s_2^{n_1} + p s_2 - m = c_0 s_2^{n_0} - \frac{f}{r+\lambda_0} + k_2 \tag{3.18}$$

$$c_1 n_1 s_1^{n_1 - 1} + p = c_0 n_0 s_1^{n_0 - 1} \tag{3.19}$$

$$c_1 n_1 s_2^{n_1 - 1} + p = c_0 n_0 s_2^{n_0 - 1} \tag{3.20}$$

By inspection, these equations are linear in c_0 and c_1. An exact solution can be readily formulated from any 2 of the above equations, for those 2 unknowns. The trick is to choose the 2 equations for c_0 and c_1 such that the "simpler" equations are left to solve for s_1 and s_2. Thus it is proposed that the first 2 equations (3.17) and (3.18) are solved for c_0 and c_1, and accordingly, (3.19) and (3.20) are solved for s_1 and s_2.

Explicitly solving for c_0 and c_1, equations (3.17) and (3.18) then yield:

$$c_1 s_1^{n_1} + p s_1 - m = c_0 s_1^{n_0} - \frac{f}{r+\lambda_0} - k_1$$

$$\Rightarrow c_1 s_1^{n_1} = c_0 s_1^{n_0} - \frac{f}{r+\lambda_0} - k_1 + m - p s_1$$

$$\Rightarrow c_1 = \left(c_0 s_1^{n_0} - \frac{f}{r+\lambda_0} - k_1 + m - p s_1 \right) s_1^{-n_1} \tag{3.21}$$

Substituting (3.21) back into equation (3.20) yields:

$$c_1 s_2^{n_1} + p s_2 - m = c_0 s_2^{n_0} - \frac{f}{r+\lambda_0} + k_2$$

$$\Rightarrow \left(c_0 s_1^{n_0} - \frac{f}{r+\lambda_0} - k_1 + m - p s_1 \right) s_1^{-n_1} s_2^{n_1} + p s_2 - m = c_0 s_2^{n_0} - \frac{f}{r+\lambda_0} + k_2$$

$$\Rightarrow \left(c_0 s_1^{\,n_0} - \frac{f}{r+\lambda_0} - k_1 + m - ps_1 \right) s_2^{\,n_1} + ps_2 s_1^{\,n_1} - ms_1^{\,n_1} = c_0 s_2^{\,n_0} s_1^{\,n_1} - \frac{fs_1^{\,n_1}}{r+\lambda_0} + k_2 s_1^{\,n_1}$$

$$\Rightarrow c_0 s_1^{\,n_0} s_2^{\,n_1} - c_0 s_2^{\,n_0} s_1^{\,n_1} = \frac{fs_2^{\,n_1}}{r+\lambda_0} + k_1 s_2^{\,n_1} - ms_2^{\,n_1} + ps_1 s_2^{\,n_1} - ps_2 s_1^{\,n_1} + ms_1^{\,n_1} - \frac{fs_1^{\,n_1}}{r+\lambda_0} + k_2 s_1^{\,n_1}$$

$$\Rightarrow c_0 (s_1^{\,n_0} s_2^{\,n_1} - s_2^{\,n_0} s_1^{\,n_1}) = \frac{f(s_2^{\,n_1} - s_1^{\,n_1})}{r+\lambda_0} + k_1 s_2^{\,n_1} + k_2 s_1^{\,n_1} + p(s_1 s_2^{\,n_1} - s_2 s_1^{\,n_1}) + m(s_1^{\,n_1} - s_2^{\,n_1})$$

$$\Rightarrow c_0 = \frac{\dfrac{f(s_2^{\,n_1} - s_1^{\,n_1})}{r+\lambda_0} + k_1 s_2^{\,n_1} + k_2 s_1^{\,n_1} + p(s_1 s_2^{\,n_1} - s_2 s_1^{\,n_1}) + m(s_1^{\,n_1} - s_2^{\,n_1})}{(s_1^{\,n_0} s_2^{\,n_1} - s_2^{\,n_0} s_1^{\,n_1})} \qquad (3.22)$$

Thus equations (3.22) and (3.21) provide an exact formula for c_0 and c_1 for any given estimates to s_1 and s_2.

Equations (3.19) and (3.20) are clearly non-linear in s_1 and s_2. Thus analytic formulae for s_1 and s_2 are not possible. Consequently, generating a solution via an appropriately stable numerical technique is the only way to determine s_1 and s_2. Given the non-linearity of equations (3.19) and (3.20), iterative techniques such as the Gauss-Seidel (see, for example, Spenser et al., 1977a) or the Newton-Raphson methods (see, for example, Spencer et al., 1977b) are appropriate.

3.7 Chapter Summary

In this chapter a new model, termed the RRM, was developed to evaluate renewable resource assets based on the framework of the real options pricing model of Brennan and Schwartz (1985b). The RRM real option values of an asset, which derives its value from an underlying renewable resource, are derived for both operational and decommissioned states of the asset.. By incorporating the relative costs of exercising these managerial options into the boundary conditions of the model, the critical output prices at which it becomes optimal to (1) open a decommissioned asset; (2) decommission an operational asset; and (3) abandon the asset are determined. The closed form solutions were then derived analytically for the RRM.

The implementation of a numerical algorithm to these closed-form solutions of the RRM is presented in the next chapter. This includes a validation of the model and solution methods by comparison with the published results of the Brennan and Schwartz (1985b) case study example. Some theoretical work is also presented which develops a method to determine an exact risk-adjusted discount rate for a renewable resource asset as a function of the spot price under the assumptions of the RRM.

Chapter 4

Solution Implementation and Validation of the RRM

This chapter presents details of the solution implementation and validation of the RRM developed in Chapter 3. That Chapter developed a closed form solution to the RRM analytically. This chapter develops an algorithm and implementation, via a numerical solution, to that closed-form solution. The chapter begins with a review of the numerical techniques that will be used in the solution implementation algorithm. This is followed by the development of the algorithm details. The implementation is then presented and illustrated via some typical parameter values. The correctness of the implementation is validated by comparison to the published finite-inventory copper mine example of Brennan and Schwartz (1985b). Finally, some theoretical work is undertaken to develop a method by which financial practitioners may objectively arrive at an exact risk-adjusted discount rate for a renewable resource asset under the assumptions of the RRM.

4.1 Numerical Methods Utilized

This section discusses the numerical methods used in the implementation of the closed form solution.

4.1.1 Gauss-Seidel iteration

In solving complex dynamic models, such as the RRM, one can end up with a closed form analytic solution which itself comprises a set of non-linear equations. To obtain specific values for the solution in such a case it is necessary to solve the system of non-linear equations numerically. There are a number of such methods for numerically solving systems of non-linear equations. Among the best known methods for numerically solving systems of non-linear equations are the Gauss-Seidel and Newton-Raphson methods.

The Gauss-Seidel iteration method may be written as (Spenser et al., 1977a):

$$x^{(n+1)} = D + Lx^{(n+1)} + Ux^{(n)}$$

where

x^n is the *n*th approximation to the solution

D is the diagonal matrix,

L is the strictly lower triangular matrix, and

U is the strictly upper triangular matrix.

This is a system of equations which are iterated, in sequence, until convergence to an appropriate error level is achieved. The rate of convergence is known to be first order and consequently solution stability is conditional upon the starting values chosen.

The sequence in which the system of equations is calculated is as follows. The first equation determines $x_1^{(n+1)}$ and this new value replaces $x_1^{(n)}$ in the second equation to give a value of $x_2^{(n+1)}$. Subsequently, both values for $x_1^{(n+1)}$ and $x_2^{(n+1)}$ are used in the next iteration until convergence is complete. This continues for all x_i.

In the current application the Gauss-Seidel iteration is implemented in the following scheme to solve (3.19) and (3.20) for s_1 and s_2:

$$c_1 n_1 s_1^{\;n_1 - 1} + p = c_0 n_0 s_1^{\;n_0 - 1}$$

$$\Rightarrow c_1 n_1 s_1^{\;n_1} + p s_1 = c_0 n_0 s_1^{\;n_0}$$

$$\Rightarrow p s_1 = c_0 n_0 s_1^{\;n_0} - c_1 n_1 s_1^{\;n_1}$$

$$\Rightarrow s_1 = \frac{c_0 n_0 s_1^{\;n_0} - c_1 n_1 s_1^{\;n_1}}{p}$$

Similarly for s_2:

$$s_2 = \frac{c_0 n_0 s_2^{\;n_0} - c_1 n_1 s_2^{\;n_1}}{p}$$

This presents a problem as the same iterative sequence has been established. That is, we will have to zero in on one of the solutions, and then continue from a different starting point to move to a different solution.

The above equations, when implemented as described, unfortunately did not yield a stable convergent system. Only for a sparse region of the phase space was convergence achieved. It thus became necessary to consider using a different, higher-order convergence technique. As such the Newton-Raphson method was next investigated.

4.1.2 Newton-Raphson iteration

The second order Newton-Raphson method is a root-finding algorithm that uses the first two terms of the Taylor series of a function $f(x)$ in the vicinity of a root. This technique involves the iterative convergence to locate the zeros of a function, $f(x)$, by stepping through the following sequence (Spencer et al., 1977b):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Thus given an initial approximation to the zero of a function, $f(x)$, the above sequence is applied iteratively to successively improve the approximation. Iteration is terminated upon a desired level of accuracy being reached.

Given the need to find more than one solution of the non-linear equation (3.19)

$$c_1 n_1 s^{\,n_1 - 1} + p = c_0 n_0 s^{\,n_0 - 1}$$

along with the failure of the Gauss-Seidel (a first order technique) implementation, the Newton-Raphson method (being a second order technique) was implemented and tested.

In this case our independent variables that we wish to solve for are s_1 and s_2 via equations (3.19) and (3.20). As these 2 equations are identical (as noted above) we thus wish to find the zeros of the function:

$$f(s) = c_1 n_1 s^{\,n_1 - 1} + p - c_0 n_0 s^{\,n_0 - 1}$$

Differentiating this function with respect to s gives

$$f'(s) = c_1 n_1 (n_1 - 1) s^{n_1 - 2} - c_0 n_0 (n_0 - 1) s^{n_0 - 2}$$

Consequently our iterative Newton-Raphson sequence is given by:

$$s_{n+1} = s_n - \frac{f(s_n)}{f'(s_n)}$$

$$\Rightarrow s_{n+1} = s_n - \frac{c_1 n_1 s^{n_1 - 1} + p - c_0 n_0 s^{n_0 - 1}}{c_1 n_1 (n_1 - 1) s^{n_1 - 2} - c_0 n_0 (n_0 - 1) s^{n_0 - 2}} \qquad (4.01)$$

This second order convergent technique, when implemented, quickly yielded satisfactory results with convergence in all physical cases examined. In particular, by starting at low spot prices it allowed rapid convergence to s_1 and then, by restarting the sequence at high spot prices, rapid convergence to s_2. Details of this are given below.

4.1.3 Convergence of the entire system

The solution to equations (3.17) to (3.20) was achieved by setting up a Gauss-Seidel style iteration around equations (3.21), (3.22) and (4.01) in which, at each step in that sequence, the following calculations were performed:

1. the values of c_0 and c_1 were determined by (3.21) and (3.22)

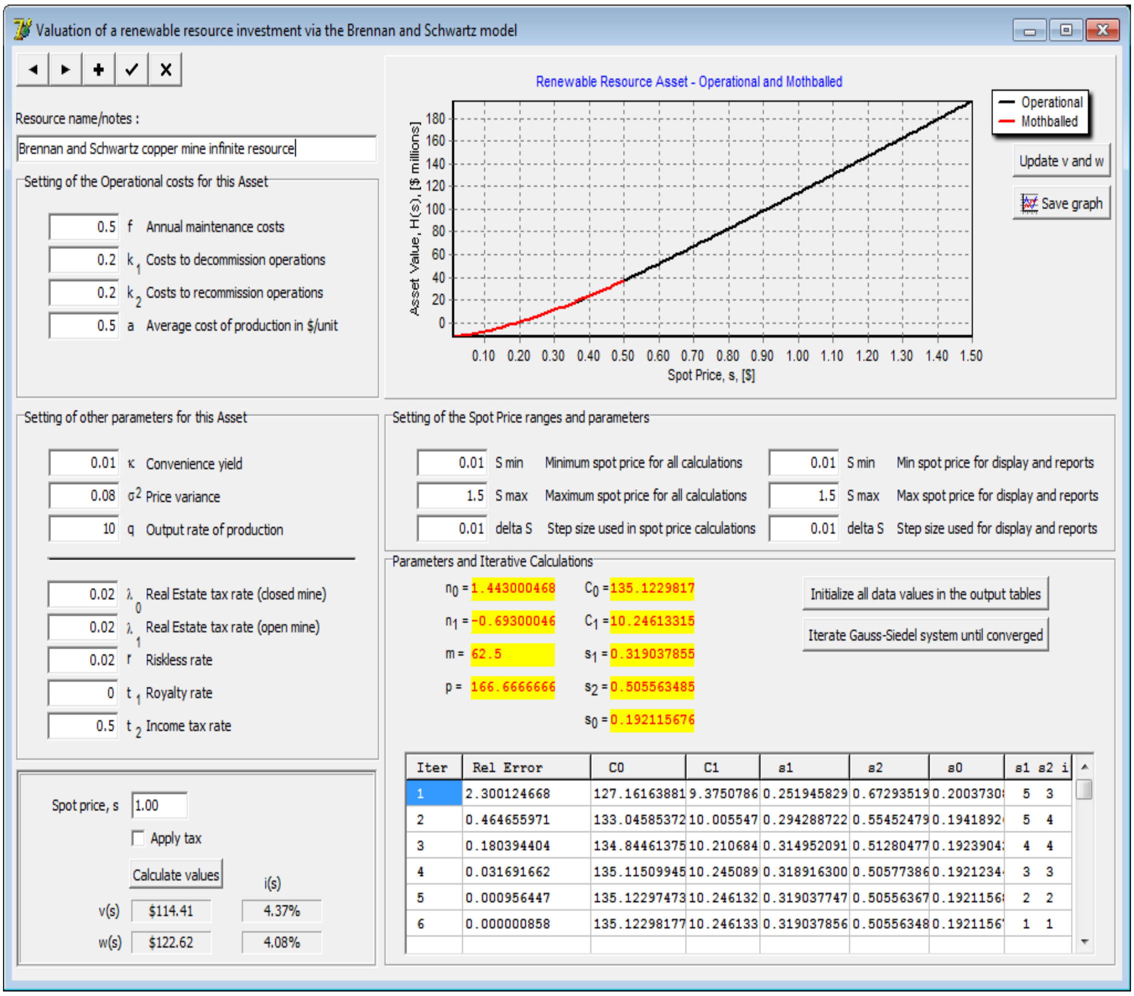
2. these values were then substituted to find s_1 and s_2 via equation (4.01)

The next sections describe how these calculations were implemented with the use of a computer program along with the numerical results obtained for a representative set of data.

4.2 Computer Implementation of Model Solution

The non-linear system of equations (3.17) to (3.20) was implemented in a Delphi program. The source code for the program is given in Appendix B. Figure 4.1 below displays the boot-up screen and explains the overall layout of the implementation.

Figure 4.1 – The RRM Solution Screen for a Renewable Resource Asset

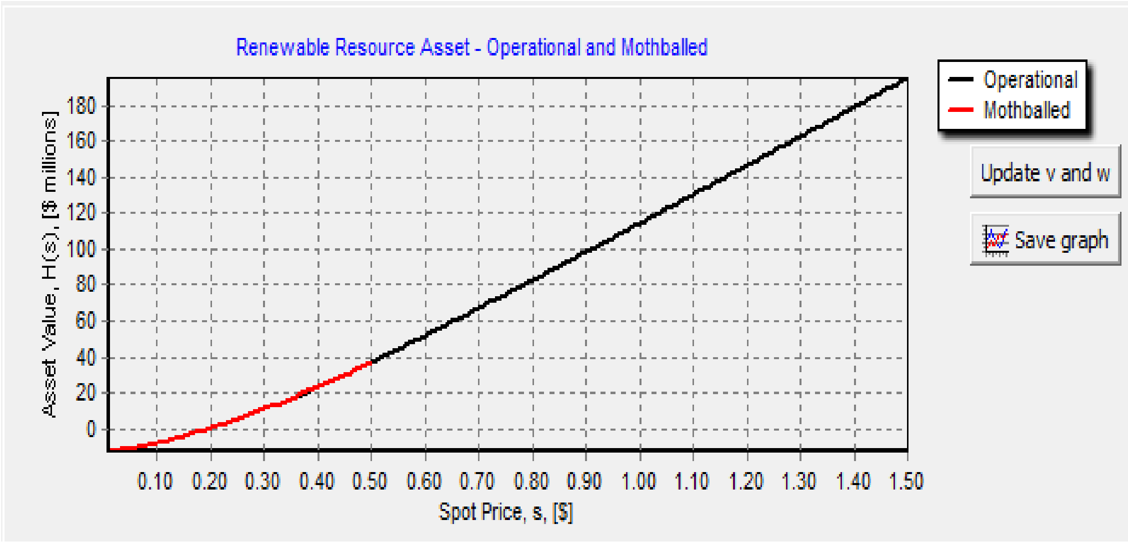


As shown in Figure 4.1, the screen design layout consists of a separation of the input and output parameters in the left- and right-hand sides of the screen respectfully. The data input parameters that are listed on the left-hand side of the screen are grouped into several sections including the operational costs and other input parameters used to evaluate the asset. Once the data parameters have been input the next step is to initialise the data values in the output table by clicking the *initialise all data values in the output tables* button on the right-hand side of the screen. Once these values are set, clicking the *calculate another iteration on these values* button starts the iteration process.

Once the algorithm has been activated the screen is updated as shown at the bottom right hand side of the screen in Figure 4.1. The values n_0, n_1, m and p are exact calculations of the constants that define both the value of the mine in addition to the critical spot prices that determines the optimal management operational policy. In particular, for the algorithm described in the preceding section, it is only necessary to initialise the values s_0, s_1 and s_2. As discussed above, these variables are determined by Gauss-Seidel iteration involving 2 sets of Newton-Raphson solutions in each of those iterations. The program has been set to automatically converge to eight significant digits of accuracy.

Using the published data from Brennan and Schwartz (1985b) copper mine example presented in Figure 4.01 the entire system of equations requires 6 iterations of the Gauss-Seidel loop to achieve 8 significant digits of accuracy in each of the 5 variables being calculated. Once all the relevant parameters have been calculated from the iterative process, activating the *update v and w* operation at the top of the right-hand side of the screen provides a graphical representation of the output functions. The settings of the graphical representation of the phase space can also be adjusted by varying the relevant spot price ranges and the discretised step size. The application also allows the user to zoom into any desired position across the range of the phase space.

Figure 4.2 – Graphical Display of the Results for the Copper Mine Example of Brennan and Schwartz Solved Analytically via the RRM



The phase space in the above figures is defined by the mine value in millions of dollars on the vertical axis and the current spot price in dollars per pound on the horizontal axis. Running the data through the implementation process described above, the model determines the value of that mine (with an inexhaustible inventory) by the following equations:

$$v(s) = 8.9638 s^{-0.693} + \frac{q(1-t_1)(1-t_2)}{(r+\lambda_1)} s - \frac{qa(1-t_2)}{(r+\lambda_1)} \qquad (4.02)$$

and

$$w(s) = 139.749 s^{1.443} - \frac{f}{r+\lambda_0} \qquad (4.03)$$

Finally, the operation on the bottom-left side of the implementation shown in Figure 4.01 calculates the value of the mine in both open and mothballed states at any given spot price. The operation also includes a calculation of the appropriate risk-adjusted discount rate which is applicable for a DCF valuation. The details of this calculation are the subject of Section 4.4.

4.3 Results and Discussion

The results of the calculations for the value of the mine and the critical output prices are presented in Table 4.1 below. By comparison, the results of both cases indicate that the exhaustibility of the mines resource inventory does have a significant impact on the mine value and the operational prices to open and close the mine. As illustrated the values for a renewable (i.e. infinite) resource mine are as high as triple the values for a finite resource asset with an inventory of 150 million pounds whose total life is thus 15 years at an annual extraction rate of 10 million pounds.

Table 4.1 – Results of Implementation and Comparison with Results Published in Brennan and Schwartz (1985b) Example

Copper Price ($/pound)	Value of Open Mine ($millions)	Value of Closed Mine ($millions)	B&S Value of Open Mine ($millions)	B&S Value of Closed Mine ($millions)
0.30	11.10	11.28	1.25	1.45
0.40	23.50	23.52	4.15	4.35
0.50	37.40	37.20	7.95	8.11
0.60	52.10	52.15	12.52	12.49
0.70	67.29	68.26	17.56	17.38
0.80	82.79	85.42	22.88	22.68
0.90	98.52	103.56	28.38	28.18
1.00	114.41	122.62	34.01	33.81
1.25	154.61	173.95	N/A	N/A
1.50	195.24	230.07	N/A	N/A

For the mine with the hypothetically renewed copper inventory, the optimal operational policy is as follows:

1. open the closed mine once the spot price goes above 50.5 cents,

2. shut down an open mine once the spot price drops below 31.9 cents, and,

3. abandon the mine if the spot price of copper drops below 19.2 cents.

Brennan and Schwartz' (1985b) calculations for the critical prices which determine the optimal operation policy for the finite copper mine were as follows: it only becomes optimal to incur the opening costs of the mine when the copper price reaches 76 cents; it is optimal to close down operations and mothball the mine when the copper price falls to 44 cents; and it becomes optimal to abandon the mine when the copper price reaches 20 cents.

Accordingly, extending the natural life of the asset to allow for an infinite extraction of the resource has, in addition to increasing the value of the asset, reduced the spot price at which it becomes optimal to open the mine, and also reduced the critical price at which to decommission operations. It is interesting to note, however, that the results further indicate that whether or not the resource inventory is exhaustible does not impact significantly on the critical price at which to abandon the mine. This finding suggests that in the particular case of this application the critical price at which to abandon the asset does appear to be, for practical purposes, independent of whether the resource is renewable or not.

4.4 The Exact DCF Risk-Adjusted Discount Rate for Renewable Resource Assets

This section examines the application of the DCF method to a renewable resource asset. The DCF approach determines the present value of an asset by discounting its expected cash flows using a risk-adjusted discount rate. Anecdotally, the DCF model is cited as the approach most commonly used to evaluate potential investment decisions in real-world settings (Kelly, 1998). This is not surprising given that the model is familiar to financial practitioners, the modelling is relatively simple to apply (compared to real options modelling) and the results are relatively easy to interpret.

However, well-known shortcomings of the DCF approach include the requirements to forecast future cash flows in addition to the future operating state of the asset at the outset and the subjective manner of assigning appropriate risk-adjusted discount rates of return for these assets. The discount rate used to determine the present value of the future cash flows of a project represents the opportunity cost of investing in a particular project rather than in another with similar risk characteristics (Pindyck & Rubinfeld, 2009). Moreover, assigning an appropriate discount rate requires an ad hoc approach to incorporate the value of any imbedded managerial and strategic options specific to a particular project.

Seminal work in the area of evaluating natural resource assets is attributed to Hotelling (1931) (see, for example, McNamara, 2003). The Hotelling model is a special case of the DCF method adapted for use to value natural resource assets with known and finite reserves. An expression of the Hotelling Valuation Principle model is given by[41]:

$$V_0 = \left[(S_0 - a_0) \sum_{t=0}^{N} q_t \right] (1 - t_c) = [S_0 - a_0) Q_0] (1 - t_c) \qquad (4.04)$$

where

V_0 is the present value of total reserves at time 0.

[41] See McNamara, 2003.

S_0 is the current spot price of the commodity at time 0.

a_0 is the constant average unit cost of extraction in period 0.

q_0 is the number of units extracted in period 0.

Q_0 is the quantity of reserves in inventory.

t_c is the current corporate tax rate.

The key insight of the Hotelling model is the idea that the price of a *nonrenewable* natural resource asset can be expected to rise at a rate which is equal to the opportunity cost of holding an inventory (McTaggart, Findlay & Parkin, 2010). Accordingly, since "the price of an exhaustible natural resource, net of extraction costs, should grow at a rate equal to the rate of interest…this implies the irrelevancy of the interest rate and future commodity prices" (McNamara, 2003:18). However, the obvious difficulties with applying this valuation procedure to renewable resource assets is that the quantity of reserves are, for all practical purposes, infinite.

4.4.1 Calculating the analytic value for the risk-adjusted discount rate as a function of the spot price for an unlevered firm

In Chapter 3 an analytic expression for the present value of a renewable resource based asset was derived under the real options approach. In this section that expression is equated to the traditional DCF model expressed as an infinite series of cash flows. This enables an analytic value for the risk-adjusted rate of return for an unlevered firm for each and every value for the spot price to be determined. That is, the risk adjusted discount rate is analytically expressed as a function of the spot price.

The significance of this work is that an analytic calculation of the appropriate discount rate to be used in DCF valuations can be produced, under general cash flow specifications, which will result in those valuations being functionally equivalent to a complete real options modelling of that valuation. This allows practitioners to be able to explain the real options valuation within the framework of a DCF approach via the specification of a precise risk-adjusted discount rate of return for each value of the spot price in full knowledge that the power of the real options theory has been embedded in their DCF calculations.

Under the assumptions of the RRM derived in Chapter 3, the present value of an asset is calculated as an analytic expression for the value of a perpetual cash flow stream. To enable these valuation models to be compatible it is therefore necessary to express the DCF model in an appropriate form that makes it possible to calculate an exact discount rate for each and every value for the spot price, *s*. Thus, in the case where the present value of an asset is equal to the discounted value of an infinite cash flow stream, the DCF model is expressed as

$$PV = \frac{C_1}{1+i} + \frac{C_2}{(1+i)^2} + \ldots + \frac{C_\infty}{(1+i)^\infty} \qquad (4.05)$$

where

PV is present value,

C_t is the expected incremental net cash flows in period t, and,

i is the appropriate risk-adjusted rate of return.

Given the well-known formula for the sum of a geometric series

$$c + cp + cp^2 + cp^3 + \ldots + cp^n = \frac{c}{p-1}\left(p^{n+1} - 1\right)$$

then, for our application, we need the limit of this series as $n \to \infty$ which results in a value of

$$\sum_{n\to\infty} cp^n = \frac{c}{p-1}\left(0 - 1\right) = \frac{c}{1-p}$$

where $0 \le p < 1$ for convergence to a finite value.

Thus calculating the total value of the cash flow stream is determined by calculating the present value of the cash flows as follows:

$$PV = \left[\sum_{t=1}^{\infty} (s-a)q(1-t_c)\frac{1}{\left(1+i\right)^t}\right]$$

$$= (s-a)q(1-t_c)\sum_{t=1}^{\infty} p^t \text{, where } p = \frac{1}{1+i}$$

$$= (s-a)q(1-t_c)\frac{1}{i} \qquad (4.06)$$

where

s is the current spot price of the commodity.

a is the constant average unit cost of extraction.

q is the constant average number of units extracted.

i is the appropriate risk-adjusted discount rate

t_c is the current corporate tax rate.

We note that equation (4.06) is analogous to valuing an unlevered firm by using the following equation[42]:

$$V_U = \left[EBITDA(1-T_C)\right]/R_U \qquad (4.07)$$

where

V_U is the value of an unlevered firm,

$EBITDA$ is the earnings before interest, tax, depreciation and amortisation,

T_C is the corporate tax rate, and,

R_U is the unlevered cost of capital.

By setting equation (4.06) equal to the analytic formulas derived in Chapter 3, $v(s)$ and $w(s)$, then the exact discount rates for an unlevered firm, i, can be calculated. Hence, for the operating asset $v(s)$, the risk-adjusted discount rates for the cash flow stream becomes:

$$(s-a)q(1-t_c)\frac{1}{i} = v(s)$$

$$i(s) = \frac{(s-a)q(1-t_c)}{v(s)} \qquad (4.07)$$

For the decommissioned asset $w(s)$

[42] See Ross et al., 2007.

$$(s-a)q(1-t_c)\frac{1}{i}=w(s)$$

$$i(s)=\frac{(s-a)q(1-t_c)}{w(s)} \qquad (4.08)$$

4.5 Chapter Summary

This chapter developed a numerical algorithm for the closed form solution to the RRM derived in Chapter 3. The implementation of that algorithm was described and the correctness of the implementation tested by direct comparison against the published work of Brennan and Schwartz (1985b). The results obtained demonstrated the correctness of the implementation by validating our expectations that the renewable resource asset would have a higher value at all spot price intervals compared to a finite-resource asset, all other things equal. An interesting finding is that optimal operational policies involving when to close and open the finite resource asset operate in a higher and wider range of spot prices, however, the prices at which it is optimal to abandon the asset in both cases are not significantly different.

In this chapter some theoretical work was also undertaken to develop a method by which financial practitioners may objectively arrive at an exact unlevered risk-adjusted discount rate for a renewable resource asset under the assumptions of the RRM. The closed form solutions to the RRM were equated to a DCF model with perpetual cash flows. The resulting equation was then solved producing an analytic formula which calculates the exact risk-adjusted rate of return, for a renewable resource model, as a function of the spot price. The implications of this ability to ***exactly*** specify, within the real options framework, a precise discount rate for use in the classical DCF model was presented and discussed.

The significance of this work is that it incorporates a risk-neutral valuation using the real options approach into the DCF framework. Thus the advantage in using this method is that it overcomes several key difficulties associated with using the conventional DCF approach. As such, it obviates the need to estimate future cash flows and assign an appropriate discount rate to determine the present value of an asset's cash flows due to the risk associated with output price volatility being explicitly accounted for in the real options valuation. In addition, it allows for the value of real options to be explicitly incorporated into this evaluation.

In the next chapter a case study is presented which applies all of the work in this research to a set of hypothetical corn farms in the United States (U.S.) Corn Belt. The value of the corn crop is evaluated as well as the unlevered risk adjusted discount rate for an equivalent DCF model.

Chapter 5

Evaluating Corn Crop Investments Under Uncertainty in the U.S. Corn Belt: A Case Study Application of the RRM

5.1 Introduction and Background

This chapter addresses the second research question of the study: it presents a case study application of the RRM developed in Chapter 3 and demonstrates the feasibility and suitability of this model to the valuation of renewable resource investments. The RRM is applied to price cropland assets across the U.S. Corn Belt as well as to outline the appropriate operational policy that farmers should follow to manage these assets optimally. Whereas some previous related studies have examined entry and exit decisions regarding new investments in agricultural assets (see, for example, Tauer (2006)), this research evaluates the managerial options available to a particular perennial agricultural operation whereby the initial decision to invest has already been made.

The applicability of the RRM to evaluating investments in agricultural croplands is initially dependent on establishing to what extent the nature of the underlying resource inventory is renewable. It is important to note here that, essentially, the RRM evaluates an asset on the basis that any exploitation of its current inventory does not impact on inventory levels in future periods, which is the case for natural resource assets with a finite inventory. Conceptually, this is the rationale behind excluding the term $-qv_Q$

from the original Brennan and Schwartz (1985b) model as current levels of extraction are assumed to not affect future levels of inventory and thus not impact on the overall value of the cropland asset. This implies that the inventory may be exploited at a constant rate of extraction on an annual basis indefinitely (i.e. over an infinite economic horizon) and that the optimal rate of extraction is at full capacity. A suitable application of the RRM, therefore, does not necessarily require that the underlying resource be replenished on an annual basis due to natural processes (which ideally could be considered infinite resources), it is equally applicable to cases where levels of a physical resource inventory are maintained annually (i.e. replanting the croplands) by the managers of the asset.

As well as making the RRM accessible to financial practitioners, we also offer farmers an approach to evaluate investments in agricultural assets that is superior in many respects to using a traditional DCF methodology. While traditional DCF method remains the most widely-used valuation model in real-world practice[43] there are well-known difficulties associated with incorporating the effects of future output price volatility into the model due to the need to assume future operating conditions from the outset. While the RRM recognises the irreversibility of agricultural investments it provides an objective means to determine the optimal operational policy at any given spot price. This has practical implications which may assist farmers to make optimal decisions regarding whether or not to plant their field crops for the current season and at which price it is worth considering alternative uses for their crop land such as rotating

[43] Chapter 2 discussed in detail the well-known difficulties associated with applying a DCF analysis to evaluate investment decisions under uncertainty.

their crops or converting their land for other industrial uses. These managerial decisions have significant implications for valuing agricultural assets that are difficult to incorporate into DCF valuations.

For instance, following the Brennan and Schwartz (1985b) example of a copper mine discussed in Chapter 4, should the price of copper fall to a sufficiently low level (the critical price S1), then the mine owner should cease operating the mine and incur the costs of closure and annual maintenance on the mine. A farmer faces a similar dilemma as the resource miner regarding their decisions to choose the appropriate operational policy that optimises the value of their investment. Each year farmers must consider whether or not the current spot price is appropriate to warrant exercising the option to outlay the costs to plant their fields for the coming season. Thus for this case study the option associated with the price S1 is interpreted as one where the farmer should forego planting for the current season whilst maintaining the crop land until prices improve and reach the critical price S2. At the critical price S2, it is appropriate to outlay the costs of production to plant the crop. If, however, the option to delay planting has already been exercised, and the price should continue to fall further to the critical price S0, then the farmer should allow the land to lay fallow whilst considering alternative uses for their cropland.

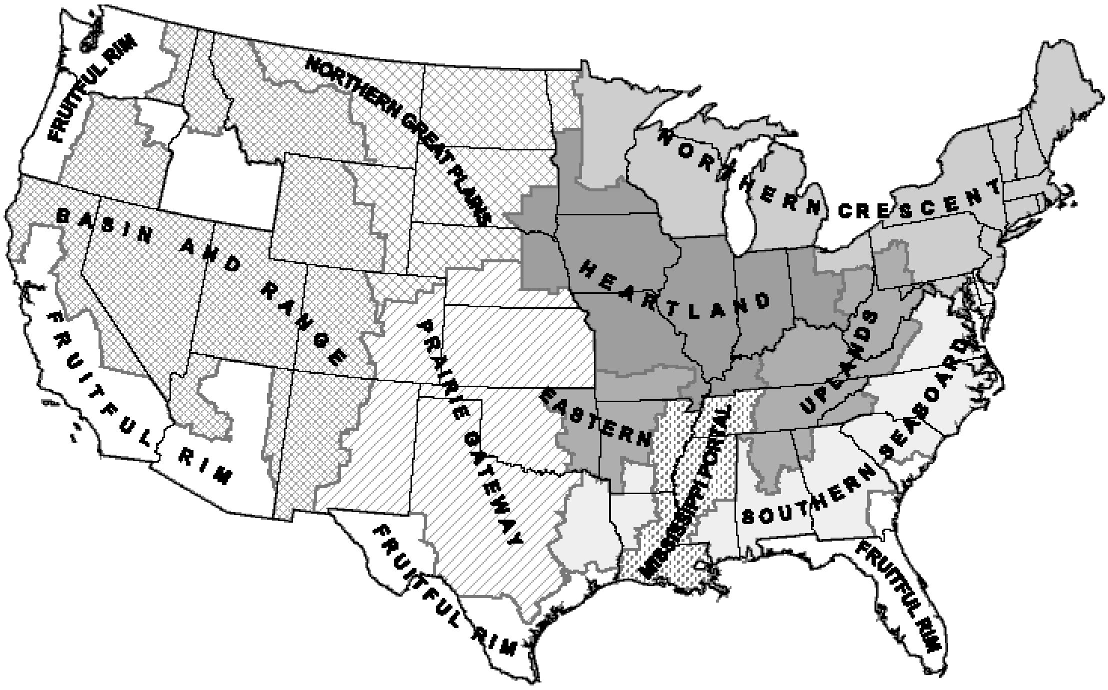
The remainder of the chapter is organised as follows. Section 5.2 introduces the case study. The various model parameters to be used in the study across the sample Corn Belt states are calculated in Section 5.3 which includes some discussion on the relevant assumptions of the RRM. The results of applying the RRM is then presented in Section 5.4. Section 5.5 reports empirical results for the calculations of the critical corn prices at which it becomes appropriate to exercise various real options over these cropland assets. Section 5.6 concludes the chapter providing a summary of the main findings.

5.2 The Setting for the Case Study

Corn is selected as a suitable agricultural commodity to illustrate the application of the model for several reasons. First, corn is the largest component of the world's coarse grain trade and is therefore an economically important soft commodity that is cultivated for a variety of purposes including food, feed and industrial uses (United States Department of Agriculture (USDA), 2011a). Second, since corn is a perennial crop (as such its inventory may be harvested and subsequently replenished on an annual basis), and has well-developed markets, it is deemed appropriate to fit with various assumptions underpinning the model's applicability to a real-world setting. Moreover, if the tenancy of the farm is free from constraints imposed by any specific lease period the investment horizon can be assumed to be infinite (thus satisfying a key assumption of the model). Third, a real options analysis is appropriate for assets such as perennial crops because the farmer holds various operational options over these assets which, as previously discussed, may be exercised at the farmer's discretion.

Since it is well-known that the U.S. is the world's top corn producer, the setting chosen for the case study is the U.S. Corn Belt. Corn is a primary feed grain grown in most states in the U.S. with the planting season typically occurring between early April and June, while the harvesting period lasts from October to December. Corn production is concentrated in the Heartland Region which consists of Illinois, Iowa, Indiana, eastern portions of South Dakota and Nebraska, western Kentucky and Ohio, and the northern two-thirds of Missouri (see Foreman, 2001). Figure 5.1 below displays the geographic location of the Heartland Region among the various U.S. farm resource regions.

Figure 5.1: U.S. Farm Resource Regions



Source: Foreman, 2001.

Accordingly, the focus of this case study application of the RRM is to evaluate hypothetical investments in corn cropland assets across the Heartland states of Illinois, Indiana, Iowa, Missouri and Ohio. Valuations are calculated based on cropland consisting of 250 and 1000 acres for which the total area is assumed amenable to crop production[44]. Parameters used in the study are average values calculated using data sourced from various U.S. Government agencies over the 10-year period 2000 to 2009.

[44] Foreman (2001) notes that farms with a corn acreage of 250 acres or less comprise 75% of all U.S. corn farms.

5.3 Estimating the Data Parameters for the Case Study

The parameters used in this study are estimated using USDA census data. The USDA collects and collates farm data across a broad range of commodities by conducting producer surveys about every 4-8 years. The survey involves field enumerators conducting personal interviews with farmers to gather detailed information about input use, field operations and production costs of the particular commodity. (For a summary of methods and procedures used by USDA to compile the costs and return data used in this study see USDA, 2010a).

5.3.1 Estimating the historical commodity price variance of average U.S. corn prices

The model requires the variance of the corn price to be calculated. A key assumption of the RRM is that the prices of the underlying commodity follow a random walk and are log-normally distributed (see Appendix A.1). Thus, using monthly U.S. corn prices data[45] from January 2000 to December 2009, the statistic $d_t = \ln(\frac{s_{t+1}}{s_t})$ is calculated resulting in a mean of 0.00568, variance of 0.00298 and standard deviation of 0.0546. Annualising these by multiplying by 12 gives an annually adjusted mean (d_t) of 0.0682 and variance of 0.03577, hence a standard deviation of 0.18914.

[45] Average prices received by farmers, see USDA (2011b). All price data used in this case study is tabled in Appendix C.1.

Whether the long-term average price of corn is stationary was tested using a Dickey-Fuller test[46]. Using USDA data for average monthly corn prices received by U.S. farmers over the period January 1975 to December 2010 yields 420 observations. The null hypothesis that the corn price is stationary could not be rejected at the 95% confidence level (see Appendix C for the details of this analysis).

5.3.2 Annual production rates and operating costs for Corn Belt states in the Heartland Region

The model requires estimates for the average rates of production and the associated costs per bushel of corn. The USDA publishes a detailed breakdown of cost data consisting of both the variable and fixed cost components of corn farm production compiled on a regionalised basis. The latest estimates for average total economic costs reported by the USDA for the Heartland Region are for the year 2009, which is $555.15 per planted acre (USDA 2011c). The average production costs for each of the states is calculated by dividing the total costs by the average yields[47] reported for each state over the decade 2000 to 2009 (USDA 2011d). Calculations for the annual rates of production and the resulting production costs per bushel of corn are presented below in Table 5.01.

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[46] This entails regressing the first differences with the observations of the time series in sequence.

[47] Average yields are used in the study to account for seasonal output variations due to prevailing weather conditions.

Table 5.01: US Corn Belt Average Annual Harvest Yields

	Average Yield by State* (bu/planted acre)				
Year	Illinois	Indiana	Iowa	Missouri	Ohio
2000	151	147	145	143	147
2001	150	160	147	136	144
2002	136	121	165	105	88
2003	169	150	159	109	156
2004	180	168	181	162	158
2005	145	151	175	108	143
2006	165	159	163	142	161
2007	175	155	171	142	150
2008	179	160	172	140	140
2009	175	166	183	151	166
Average	162.5	153.7	166.1	133.8	145.3
Production Cost** ($/bu)	$3.42	$3.61	$3.34	$4.15	$3.82

** United States Department of Agriculture (2010d)*

*** Assuming total costs of $555.15 per planted acre (USDA, 2010c), Average cost = average total economic costs per planted acre / average yield per planted acre*

The results of these calculations show that farmers with corn crops located in the Heartland states of Iowa and Illinois are relatively low-cost producers, while farmers in Missouri have relatively high costs of production. It is not surprising therefore that farms located in the states of Iowa and Illinois typically produce in excess of one-third of the total U.S corn crop (USDA, 2011a).

5.3.3 Estimating the real interest rate and convenience yield for corn

Since the actual risk-free rate is not readily observable (and thus known) the standard approach in the literature is to use the yield on U.S. Treasury securities as a proxy for the risk-free rate. The 10-year U.S. Treasury bond rate was chosen for use in this study as the nominal risk-free rate and monthly interest rate data was obtained from the Board of Governors of the Federal Reserve System for the period 2000 to 2009. These interest rates were matched with monthly changes in the U.S. Consumer Price Index (CPI) data that was sourced from the U.S. Department of Labor. Calculations of the real interest rate for each month resulted in an average annualised rate of 2.39% over the period[48]. Details of these calculations are presented below in Table 5.02.

[48] The real interest rate is calculated as $r = ((1+i) / (1+\pi)) - 1$. Paddock et al., 1988 note that a common estimate of the (pre-tax) annual real riskless rate is 2% and, furthermore, using real estimates gives more plausibility to the assumption of a constant risk-free rate.

Table 5.02: Calculation for Average Real Interest Rate 2000-2009.

Year-Month	US CPI* (all items)	Inflation (annualised)	US 10 Year Bond Rate**	Real Interest Rate***	Year-Month	US CPI* (all items)	Inflation (annualised)	US 10 Year Bond Rate**	Real Interest Rate***
1999-12	504.1				2005-01	571.2	0.023153833	0.0422	0.018615155
2000-01	505.8	0.040468161	0.0666	0.025115462	2005-02	574.5	0.069327731	0.0417	-0.025836542
2000-02	508.7	0.068801898	0.0652	-0.00337003	2005-03	579	0.093994778	0.045	-0.044785203
2000-03	512.8	0.096717122	0.0626	-0.03110841	2005-04	582.9	0.080829016	0.0434	-0.034629914
2000-04	513.2	0.009360374	0.0599	0.050070943	2005-05	582.4	-0.010293361	0.0414	0.052230993
2000-05	513.6	0.009353079	0.0644	0.054536834	2005-06	582.6	0.004120879	0.04	0.035731874
2000-06	516.5	0.067757009	0.061	-0.00632823	2005-07	585.2	0.053553038	0.0418	-0.011155621
2000-07	517.5	0.023233301	0.0605	0.03642053	2005-08	588.2	0.06151743	0.0426	-0.01782112
2000-08	517.6	0.002318841	0.0583	0.055851648	2005-09	595.4	0.146888813	0.042	-0.091455084
2000-09	520.3	0.0625966	0.058	-0.00432582	2005-10	596.7	0.026200873	0.0446	0.017929362
2000-10	521.2	0.020757255	0.0574	0.035897609	2005-11	592	-0.094519859	0.0454	0.154525597
2000-11	521.5	0.006907137	0.0572	0.049947866	2005-12	589.4	-0.052702703	0.0447	0.102821683
2000-12	521.1	-0.00920422	0.0524	0.062176505	2006-01	593.9	0.091618595	0.0442	-0.043438794
2001-01	524.5	0.078295912	0.0516	-0.0247575	2006-02	595.2	0.026267048	0.0457	0.01893557
2001-02	526.7	0.050333651	0.051	0.000634416	2006-03	598.6	0.068548387	0.0472	-0.019978868
2001-03	528	0.029618379	0.0489	0.018726959	2006-04	603.5	0.098229201	0.0499	-0.04400648
2001-04	529.9	0.043181818	0.0514	0.007877996	2006-05	606.5	0.05965203	0.0511	-0.008070602
2001-05	532.2	0.052085299	0.0539	0.001724861	2006-06	607.8	0.025721352	0.0511	0.024742244
2001-06	533.3	0.024802706	0.0528	0.027319692	2006-07	609.6	0.035538006	0.0509	0.014834795
2001-07	531.6	-0.03825239	0.0524	0.094257984	2006-08	610.9	0.025590551	0.0488	0.022630326
2001-08	531.8	0.004514673	0.0497	0.044982247	2006-09	607.9	-0.058929448	0.0472	0.11277523
2001-09	534	0.049642723	0.0473	-0.00223192	2006-10	604.6	-0.065142293	0.0473	0.120277441
2001-10	532.2	-0.04044944	0.0457	0.08978103	2006-11	603.6	-0.019847833	0.046	0.067181235
2001-11	531.3	-0.02029312	0.0465	0.06817664	2006-12	604.5	0.017892644	0.0456	0.027220313
2001-12	529.2	-0.04743083	0.0509	0.103226971	2007-01	606.348	0.036684864	0.0476	0.010528886
2002-01	530.6	0.031746032	0.0504	0.01808	2007-02	609.594	0.064240337	0.0472	-0.016011738
2002-02	532.7	0.047493404	0.0491	0.001533753	2007-03	615.145	0.109272729	0.0456	-0.057400428
2002-03	535.5	0.063074901	0.0528	-0.00966527	2007-04	619.14	0.077932845	0.0469	-0.028789219
2002-04	538.6	0.069467787	0.0521	-0.01623965	2007-05	622.921	0.073282295	0.0475	-0.024021914
2002-05	538.5	-0.002228	0.0516	0.053948195	2007-06	624.129	0.023271009	0.051	0.027098385
2002-06	538.9	0.008913649	0.0493	0.040029542	2007-07	623.97	-0.00305706	0.05	0.053219756
2002-07	539.5	0.013360549	0.0465	0.032702527	2007-08	622.827	-0.021981826	0.0467	0.070225511
2002-08	541.2	0.03781279	0.0426	0.004612788	2007-09	624.543	0.03306215	0.0452	0.011749389
2002-09	542.1	0.019955654	0.0387	0.018377609	2007-10	625.879	0.02566997	0.0453	0.019138739
2002-10	543.2	0.024349751	0.0394	0.014692491	2007-11	629.598	0.071304517	0.0415	-0.027820771
2002-11	543.1	-0.00220913	0.0405	0.04280369	2007-12	629.174	-0.008081347	0.041	0.049481222
2002-12	541.9	-0.02651445	0.0403	0.068634254	2008-01	632.301	0.0596401	0.0374	-0.020988352
2003-01	544.2	0.050931906	0.0405	-0.00992634	2008-02	634.139	0.034882121	0.0374	0.00243301
2003-02	548.5	0.094818082	0.039	-0.05098389	2008-03	639.636	0.104021358	0.0351	-0.062427559
2003-03	551.8	0.072196901	0.0381	-0.03180097	2008-04	643.515	0.072772639	0.0368	-0.033532398
2003-04	550.5	-0.02827111	0.0396	0.069845729	2008-05	648.933	0.10103261	0.0388	-0.056522041
2003-05	549.7	-0.01743869	0.0357	0.054081808	2008-06	655.474	0.120955476	0.041	-0.071327968
2003-06	550.4	0.015281062	0.0333	0.017747733	2008-07	658.915	0.062995634	0.0401	-0.021538784
2003-07	550.9	0.010901163	0.0398	0.028587203	2008-08	656.284	-0.047915133	0.0389	0.091184238
2003-08	553	0.045743329	0.0445	-0.00118894	2008-09	655.376	-0.016602568	0.0369	0.054405845
2003-09	554.7	0.036889693	0.0427	0.005603593	2008-10	648.758	-0.121176241	0.0381	0.181237978
2003-10	554.3	-0.00865333	0.0429	0.052003328	2008-11	636.332	-0.229842252	0.0353	0.344270058
2003-11	552.7	-0.03463828	0.043	0.080424033	2008-12	629.751	-0.124105027	0.0242	0.169318276
2003-12	552.1	-0.01302696	0.0427	0.056462493	2009-01	632.491	0.052211112	0.0252	-0.02567081
2004-01	554.9	0.06085854	0.0415	-0.01824799	2009-02	635.637	0.059687806	0.0287	-0.029242392
2004-02	557.9	0.064876554	0.0408	-0.02260971	2009-03	637.182	0.029167591	0.0282	-0.000940168
2004-03	561.5	0.077433232	0.0383	-0.0363208	2009-04	638.771	0.029925516	0.0293	-0.000607341
2004-04	563.2	0.036331256	0.0435	0.006917426	2009-05	640.616	0.034660309	0.0329	-0.00170134
2004-05	566.4	0.068181818	0.0472	-0.01964255	2009-06	646.121	0.103119497	0.0372	-0.05975735
2004-06	568.2	0.038135593	0.0473	0.008827755	2009-07	645.096	-0.019036682	0.0356	0.055696967
2004-07	567.5	-0.01478353	0.045	0.0606806	2009-08	646.544	0.026935526	0.0359	0.008729345
2004-08	567.6	0.002114537	0.0428	0.040599613	2009-09	646.948	0.00749833	0.034	0.026304431
2004-09	568.7	0.023255814	0.0413	0.017634091	2009-10	647.57	0.011537249	0.0339	0.022107689
2004-10	571.9	0.06752242	0.041	-0.02484484	2009-11	648.028	0.008487113	0.034	0.025298178
2004-11	572.2	0.006294807	0.0419	0.035382467	2009-12	646.887	-0.021128717	0.0359	0.058259669
2004-12	570.1	-0.04404055	0.0423	0.090318208	Average	N/A	2.47%	4.46%	2.39%

** United States Department of Labor (2011)*

*** Board of Governors of the Federal Reserve System (2011)*

*** Calculated as $r = ((1+i)/(1+\pi))-1$

A key assumption of the model is that the future price of a commodity is a function of the spot price and time to maturity. Because the model assumes the net convenience yield[49] to be proportional to the price it may be estimated directly from the following relationship:

$$F(T) = S(t)e^{(r-\kappa)(T-t)}$$

where *F(T)* is the futures price, *S(t)* is the spot price, *r* is the nominal interest rate and κ is the convenience yield. Thus monthly convenience yields are calculated[50] and annualised following the approach adopted by Gibson and Schwartz (1990):

$$\kappa = r - \frac{1}{\Delta t} * \ln\left[\frac{F(T)}{S(t)}\right]$$

where Δt is equal to 1/12.

The average annualised convenience yield over the period January 2000 to December 2009 was calculated to be 4.39%. The results of these calculations are tabled in Table 5.03 below.

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[49] The convenience yield is defined as the benefit that accrues to the owner of an asset, but not to the holder of a futures contract over the underlying commodity (Brennan and Schwartz, 1985b). Section 2.7.1 provides a detailed description of the convenience yield.

[50] The log changes in the corn price were calculated using average monthly prices received by U.S. farmers published by the USDA.

Table 5.03: Calculation for Average Convenience Yield (Corn) 2000-2009.

Year-Month	Average US Corn Price* $/bushel	$\ln(S_1/S_0)$	10yr US Bond Rate** % per year	Convenience Yield*** % (annualised)	Year-Month	Average US Corn Price* $/bushel	$\ln(S_1/S_0)$	10yr US Bond Rate** % per year	Convenience Yield*** % (annualised)
1999-12	1.82				2005-01	2.12	0.0385	4.22	3.76
2000-01	1.91	0.0483	6.66	6.08	2005-02	1.95	-0.0836	4.17	5.17
2000-02	1.98	0.0360	6.52	6.09	2005-03	2.02	0.0353	4.5	4.08
2000-03	2.03	0.0249	6.26	5.96	2005-04	2	-0.0100	4.34	4.46
2000-04	2.03	0.0000	5.99	5.99	2005-05	1.98	-0.0101	4.14	4.26
2000-05	2.11	0.0387	6.44	5.98	2005-06	2.03	0.0249	4	3.70
2000-06	1.91	-0.0996	6.1	7.30	2005-07	2.11	0.0387	4.18	3.72
2000-07	1.64	-0.1524	6.05	7.88	2005-08	1.95	-0.0789	4.26	5.21
2000-08	1.52	-0.0760	5.83	6.74	2005-09	1.9	-0.0260	4.2	4.51
2000-09	1.61	0.0575	5.8	5.11	2005-10	1.82	-0.0430	4.46	4.98
2000-10	1.74	0.0777	5.74	4.81	2005-11	1.77	-0.0279	4.54	4.87
2000-11	1.86	0.0667	5.72	4.92	2005-12	1.92	0.0813	4.47	3.49
2000-12	1.97	0.0575	5.24	4.55	2006-01	2	0.0408	4.42	3.93
2001-01	1.98	0.0051	5.16	5.10	2006-02	2.02	0.0100	4.57	4.45
2001-02	1.96	-0.0102	5.1	5.22	2006-03	2.06	0.0196	4.72	4.48
2001-03	1.96	0.0000	4.89	4.89	2006-04	2.11	0.0240	4.99	4.70
2001-04	1.89	-0.0364	5.14	5.58	2006-05	2.17	0.0280	5.11	4.77
2001-05	1.82	-0.0377	5.39	5.84	2006-06	2.14	-0.0139	5.11	5.28
2001-06	1.76	-0.0335	5.28	5.68	2006-07	2.14	0.0000	5.09	5.09
2001-07	1.87	0.0606	5.24	4.51	2006-08	2.09	-0.0236	4.88	5.16
2001-08	1.9	0.0159	4.97	4.78	2006-09	2.2	0.0513	4.72	4.10
2001-09	1.91	0.0052	4.73	4.67	2006-10	2.55	0.1476	4.73	2.96
2001-10	1.84	-0.0373	4.57	5.02	2006-11	2.88	0.1217	4.6	3.14
2001-11	1.85	0.0054	4.65	4.58	2006-12	3.01	0.0441	4.56	4.03
2001-12	1.98	0.0679	5.09	4.28	2007-01	3.05	0.0132	4.76	4.60
2002-01	1.97	-0.0051	5.04	5.10	2007-02	3.44	0.1203	4.72	3.28
2002-02	1.93	-0.0205	4.91	5.16	2007-03	3.43	-0.0029	4.56	4.59
2002-03	1.94	0.0052	5.28	5.22	2007-04	3.39	-0.0117	4.69	4.83
2002-04	1.91	-0.0156	5.21	5.40	2007-05	3.49	0.0291	4.75	4.40
2002-05	1.93	0.0104	5.16	5.03	2007-06	3.53	0.0114	5.1	4.96
2002-06	1.97	0.0205	4.93	4.68	2007-07	3.32	-0.0613	5	5.74
2002-07	2.13	0.0781	4.65	3.71	2007-08	3.26	-0.0182	4.67	4.89
2002-08	2.38	0.1110	4.26	2.93	2007-09	3.28	0.0061	4.52	4.45
2002-09	2.47	0.0371	3.87	3.42	2007-10	3.29	0.0030	4.53	4.49
2002-10	2.34	-0.0541	3.94	4.59	2007-11	3.44	0.0446	4.15	3.61
2002-11	2.28	-0.0260	4.05	4.36	2007-12	3.77	0.0916	4.1	3.00
2002-12	2.32	0.0174	4.03	3.82	2008-01	3.98	0.0542	3.74	3.09
2003-01	2.33	0.0043	4.05	4.00	2008-02	4.54	0.1316	3.74	2.16
2003-02	2.34	0.0043	3.9	3.85	2008-03	4.7	0.0346	3.51	3.09
2003-03	2.33	-0.0043	3.81	3.86	2008-04	5.14	0.0895	3.68	2.61
2003-04	2.34	0.0043	3.96	3.91	2008-05	5.27	0.0250	3.88	3.58
2003-05	2.38	0.0169	3.57	3.37	2008-06	5.47	0.0372	4.1	3.65
2003-06	2.34	-0.0169	3.33	3.53	2008-07	5.25	-0.0411	4.01	4.50
2003-07	2.17	-0.0754	3.98	4.89	2008-08	5.26	0.0019	3.89	3.87
2003-08	2.15	-0.0093	4.45	4.56	2008-09	5.01	-0.0487	3.69	4.27
2003-09	2.2	0.0230	4.27	3.99	2008-10	4.37	-0.1367	3.81	5.45
2003-10	2.12	-0.0370	4.29	4.73	2008-11	4.26	-0.0255	3.53	3.84
2003-11	2.2	0.0370	4.3	3.86	2008-12	4.11	-0.0358	2.42	2.85
2003-12	2.31	0.0488	4.27	3.68	2009-01	4.36	0.0590	2.52	1.81
2004-01	2.39	0.0340	4.15	3.74	2009-02	3.87	-0.1192	2.87	4.30
2004-02	2.61	0.0881	4.08	3.02	2009-03	3.85	-0.0052	2.82	2.88
2004-03	2.75	0.0523	3.83	3.20	2009-04	3.85	0.0000	2.93	2.93
2004-04	2.89	0.0497	4.35	3.75	2009-05	3.96	0.0282	3.29	2.95
2004-05	2.87	-0.0069	4.72	4.80	2009-06	4.01	0.0125	3.72	3.57
2004-06	2.79	-0.0283	4.73	5.07	2009-07	3.6	-0.1079	3.56	4.85
2004-07	2.51	-0.1058	4.5	5.77	2009-08	3.33	-0.0780	3.59	4.53
2004-08	2.34	-0.0701	4.28	5.12	2009-09	3.25	-0.0243	3.4	3.69
2004-09	2.2	-0.0617	4.13	4.87	2009-10	3.61	0.1051	3.39	2.13
2004-10	2.14	-0.0277	4.1	4.43	2009-11	3.65	0.0110	3.4	3.27
2004-11	2.05	-0.0430	4.19	4.71	2009-12	3.6	-0.0138	3.59	3.76
2004-12	2.04	-0.0049	4.23	4.29	**Average**	**$2.68**	**0.0057**	**4.46%**	**4.39%**

** United States Department of Agriculture (2011b)*

*** Board of Governors of the Federal Reserve System (2011)*

**** Calculated as cy=r-12*ln(s1/s0), see Gibson & Schwartz (1990)*

5.3.4 *Estimates for federal income taxes, state property taxes and the royalty rate*

The USDA Economic Research Service classifies farms as rural residence (lifestyle, residence and limited-resource farms), intermediate farms (annual sales less than $250,000 and primary occupation is farming), and commercial farms (annual sales greater than $250,000). According to Durst (2009), 99% of U.S. farm income is taxed at the individual tax rate rather than the corporate tax rate since most farms are operated as sole proprietors, partnerships, or small business corporations. For this study we apply a tax rate of 15% which is reported by the USDA as the average effective Federal income tax rate for 2004. The USDA notes that 2004 is the latest year for which complete tax data is available for farmers (see USDA, 2010b).

State property taxes in the U.S. vary across states. The average state property tax rates for each state is taken from a 2007 article in *The New York Times*: Illinois (1.79%), Indiana (2.12%), Iowa (2.15%), Missouri (1.42%) and Ohio (1.81%)[51]. Royalty rates do not apply and are set to zero percent. Furthermore, for simplicity we have ignored the various levels of government subsidies that apply to farming.

5.3.5 *Maintenance costs, costs of closure and costs of reinstating operations*

Costs for cropland that is currently being maintained as non-operational are estimated using only those recurring annual costs of the farm published by the USDA for the Heartland Region. Included in these cost estimates are non-real estate taxes and

[51] See The New York Times, 2007.

insurance and general farm overhead which are reported by the USDA to total $24 per planted acre for 2009.

Costs associated with exercising the option to temporarily delay operations, k_1, may include the costs of storing machinery (and silage if appropriate), paying out relevant redundancies and termination payments on current contracts, as well as any outstanding general liabilities. While the costs to exercise the option to reinstate operations, k_2, may involve hiring and training costs and other miscellaneous recommissioning costs due on the farm. For this study we assume the costs of transitioning between each operating state to be $10,000 each[52]. Costs of abandoning the crop to let the land lie fallow are assumed to be zero.

Transition costs may be set higher for larger sized enterprises than for small sized enterprises if the larger scale operations are considered to incur higher costs associated with greater labour requirements and additional machinery and inventory to store. We note here that sensitivity tests performed[53] by varying these costs show the option value for cropland consisting of 250 acres to be relatively more sensitive to changes in operational transition costs than is the value for 1000 acres[54]. However, results further show that the increasing the transition costs decreases the critical prices at which it becomes appropriate to exercise the options to temporarily cease and resume operations.

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[52] The value of $10,000 is an estimate made due to a lack of hard data.

[53] These tests were performed using parameters for Illinois.

[54] For instance, doubling the transition costs to $20,000 for the 1000 acre crop only results in a 3% decrease in cropland value.

It is noted that if it is costless to transition between operating states then the prices at which it becomes optimal to exercise these options approach the same value. Furthermore, if no maintenance costs are incurred (explicit or implicit) when delaying production and keeping the cropland idle, these critical prices are equal to the average cost of production.

Having now calculated all of the required parameters for use in the model, a summary of the parameters are detailed in the following Table 6.4.

Table 5.04: Data Parameters Summary

Parameter		Value
Real Interest Rate		2.39% per year
Annual Volatility*		18.91% per year
Average Convenience Yield		4.39% per year
Average Annual Yield		
	Illinois	162.5 bu/planted acre
	Indiana	153.7 bu/planted acre
	Iowa	166.1 bu/planted acre
	Missouri	133.8 bu/planted acre
	Ohio	145.3 bu/planted acre
Average Production Cost		
	Illinois	$3.42 per bushel
	Indiana	$3.61 per bushel
	Iowa	$3.24 per bushel
	Missouri	$4.15 per bushel
	Ohio	$3.82 per bushel
Maintenance Cost		$24 per planted acre
Cost to Temporarily Close		$10,000
Cost to Reinstate Operations		$10,000
Federal Income Tax Rate		15%
State Propoerty Tax Rate		
	Illinois	1.79% per year
	Indiana	2.12% per year
	Iowa	2.15% per year
	Missouri	1.42% per year
	Ohio	1.81% per year
Royalty Rate		0%

** Variance = 0.0358*

5.4 Results for Cropland Evaluation

In this section we present the results following the application of the RRM using the parameters discussed in the previous section. For cropland that is currently being operated, its value, *v(s)*, is defined by the following differential equation

$$\frac{1}{2}\sigma^2 s^2 \frac{d^2 v}{ds^2} + \left(- \kappa s\right)\frac{dv}{ds} + q\left(- a\right) - \tau - \left(+ \lambda_1\right) v = 0 .$$

For cropland that is currently not being operated but is rather being maintained until prices improve, its value, *w(s)*, is defined by the differential equation

$$\frac{1}{2}\sigma^2 s^2 \frac{d^2 w}{ds^2} + \left(- \kappa s\right)\frac{dw}{ds} - f - \left(+ \lambda_0\right) w = 0 .$$

The model is subject to the following boundary conditions that determine the optimal operating policy (considering the costs to transition between operating states) in addition to ensuring smoothness over the transition points.

$$w(s_0) = 0$$

$$v(s_1) = w(s_1) - k_1$$

$$v(s_2) = w(s_2) + k_2$$

$$v_s(s_1) = w_s(s_1)$$

$$v_s(s_2) = w_s(s_2)$$

The interpretation of the various terms above is straightforward and similar for most applications of continuous-time real options models (see, for example, McNamara, 2003). For a perennial crop the first term $\frac{1}{2}\sigma^2 s^2 \frac{d^2 v}{ds^2}$ captures the rate of change in value of the cropland as a result of the price volatility for corn. Higher levels of uncertainty in the corn price have a positive affect on the option value (all other things being equal). A larger range in the corn price implies the potential for a farmer to receive higher returns when prices are at relatively high levels while, at the same time, holding the option to avoid operating losses by delaying planting and harvesting or, if appropriate, letting the land lie fallow when corn prices are at relatively low levels.

The second term $(r - \kappa)s \frac{dv}{ds}$ captures the change in value of the cropland as an increasing function of the basis between the real interest rate and the convenience yield. High convenience yields correspond with relatively high spot prices (relative to the future price) while low convenience yields correspond with relatively low spot prices. A higher (lower) convenience yield relative to the interest rate increases the opportunity cost of not farming the crop and thus decreases (increases) the value of the options to temporarily delay or abandon production.

The term $q(s - a) - \tau$ captures the value of the current period cash flows, net of federal income taxes and royalties[55], while the final term $-(\phi + \lambda_1)v$ incorporates the state property taxes due on the value of the asset regardless of its operating state. The distinction between operating states is that since cropland presently being maintained accrues zero current period cash flows (i.e. $q(s - a) - \tau = 0$) it instead incurs maintenance costs of $-f$.

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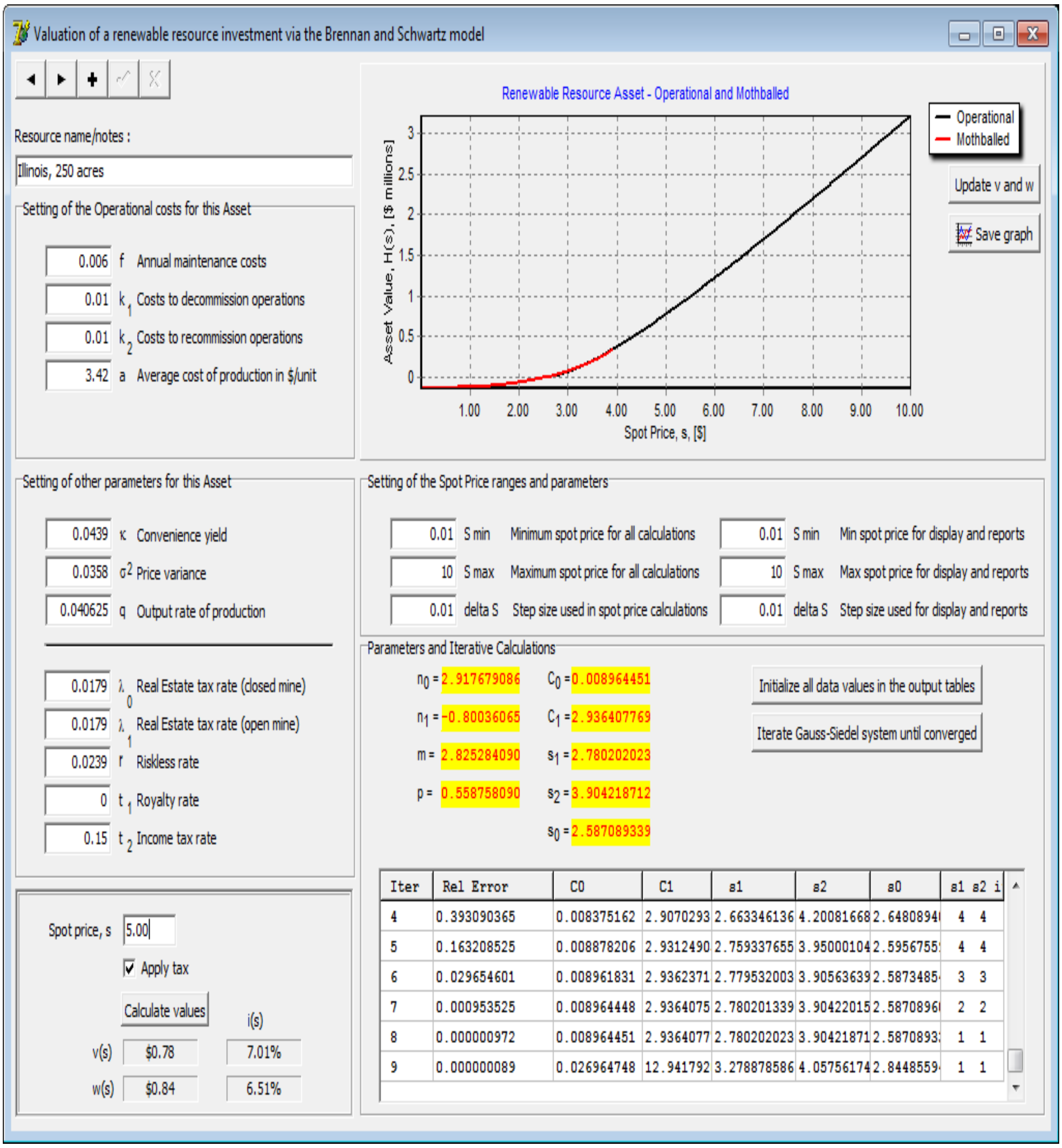
[55] McNamara (2003) notes that this is analogous to a continuous dividend rate.

Figure 5.2 below displays the output screen of the model implementation applied to a 250 acre cropland in Illinois. (Similar output screens for other states are located in Appendix D.) The computer code implements the solution methods described in Section 4.2 of Chapter 4.

Figure 5.2: Evaluation for an Average Corn Cropland Asset in Illinois, 250 Acres



As described in Chapter 4 the screen design layout consists of a separation of the input and output parameters. Once the data parameters have been input and initialised, the algorithm calculates the cropland value in addition to the critical spot prices that determines the optimal management operational policies s_0 (abandon operations), s_1 (cease operations) and s_2 (commence operations) - which are \$2.59, \$2.78 and \$3.90 respectively.

The specific cropland values and their relative risk-adjusted discount rates are shown when a particular spot price is entered into the operation located at the bottom left of screen. The value of an asset currently being operated is displayed as *v(s)*, and an asset that is currently untapped as *w(s)* in millions of dollars to the nearest ten thousand dollars. For instance, Figure 6.2 above shows that for a corn price of \$5.00, and following the appropriate managerial policies outlined above (the asset is assumed to be operational since this price is above all critical spot prices), the 250-acre cropland in Illinois is valued at \$780,000 which implies a 7.01% risk-adjusted discount rate for that particular value.

A graphical representation of the two separate output functions for both operating states *v(s)* (operational) and *w(s)* (mothballed) are displayed at the top right-hand side of the screen. The phase space is defined by the cropland value in millions of dollars on the vertical axis and the current corn spot price in dollars per bushel on the horizontal axis providing the full range of cropland values for corn prices in the range of \$0.00 - \$10.00 per bushel. In keeping with the specifications of the model, by which it is assumed the managerial policies are being followed appropriately in order to optimise the value of the asset, only the portion of the *v(s)* function (displayed in black) above the critical price S1, and the portion of the function *w(s)* (displayed in red) below the critical price S2 are shown.

For this specific application the black section displays the value of the cropland asset *v(s)* when being operated above the critical spot price \$2.78. It is optimal to maintain operations of the cropland until the spot price falls below this price where the value of the operational cropland is worth less than the value of the untapped (mothballed) cropland plus the \$10,000 cost to close down operations. The red section shows the value of a currently untapped asset below a corn price of \$3.90. It is optimal to begin operating an untapped asset above this price because the value of the operating asset is worth more than to the value of the untapped asset plus the \$10,000 cost to commence operations.

5.4.1 Results for RRM Cropland Valuations

Table 5.05 below presents the results for the calculations of cropland value as a function of the corn price for the various states. The table also presents values of the associated unlevered risk-adjusted discount rates that would be applicable in a corresponding DCF analysis which would produce identical results for this particular range of spot prices. These discount rates are calculated using the novel incorporation of the real options approach into the traditional present value framework for a perpetuity[56].

[56] See Chapter 4 for the derivation of the risk-adjusted discount rate for an unlevered firm.

Table 5.05: RRM Evaluation of Corn Cropland and Unlevered Risk-adjusted Discount Rates at Various Spot Prices

250 acres	Illinois		Indiana		Iowa		Missouri		Ohio	
Spot Price ($/bu)	Crop Value ($/acre)	i(s) (%)	Crop Value ($/acre)	i(s) (%)	Crop Value ($/acre)	i(s) (%)	Crop Value ($/acre)	i(s) (%)	Crop Value ($/acre)	i(s) (%)
$3.50*	840	1.33	560	-2.52	960	3.77	200	-37.90	400	-9.69
$4.00**	1520	5.30	1160	4.45	1680	6.40	640	-2.70	920	2.39
$4.50	2280	6.54	1800	6.43	2440	7.24	1160	3.39	1560	5.43
$5.00	3120	7.01	2520	7.17	3280	7.54	1800	5.38	2240	6.53
$5.50	4000	7.20	3320	7.47	4160	7.65	2480	6.21	2960	6.99
$6.00	4920	7.26	4120	7.58	5080	7.67	3200	6.58	3760	7.19
1000 acres	Illinois		Indiana		Iowa		Missouri		Ohio	
Spot Price ($/bu)	Crop Value ($/acre)	i(s) (%)	Crop Value ($/acre)	i(s) (%)	Crop Value ($/acre)	i(s) (%)	Crop Value ($/acre)	i(s) (%)	Crop Value ($/acre)	i(s) (%)
$3.50*	890	1.24	630	-2.29	1020	3.58	260	-28.14	470	-8.42
$4.00**	1560	5.13	1200	4.26	1720	6.23	690	-2.46	980	2.26
$4.50	2330	6.41	1850	6.27	2500	7.12	1230	3.23	1600	5.26
$5.00	3160	6.92	2580	7.05	3330	7.46	1850	5.23	2280	6.39
$5.50	4030	7.12	3350	7.38	4210	7.58	2520	6.09	3010	6.89
$6.00	4950	7.21	4150	7.52	5110	7.62	3240	6.49	3790	7.11

** Spot price is below average cost of production for Indiana, Missouri and Ohio*

*** Spot price is below average production cost for Missouri*

The results contained in Table 5.05 show that cropland values per planted acre vary significantly according to location in the U.S. while the size of the cropland has a rather smaller impact. It can be seen that cropland located in the states of Iowa and Illinois have the highest values per-acre at each price interval, while values in Indiana and Ohio constitute an apparent mid-range across all states, and values calculated for Missouri are significantly lower. This is unsurprising given that farms in Iowa and Illinois were shown earlier in the chapter to have relatively low production costs while Missouri farms have higher costs.

Note that although the hypothetical spot price $3.50 is below the actual average costs of production for several states (i.e. on average farmers in these states are currently out-of-the-money) the option values of the cropland remain positive at this price. The observable negative discount rates associated with this price level are the result of equating the DCF valuation with the real options valuation. Given a positive options value when the free cash flows from the DCF model are negative this must mathematically result in a corresponding negative discount rate.

This presents a clear distinction between using the DCF method and the real options approach. For instance, at a current spot price below the average costs of production a standard NPV analysis would reject the cropland as a viable investment. In contrast, the real options approach explicitly accounts for the effects of output price volatility, so that the probability of prices improving in the future implies that the option values for these cropland assets still have a positive value.

It is of interest to compare these model results with the official estimates of average cropland value published by the USDA for each State. The USDA estimates are calculated using a complete, probability-based land-area sampling frame based on annual survey data which includes a stratified sample of land areas averaging approximately one square mile in size gathered during the first two weeks of June (USDA, 2011e). Table 5.06 below shows the most recent cropland values reported by the USDA for the years 2006 to 2010.

Table 5.06: USDA Average Cropland Values by State

Year	Illinois	Indiana	Iowa	Missouri	Ohio
2006	$ 3,640	$ 3,250	$ 3,100	$ 2,010	$ 3,470
2007	$ 4,150	$ 3,640	$ 3,600	$ 2,330	$ 3,820
2008	$ 4,850	$ 4,140	$ 4,260	$ 2,500	$ 4,140
2009	$ 4,670	$ 3,950	$ 4,050	$ 2,540	$ 3,900
2010	$ 4,820	$ 4,030	$ 4,100	$ 2,670	$ 3,950

Source: USDA, 2011e

Unfortunately, the USDA does not disaggregate their data to distinguish between specific uses of cropland (for example, corn, sorghum, or wheat) within each of the states. Nevertheless, it is noted that a similar pattern can be observed between the USDA values and the model values, in that Illinois and Iowa have the highest cropland values, while Missouri has significantly lower values. This raises an interesting question: what is the appropriate spot price at which the values obtained by implementing the RRM match the official USDA estimates? In Table 5.07 we report the appropriate spot price at which the model produces cropland values equivalent to those reported by the USDA for 2010.

Table 5.07: Results for the Appropriate Corn Prices at which the RRM Matches the USDA Estimates for Cropland Values

	Spot Prices ($/bu)				
Crop Size	Illinois	Indiana	Iowa	Missouri	Ohio
250 acres	$5.95	$5.95	$5.45	$5.64	$6.12
1000 acres	$5.93	$5.93	$5.44	$5.60	$6.10

It can be seen in the table above that a spot price in the range $5.45 to $6.12 would match the official cropland values published by the USDA for 2010. To see how these calculations compare with observed monthly corn prices received by U.S. farmers as well as general market prices indicative of the Dow Jones Industrial index (DJI) [57], a time-series plotting monthly prices from January 2010 to June 2011 is presented below in Figure 5.3.

[57] DJI prices sourced from Yahoo Finance, 2012.

Figure 5.3: U.S. Average Monthly Corn Prices, January 2010 – June 2011

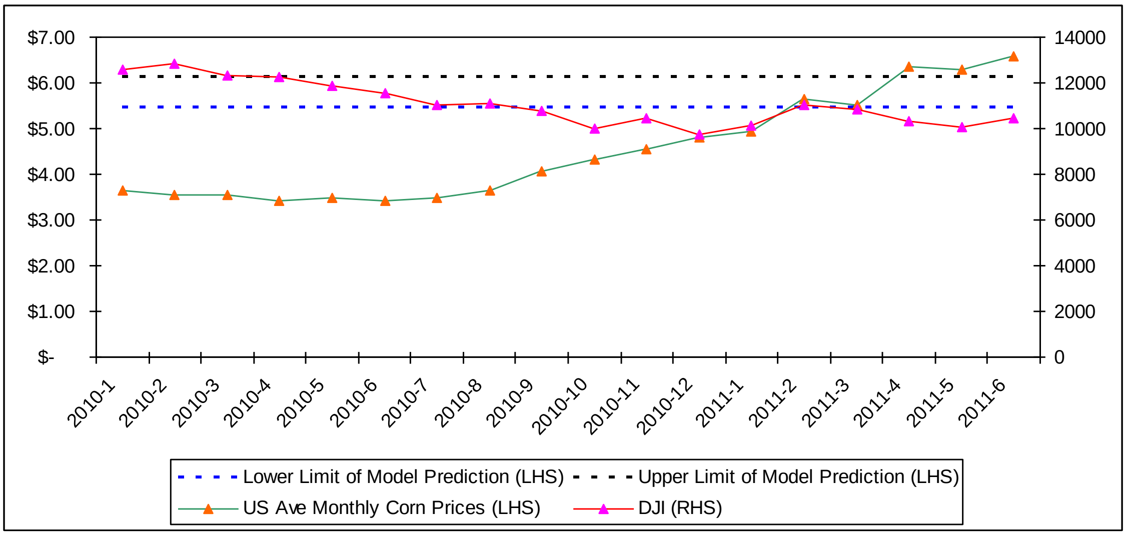


Figure 5.03 shows that during 2010 corn prices were below the calculations of the appropriate prices at which the RRM matches the USDA values. However, prices observed in 2011 from April to June are found to be higher than these "benchmark" prices. Intuitively, since the planting season for corn typically occurs between early April and June, the period preceding planting is when there is most uncertainty in the future availability of the commodity (hence the convenience yield is highest). Upon examining the average of monthly prices for each month from January, 1975 to December, 2010 (see Table C.1, Appendix C), it is found that monthly prices over the period January to June are on average 7.48% higher than monthly prices over the period July to December. This may explain, at least partially, the discrepancies between the RRM valuations of corn croplands and the USDA estimates. The discrepancies (in percentage terms) for monthly corn prices and the results for the appropriate corn price at which the RRM and USDA estimates match are shown for each state in Table 5.08 below.

Table 5.08: Discrepancies (in percentage terms) for Average Monthly Corn Prices and Appropriate Prices at which the RRM Matches the USDA Estimates for Cropland Values

Year-month	Corn price	Illinois (%)		Indiana (%)		Iowa (%)		Missouri (%)		Ohio (%)	
	$/bu	250 acre	1000 acre	250 acre	1000 acre	250 acre	1000 acre	250 acre	1000 acre	250 acre	1000 acre
2010-1	$ 3.66	-38.49	-38.28	-38.49	-38.28	-32.84	-32.72	-35.11	-34.64	-40.20	-40.00
2010-2	$ 3.55	-40.34	-40.13	-40.34	-40.13	-34.86	-34.74	-37.06	-36.61	-41.99	-41.80
2010-3	$ 3.55	-40.34	-40.13	-40.34	-40.13	-34.86	-34.74	-37.06	-36.61	-41.99	-41.80
2010-4	$ 3.41	-42.69	-42.50	-42.69	-42.50	-37.43	-37.32	-39.54	-39.11	-44.28	-44.10
2010-5	$ 3.48	-41.51	-41.32	-41.51	-41.32	-36.15	-36.03	-38.30	-37.86	-43.14	-42.95
2010-6	$ 3.41	-42.69	-42.50	-42.69	-42.50	-37.43	-37.32	-39.54	-39.11	-44.28	-44.10
2010-7	$ 3.49	-41.34	-41.15	-41.34	-41.15	-35.96	-35.85	-38.12	-37.68	-42.97	-42.79
2010-8	$ 3.65	-38.66	-38.45	-38.66	-38.45	-33.03	-32.90	-35.28	-34.82	-40.36	-40.16
2010-9	$ 4.08	-31.43	-31.20	-31.43	-31.20	-25.14	-25.00	-27.66	-27.14	-33.33	-33.11
2010-10	$ 4.32	-27.39	-27.15	-27.39	-27.15	-20.73	-20.59	-23.40	-22.86	-29.41	-29.18
2010-11	$ 4.55	-23.53	-23.27	-23.53	-23.27	-16.51	-16.36	-19.33	-18.75	-25.65	-25.41
2010-12	$ 4.82	-18.99	-18.72	-18.99	-18.72	-11.56	-11.40	-14.54	-13.93	-21.24	-20.98
2011-1	$ 4.94	-16.97	-16.69	-16.97	-16.69	-9.36	-9.19	-12.41	-11.79	-19.28	-19.02
2011-2	$ 5.64	-5.21	-4.89	-5.21	-4.89	3.49	3.68	0.00	0.71	-7.84	-7.54
2011-3	$ 5.53	-7.06	-6.75	-7.06	-6.75	1.47	1.65	-1.95	-1.25	-9.64	-9.34
2011-4	$ 6.35	6.72	7.08	6.72	7.08	16.51	16.73	12.59	13.39	3.76	4.10
2011-5	$ 6.30	5.88	6.24	5.88	6.24	15.60	15.81	11.70	12.50	2.94	3.28
2011-6	$ 6.58	10.59	10.96	10.59	10.96	20.73	20.96	16.67	17.50	7.52	7.87

These results can be interpreted in at least several ways. First, they suggest that, if the RRM's assumptions hold and if the model parameters adopted above are a valid representation of the true situation, then the market valuations of corn cropland in the states studied may have involved some over-valuation during 2010, in that the corn prices which prevailed during the period implied (according to the model) lower cropland values. By the same token, the market valuations in much of 2011 appear to have been somewhat lower than indicated by the model.

Second, the above characterisation of over- and under-valuation needs to be qualified by the fact that USDA valuations of cropland related to the average for *all* crops, whereas the model-generated valuations are geared to *corn* croplands only. It is possible, for example, that during 2010 the returns on crops other than corn may have been, and were expected to remain, higher than returns on corn, so that market values of all croplands combined would be higher than those for corn croplands. In that case, market values of corn croplands may have been far closer to the model-based values than the above comparison may suggest.

Third, the discussion in the above two paragraphs notwithstanding, it is of interest to note how close the model's calculations turn out to be relative to market valuations, as indicated by USDA estimations. Recall that in parameterising the model and implementing the solution methods in this case study, we have had to make a number of simplifying assumptions and work with "ballpark" estimation, such as those for production costs. Despite these approximations, the results turn out to be certainly quite close to USDA estimates. This suggests that, in a real-world situation, it may well be worth investing additional resources into refining the estimates of the relevant model parameters in order to obtain more accurate RRM valuations. In particular, compiling cost and tax data that is more specific for individual farms would provide a more robust empirical analysis, and would seem a worthwhile avenue for future research.

5.4.2 Results for the Risk-Adjusted Rates of Return

The RRM calculates a risk-adjusted discount rate of return (exactly within the model) at any given spot price, given the assumptions of the model. The RRM discount rates may be interpreted as the return on equity that a farmer can expect to receive when adjustments for output price uncertainty have been accounted for in determining the present value of the future cash flows of the cropland. These calculated unlevered discount rates are thus useful for farmers, and investors in general, to compare with returns on similar investment opportunities. Importantly, however, one needs to keep in mind that the portfolio constructed to replicate the value of the investment opportunity in the cropland asset *does not* hedge the returns for operational risk.

As noted in Chapter 2, firms often use the WACC as the appropriate nominal discount rate for a levered firm. How the WACC calculations may be interpreted is described succinctly by Nordon (2009:1):

> “The risks assumed in an agricultural firm are assumed to be reflected in the actual cost of capital for the firm. The cost of capital is the actual or opportunity cost incurred by an agricultural firm when using debt and or equity financing. It is typically stated as an interest rate and is typically the rate of return an agricultural firm would receive if investing in a risk similar firm”.

Durst (2009) notes, however, that most farms in the U.S. are operated as sole proprietors, partnerships, or small business corporations. It is thus reasonable to assume the appropriate risk-adjusted rates of return on the cropland for the purpose of this case study application are unlevered. Accordingly, given the price range between $3.50 and $6.00 reported in Table 5.05, the calculations for the appropriate unlevered discount rates of return for cropland in each state would range between -37.9% to 7.67%.

These results show also that between the price range of $4.50 to $6.00 (over which all farmers are currently in the money) the calculations for the appropriate discount rates in each state range between 3.23% and 7.67%. We note that the RRM discount rates are relatively low when firms are near-the-money, which was the case throughout the early part of 2010 where the price received by U.S. farmers ranged between $3.41 and $3.66 per bushel from January to August before rising later in the year to between $4.08 and $4.82 from September to December as shown in Table 5.08. These results appear reasonable when compared with similar calculations for the WACC for crop farming in the U.S. when the cropland assets are in the money. For example, Nordon (2009) calculated the average WACC to be 6.015% across a 20 year study which included 398 Kansas farms over the period 1985 to 2005. Similarly, Iwai et al. (2009) calculated a WACC of 8.1% in their recent study which focussed on the Florida Citrus industry.

Nordon's (2009:32) study observes that "comparing the returns for individual farms as an asset class with other returns for asset classes in the market index indicate that on average individual farms provide less return than corporate equities, corporate bonds, treasury securities, and municipal securities but more return than agricultural land values". In examining to what extent Nordon's observations are consistent with the results from this study, Table 5.09 below compares the unlevered returns on corn cropland assets consisting of 250 acres[58] in each state to average returns on other asset classes including returns on corn as a commodity, returns on the market index (as reflected in changes to the DJI), and 10 year U.S. Treasury yields.

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[58] We recall from Section 5.2 that according to Foreman (2001) farms with a corn acreage of 250 acres or less comprise 75% of all U.S. corn farms

Table 5.09: A Comparison of the Unlevered Returns on Corn Cropland to Returns on Other Asset Classes Over the Period January 2010 to May 2011

Year-month	2009-12	2010-1	2010-2	2010-3	2010-4	2010-5	2010-6	2010-7	2010-8	2010-9
Corn price	$ 3.60	$ 3.66	$ 3.55	$ 3.55	$ 3.41	$ 3.48	$ 3.41	$ 3.49	$ 3.65	$ 4.08
returns*		1.67%	-3.01%	0.00%	-3.94%	2.05%	-2.01%	2.35%	4.58%	11.78%
annualised**		20.00%	-36.07%	0.00%	-47.32%	24.63%	-24.14%	28.15%	55.01%	141.37%
DJI***		12569.79	12810.54	12319.73	12226.34	11891.93	11577.51	11006.02	11118.4	10788.05
returns*		1.92%	-3.83%	-0.76%	-2.74%	-2.64%	-4.94%	1.02%	-2.97%	-7.17%
annualised**		22.98%	-45.98%	-9.10%	-32.82%	-31.73%	-59.23%	12.25%	-35.65%	-86.02%
10 Year US Bond Yield		3.73%	3.69%	3.73%	3.85%	3.42%	3.20%	3.01%	2.70%	2.65%
RRM Unlevered returns, i(s)										
Illinois		3.60%	2.10%	2.10%	-1.90%	1.30%	-1.90%	1.80%	3.10%	5.59%
Indiana		0.88%	-1.26%	-1.26%	-5.46%	-3.90%	-5.46%	-2.80%	0.72%	4.93%
Iowa		5.00%	4.22%	4.22%	2.80%	3.58%	2.80%	3.68%	4.93%	6.60%
Missouri		-17.37%	-29.32%	-29.32%	-64.22%	-42.24%	-64.22%	-39.99%	-18.20%	-0.11%
Ohio		-3.52%	-7.34%	-7.34%	-15.47%	-10.78%	-15.47%	-10.22%	-3.80%	3.14%

Year-month	2010-10	2010-11	2010-12	2011-1	2011-2	2011-3	2011-4	2011-5	Average
Corn price	$ 4.32	$ 4.55	$ 4.82	$ 4.94	$ 5.64	$ 5.53	$ 6.35	$ 6.30	**$ 4.40**
returns*	5.88%	5.32%	5.93%	2.49%	14.17%	-1.95%	14.83%	-0.79%	**3.49%**
annualised**	70.59%	63.89%	71.21%	29.88%	170.04%	-23.40%	177.94%	-9.45%	**41.90%**
DJI***	10014.72	10465.94	9774.02	10136.63	11008.61	10856.63	10325.26	10067.33	**11115.14**
returns*	4.51%	-6.61%	3.71%	8.60%	-1.38%	-4.89%	-2.50%	3.58%	**-1.01%**
annualised**	54.07%	-79.33%	44.52%	103.23%	-16.57%	-58.73%	-29.98%	43.00%	**-12.06%**
10 Year US Bond Yield	2.54%	2.76%	3.29%	3.39%	3.58%	3.41%	3.46%	3.17%	**3.27%**
RRM Unlevered returns, i(s)									
Illinois	6.23%	6.61%	6.89%	6.97%	7.22%	7.20%	7.27%	7.27%	**4.20%**
Indiana	5.95%	6.54%	6.98%	7.11%	7.51%	7.48%	7.62%	7.61%	**2.54%**
Iowa	7.03%	7.29%	7.47%	7.52%	7.66%	7.65%	7.66%	7.66%	**5.75%**
Missouri	1.99%	3.69%	4.87%	5.23%	6.34%	6.24%	6.72%	6.71%	**-15.48%**
Ohio	4.70%	5.59%	6.24%	6.44%	7.06%	7.00%	7.25%	7.24%	**-1.13%**

* Monthly returns

** Annualised by multiplying by 12

*** Taken from first trading day of the month (Yahoo Finance, 2012)

Table 5.09 above shows that the highest average annualised return over the period January 2010 to May 2011 was from holding corn as a physical commodity (41.9%). Positive average annual returns are also observable for unlevered returns on cropland in Iowa (5.75%) and Illinois (4.2%), 10 year U.S. government bond yields (3.27%), and cropland in Indiana (2.54%). Conversely, negative average returns are shown for corn cropland in Ohio (-1.13%), the DJI (-12.06%) and cropland in Missouri (-15.48%). Another interesting finding from the results is that, while yields on 10 year U.S. bonds are the least volatile (σ = 0.41%), the unlevered returns on corn cropland assets are less volatile (σ = 3.19%; 5.05%; 1.9%; 25.18%; 8.51%)[59] than the annualized returns of corn as a physical commodity (σ = 68.61%) and the DJI (σ = 51.84%).

5.5 Results for Exercise Prices of Operational Options

In this section we present the computations for the appropriate prices at which the farmer should exercise their various options to optimise cropland value. Under the RRM, it is optimal to continue perennial corn farming operations on the crop land until the price falls to the critical price S1. If the price falls below this level, it is optimal to delay[60] planting for the coming season and instead incur the costs of closure and maintenance of the land to minimise operational losses which would have resulted from going ahead with planting. It is optimal to abandon production of corn entirely when the price falls to S0 where the option value to operate a corn crop on this land is zero. Conversely, if the crop land is currently not operational (that is the spot price has been low enough to warrant delaying planting this seasons corn crop), and the price rises above the critical price level S2, it is optimal to return to normal operating conditions and incur the associated production costs. The results of the calculations for these critical prices is presented in Table 5.10 below.

[59] Illinois; Indiana; Iowa; Missouri; and Ohio, respectively.

[60] For the case of a perennial crop this can be also interpreted as the farmer should forego delivery of product to market whilst maintaining the inventory should the crop be harvested, or allow the crop to continue growing if it has already been planted. For the former it may be worthwhile incorporating the relevant costs associated with silage into the calculations for k1 and k2.

Table 5.10: Real Options Analysis of Critical Option Prices

250 acres	S0*	S1**	S2***
Illinois	$2.59	$2.78	$3.90
Indiana	$2.74	$2.93	$4.12
Iowa	$2.48	$2.62	$3.70
Missouri	$3.14	$3.37	$4.74
Ihio	$2.89	$3.10	$4.36
1000 acres	**S0**	**S1**	**S2**
Illinois	$2.55	$2.94	$3.63
Indiana	$2.69	$3.10	$3.83
Iowa	$2.44	$2.77	$3.44
Missouri	$3.08	$3.56	$4.41
Ihio	$2.84	$3.28	$4.06

** S0 - abandon*

*** S1 - temporarily cease production*

**** S2 - recommence production*

Illinois and Iowa are found to have lower price thresholds at which it becomes optimal to exercise the various operational options over a farm. This can be expected given that our calculations showed farmers in these states to have the lowest costs of production in the Heartland Region. It can be seen that the larger the farm size, the lower the critical price, S1, at which it becomes optimal to abandon corn-farming operations. Similarly, a larger farm size reduces the critical price to reinstate operations, S2, over a previously mothballed crop. A larger farm size, on the other hand, increases the price at which it becomes optimal to temporarily delay corn farming. In other words, the larger the corn crop size, the smaller the price range between critical prices S1 and S2, and the larger the price range between critical prices S0 and S1.

In view of the monthly USDA data for average U.S. corn prices received by farmers over the period January 2010 to June 2011 there were no instances where the corn price fell below the highest calculated abandonment price of $3.14 (for both small and large farms). In fact the last time the monthly price was below this level was in January 2007. It should be noted, however, that recently the monthly corn price fell below the critical price to cease production, S1, for both small and large farms in Missouri in 2009 during the months of August and September. Similarly in 2010 the price fell below these prices for large farms on consecutive months from February to July.

Clearly inventory can be stockpiled by farmers for future delivery when market conditions improve or may not even be ready for market during such months. Furthermore, for many farms the decision to continue unprofitable operations may be influenced by both contractual obligations with various end-users of corn and with efforts to ensure marketing networks are maintained. Investigating how these relatively low prices affected farming operations amongst individual farms in Missouri (as the model suggests was appropriate) and, more generally, looking for evidence of hysteresis inertia in output level affecting operational decision-making in agriculture, would be an interesting avenue for further research.

Durst (2009) notes that nearly three-quarters of farm sole proprietors in the U.S. reported a farm loss in 2006 and that for small residential farms in particular the return to farming from the tax code may partially explain continued farm production despite the persistence of farm losses. Losses from farming are used to reduce taxes on other income which is especially the case for individuals who report their primary occupation to be something other than farming. Similarly, large commercial farms may not be stand alone businesses, but rather part of a broader portfolio of assets for which losses may be offset against other parts of the firm (see Moel & Tufano, 2002). This implies that for large, well-diversified firms, operating policy decisions may be made at the firm level rather than at the operational level. Other non-financial considerations may also influence the investment decision process. For instance, Tauer (2006) posited that farmers often hold optimistic expectations of the future in that next month's price might be better than that at present and thus continue operations despite an objective financial analysis indicating that it would be appropriate to do otherwise.

5.6 Conclusion

This chapter has specifically addressed the second research question for this study. In this chapter the RRM developed in Chapter 3 was applied to evaluate an agricultural investment in a real-world setting, namely corn-crop farming in the U.S. Corn Belt.

Calculations for cropland values showed that the States of Illinois and Iowa had the highest cropland values, while calculations for Missouri were significantly lower. The model valuations turned out to be generally consistent with official estimates of cropland values published by the USDA, although there were differences which can be explained in terms of fairly plausible factors, and which should be further investigated in the future.

Our results further indicate significant differences between optimal operational decision-making for relatively small and large corn farms. For larger farms the trigger prices at which it becomes appropriate to recommence operations on farms in which operations had been temporarily mothballed were relatively lower than for smaller sized farms while the appropriate crop abandonment prices were also lower for large farms. Prices at which it is appropriate to delay production to avoid operating losses, however, were higher for larger farms. As one may expect, seasonal entry and exit prices as well as crop abandonment prices were found to be lower for relatively low-cost producers than high-cost producers

The implications of these findings for agricultural investments were further discussed in the context of potential sources of hysteresis inertia in output level affecting operational decision-making. It was noted that, although our model provides an objective approach to determining the appropriate operating policy for these farmers, their decisions may ultimately be affected by considerations that include subjective factors outside the scope of the RRM. The key contribution of the current research is to provide farmers with a means to inform their overall decisions using a purely objective financial analysis.

Chapter 6

Conclusion and Future Research

This chapter summarises and concludes the study in addition to providing suggestions for avenues for future research. A brief review of the objectives and motivation of the research is discussed in Section 6.1. A summary of the main findings and key contributions of the study are presented in Section 6.2. The theoretical and practical implications of the research are outlined in Section 6.3 while Section 6.4 concludes the chapter by identifying avenues for future research.

6.1 Overview of Research

This research applied real options theory to evaluating investments in renewable resource assets under uncertainty. Although a good deal of work reviewed in the academic literature has examined applications of real option theory to assets which have a finite inventory (and investment horizon), examples of research specifically geared towards evaluating assets that have a renewable inventory are sparse. Thus, the key motivation of this research was to develop a real options model to evaluate renewable resource assets that: (1) has academic appeal; (2) lends itself to practical use for investors, financial practitioners and policy-makers; and (3) is a superior approach to evaluate this special class of assets compared to the traditional and widely-used DCF method.

There is a clear consensus in the academic literature that the DCF valuation method is unsatisfactory when being used to evaluate investment under uncertainty (see, for example, Brennan & Schwartz, 1985a; Kelly, 1998; Cortazar, Schwartz & Salinas, 1998; Duku-Kaakyire & Nanang, 2002; Colwell et al., 2003; McNamara, 2003; Schwartz & Trigeorgis, 2004). This is on the basis of such difficulties as predicting future cash flows when output prices are volatile and, furthermore, these models are ill equipped to systematically factor in managerial flexibility to change future operating conditions in response to fluctuations in profitability. Where these issues are ignored or not addressed appropriately, the mispricing of an asset may imply the failure to identify an otherwise profitable investment opportunity.

The economic importance of renewable resources such as water, food and renewable energy to the world's population warrants the use of appropriate financial models to accurately evaluate potential investment opportunities in renewable resource assets. Furthermore, these evaluations can also be used to determine operational policies that can be followed to ensure current assets are being managed optimally. It is argued that the RRM developed offers a number of advantages over using the DCF method. First, the present value of an asset can be calculated which incorporates the effects of output price volatility at any point in time under various operational states. Second, the appropriate operational policies to optimise the value of an asset can be calculated. Third, a method was developed by which financial practitioners may objectively arrive at an exact risk-adjusted discount rate for a renewable resource asset under the assumptions of the RRM.

Following its development, testing of the validity of the RRM involved two separate cases. An initial comparison was made to the published results of the Brennan and Schwartz (1985b) copper mine example, and subsequently, a specific case-study was undertaken involving a stylised application of the model and the results compared to official market valuations obtained from a published source. This case study demonstrated how the RRM may be applied to evaluate real-world renewable resource assets such as perennial agricultural investments. The application selected was cropland assets that derive their value from an underlying perennial crop of corn. The nature of corn as a well-traded commodity enabled ready access to the relevant data required to conduct the real options analysis and thus apply the RRM in a real-world setting. Real options evaluations for stylised cropland assets consisting of 250 and 1000 acres were analysed across 5 States in the U.S. Corn Belt.

6.2 Main Findings and Contributions of Research

In addressing the first research question of this study, in Chapter 3 the RRM was developed within the real options framework, based on the Brennan-Schwartz model and incorporating the assumption that the resource is never fully depleted at any point in time. The equations of the RRM were then solved analytically to produce a closed-form solution. In Chapter 4 an algorithm and implementation via a numerical solution to the closed-form solution of the RRM were developed.

Some original theoretical work was also undertaken in Chapter 4 which incorporates the real options approach into a traditional DCF valuation framework. While the DCF method remains the conventional approach to evaluating investment opportunities despite its well-known shortcomings (such as the difficulty in predicting future output prices and assigning an appropriate risk-adjusted discount rate), this research showed how the discount rate can be derived as a function of the spot price to provide an objective means by which to calculate an exact unlevered risk-adjusted discount rate applicable to DCF valuations.

The correctness of the implementation of the RRM was validated by comparison to the published results from the finite-inventory copper mine example in Brennan and Schwartz (1985b). The results of this comparison showed that the exhaustibility of the mine's resource inventory does indeed have a significant impact on the mine value and the operational prices at which to open and close the mine. The values for the RRM mine were found to be as high as triple the values for the finite resource asset of that example (which had an inventory of 150 million pounds of copper). While extending the natural life of the asset increased the value of the asset at every spot price, it reduced the critical option price at which it becomes optimal to open the mine, as well as the critical price at which to decommission operations. An interesting finding for this particular application was the results indicated that the critical price at which to abandon the asset appeared to be, for practical purposes, independent of the level of inventory.

The results of the case study of cropland assets across the U.S. Corn Belt in Chapter 5 showed that the crop size as well as the location of a farm are both relevant factors in determining cropland asset values. By comparison, cropland values were found to be higher for farms with relatively low costs of production such as Illinois and Iowa than for those with relatively high costs such as Missouri, as one may expect.

The results suggest that, under the assumptions of the RRM and if the model parameters adopted are a valid representation of the true situation, then the USDA market valuations of corn cropland appeared over-valued during 2010, while the valuations in much of 2011 appear to have been somewhat under-valued compared with the values produced by the RRM. Given the fact that USDA valuations of cropland in the five states examined are based on survey data which relate to the average for *all* crops in each of those states, whereas the RRM-generated valuations are geared to *corn* croplands only, the returns on crops other than corn may have had some impact on these results. In that case, market values of corn croplands may have been far closer to the RRM-based values than the results of this comparison suggests notwithstanding how close the RRM's calculations were relative to the USDA market value estimates.

The lowest discrepancies (in percentage terms) between the RRM and USDA estimates were observed during the period January to June 2011. An examination of average monthly prices for each month from January, 1975 to December, 2010 found that monthly prices over the period January to June are on average 7.48% higher than monthly prices over the period July to December. Since the period preceding planting (planting typically occurs between early April and June) is when there is most uncertainty in the future availability of the commodity (hence the convenience yield is highest) it is thus worth further investigating to what extent this may explain the discrepancies between the RRM valuations of corn croplands and the USDA estimates.

The location and relative size of corn croplands were also found to be important factors affecting the calculations of the critical prices at which to exercise various operational options in order to manage operations optimally. Illinois and Iowa, which are relatively low-cost producers, were found to have lower price thresholds at which it becomes optimal to exercise all three options. Across all States, larger croplands were found to have lower critical prices at which it becomes optimal to abandon corn farming operations (S0) than smaller croplands. The differences between 250- and 1000-acre croplands ranged from 4 to 6 cents.

Larger croplands were found also to have higher prices at which it becomes optimal to temporarily cease corn farming (S1) in all cases. The relatively larger price ranges between the critical prices S1 and S0 show that it is optimal to exercise the option to cease operating larger croplands at a higher price but abandon at a lower price than for small croplands. Conversely, smaller croplands had higher critical prices to exercise the option to commence operations (S2). Although small farms require higher price thresholds at which it becomes optimal to commence operating their croplands, should the price fall once operations begin, it is optimal to sustain operations at lower price thresholds before exercising the option to cease operations.

In summary, this study makes a number of contributions to the literature, particularly the growing body of research on real options. First, it provided an innovative extension to the seminal academic work of Brennan and Schwartz (1985b) by developing a real options model, the RRM, which is generally applicable to evaluating renewable resource investments. Second, a practical solution and implementation of the RRM was developed with a view to making it accessible to practitioners. Third, some theoretical work was presented which equated the real options approach to a traditional valuation framework in order to calculate an exact risk-adjusted discount rate applicable to traditional discounted cash flow valuations for an unlevered firm. Fourth, it demonstrated how the RRM can be applied generally to renewable resource assets using a real-world example.

6.3 Implications of the Research

The findings of the research presented in this thesis have theoretical implications for the growing body of real options literature as well as practical implications for investors, financial practitioners and policy–makers.

6.3.1 Theoretical Implications

A key theoretical implication of this research is that it extends real options theory to modelling investment opportunities in assets that have a renewable, or indeed infinite, resource inventory. As was discussed in Section 2.8, very few research efforts have been published involving theoretical follow-on work to the original Brennan and Schwartz (1985b) Model despite the seminal nature of their work in the area of evaluating natural resource assets. This research makes contributions to real options theory by extending their work to produce a new model, the RRM, that may be applied under general conditions to evaluate assets that have a renewable resource inventory. Moreover, the research describes and demonstrates, in sufficient detail so that it may be reproduced, how the model can be solved mathematically and subsequently implemented in a computer program.

6.3.2 Practical Implications

In addition to its theoretical contributions, this research has several practical implications for investors, financial practitioners and policy–makers. From an investor's perspective, once the decision has been made to invest in a renewable resource asset that is unviable or indeed unfeasible to reverse, subsequent decisions involving the appropriate operating policy are critical to maximise the returns on the investment. A key implication that results from this research is that the RRM provides managers of renewable resource assets the present value of the asset in its current state of operations as well as an objective calculation of the critical output prices at which it becomes optimal to exercise various operational policies over the asset to maximise its value.

The RRM is potentially a valuable tool that managers can readily employ to calculate the optimal operating policy when facing uncertain future market conditions. For instance, once an farmer initially invests in a cropland asset, subsequent operation decisions are required to maximise the profitability of the cropland. If current spot prices are relatively low the farmer has the option to mothball current cropping operations. In the meantime, if prices improve, the farmer has the option to reinstate cropping operations. Otherwise, if prices continue to fall further, the farmer has the option to abandon operations of the cropland altogether or considering rotating their cropland and introducing break species which are presently more viable.

In viewing agricultural assets as one specific class of renewable resource investments where the inventory is physically replenished on a perennial basis, the RRM developed in this study may also be applied to a multitude of other renewable and infinite resource assets. For example, the model is applicable to evaluate fish farms with a breeding program in place that results in an annual extraction rate that does not exceed the rate of replenishment. In addition to valuing the asset in both an operating and mothballed state of operations, the RRM would outline the optimal harvesting policies in light of current market prices.

Similarly, the RRM may be applicable to evaluate investments in desalination plants and energy utilities such as wind and solar farms whose underlying resource is replenished naturally (which implies that the underlying resource is infinite). If varying the state of operations is a feasible option for managers of these utilities, and costs associated with implementing the various policies can be estimated, the RRM is applicable to these types of renewable resource assets.

There are also commercial implications of this research. Financial practitioners concerned with the difficulties faced when using traditional DCF methods may wish to adopt the RRM to evaluate investments in renewable resource assets under uncertainty. Unlike the DCF methods, the RRM does not require that practitioners attempt to forecast future cash flows and assign a risk-adjusted discount rate. While estimating parameters such as these is difficult at best when conducting a DCF analysis for an asset with a finite life, this becomes even more problematic when considering an asset that can be assumed to have an infinite life. The RRM has been specifically designed to include a user-friendly interface which is useful for practitioners to readily perform tests on the sensitivity of asset value to changes in the market parameter estimates such as interest rates, output price volatility and costs.

Finally, the RRM can be used to determine effects from changes in the tax or regulatory environment on asset value as well as the optimal operational decisions of firms in a particular industry. This is useful to policy-makers who are considering reforms to current tax and industry subsidy policies. Although the tax function in its current form does not include a term for government subsidies, it is relatively straightforward to amend the function to incorporate this as well as other changes to the tax code that would impact on the value of the asset (such as a carbon tax).

6.4 Directions for Future Research

There are several potential avenues to furthering the research presented in this thesis. First, taking into account the private nature of firm data, compiling cost and tax data that is more specific for individual farms would provide a more robust empirical analysis than relying solely on official publications based on estimations of industry averages generated from intermittent survey data. Investing additional resources towards obtaining this data for any future work in order to refine the estimates of the model parameters should enhance the accuracy of the model evaluations.

Second, while this study specifically examined corn cropland assets, similar applications can be undertaken on other agricultural cropland assets, for example, wheat, sorghum and soybeans as well as a multitude of renewable resource assets including fish farms, desalination plants and energy utilities as discussed in section 6.3.2. Accordingly, the broad array of renewable resource assets in which the RRM can theoretically be applied illustrates the fruitfulness and scope of worthwhile avenues for future research.

Third, an interesting extension of this research would be to further investigate the feasibility of applying the RRM and similar methods to the more general case of assets with a non-renewable inventory. This would ideally involve using the analytic solution for the RRM as a limiting domain for the propagation of a precise numerical calculation. Such research would necessarily require further analytic work to bridge the gap between infinity and the large-volume resource inventory case.

Appendix A

Derivation of the Original Brennan and Schwartz (1985b) Model

A complete and thorough derivation of the Brennan and Schwartz (1985b) model requires a review of the essential mathematical techniques and processes that are assumed in their derivation. It is anecdotally accepted that the complete derivation is lengthy and hence is often glossed over. Given the lack of a text-book style presentation, the aim is to present a self-contained, complete, and rigorous derivation of this model. The mathematics that follows applies the format used by Pindyck (1991) and then maintains that rigour throughout the entire derivation.

A.1 The Mathematics of Random Processes

A random, or stochastic, process is one in which the processes involved, however simple or complex, cannot be predicted. As such, the future behaviour of commodity spot prices, S, is assumed to be determined competitively following an exogenously determined stochastic process known as a Wiener process. A Wiener process, defined here as $z(t)$, is a continuous-time Markov stochastic process, that is, the increments of each movement are independent of the past behaviour of the process. It follows that any change in this process, $\Delta z(t)$, occurring within a time interval, Δt, will satisfy the following conditions:

(i) $\Delta z(t) = \varepsilon_t \sqrt{\Delta t}$, where $\varepsilon_t \equiv N(0,1)$ is a normally distributed random variable (i.e. $\mu = 0$, $\sigma = 1$), and,

(ii) the set $\{\varepsilon_i\}$ are independent of each other and serially uncorrelated (i.e. $E(\varepsilon_i \varepsilon_j) = 0$ for $i \neq j$).

Given a finite interval of time, T, which is made up of n units of lengths in time Δt such that $n = \frac{T}{\Delta t}$, the change in $z(t)$, $\Delta z(t)$, over this time interval can be written as:

$$\Delta z(T) = z(t+T) - z(t) = \sum_{i=1}^{n} \varepsilon_i \sqrt{\Delta t}$$

Due to the second condition (ii) which maintains that each event, ε_i, is independent, the change $z(t+T) - z(t)$ is thus also normally distributed. It follows that the mean will also be 0 and the variance is:

$$Var = \frac{1}{N} \sum \left(x - \bar{x}\right)^2$$

$$= \frac{1}{N} \sum_{j=1}^{N} \left(\Delta z_j - \overline{\Delta z}\right)^2$$

$$= \frac{1}{N} \sum_{j=1}^{N} \left(\left(\sum_{i=1}^{n} \varepsilon_i \sqrt{\Delta t} \right) - \overline{\Delta z} \right)^2$$

As ε_i has a mean 0 and variance 1,

$$= \frac{1}{N} \sum_{i=1}^{n} \sum_{j=1}^{N} \Delta t$$

$$= \frac{1}{N} nN\Delta t$$

$$= n\Delta t$$

This implies that the variance of the change in a Wiener process has a linear relationship with the time interval. It follows that when Markov processes are considered, the variances of the changes in successive time periods are additive, however, the standard deviations of the changes in successive periods are not[61].

Letting the unit lengths in time, $\Delta t's$, become infinitesimally small, the increment of the Wiener process becomes:

$$dz(t) = \varepsilon(t)\sqrt{(dt)}$$

The behaviour of stock and commodity prices, S, is assumed to follow a special case of the Wiener process known as geometric Brownian motion with drift. The generalised form of the Wiener process used to model this stochastic behaviour is:

$$dS(t) = \mu\left(S(t),t\right)dt + \sigma\left(S(t),t\right)dz(t)$$

where

$S(t)$ is a diffusion process

$\mu\left(S(t),t\right)$ is the expected drift rate, and,

$\sigma\left(S(t),t\right)$ is the standard deviation of the diffusion process.

Adapting geometric Brownian motion with drift, $\mu\left(S(t),t\right)$ and $\sigma\left(S(t),t\right)$ become μS and σS independent of time respectively where μ (the drift coefficient) and σ (the

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[61] See Hull, 2009.

diffusion coefficient) are constants. We then formally model the infinitesimal change in the stock/commodity spot price, S, as:

$$dS = \mu S dt + \sigma S dz$$

$$dS = S(\mu dt + \sigma dz)$$

$$\frac{dS}{S} = \mu dt + \sigma dz$$

where

μ is the local trend (drift) in the price

dt is the time differential

σ is the instantaneous standard deviation of the spot price

dz is an increment to a standard Wiener process

The Wiener process implies that over any finite interval of time, percentage changes in the stock/commodity price ($\Delta s / s$) are normally distributed and thus absolute changes in the price (Δs) are log-normally distributed.

A.2 Ito's Lemma

An incredibly powerful tool for the calculus of continuous-time stochastic processes was developed by Ito and is known as Ito's Lemma. Ito's lemma is most intuitively explained as a Taylor series expansion for the Weiner process. In normal calculus, one would write a Taylor series expansion of the differential of a function and allow the higher order terms to vanish in the limit. Formally, Ito's lemma keeps the second order term and the differential dF is written as:

$$dF = F_x dx + F_t dt + \frac{1}{2} F_{xx} (dx)^2$$

Assuming the Ito process

$$dx = adt + bdz$$

Expanding $f(x,t)$ in a Taylor series in x and t gives

$$df = f_x dx + f_t dt + \frac{1}{2} f_{xx} dx^2 + \ldots$$

Substituting for dx

$$df = f_x (adt + bdz) + f_t dt + \frac{1}{2} f_{xx} (a^2 dt^2 + 2abdtdz + b^2 dz^2) + \ldots$$

In the limit as $dt \to 0$, the terms dt^2 and $dtdz$ vanish and $dz^2 \to dt$. Thus substituting and collecting term leaves us with

$$df = (af_x + f_t + \frac{1}{2} b^2 f_{xx}) dt + bf_x dz$$

Let us now consider the special case where changes in spot prices follow a process of geometric Brownian motion as described in the previous section

$$dS = S(\mu dt + \sigma dz)$$

Thus

$$F(S) = \log S$$

$$F_t = 0$$

$$F_s = \frac{1}{S}$$

$$F_{ss} = -\frac{1}{S^2}$$

Using Ito's lemma gives

$$\ln S = F_s dS + \frac{1}{2} F_{ss} S^2 \sigma^2 dt$$

Substituting for dS

$$\ln S = (\frac{1}{S})(\mu S dt + \sigma S dz) - (\frac{1}{2S^2}) S^2 \sigma^2 dt$$

$$\ln S = (\frac{1}{S})(\mu S dt + \sigma S dz) - \frac{1}{2} \sigma^2 dt$$

$$\ln S = \mu dt + \sigma dz - \frac{1}{2} \sigma^2 dt$$

$$\ln S = (\mu - \frac{\sigma^2}{2}) dt + \sigma dz$$

Where lnS follows a Wiener process, the change in lnS between time 0 and T is normally distributed with mean $(\mu - \frac{\sigma^2}{2})T$ and variance $\sigma^2 T$. That is

$$\ln S_T - \ln S_0 = \phi\left[(\mu - \frac{\sigma^2}{2})T, \sigma^2 T \right]$$

and

$$\ln S_T = \ln S_0 + (\mu - \frac{\sigma^2}{2})T + \sigma dz_T$$

$$S_T = S_0 \exp((\mu - \frac{\sigma^2}{2})T + \sigma dz_T).$$

A.3 The Brennan and Schwartz (1985b) Model

From the previous sections we model the stochastic process for the spot price as

$$\frac{dS}{S} = \mu dt + \sigma dz \qquad (1)$$

The Brennan and Schwartz (1985b) model calculates the value of a mine, H, by assuming that it is a function of the current spot price of the underlying commodity, S, the mine's resource inventory, Q, and time, t. The value of the mine is then denoted as

$$H(S, Q, t)$$

Applying Ito's lemma to find the total differential of this function results in the instantaneous change in the value of the mine given by:

$$dH(t) = \frac{\delta H}{\delta S} dS + \frac{\delta H}{\delta Q} dQ + \frac{\delta H}{\delta t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 H}{\delta S^2} dt \qquad (2)$$

The mine is in one of two possible states – either it is open or closed. The function determining the output rate when the mine is open is given by the expression $q(S, Q, t)$. In the case of the mine being open the output production rate of the mine is assumed to be able to be varied without cost between an upper and lower bound, while for a closed mine, the output rate is zero. The instantaneous change in the mine inventory is determined by the output rate and given by

$$dQ = -q dt \qquad (3)$$

where

q is the annual output rate of production.

The value of the after-tax cash flows (analogous to a continuous dividend rate on a stock) yielded from the mine is given by the expression

$$q(S-A)-M(1-j)-\lambda_j H-T \qquad (4)$$

where

$A(q,Q,t)$ is the average cost of extracting at the output rate q at time t and inventory Q;

$M(t)$ is the after-tax fixed cost rate of maintaining the closed mine at time t;

λ_j is the proportional rate of tax on the mine (i.e. property tax);

j is the indicator variable given the relative status of the mine, taking a value of 1 for an open mine and 0 for a closed mine; and

$T(q,Q,S,t)$ is the total income tax and royalties levied on the operating mine.

Brennan and Schwartz note that although the tax function is arbitrary, it is assumed in this case to be given as:

$$T(q,Q,S,t)=t_1qS+\max\{t_2q[S(1-t_1)-A],0\}$$

where

t_1 is the royalty rate, and,

t_2 is the income tax rate.

Let us first consider a portfolio consisting of a long position in the mine which is instantaneously hedged for dt with a short position in $\frac{\delta H}{\delta S} / \frac{\delta F}{\delta S}$ futures (contracts) written on the underlying commodity where, by definition, F is the futures price.

The instantaneous rate of return earned by an individual who purchases (goes long) one unit of the commodity is given by the expression

$$\frac{dS}{S} + \frac{\kappa(S)dt}{S} \qquad (5)$$

where

$\kappa(S)$ is the net convenience yield of the commodity

Brennan and Schwartz assume that the convenience yield is a function only of the current spot price. An investor, long one unit of the commodity, can hedge this position by writing

$$1/\frac{\delta F}{\delta S} \qquad (6)$$

futures contracts. Thus the instantaneous return on a portfolio consisting of a long position of one unit of the commodity (5) and a short position of $(F_s)^{-1}$ futures contracts (6) results in the instantaneous return per dollar of investment (including the net convenience yield) being:

$$\frac{dS}{S} + \frac{\kappa(S)dt}{S} - \frac{1}{S\frac{\delta F}{\delta S}} dF \qquad (7)$$

The instantaneous change in the futures price, dF, can be derived as follows.

Let $F(S,\psi)$ represent the futures price

where

$\psi = T - t$ represents time t for delivery of one unit of the commodity at time T (effectively the time left to expiry of the futures contract).

Thus applying Ito's lemma gives the instantaneous change in the futures price as

$$dF = (-\frac{\delta F}{\delta \psi} + \frac{1}{2}\frac{\delta^2 F}{\delta S^2}\sigma^2 s^2)dt + \frac{\delta F}{\delta S}ds \qquad (8)$$

Since the return on the portfolio is known with certainty it must therefore be equal to the risk free interest rate, ρdt. Brennan and Schwartz make the assumption that the riskless interest rate is constant. Thus substituting (8) into (7) and setting the right hand side equal to ρdt gives the following partial differential equation:

$$\frac{dS}{S} + \frac{\kappa(S)dt}{S} - \frac{1}{S\frac{\delta F}{\delta S}}\left[(-\frac{\delta F}{\delta \psi} + \frac{1}{2}\frac{\delta^2 F}{\delta S^2}\sigma^2 S^2)dt + \frac{\delta F}{\delta S}dS\right] = \rho dt$$

simplifying this yields

$$\Rightarrow dS + \kappa(S)dt - \frac{1}{\frac{\delta F}{\delta S}}\left[(-\frac{\delta F}{\delta \psi} + \frac{1}{2}\frac{\delta^2 F}{\delta S^2}\sigma^2 S^2)dt + \frac{\delta F}{\delta S}dS\right] = S\rho dt$$

$$\Rightarrow dS + \kappa(S)dt + \frac{1}{\frac{\delta F}{\delta S}}\frac{\delta F}{\delta \psi}dt - \frac{1}{\frac{\delta F}{\delta S}}\frac{1}{2}\frac{\delta^2 F}{\delta S^2}\sigma^2 S^2 dt - \frac{1}{\frac{\delta F}{\delta S}}\frac{\delta F}{\delta S}dS = S\rho dt$$

$$\Rightarrow \frac{\delta F}{\delta S}dS + \frac{\delta F}{\delta S}\kappa(S)dt + \frac{\delta F}{\delta \psi}dt - \frac{1}{2}\frac{\delta^2 F}{\delta S^2}\sigma^2 S^2 dt - \frac{\delta F}{\delta S}dS = \frac{\delta F}{\delta S}S\rho dt$$

$$\Rightarrow \frac{\delta F}{\delta S} S\rho dt - \frac{\delta F}{\delta S}\kappa(S)dt - \frac{\delta F}{\delta \psi}dt + \frac{1}{2}\frac{\delta^2 F}{\delta S^2}\sigma^2 S^2 dt = 0$$

$$\Rightarrow \left[\frac{\delta F}{\delta S}(S\rho dt - \kappa(S)) - \frac{\delta F}{\delta \psi} + \frac{1}{2}\frac{\delta^2 F}{\delta S^2}\sigma^2 S^2\right] dt = 0$$

$$\Rightarrow \frac{1}{2}\frac{\delta^2 F}{\delta S^2}\sigma^2 S^2 + \frac{\delta F}{\delta S}(S\rho dt - \kappa(S)) - \frac{\delta F}{\delta \psi} = 0 \qquad (9)$$

The futures price is given by the solution to (9) subject to the boundary condition, $F(S,0) = S$. Because the futures price is a function of the current spot price and the interest rate, ρ, is assumed constant, it is possible to estimate the parameters of the convenience yield directly from this relationship. If the convenience yield is proportional to the spot price then

$$\kappa(S,t) = \kappa S$$

The futures price, independent of the stochastic process of the spot price, is given by

$$F(S,\tau) = Se^{(\rho-\kappa)\tau}$$

Substituting (9) into (8) the instantaneous change in the futures price may then be expressed in terms of the convenience yield and the instantaneous change in the spot price as follows:

$$dF = \frac{\delta F}{\delta S}\left[S(\mu - \rho) + \kappa\right]dt + \frac{\delta F}{\delta S} S\sigma dZ \qquad (10)$$

We are now in a position to derive the return on the hedged mine portfolio. First consider that the rate of return on a long investment in stock will stem from two sources - the capital appreciation of the stock price and the dividend income. Similarly the return on the mine consists of the instantaneous change in the value of the mine, which is given by combining equations (2) and (3):

$$\frac{\delta H}{\delta S}dS - q\frac{\delta H}{\delta Q}dt + \frac{\delta H}{\delta t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 H}{\delta S^2}$$

and the value of the after-tax cash flows (analogous to a continuous dividend rate on a stock) yielded from the mine which is given by equation (4):

$$q(S-A) - M(1-j) - \lambda_j H - T$$

Now consider that the long position of the portfolio is hedged by a short position in $\frac{\delta H}{\delta S} / \frac{\delta F}{\delta S}$ futures contracts over the underlying commodity. Together with the change in the futures price given by equation (10) yields the following expression:

$$-\frac{\delta H}{\delta S} / \frac{\delta F}{\delta S}\left[\frac{\delta F}{\delta S}\left[S(\mu-\rho)+\kappa\right]dt + \frac{\delta F}{\delta S}S\sigma dZ\right]$$

Combining these expressions the return on the portfolio is written as follows:

$$\left[\frac{\delta H}{\delta S}dS - q\frac{\delta H}{\delta Q}dt + \frac{\delta H}{\delta t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 H}{\delta S^2}\right]dt + \left[q(S-A) - M(1-j) - \lambda_j H - T\right]dt$$

$$+\left[-\frac{\delta H}{\delta S} / \frac{\delta F}{\delta S}\left[\frac{\delta F}{\delta S}\left[S(\mu-\rho)+\kappa\right]dt + \frac{\delta F}{\delta S}S\sigma dZ\right]\right]$$

and simplifying yields

$$\frac{1}{2}\sigma^2 S^2 \frac{\delta^2 H}{\delta S^2}dt - q\frac{\delta H}{\delta Q}dt + \frac{\delta H}{\delta t}dt + (\rho S - \kappa)\frac{\delta H}{\delta S}dt + q(S-A)dt - M(1-j)dt - Tdt - \lambda_j Hdt$$

(11)

Ignoring the possibility of expropriation, the return of the hedged portfolio is non-stochastic (it is known with certainty). In the absence of arbitrage opportunities between the mine and the portfolio, the return on the investment must be equal to the current market riskless rate of interest. Thus equating (11) to the riskless interest rate, $\rho H dt$, gives

$$\frac{1}{2}\sigma^2 S^2 \frac{\delta^2 H}{\delta S^2} dt - q\frac{\delta H}{\delta Q} dt + \frac{\delta H}{\delta t} dt + (\rho S - \kappa)\frac{\delta H}{\delta S} dt + q(S - A)dt$$

$$- M(1 - j)dt - Tdt - \lambda_j H dt = \rho H dt$$

simplifying yields

$$\frac{1}{2}\sigma^2 S^2 \frac{\delta^2 H}{\delta S^2} + (\rho S - \kappa)\frac{\delta H}{\delta S} - q\frac{\delta H}{\delta Q} + \frac{\delta H}{\delta t} + q(S - A) - M(1 - j) - T - (p + \lambda_j)H = 0 \quad (12)$$

Where, as before, $j = 1$ for the open mine and $j = 0$ for the closed mine.

The mine operator wields several managerial options pertaining to the operational status of the mine that may be exercised as uncertainty evolves over time. The Brennan and Schwartz model analyses three specific options including the decision to close the open mine, to open the closed mine or abandon the closed mine. The relevant operating policy is determined by the following critical commodity output prices:

$S_1(Q,t)$ is the output price at which the open mine is closed down or abandoned;

$S_2(Q,t)$ is the output price at which closed mine is opened; and

$S_0(Q,t)$ is the output price at which to abandon a previously closed mine.

The value of the mine satisfies equation (15) for any operating policy. The value-maximising output and the value of the mine under the value maximising policy thus satisfy the following two equations.

For the open mine $V(S,Q,t)$:

$$\max_{q\in(0,\bar{q})}\left[\frac{1}{2}\sigma^2 S^2\frac{\delta^2 V}{\delta S^2}+(\rho S-\kappa)\frac{\delta V}{\delta S}-q\frac{\delta V}{\delta Q}+q(S-A)-T-(\rho+\lambda_1)V\right]=0 \qquad (16)$$

and for the closed mine $W(S,Q,t)$:

$$\frac{1}{2}\sigma^2 S^2\frac{\delta^2 W}{\delta S^2}+(\rho-\kappa)\frac{\delta W}{\delta S}+\frac{\delta W}{\delta t}-M-(\rho+\lambda_0)W=0 \qquad (17)$$

Brennan and Schwartz specify a number of boundary conditions applicable to their model. Importantly, these boundary conditions are instrumental in determining the critical output prices at which to exercise various operational policies. The respective costs of exercising each option are denoted by K. Costs of closing the open mine, K_1, include, for example, costs payable such as redundancies and storage costs while the relevant costs of opening a closed mine, K_2, include recommissioning expenses such as hiring and training costs. An abandoned mine incurs no ongoing costs but cannot be reopened.

Accordingly, since the operational policies are assumed to be known by investors, we have

$$W(S_0^*,Q,t)=0$$

which stipulates that the value of a closed mine that is abandoned at the critical price s_0^* is zero;

$$V(S_1^*, Q, t) = \max[W(S_1^*, Q, t) - K_1(Q, t), 0]$$

which stipulates that at the critical price s_1^* the difference between the values of the open and closed mines is the costs of closing the open mine; and

$$W(S_2^*, Q, t) = V(S_2^*, Q, t) - K_2(Q, t)$$

which stipulates that at the critical price s_2^* the difference between the values of the open and closed mines is the costs of opening the closed mine.

Assuming that the value of the exhausted mine and abandoned mine is zero we have, respectfully,

$$W(S, 0, t) = V(S, 0, t) = 0$$

and

$$\frac{\delta W}{\delta S}(S_0^*, Q, t) = 0;$$

To ensure smoothness and continuity of the functions at the physical boundary and transition points we have

$$\frac{\delta V}{\delta S}(S_1^*, Q, t) = \begin{cases} \frac{\delta W}{\delta S}(S_1^*, Q, t) & if \quad W(S_1^*, Q, t) - K_1(Q, t) \geq 0, \\ 0 & if \quad W(S_1^*, Q, t) - K_1(Q, t) < 0; \end{cases}$$

$$\frac{\delta W}{\delta S}(S_2^*, Q, t) = \frac{\delta V}{\delta S}(S_2^*, Q, t).$$

Finally, the mines values depend on calendar time only because the costs A, M , K_1 and K_2 as well as the convenience yield κ depend on time. Brennan and Schwartz make the assumption that the inflation rate increases uniformly at a constant rate across all prices and that the convenience yield is a linear function of the commodity price, κS, as to remove the time dependence, t.

Consequently, we can define the following equations for the deflated variables:

$$A(q,Q,t)e^{-\pi t} = a(q,Q) \,,$$

$$M(t)e^{-\pi t} = f \,,$$

$$K_1(Q,t)e^{-\pi t} = k_1(Q) \,,$$

$$K_2(Q,t)e^{-\pi t} = k_2(Q) \,,$$

$$Se^{-\pi t} = s \,,$$

$$V(S,Q,t)e^{-\pi t} = v(s,Q) \,,$$

$$W(S,Q,t)e^{-\pi t} = w(s,Q) \,.$$

Using these in conjunction with the chain rule (for partial derivatives) results in:

$$\frac{\delta V}{\delta S} = \frac{\delta v}{\delta s}$$

$$\frac{\delta^2 V}{\delta S^2} = e^{-\pi t}\frac{\delta^2 v}{\delta s^2}$$

$$\frac{\delta V}{\delta Q} = e^{\pi t}\frac{\delta v}{\delta Q}$$

$$\frac{\delta V}{\delta t} = e^{\pi t}\pi s\frac{\delta v}{\delta s} + \pi e^{\pi t} v$$

$$\frac{\delta W}{\delta S} = \frac{\delta w}{\delta s}$$

$$\frac{\delta^2 W}{\delta S^2} = e^{-\pi t}\frac{\delta^2 w}{\delta s^2}$$

$$\frac{\delta W}{\delta t} = e^{\pi t}\pi s\frac{\delta w}{\delta s} + \pi e^{\pi t} w$$

Substitution of these results into the inflated expressions gives the Brennan and Schwartz model equations for the present value of the mine for both open $v(s,Q)$ and closed $w(s,Q)$ states:

$$\max_{q\in\{0,\bar{q}\}}\left[\frac{1}{2}\sigma^2 s^2\frac{\delta^2 v}{\delta s^2} + (r-\kappa)\frac{\delta v}{\delta s} - q\frac{\delta v}{\delta Q} + q(s-a) - \tau - (r+\lambda_1)v\right] = 0$$

$$\frac{1}{2}\sigma^2 s^2\frac{\delta^2 w}{\delta s^2} + (r-\kappa)\frac{\delta w}{\delta s} - f - (r+\lambda_0)w = 0$$

where

$r = \rho - \pi$ is the real riskless rate,

$\tau = t_1 qs + \max\left[t_2 q(s(1-t_1)-a),0\right].$

Appendix B

Source Code for the Renewable Resource Model Implementation

```
unitRRMainForm;


interface


uses
  Windows, Messages, SysUtils, Variants, Classes, Graphics, Controls, Forms,dmComms,
  Dialogs, ComCtrls, ExtCtrls, DB, DBCtrls,Math,
nxdb, Grids, Wwdbigrd, Wwdbgrid, StdCtrls, wwdblook, Buttons,
WallPaper, Shader, LayeredForm, BtnListB, fcButton, fcImgBtn, fcShapeBtn,
wwrcdpnl, DBGrids, TeEngine, TeeSurfa, TeeProcs, Chart, DBChart, rtflabel,
TeeTools, Series, ppDB, ppBands, ppCache, ppClass, ppProd, ppReport,
ppComm, ppRelatv, ppDBPipe, ppDBBDE, Menus, Spin, Mask;

type
TMainformDialog = class(TForm)
tblMines: TnxTable;
dsMines: TDataSource;
tblValueSQ: TnxTable;
tblValueQS: TnxTable;
tblInfinteResource: TnxTable;
```

```
dsInfinteResource: TDataSource;
    SaveDialog1: TSaveDialog;
dsValueQS: TDataSource;
tblInfinteResourceMineID: TIntegerField;
tblInfinteResourceiS: TIntegerField;
tblInfinteResources: TFloatField;
tblInfinteResourcevs: TFloatField;
tblInfinteResourcews: TFloatField;
tblInfinteResourcevApprox: TFloatField;
tblInfinteResourcewApprox: TFloatField;
    Label13: TLabel;
    GroupBox2: TGroupBox;
    Label17: TLabel;
    Label14: TLabel;
    Label12: TLabel;
    Label19: TLabel;
    Label20: TLabel;
    Label22: TLabel;
    Label23: TLabel;
    Label24: TLabel;
    Label25: TLabel;
    Label26: TLabel;
    Label27: TLabel;
    Label28: TLabel;
    DBEdit6: TDBEdit;
```

```
DBEdit3: TDBEdit;
DBEdit1: TDBEdit;
DBEdit5: TDBEdit;
DBEdit7: TDBEdit;
DBEdit8: TDBEdit;
DBEdit2: TDBEdit;
GroupBox3: TGroupBox;
Label35: TLabel;
Label37: TLabel;
Label38: TLabel;
Label40: TLabel;
Label29: TLabel;
Label30: TLabel;
Label33: TLabel;
Label34: TLabel;
Label53: TLabel;
Label54: TLabel;
Label32: TLabel;
Label36: TLabel;
DBEdit12: TDBEdit;
DBEdit14: TDBEdit;
DBEdit9: TDBEdit;
DBEdit11: TDBEdit;
DBEdit21: TDBEdit;
GroupBox4: TGroupBox;
```

```
Label41: TLabel;
Label42: TLabel;
Label43: TLabel;
Label44: TLabel;
Label45: TLabel;
Label46: TLabel;
Label47: TLabel;
Label48: TLabel;
Label49: TLabel;
Label50: TLabel;
Label51: TLabel;
Label52: TLabel;
Label31: TLabel;
Label15: TLabel;
Label16: TLabel;
Label18: TLabel;
Label39: TLabel;
Label55: TLabel;
Label56: TLabel;
Label57: TLabel;
Label58: TLabel;
DBEdit15: TDBEdit;
DBEdit16: TDBEdit;
DBEdit17: TDBEdit;
DBEdit18: TDBEdit;
```

```
DBEdit19: TDBEdit;
DBEdit20: TDBEdit;
Panel4: TPanel;
DBEdit4: TDBEdit;
DBEdit10: TDBEdit;
GroupBox1: TGroupBox;
RTFLabel1: TRTFLabel;
RTFLabel2: TRTFLabel;
RTFLabel3: TRTFLabel;
RTFLabel4: TRTFLabel;
RTFLabel5: TRTFLabel;
RTFLabel6: TRTFLabel;
RTFLabel7: TRTFLabel;
DBText1: TDBText;
DBText2: TDBText;
DBText3: TDBText;
DBText4: TDBText;
DBText5: TDBText;
DBText6: TDBText;
DBText7: TDBText;
SpeedButton2: TSpeedButton;
SpeedButton3: TSpeedButton;
DBText8: TDBText;
DBText9: TDBText;
RTFLabel8: TRTFLabel;
```

```
RTFLabel9: TRTFLabel;
StringGrid1: TStringGrid;
DBChart2: TDBChart;
SpeedButton4: TSpeedButton;
SpeedButton1: TSpeedButton;
Series2: TLineSeries;
Series3: TLineSeries;
DBNavigator1: TDBNavigator;
Panel1: TPanel;
Label1: TLabel;
Edit1: TEdit;
SpeedButton5: TSpeedButton;
StaticText1: TStaticText;
StaticText2: TStaticText;
StaticText3: TStaticText;
Label3: TLabel;
Label4: TLabel;
Label5: TLabel;
CheckBox1: TCheckBox;
StaticText4: TStaticText;
nxTable1: TnxTable;
IntegerField1: TIntegerField;
IntegerField2: TIntegerField;
FloatField1: TFloatField;
FloatField2: TFloatField;
```

```
    FloatField3: TFloatField;
    FloatField4: TFloatField;
    FloatField5: TFloatField;
    nxTable2: TnxTable;
    IntegerField3: TIntegerField;
    IntegerField4: TIntegerField;
    FloatField6: TFloatField;
    FloatField7: TFloatField;
    FloatField8: TFloatField;
    FloatField9: TFloatField;
    FloatField10: TFloatField;
procedureFormCreate(Sender: TObject);
procedure SpeedButton2Click(Sender: TObject);
    Function Vinf(s:Double):Double;
    Function Vsinf(s:Double):Double;
    Function Vssinf(s:Double):Double;
    Function Vzinf(s,z:Double):Double;
    Function Winf(s:Double):Double;
    Function CalcC1A:Double;
    Function CalcC0A:Double;
    Function CalcC1B:Double;
    Function CalcC0B:Double;
    Function CalcS1ANR(S1:Double):Double;
    Function CalcS2ANR(S2:Double):Double;
    Function CalcSBNR(S:Double):Double;
```

```
   Function CalcSBGS(S:Double):Double;
   Function CalcS0:Double;
procedure SpeedButton1Click(Sender: TObject);
procedure SpeedButton3Click(Sender: TObject);
procedure SpeedButton4Click(Sender: TObject);
procedure SpeedButton5Click(Sender: TObject);
private
protected
public
{ Public declarations }
s,delS,delS2,maxS,firstS,firstQ,delQ,Q0:Double;
nS,MineID:Integer;
nIters,nQ,nZ,MidRow,nSkipForDisplay:Integer;
    L0,L1,f,r,Kappa,Sigma2,Rho,t1,t2,k1,k2,s0,s1,s2,rq,a0,C0,C1,n0,n1,api:Double;
p,m,RelErr,h,k,Z0,delZ:Double;
vQ:array[0..11000,0..500] of Double;
    Function Dlr(amount:Double):String;
    Function Num(aNumber:Double):String;
end;

var
MainformDialog: TMainformDialog;

implementation
```

```
{$R *.dfm}


Procedure SD1toDMY(var SD1,d,m,y:Integer);
var
i:Integer;
x:array[1..12] of integer;
begin
y:=trunc(sd1/365.25);
d:=sd1-y*365-(y div 4);
if (y mod 4)=0 then d:=d+1;
for i:=1 to 12 do x[i]:=31;
x[4]:=30;x[6]:=30;x[9]:=30;x[11]:=30;
if (y mod 4)=0 then x[2]:=29 else x[2]:=28;
m:=1;
while (m<12) and (x[m]<=d) do
begin
d:=d-x[m];
inc(m);
end;
if d=0 then
begin
dec(m);
d:=x[m];
end;
y:=2000+y;
```

```
end;

function YMDfrom2K(JD:integer):string;
var Y,M,D,N,L: Longint;
Begin
  L:= JD + 2520114;
  N:=trunc(4*L/146097);
  L:= L - trunc((146097*N + 3) / 4);
  Y:=trunc(4000*(L + 1)/1461001);
  L:= L - trunc(1461*Y / 4) + 31;
  M:=trunc(80*L/2447);
  D:= L - trunc(2447*M / 80);
  L:=trunc(M/11);
  M:= M + 2 - 12*L;
  Y:= 100*(N - 49) + Y + L;
result:=inttostr(D)+'/'+inttostr(M)+'/'+inttostr(Y);
end;

procedureTMainformDialog.FormCreate(Sender: TObject);
begin
Application.CreateForm(TDataModule4,DataModule4);
  DataModule4.nxDatabase1.AliasPath:=extractFilePath(ParamStr(0));
```

```
RelErr:=0.00001;
tblMines.Open;
tblInfinteResource.Open;
tblValueQS.Open;
  nxTable1.Open;
  nxTable2.Open;
  With StringGrid1 do
begin
Cells[0,0]:=' Iter';
Cells[1,0]:=' Rel Error';
Cells[2,0]:='  C0';
Cells[3,0]:='  C1';
Cells[4,0]:='  s1';
Cells[5,0]:='  s2';
Cells[6,0]:='  s0';
Cells[7,0]:=' s1 s2 iters';
end;
end;

procedure TMainformDialog.SpeedButton2Click(Sender: TObject);
begin
withtblMines do
begin
if State in dsEditModes then Post;
MineID:=FieldBYName('MineID').asInteger;
```

```
delS:=FieldByName('delS').asFloat;
maxS:=FieldByName('maxS').asFloat;
firstS:=FieldByName('firstS').asFloat;
nS:=Round(maxS/delS);
    L0:=FieldByName('Lamda0').asFloat;
    L1:=FieldByName('Lamda1').asFloat;
f:=FieldByName('f').asFloat;
    Rho:=FieldByName('Rho').asFloat;
aPi:=FieldByName('Pi').asFloat;
//    r:=rho-aPi;                    leaving in case we need to put back in for whatever
reason
r:=rho;
    Kappa:=FieldByName('Kappa').asFloat;
    Sigma2:=FieldByName('Sigma2').asFloat;
    t1:=FieldByName('t1').asFloat;
    t2:=FieldByName('t2').asFloat;
    k1:=FieldByName('k1').asFloat;
    k2:=FieldByName('k2').asFloat;
rq:=FieldByName('q').asFloat;
    a0:=FieldByName('a').asFloat;
    C0:=FieldByName('C0').asFloat;
    C1:=FieldByName('C1').asFloat;
n1:=-((r-Kappa-Sigma2/2)+sqrt(Sqr(r-Kappa-Sigma2/2)+2*Sigma2*(r+L1)))/Sigma2;
n0:=(-(r-Kappa-Sigma2/2)+sqrt(Sqr(r-Kappa-Sigma2/2)+2*Sigma2*(r+L0)))/Sigma2;
p:=rq*(1-t1)*(1-t2)/(Kappa+L1);
```

```
m:=rq*a0*(1-t2)/(r+L1);
    s2:=2.5*m/p;
    s1:=0.5*m/p;
  //  s1:=max(0.01,m-f/(r+L0)-k1);
  //  s2:=max(s1*1/5,m-f/(r+L0)+k2);
    s0:=s1/4;
    Edit;
FieldByName('r').asFloat:=r;
FieldByName('p').asFloat:=p;
FieldByName('m').asFloat:=m;
FieldByName('n0').asFloat:=n0;
FieldByName('n1').asFloat:=n1;
FieldByName('s0').asFloat:=s0;
FieldByName('s1').asFloat:=s1;
FieldByName('s2').asFloat:=s2;
    Post;
end;
tblInfinteResource.DeleteRecords;
end;

functionTMainformDialog.Vinf(s: Double): Double;
begin
  Result:=C1*Power(s,n1)+p*s-m;
end;
```

```
functionTMainformDialog.Vsinf(s: Double): Double;
begin
  Result:=C1*n1*Power(s,n1-1)+p;
end;

functionTMainformDialog.Vssinf(s: Double): Double;
begin
  Result:=C1*n1*(n1-1)*Power(s,n1-2);
end;

functionTMainformDialog.Vzinf(s,z: Double): Double;
var
tau:Double;
begin
tau:=t1*rq*s+t2*rq*(s*(1-t1)-a0);
Result:=-(sigma2*sqr(s)*Vssinf(s)/2+(r-kappa)*s*Vsinf(s)+rq*(s-a0)-tau-(r+L1)*Vinf(s))/(sqr(z)*rq);
end;

functionTMainformDialog.Winf(s: Double): Double;
begin
  Result:=C0*Power(s,n0)-f/(r+L0);
end;

procedure TMainformDialog.SpeedButton1Click(Sender: TObject);
var
```

```
i:Integer;
s,v,w:Double;
begin
withtblInfinteResource do
begin
DeleteRecords;
Database.StartTransaction;
for i:=1 to nS do
begin
s:=i*delS;
if s>=FirstS then
begin
ifFindKey([MineID,i]) then Edit else Insert;
FieldByName('iS').AsInteger:=i;
FieldByName('s').asFloat:=s;
v:=Vinf(s);
w:=Winf(s);
FieldByName('v(s)').asFloat:=trunc(v*100)/100;
FieldByName('w(s)').asFloat:=trunc(w*100)/100;
        Post;
end;
end;
Database.Commit;
end;
with nxTable1 do
```

```
begin
DeleteRecords;
Database.StartTransaction;
for i:=1 to nS do
begin
s:=i*delS;
if s>=S1 then
begin
ifFindKey([MineID,i]) then Edit else Insert;
FieldByName('iS').AsInteger:=i;
FieldByName('s').asFloat:=s;
v:=Vinf(s);
w:=Winf(s);
FieldByName('v(s)').asFloat:=trunc(v*100)/100;
FieldByName('w(s)').asFloat:=trunc(w*100)/100;
        Post;
end;
end;
Database.Commit;
end;
with nxTable2 do
begin
DeleteRecords;
Database.StartTransaction;
for i:=1 to nS do
```

```
begin
s:=i*delS;
if (s>=FirstS) and (s<=S2) then
begin
ifFindKey([MineID,i]) then Edit else Insert;
FieldByName('iS').AsInteger:=i;
FieldByName('s').asFloat:=s;
v:=Vinf(s);
w:=Winf(s);
FieldByName('v(s)').asFloat:=trunc(v*100)/100;
FieldByName('w(s)').asFloat:=trunc(w*100)/100;
        Post;
end;
end;
Database.Commit;
end;
withtblValueQS do
begin
DeleteRecords;
Database.StartTransaction;
for i:=1 to nS do
begin
s:=i*delS;
if s>=FirstS then
begin
```

```
ifFindKey([MineID,i]) then Edit else Insert;
FieldByName('iQ').AsInteger:=0;
FieldByName('iS').AsInteger:=i;
FieldByName('s').asFloat:=s;
v:=Vinf(s);
w:=Winf(s);
FieldByName('v(s)').asFloat:=trunc(v*100)/100;
FieldByName('w(s)').asFloat:=trunc(w*100)/100;
        Post;
end;
end;
Database.Commit;
end;
end;

function TMainformDialog.CalcC0A: Double;
begin
  Result:=p*(Power(s2,n1-1)-Power(s1,n0-1))/(Power(s1,n0-1)*Power(s2,n1-1)-Power(s2,n0-
1)*Power(s1,n1-1));
end;

function TMainformDialog.CalcC1A: Double;
begin
Result:=(C0*n0*Power(s1,n0-1)-p)/(n1*Power(s1,n1-1));
end;
```

```
function TMainformDialog.CalcC0B: Double;
begin
  Result:=(k1*Power(s2,n1)+k2*Power(s1,n1)+p*(s1*Power(s2,n1)-
s2*Power(s1,n1))+f*(Power(s2,n1)-Power(s1,n1))/(r+L0)
        +m*(Power(s1,n1)-Power(s2,n1)))/(Power(s1,n0)*Power(s2,n1)-
Power(s2,n0)*Power(s1,n1));
end;

function TMainformDialog.CalcC1B: Double;
begin
Result:=(C0*Power(s1,n0)-f/(r+L0)-k1+m-p*s1)/Power(s1,n1);
end;

function TMainformDialog.CalcS1ANR(S1: Double): Double;
begin
  Result:=S1;
  Repeat
    s1:=max(0.01,Result);
    Result:=s1-(C1*Power(s1,n1)+p*s1-C0*Power(s1,n0)-m+f/(r+L0)+k1)/(C1*n1*Power(s1,n1-
1)+p-C0*n0*Power(s1,n0-1));
  Until abs(Result-s1)<RelErr;
end;

function TMainformDialog.CalcS2ANR(S2: Double): Double;
```

```
begin
  Result:=S2;
  Repeat
    s2:=max(0.01,Result);
    Result:=s2-(C1*Power(s2,n1)+p*s2-C0*Power(s2,n0)-m+f/(r+L0)-k2)/(C1*n1*Power(s2,n1-
1)+p-C0*n0*Power(s2,n0-1));
  Until abs(Result-s2)<RelErr;
end;

functionTMainformDialog.CalcSBNR(S: Double): Double;
begin
  Result:=S;
nIters:=0;
  Repeat
inc(nIters);
s:=max(0.01,Result);
    Result:=s-(C1*n1*Power(s,n1-1)+p-C0*n0*Power(s,n0-1))/(C1*n1*(n1-1)*Power(s,n1-2)-
C0*n0*(n0-1)*Power(s,n0-2));
  Until abs((Result-s)/Result)<RelErr;
end;

functionTMainformDialog.CalcSBGS(S: Double): Double;
begin
  Repeat
Result:=(C0*n0*Power(s,n0)-C1*n1*Power(s,n1))/p;
```

```
  Until abs((Result-s)/Result)<RelErr;
end;


function TMainformDialog.CalcS0: Double;
begin
  Result:=Power(f/(C0*(r+L0)),1/n0);
end;


procedure TMainformDialog.SpeedButton3Click(Sender: TObject);
var
i,ns1,ns2:Integer;
  Err,C0old,C1old,s1old,s2old:Double;
begin
with StringGrid1 do
begin
RowCount:=2;
for i:=0 to 7 do cells[i,1]:='';
end;
i:=0;
repeat
    C0old:=C0;
    C1old:=C1;
    s1old:=s1;
    s2old:=s2;
    C0:=CalcC0B;
```

```
      C1:=CalcC1B;
      s1:=CalcSBNR(S1);
      ns1:=nIters;
      s2:=CalcSBNR(S2);
      ns2:=nIters;
      s0:=CalcS0;
//      s0:=Min(S1,CalcS0);
      err:=abs((C0old-C0)/C0)+abs((C1old-C1)/C1)+abs((s1old-s1)/s1)+abs((s2old-s2)/s2);
with StringGrid1 do
begin
inc(i);
RowCount:=RowCount+1;
Cells[0,i]:=' '+IntToSTr(i);
Cells[1,i]:=FloatToStrF(err,ffFixed,15,9);
Cells[2,i]:=FloatToStrF(C0,ffFixed,15,9);
Cells[3,i]:=FloatToStrF(C1,ffFixed,15,9);
Cells[4,i]:=FloatToStrF(s1,ffFixed,15,9);
Cells[5,i]:=FloatToStrF(s2,ffFixed,15,9);
Cells[6,i]:=FloatToStrF(s0,ffFixed,15,9);
Cells[7,i]:='  '+IntToStr(ns1)+'  '+IntToStr(ns2);
end;
until err<RelErr;
withtblMines do
begin
      Edit;
```

```
FieldByName('C0').asFloat:=C0;
FieldByName('C1').asFloat:=C1;
FieldByName('s0').asFloat:=s0;
FieldByName('s1').asFloat:=s1;
FieldByName('s2').asFloat:=s2;
    Post;
end;
  SpeedButton1.enabled:=True;
end;

procedure TMainformDialog.SpeedButton4Click(Sender: TObject);
begin
with SaveDialog1 do
if Execute then dbChart2.SaveToBitmapFile(Filename);
end;

procedure TMainformDialog.SpeedButton5Click(Sender: TObject);
var
s,i,w,v:Double;
begin
s:=StrToFloat(Edit1.Text);
v:=Vinf(s);
w:=Winf(s);
  StaticText1.Caption:=FloatToStrF(v,ffCurrency,15,2);
  StaticText2.Caption:=FloatToStrF(w,ffCurrency,15,2);
```

```
  if not checkbox1.checked then i:=(s-a0)*rq/v else i:=(s*(1-t1)-a0)*rq*(1-t2)/v;
  StaticText3.Caption:=FloatToStrF(i*100,ffFixed,15,2)+'%';
  if not checkbox1.checked then i:=(s-a0)*rq/w else i:=(s*(1-t1)-a0)*rq*(1-t2)/w;
  StaticText4.Caption:=FloatToStrF(i*100,ffFixed,15,2)+'%';
end;

functionTMainformDialog.Dlr(amount: Double): String;
begin
  Result:=FloatToStrF(amount,ffCurrency,15,2);
end;

functionTMainformDialog.Num(aNumber: Double): String;
begin
  Result:=FloatToStrF(aNumber,ffExponent,5,2);
end;

end.
```

Appendix C

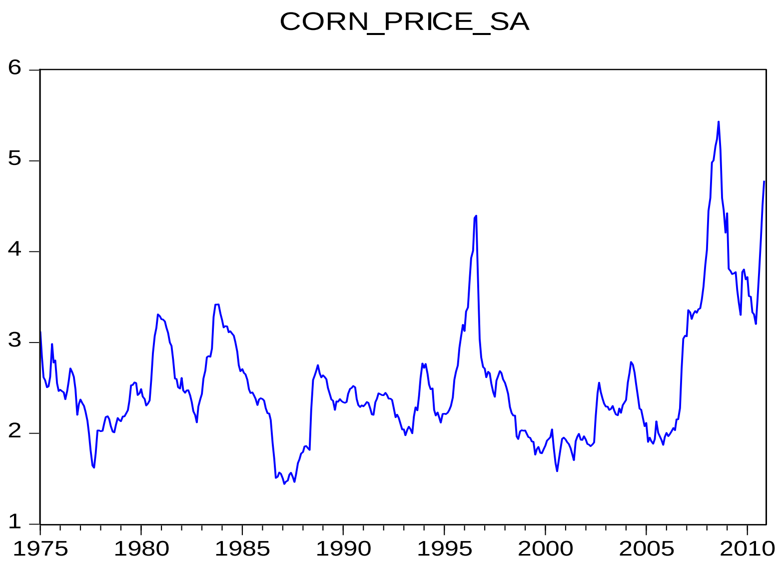
Data for the Case Study

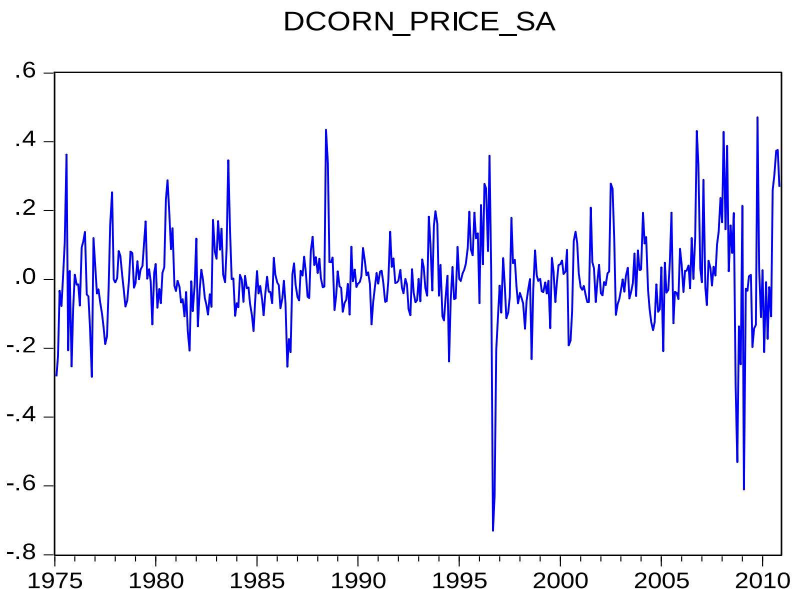
Figure C.1: U.S. Average Monthly Corn Prices, January 1975 – June 2011

Corn: Average prices received by farmers, United States												
Year	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
1975	3.07	2.86	2.67	2.68	2.66	2.68	2.72	2.95	2.76	2.62	2.33	2.37
1976	2.44	2.48	2.50	2.46	2.61	2.74	2.82	2.64	2.60	2.33	2.02	2.24
1977	2.34	2.34	2.35	2.31	2.25	2.12	1.88	1.63	1.60	1.67	1.88	1.97
1978	2.00	2.03	2.15	2.24	2.29	2.28	2.16	2.01	1.98	1.97	2.02	2.09
1979	2.11	2.18	2.22	2.27	2.35	2.49	2.64	2.54	2.51	2.41	2.27	2.38
1980	2.45	2.39	2.40	2.36	2.42	2.49	2.73	2.92	3.01	2.99	3.10	3.19
1981	3.19	3.22	3.25	3.24	3.24	3.17	3.14	2.87	2.55	2.45	2.34	2.39
1982	2.54	2.44	2.46	2.55	2.60	2.57	2.50	2.30	2.15	1.98	2.13	2.26
1983	2.36	2.56	2.71	2.95	3.03	3.04	3.13	3.35	3.32	3.15	3.17	3.15
1984	3.15	3.11	3.21	3.32	3.34	3.36	3.30	3.12	2.90	2.65	2.55	2.56
1985	2.64	2.62	2.67	2.70	2.68	2.64	2.60	2.44	2.29	2.11	2.21	2.29
1986	2.33	2.32	2.29	2.30	2.39	2.32	2.00	1.73	1.45	1.40	1.47	1.50
1987	1.48	1.42	1.47	1.52	1.66	1.69	1.60	1.47	1.49	1.55	1.61	1.72
1988	1.77	1.83	1.86	1.88	1.94	2.41	2.72	2.65	2.60	2.58	2.51	2.53
1989	2.60	2.59	2.60	2.56	2.58	2.52	2.47	2.27	2.29	2.22	2.24	2.27
1990	2.31	2.32	2.37	2.51	2.62	2.63	2.62	2.51	2.32	2.19	2.16	2.22
1991	2.27	2.32	2.39	2.42	2.38	2.31	2.27	2.33	2.33	2.31	2.29	2.33
1992	2.40	2.46	2.49	2.48	2.49	2.47	2.33	2.15	2.16	2.05	1.98	1.97
1993	2.03	2.00	2.10	2.16	2.14	2.09	2.22	2.25	2.21	2.28	2.45	2.67
1994	2.70	2.79	2.74	2.65	2.60	2.61	2.29	2.16	2.19	2.06	1.99	2.13
1995	2.19	2.23	2.30	2.36	2.42	2.51	2.63	2.63	2.69	2.79	2.87	3.07
1996	3.09	3.37	3.51	3.85	4.14	4.20	4.43	4.30	3.56	2.88	2.66	2.63
1997	2.69	2.65	2.79	2.80	2.69	2.56	2.42	2.50	2.52	2.54	2.51	2.52
1998	2.56	2.55	2.55	2.41	2.34	2.28	2.19	1.89	1.84	1.91	1.93	2.00
1999	2.06	2.05	2.06	2.04	1.99	1.97	1.74	1.75	1.75	1.69	1.70	1.82
2000	1.91	1.98	2.03	2.03	2.11	1.91	1.64	1.52	1.61	1.74	1.86	1.97
2001	1.98	1.96	1.96	1.89	1.82	1.76	1.87	1.90	1.91	1.84	1.85	1.98
2002	1.97	1.93	1.94	1.91	1.93	1.97	2.13	2.38	2.47	2.34	2.28	2.32
2003	2.33	2.34	2.33	2.34	2.38	2.34	2.17	2.15	2.20	2.12	2.20	2.31
2004	2.39	2.61	2.75	2.89	2.87	2.79	2.51	2.34	2.20	2.14	2.05	2.04
2005	2.12	1.95	2.02	2.00	1.98	2.03	2.11	1.95	1.90	1.82	1.77	1.92
2006	2.00	2.02	2.06	2.11	2.17	2.14	2.14	2.09	2.20	2.55	2.88	3.01
2007	3.05	3.44	3.43	3.39	3.49	3.53	3.32	3.26	3.28	3.29	3.44	3.77
2008	3.98	4.54	4.70	5.14	5.27	5.47	5.25	5.26	5.01	4.37	4.26	4.11
2009	4.36	3.87	3.85	3.85	3.96	4.01	3.60	3.33	3.25	3.61	3.65	3.60
2010	3.66	3.55	3.55	3.41	3.48	3.41	3.49	3.65	4.08	4.32	4.55	4.82
2011	4.94	5.64	5.53	6.35	6.30	6.58						

Source: *USDA, 2011b*.

Figure C.2: Statistical Tests for Monthly U.S. Corn Price Series





Null Hypothesis: CORN_PRICE_SA has a unit root
Exogenous: Constant
Lag Length: 1 (Automatic based on SIC, MAXLAG=17)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-2.133401	0.2317
Test critical values:	1% level	-3.445373	
	5% level	-2.868058	
	10% level	-2.570306	

*MacKinnon (1996) one-sided p-values.

Null Hypothesis: CORN_PRICE_SA has a unit root
Exogenous: Constant
Bandwidth: 8 (Newey-West using Bartlett kernel)

		Adj. t-Stat	Prob.*
Phillips-Perron test statistic		-2.110415	0.2408
Test critical values:	1% level	-3.445338	
	5% level	-2.868042	
	10% level	-2.570298	

*MacKinnon (1996) one-sided p-values.

Null Hypothesis: CORN_PRICE_SA has a unit root
Exogenous: Constant
Lag Length: 1 (Automatic based on SIC, MAXLAG=17)

		t-Statistic
Elliott-Rothenberg-Stock DF-GLS test statistic		-1.987681
Test critical values:	1% level	-2.570364
	5% level	-1.941563
	10% level	-1.616204

*MacKinnon (1996)

Null Hypothesis: CORN_PRICE_SA is stationary
Exogenous: Constant
Bandwidth: 16 (Newey-West using Bartlett kernel)

		LM-Stat.
Kwiatkowski-Phillips-Schmidt-Shin test statistic		0.343402
Asymptotic critical values*:	1% level	0.739000
	5% level	0.463000
	10% level	0.347000

*Kwiatkowski, Phillips, Schmidt, Shin (1992, Table 1)

Null Hypothesis: D(CORN_PRICE_SA) has a unit root
Exogenous: Constant
Lag Length: 0 (Automatic based on SIC, MAXLAG=17)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-13.38055	0.0000
Test critical values:	1% level	-3.445373	
	5% level	-2.868058	
	10% level	-2.570306	

*MacKinnon (1996) one-sided p-values.

Null Hypothesis: D(CORN_PRICE_SA) has a unit root
Exogenous: Constant
Bandwidth: 4 (Newey-West using Bartlett kernel)

		Adj. t-Stat	Prob.*
Phillips-Perron test statistic		-13.45935	0.0000
Test critical values:	1% level	-3.445373	
	5% level	-2.868058	
	10% level	-2.570306	

*MacKinnon (1996) one-sided p-values.

Null Hypothesis: D(CORN_PRICE_SA) has a unit root
Exogenous: Constant
Lag Length: 8 (Automatic based on SIC, MAXLAG=17)

		t-Statistic
Elliott-Rothenberg-Stock DF-GLS test statistic		-1.042394
Test critical values:	1% level	-2.570466
	5% level	-1.941578
	10% level	-1.616194

*MacKinnon (1996)

Null Hypothesis: D(CORN_PRICE_SA) is stationary
Exogenous: Constant
Bandwidth: 8 (Newey-West using Bartlett kernel)

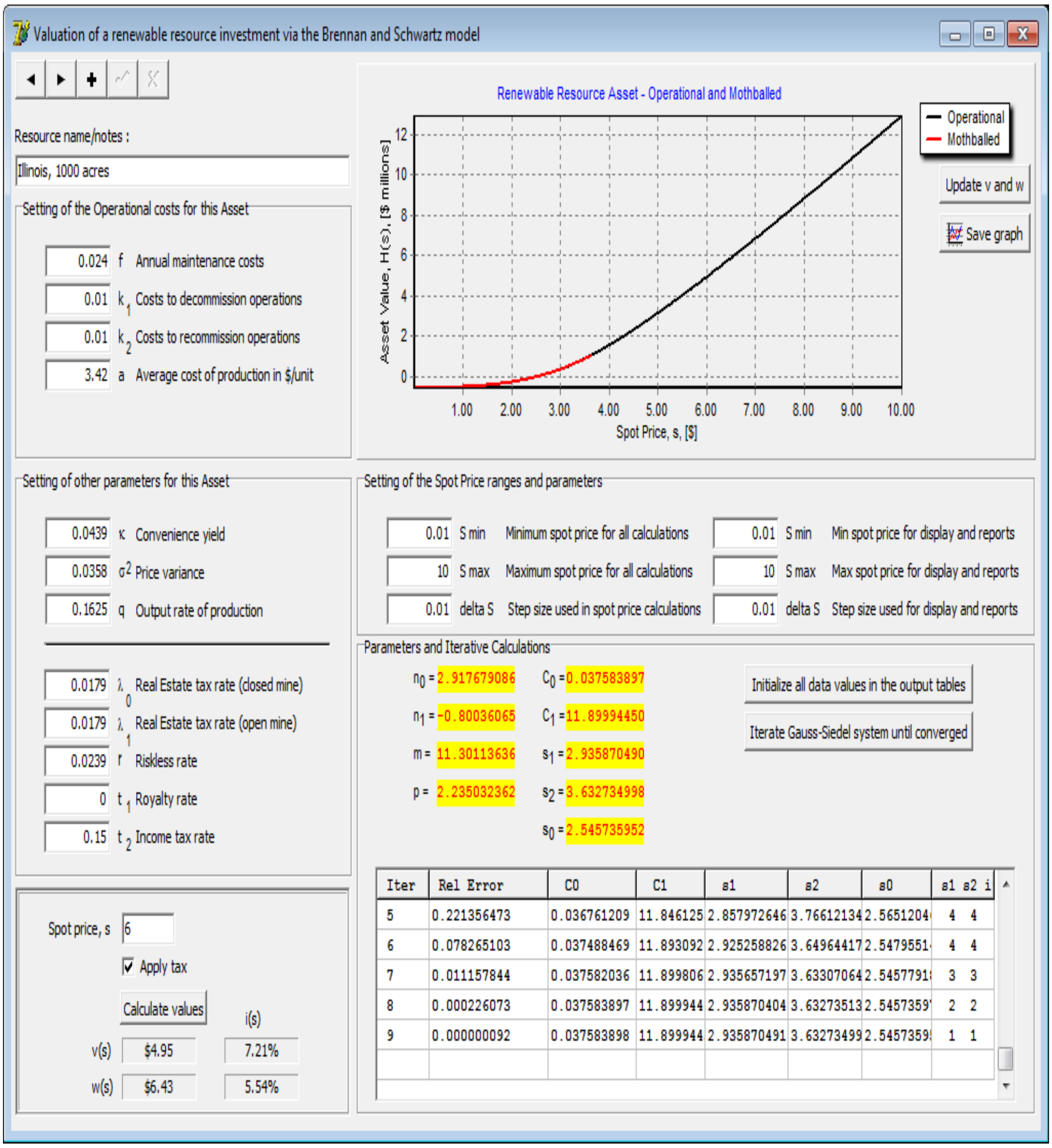
		LM-Stat.
Kwiatkowski-Phillips-Schmidt-Shin test statistic		0.158714
Asymptotic critical values*:	1% level	0.739000
	5% level	0.463000
	10% level	0.347000

*Kwiatkowski, Phillips, Schmidt & Shin (1992, Table 1)

Appendix D

Output Screens for the Case Study

Figure D.1: Evaluation for an Average Corn Cropland Asset in Illinois, 1000 Acres



Iter	Rel Error	C0	C1	s1	s2	s0	s1	s2 i
5	0.221356473	0.036761209	11.846125	2.857972646	3.76612134	2.5651204	4	4
6	0.078265103	0.037488469	11.893092	2.925258826	3.64964417	2.5479551	4	4
7	0.011157844	0.037582036	11.899806	2.935657197	3.63307064	2.5457791	3	3
8	0.000226073	0.037583897	11.899944	2.935870404	3.63273513	2.5457359	2	2
9	0.000000092	0.037583898	11.899944	2.935870491	3.63273499	2.5457359	1	1

Figure D.2: Evaluation for an Average Corn Cropland Asset in Indiana, 250 Acres

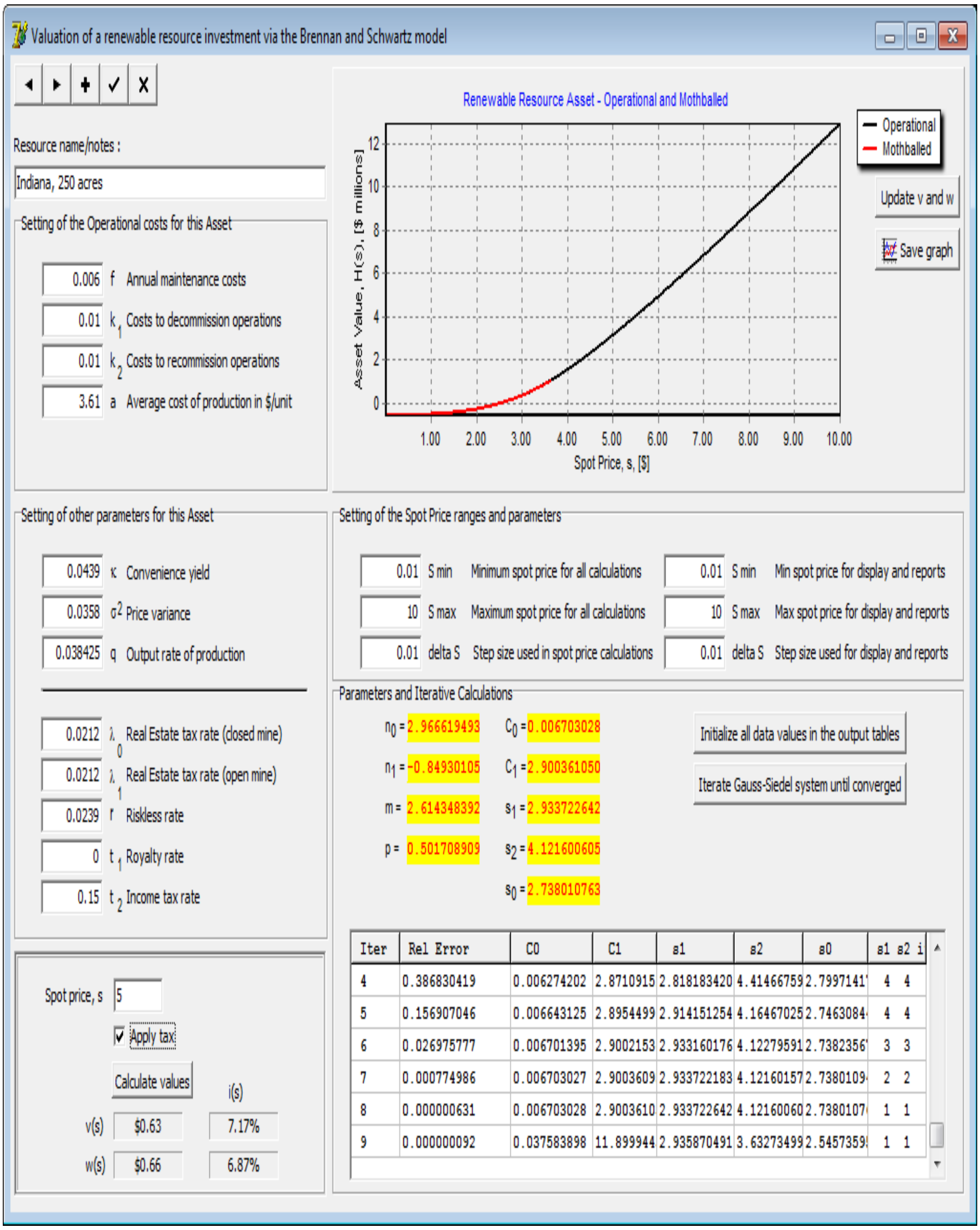


Figure D.3: Evaluation for an Average Corn Cropland Asset in Indiana, 1000 Acres

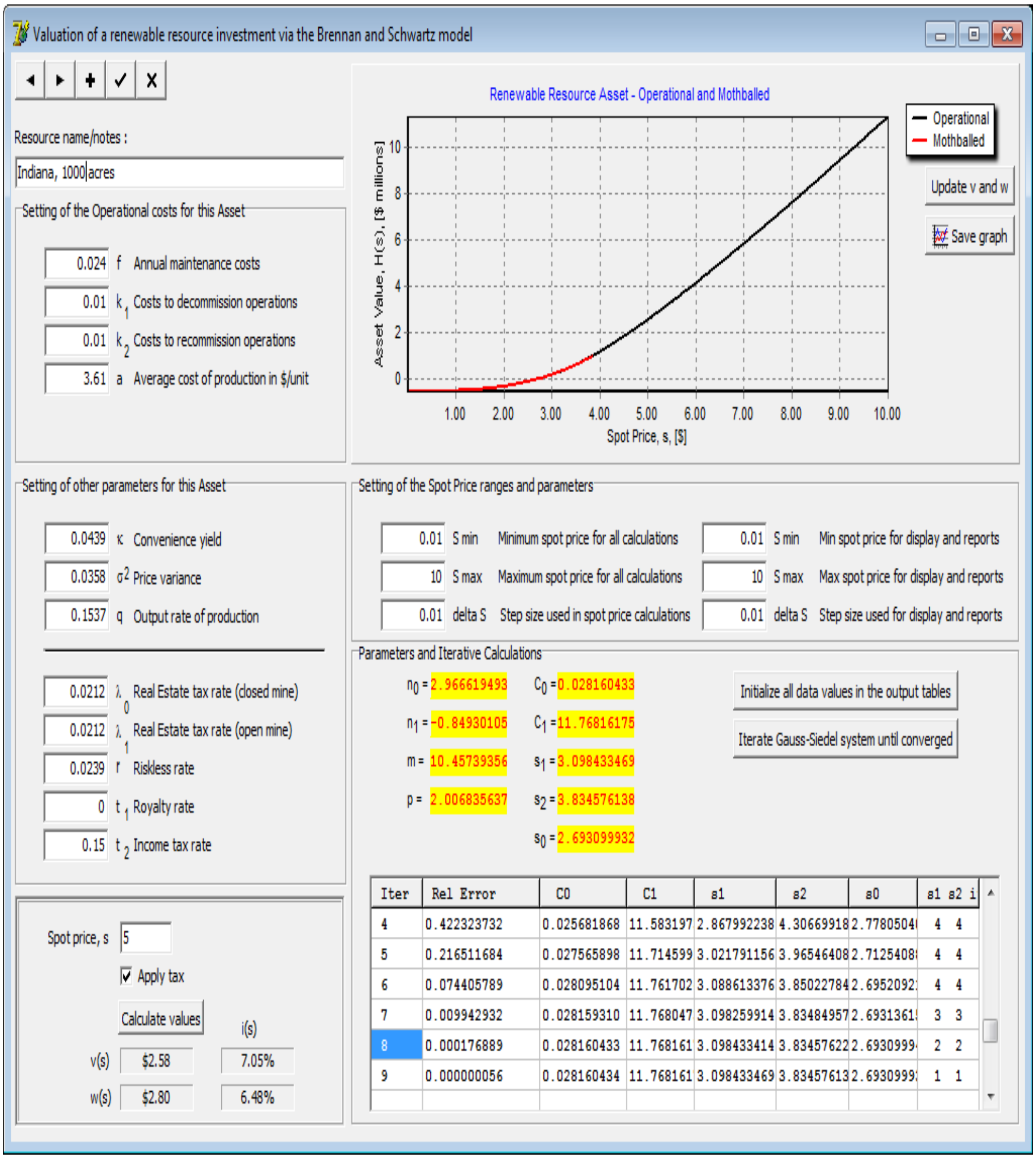


Figure D.4: Evaluation for an Average Corn Cropland Asset in Iowa, 250 Acres

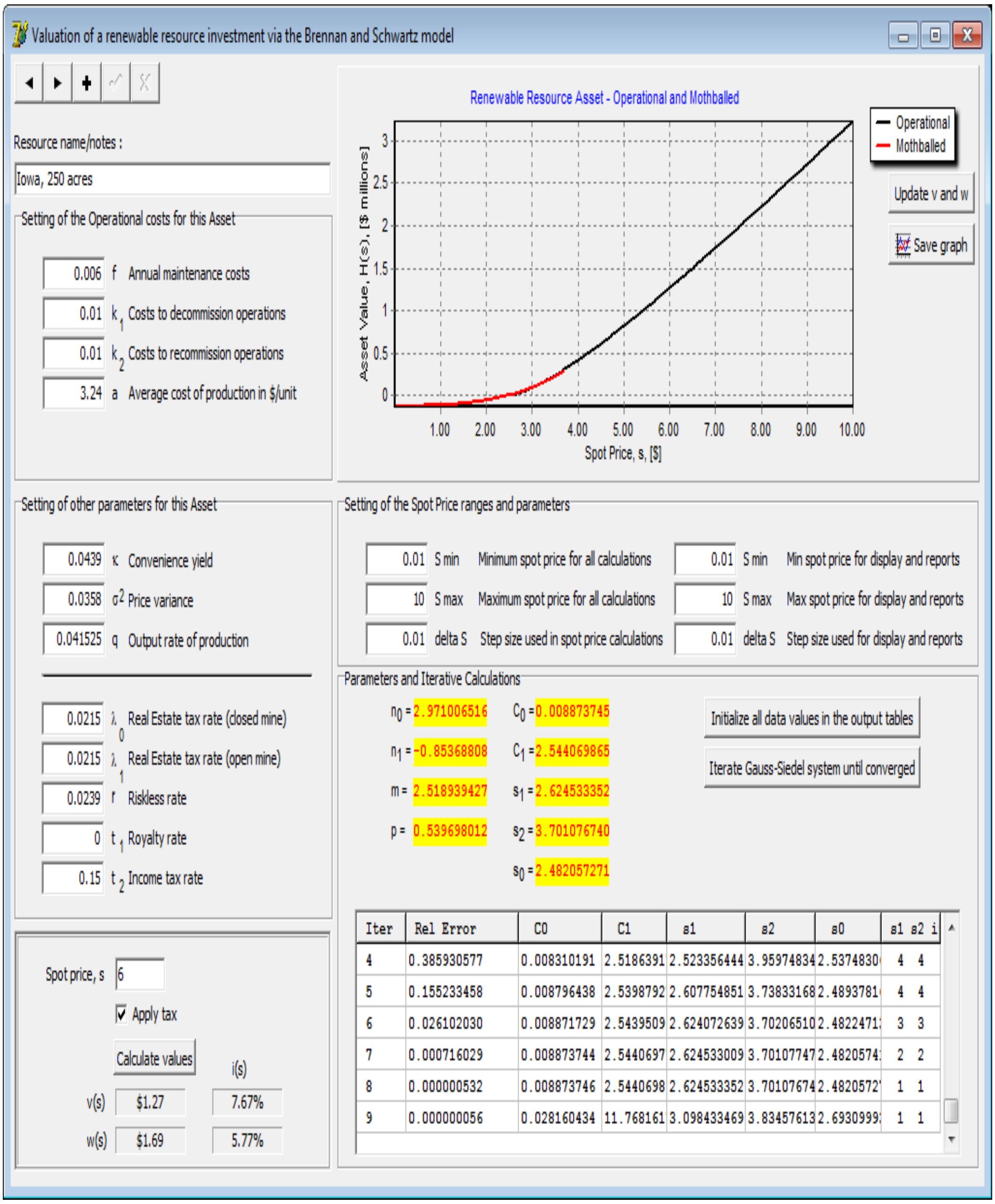


Figure D.5: Evaluation for an Average Corn Cropland Asset in Iowa, 1000 Acres

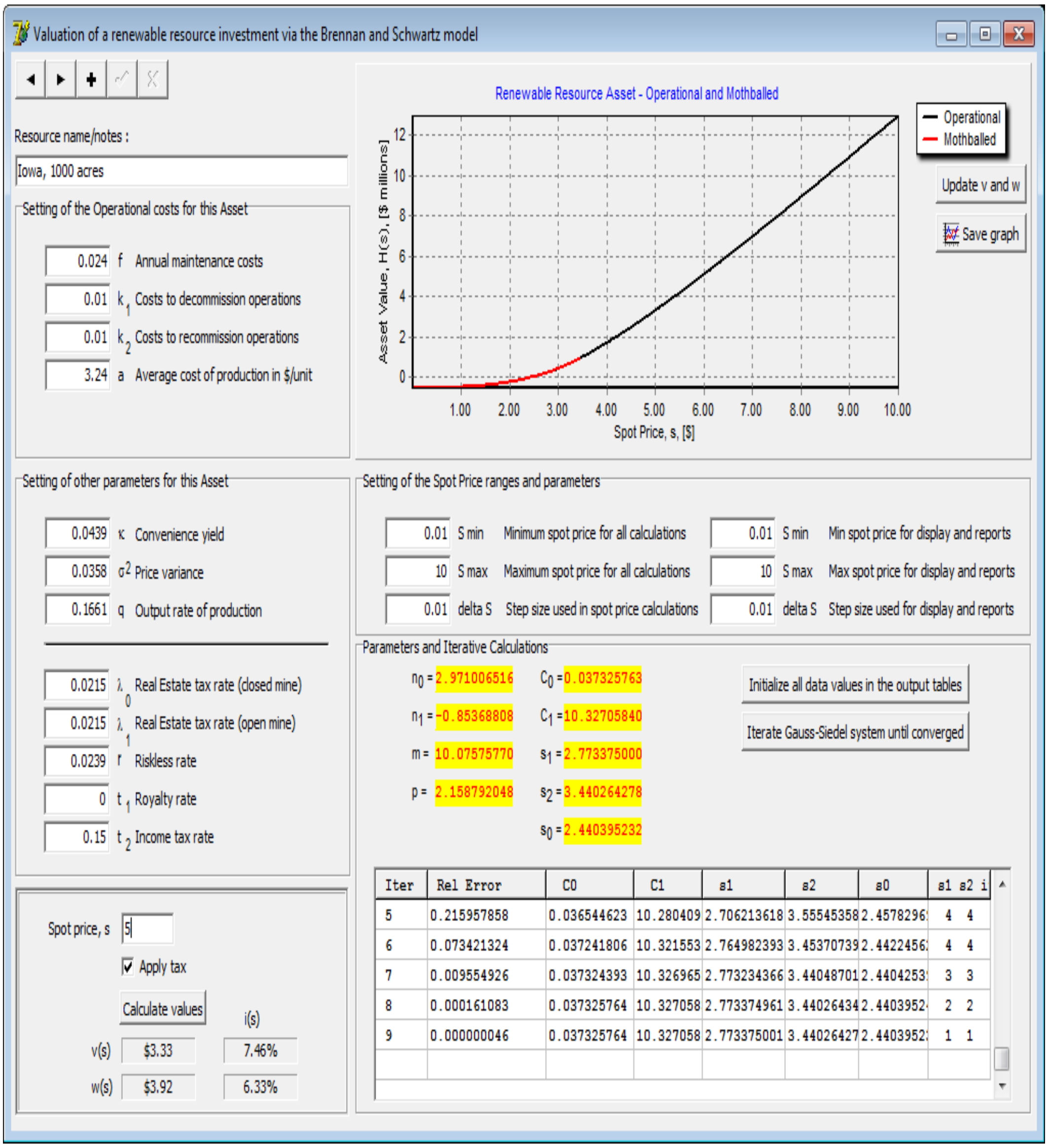
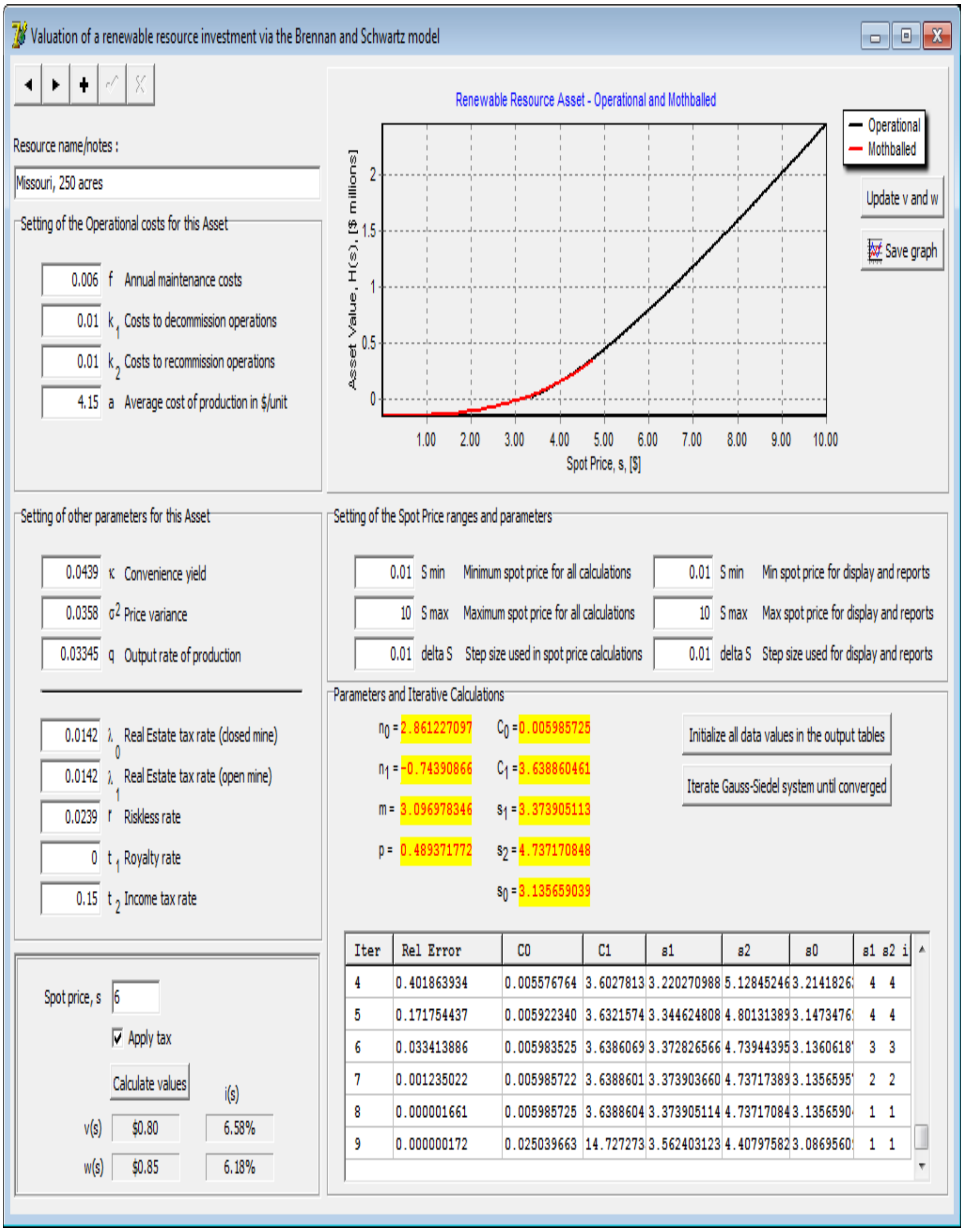
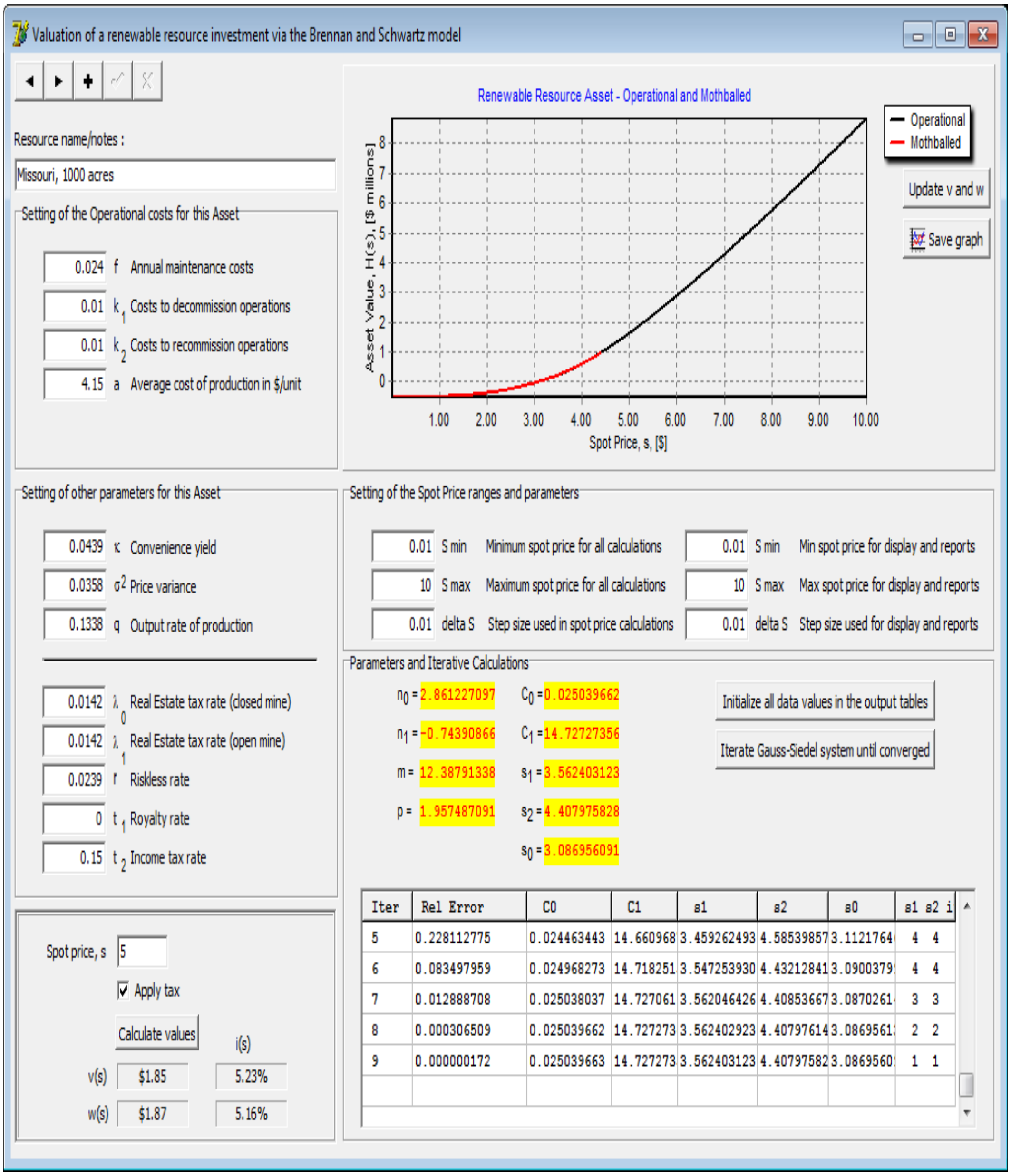


Figure D.6: Evaluation for an Average Corn Cropland Asset in Missouri, 250 Acres



Iter	Rel Error	C0	C1	s1	s2	s0	s1	s2 i
4	0.401863934	0.005576764	3.6027813	3.220270988	5.12845246	3.2141826	4	4
5	0.171754437	0.005922340	3.6321574	3.344624808	4.80131389	3.1473476	4	4
6	0.033413886	0.005983525	3.6386069	3.372826566	4.73944395	3.1360618	3	3
7	0.001235022	0.005985722	3.6388601	3.373903660	4.73717389	3.1356595	2	2
8	0.000001661	0.005985725	3.6388604	3.373905114	4.73717084	3.1356590	1	1
9	0.000000172	0.025039663	14.727273	3.562403123	4.40797582	3.0869560	1	1

Figure D.7: Evaluation for an Average Corn Cropland Asset in Missouri, 1000 Acres



Iter	Rel Error	C0	C1	s1	s2	s0	s1	s2	i
5	0.228112775	0.024463443	14.660968	3.459262493	4.58539857	3.1121764	4	4	
6	0.083497959	0.024968273	14.718251	3.547253930	4.43212841	3.090037	4	4	
7	0.012888708	0.025038037	14.727061	3.562046426	4.40853667	3.087026	3	3	
8	0.000306509	0.025039662	14.727273	3.562402923	4.40797614	3.086956	2	2	
9	0.000000172	0.025039663	14.727273	3.562403123	4.40797582	3.086956	1	1	

Figure D.8: Evaluation for an Average Corn Cropland Asset in Ohio, 250 Acres

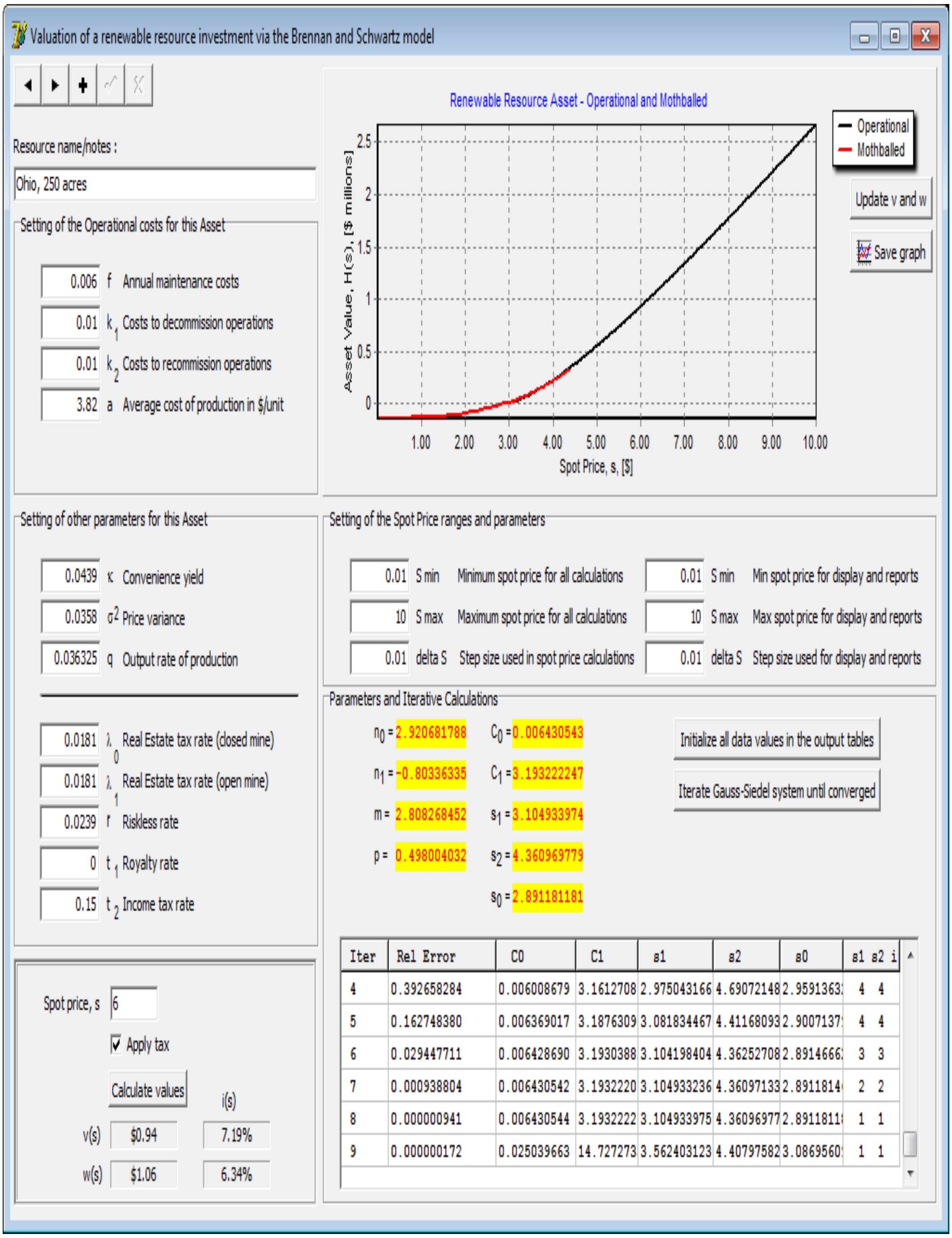
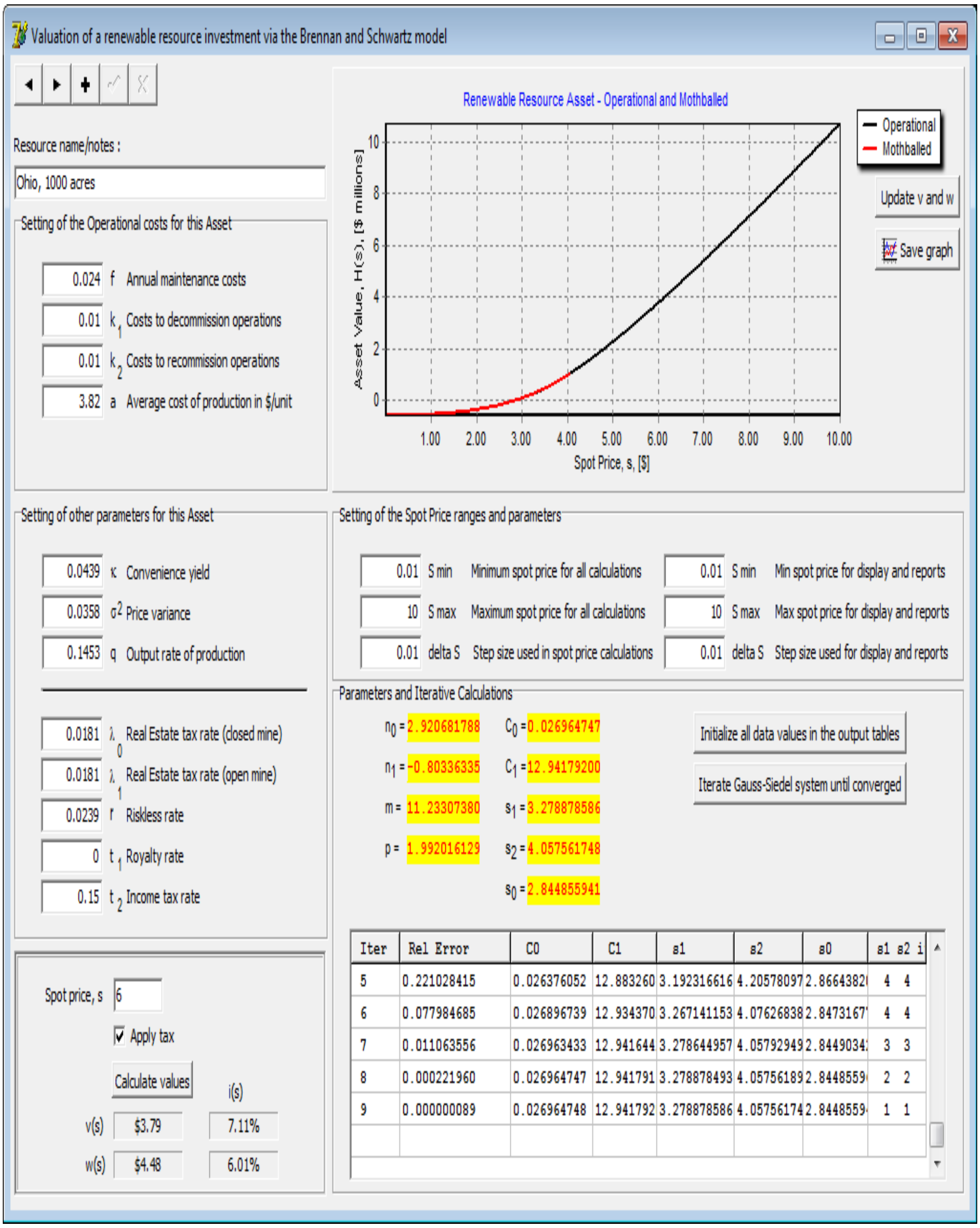


Figure D.9: Evaluation for an Average Corn Cropland Asset in Ohio, 1000 Acres



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