Three-Dimensional Wave-Structure Interaction Modelling
Using the Scaled Boundary Finite Element Method

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This thesis is dedicated to my eternally beloved father, my deeply cherished mother and my lovely younger brother

谨以此作献给我永远深爱的父亲母亲和我亲爱的弟弟
Abstract

In ocean engineering, the subject of wave-structure interaction has been the focus of heated debate for decades. Ocean engineers investigate wave behaviour in the presence of man-made structures, whilst structural engineers utilise accessible wave parameters to evaluate structural response. However, these two processes are closely related to each other, and difficulties exist if explicit wave forces are not available for structural analysis. Furthermore, differences in the theoretical background between wave and structural analyses pose additional challenges. Taking into consideration all these arguments, this PhD project aims to develop a three-dimensional wave-structure interaction model, assessing the wave field behaviour in the presence of structures, while simultaneously investigating subsequent structural response.

The Scaled Boundary Finite Element Method (SBFEM) is adopted to formulate the interaction problem, having many advantages including: significantly releasing the computational burden of three-dimensional calculations; precisely satisfying the boundary condition at infinity; not requiring fundamental solutions and being free from irregular frequencies and singular integrals.

This study is organised in order of increasing complexity of the physical problem and model formulation, from wave-monopile interaction through to wave interaction with pile group foundations, from steady scope examinations through to time-dependent investigations. In each case, waves and structures are formulated in a three-dimensional SBFEM model, which is subsequently used in well-designed parametric analyses, to derive a better understanding of the interaction mechanism.

The three-dimensional wave-structure interaction model has been established with the following advanced aspects: (1) combining an unbounded scalar field and a bounded vector field; (2) performing the investigation in three dimensions with less-demanding computational requirements; and (3) solving ordinary differential equations instead of partial differential equations, with the solutions possessing both numerical efficiency and analytical accuracy. The model has been applied to wave-monopile interaction and wave-pile group interaction in both steady scope and time-dependent context, with significant findings summarised as: (1) the response of single
pile foundations to wave forces becomes more noticeable with an increase in wave number, wave amplitude and water depth. The spatial arrangement of pile group foundations affects piles’ behaviour, and the influence is more apparent in cases with higher wave numbers than with lower ones; (2) pile members in a pile group take turns to play defensive roles when they act against waves. This prolongs the lifespan and ensures reliable functional ability of pile group foundations; and (3) piles are vibrating at the same frequency as waves taking into consideration the material damping effect.

This study increases the fundamental understanding of wave-structure interaction problems by clarifying the wave field behaviour in the presence of structures, and the structural response due to wave forces. From an engineering perspective, the comprehensive analyses presented in this study provide a valuable contribution towards structural safety and reliability assessment. In the meanwhile, it stimulates further interest in the theoretical and technical development of SBFEM.
Statement of Originality

This work has not previously been submitted for a degree or diploma in any university. To the best of my knowledge and belief, the thesis contains no material previously published or written by another person except where due reference is made in the thesis itself.

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List of Publications

Refereed Journal publications:


**Refereed Conference publications:**


Chapter 1 Introduction

1.1 Background and motivation

Oceans comprise more than two thirds of the earth’s surface, and possess a diverse range of inexhaustible natural resources that are valuable and essential to human activities. As a consequence, both offshore areas and coastal zones have attracted great attention from human beings. Various man-made structures shown in Figure 1-1, such as (a) oil and gas exploitation platforms; (b) breakwaters; (c) riverside recreation villas and (d) long-span bridges have been constructed, leading to the advent and development of ocean engineering.

Figure 1-1. Ocean installations related to human activities: (a) oil-drilling unit (www.gigant.co.uk); (b) breakwater (www.edinphoto.org.uk); (c) riverside recreation villa (www.caribbeanway.com) and (d) long-span bridge (service.photo.sina.com.cn)
In ocean engineering, the interrelationships between the natural ocean environment and man-made constructions have been identified with fundamental importance. Interpreted specifically from an engineering point of view, all structures in offshore and coastal areas are at the mercy of waves, winds, currents and earthquakes, and must survive the most severe conditions during their expected lifespan. Though most structures constructed to date have withstood the test of time, many catastrophic failures have been reported. A Norwegian semi-submersible drilling rig, the Alexander L. Kielland, collapsed while working in the Ekofisk oil field on March 27th, 1980, when the wind was gusting to 74 km/h with waves up to 12 m high (Figure 1-2 (a)). Of the 212 people on board 123 were killed, making it the worst disaster in Norwegian offshore history since World War II (Moan, 1981). Approximately two years later on February 14th, 1982, Ocean Ranger, the world's mightiest drilling rig at that time, was pounded by 190 km/h winds and 20 m high waves off the Newfoundland coast (Figure 1-2 (b)). All 84 crew members died (Royal Commission (Canada) on the Ocean Ranger Marine Disaster, 1984). More recently in early 21st century, hurricanes destroyed or damaged numerous offshore oil and gas platforms in the Gulf of Mexico, causing massive and irreparable losses (Figure 1-2 (c) and (d)). To achieve reliable and optimal usage of structure facilities, engineers must understand the physical and mechanical behaviour of each element involved in the ocean environment as well as their mutual interactions.

By exerting constant and detrimental forces onto structures, waves are the most significant environmental concern for coastal and offshore operations, and are therefore the main consideration for planning, designing and maintaining ocean structures (Benassai, 2006; Goda, 2010). Investigations of engineering failures have established that waves can be a critical factor causing accidents in ocean installations. The U.S. Coast Guard (1983) summarised that the primary reason leading to the tragedy of Ocean Ranger can be attributed to the vicious wave causing a broken portlight, consequently bringing down the Ocean Ranger. The explanation of the High Island III Jack-up incident also suspects the occurrence of an extreme wave phenomenon and the flawed estimates of wave conditions in the design process (Moan, 2005). Therefore, the analyses of waves, i.e. their behaviour in the presence of, and their impact on structures, and the corresponding structural performance, are of vital importance for the safe design and reliable utilisation of ocean constructions.
Significant research has been carried out by ocean engineers focusing on the wave field behaviour due to the existence of structures, and provides explicit wave data to structural engineers for structural design and reliability assessment. However, in situations where multiple structures or complex structural configurations are involved, the link between the wave field analysis and structural behaviour analysis becomes rather challenging. In addition, the wave field analysis is normally associated with an extensive geometric domain and the boundary condition is difficult to be defined, whereas the structural analysis can be conducted within a finite geometric scope. Furthermore, the different physical nature of waves and structures leads to the governing equations of their behaviour being expressed in a scalar field and a vector...
field, respectively. These factors render wave-structure interaction analysis difficult as theoretical differences between wave-centred study and structure-oriented study pose an impediment. The above arguments provide strong motivation for this PhD project to develop an efficient interaction model encompassing both waves and structures, to explore the wave properties in the presence of structures, and simultaneously examine the structural performance when subjected to wave forces, thereby clarifying the interaction mechanisms between the two elements.

Having identified the physical problem, the search for an appropriate methodology requires careful consideration. Normally, analytical solutions for idealised physical models offer fundamental insights into underlying problems subject to artificial simplifications. Observations from extensive laboratory experiments provide further feedback on prototypes. Of recent times, there has been an exponential advancement in the performance of numerical simulations corresponding to the development of computational techniques. However, the complex nature of the identified interaction problem, together with other issues such as the unbounded scope of the study domain and the dissimilar nature of the multi-physical field, precludes the acquisition of an analytical solution. Physical model design, performance of tests and the subsequent interpretation of results also pose difficulties in experimental analyses both in the laboratory and the field. Numerical methods, on the other hand, are free of these adverse factors and present the best option in cases where neither analytical investigation nor model examination is approachable.

Two popular and widely used computational algorithms in engineering fields are the Finite Element Method (FEM) and the Boundary Element Method (BEM). FEM is very flexible in dealing with inhomogeneous and anisotropic materials, but runs into difficulties when unbounded domains are involved. The spatial discretisation is normally terminated at an artificial boundary where the truncated domain outside the boundary up to infinity can only be represented approximately. In addition, extensive computational cost is required to achieve a satisfactory level of accuracy for large-scale problems. BEM is well suited to model unbounded domains, as its fundamental solution exactly satisfies the boundary condition at infinity. It also reduces the computational effort and data pre-processing by only discretising domain boundaries. However, the acquisition of the fundamental solution is often very difficult, and sometimes infeasible. Overall, despite much achievement in numerical algorithms, an
efficient and accurate technique is still in need of dedicated research for a precise investigation of the three-dimensional wave-structure interaction problem, in which both unbounded and bounded domains, both vector and scalar fields are involved.

A semi-analytical numerical method developed by Wolf and Song (1996), termed as the Scaled Boundary Finite Element Method (SBFEM), overcomes the disadvantages of FEM and BEM mentioned above, whilst at the same time, forming its own favoured features by combining their advantages. By performing a coordinate transformation, SBFEM solves ordinary differential equations instead of partial differential equations. Boundary conditions at infinity are satisfied accurately without the requirement for fundamental solutions. Discretisation only on the domain boundary makes three-dimensional analyses computationally feasible and efficient. In addition, SBFEM is capable of conveniently and effectively handling problems with irregular frequencies and sharp corners. SBFEM is originally developed to address wave propagation problems within the framework of dynamic unbounded medium-structure interaction, and has been applied to many topics in geotechnical engineering, ocean engineering, structural engineering, electro-magnetic engineering and hydraulic engineering. However, it has not yet been used to solve interaction problems involving waves and structures. Its favourable features provide strong motivation for this PhD project to employ SBFEM to address the specified three-dimensional wave-structure interaction problem by investigating the wave field behaviour in the presence of structures and at the same time, examining the structural response to ocean wave loads.

1.2 Research objectives

One predominant structural concept extensively used in ocean installations is associated with circular cross-sectional configurations, appearing as single standing piles for wind farms, or pile group foundations supporting various platforms. They are also widely used in ports and harbour areas in the form of piers for bridges or jetties for mooring. Therefore, this PhD research adopts pile foundations as the structural representative, and focuses on the following two major objectives:

- Develop and verify a three-dimensional SBFEM model addressing wave-structure interaction problems.
- Investigate the interaction mechanism between waves and structures.
The two objectives are achieved in the following stages, in order of increasing complexity of the problem definition and the computational model:

- Methodology investigation in terms of SBFEM’s numerical performance, to ensure the credibility of further studies.
- Performing the analysis of the wave interaction with a single pile foundation, examining the structural behaviour when the external wave pressure is analytically available.
- Advancing the analysis to wave interaction with pile group foundations, in which both the wave field behaviour and the structural response are investigated.
- Further promoting the study into time-dependent context by reformulating the SBFEM model to address the dynamic nature of the interaction problem.

The outcomes of this research are expected to attract careful consideration of the wave-structure interaction when designing ocean installations and evaluating their functional ability, and in the meanwhile, enable SBFEM for a successful extension of its application.

1.3 Thesis outline

In order to achieve successful implementation of the research targets detailed in the previous section, the remainder of the thesis consists of six chapters and is organised as follows:

Relevant literature is reviewed in Chapter 2 in three aspects: the wave-structure interaction problem, current research methodologies, and a detailed introduction of SBFEM. This chapter derives the specific problem to be solved and the particular methodology to be used in this PhD project.

Chapter 3 commences with a brief revision of basic SBFEM derivations for integrity of presentation. Subsequently, two issues associated with SBFEM’s numerical performance are addressed in two subsections, emphasising the matrix manipulation technique and matrix properties, respectively. In both cases, reasons for potential numerical difficulties are investigated. Corresponding solution schemes are discussed and benchmarked.

Chapter 4 opens SBFEM’s exploration of the proposed research by investigating the structural response of a monopile foundation subject to ocean wave loads. In this
study, the SBFEM model is only formulated for the structural part as the wave field solution is readily obtainable. Parametric analyses in terms of dominant wave properties are carried out.

Chapter 5 extends SBFEM’s investigation of the specified research objectives by addressing wave interaction with pile group foundations, in which SBFEM formulations of both wave behaviour and structural responses are established. The wave interaction with a group of two piles is considered to offer valuable insights into the physical nature of the problem. Parametric analyses under three representative situations with different incident wave angles are also provided.

Chapter 6 addresses the research objectives by further advancing SBFEM into time-dependent analysis of the wave interaction with pile foundations. SBFEM formulation of the wave field remains the same as that established in Chapter 5. The wave field results require a restoration into the time domain by analytically attaching a time-dependent term to the results for subsequent structural analyses. The solution procedure of the structural analysis needs reformulation corresponding to the differential equation governing the dynamic structural behaviour. The wave interaction with a monopile foundation is discussed first for preliminary knowledge on how piles behave when subjected to time-dependent wave pressure. Wave interaction with pile group foundations is also examined with a concurrently-performed parametric analysis to clarify the mechanism of the time-dependent interaction problem.

Finally, in Chapter 7, this thesis concludes by highlighting the significant contributions offered by this PhD study and proposing a preliminary outline for future research.
Chapter 2  Literature Review

The most relevant literature pertaining to the physical problem and the research methodologies in the context of wave-structure interaction is reviewed in this chapter to support the research objectives specified in Chapter 1. Section 2.1 reviews wave-structure interaction studies. Section 2.2 focuses on the merits and deficiencies of various research methods currently employed to solve the interaction problem. Finally, SBFEM is introduced in Section 2.3 covering its establishment, development, applications as well as limitations. By reviewing the literature, this chapter concludes with a clear picture of ‘what to do’ and ‘how to do it’ in this PhD project, inspiring the problem formulation in the subsequent chapters.

2.1  Wave-structure interaction problem

The conventionally-used term ‘wave-structure interaction’ in most reported studies refers to the examination of wave behaviour in the presence of structures, with the intention of deriving wave velocities, pressures and surface elevations. These studies only formulate the wave domain. They are categorised as wave analysis and are reviewed in Section 2.1.1. There are studies carried out by structural engineers who employ accessible wave parameters in the design and safety evaluation of ocean constructions. These studies only solve the equations governing the structural behaviour, and therefore are specified as structural analysis. They are introduced in Section 2.1.2, in which studies examining both wave and structural performance are also included.

2.1.1  Wave analysis

The wave transformation due to the presence of structures and the corresponding wave forces exerted on structures have been the main focus of intense investigations for decades. One pioneering work can be identified as the Morison equation (Morison et al., 1950), which is used to estimate the loads exerted by unbroken surface waves on a single-standing cylindrical object. The total force per unit length in the flow direction, denoted by $F$, is given by the expression:
\[
F = \frac{1}{2} \rho C_D u^2 D + \rho C_M \dot{u} \frac{\pi D^2}{4}
\]  
(2.1)

where \( \rho \) is the density of water; \( C_D \) is the drag coefficient; \( C_M \) is the inertia coefficient; \( u \) is the particle velocity in the flow direction; \( \dot{u} \) is the acceleration corresponding to \( u \); \( D \) is the pile diameter. The force consists of two components, namely a drag force proportional to the square of the velocity \( u \) and a virtual mass force proportional to the horizontal component of the accelerative force exerted on the mass of water displaced by the pile. The Morison equation is applicable when the incident wavelength is much greater than the geometric dimension of the structure, assuming that the wave field behaviour is not significantly affected by the presence of structures. However, when the size of the structure is of similar dimension to the wavelength, diffraction phenomenon must be taken into consideration. Moreover, in orbital flow which is a case of multi-directional flow encountered by a horizontal cylinder, the Morison equation does not give a good representation of the forces as a function of time (Chaplin, 1984).

MacCamy and Fuchs (1954) studied the linear diffraction of plane waves in the presence of large cylindrical piles, and presented a quantitative expression of the resulting wave forces. Au and Brebbia (1983) employed standard BEM to formulate plane wave diffraction by vertical cylinders of constant circular, square and elliptical cross-sections throughout the cylinder height. Zhu (1993) analytically solved short-crested wave diffraction with a circular cylinder. Tao et al. (2007) formulated the same problem using SBFEM. Subsequently, short-crested wave diffraction by a square caisson (Song and Tao, 2008) and an elliptical cylinder (Tao and Song, 2008) were investigated.

In cases with severe wave heights, the application of small amplitude theory and the associated linear assumption were found to be questionable. A number of studies were carried out to investigate the nonlinear wave diffraction problems (Chakrabarti, 1975; Isaacson, 1982; Eatock Taylor and Hung, 1987; Jiang and Wang, 1995). These early studies concentrated on the wave interaction with single-standing vertical cylinders of circular cross-sections (see Figure 2-1), and high-order wave forces acting on structures were formulated.
In addition to the wave diffraction with single vertical cylinders, many researchers have been engaged in the study of wave diffraction with multiple vertical cylinders (see Figure 2-2), and have seen significant advances in the last few decades. Spring and Monkmeyer (1974) extended MacCamy and Fuchs’ work (1954) to two cylinders. They analytically formulated the first-order plane wave forces on groups of two bottom-fixed, surface-piercing vertical cylinders. The unknown coefficients of the series solution were obtained in a straightforward manner from a linear algebraic matrix equation. Situations with various two-cylinder configurations and arbitrary incident wave angles were discussed. Linton and Evans (1990) simplified the theory proposed by Spring and Monkmeyer (1974) and investigated the first-order diffraction problem of a regular incident wave in the presence of $N$ bottom-mounted vertical circular cylinders. They provided a more direct evaluation of the wave forces on cylinders. However, their results for four cylinders were erroneous due to insufficient points having been used for presentation, hence had to be corrected retrospectively (Chen et al., 2009b).

An important breakthrough can be identified as the generalised wave-structure interaction theory derived by Kagemoto and Yue (1986). Their method is applicable to arrays of arbitrary structures with two geometric specifications on the structures: (1) the vertical projections of the structures on a horizontal plane cannot overlap with each other; and (2) fictitiously introduced circles on the projection plane to enclose individual structures cannot enclose centres of other circles. Yilmaz and Incecik (1998) employed Kagemoto and Yue’s (1986) formulation to address wave diffraction by a
group of truncated vertical cylinders. The mushrooming of very large floating structures supported by thousands of cylindrical legs led to the work of Maniar and Newman (1997) and Kashiwagi (2000). They employed a hierarchical procedure to first divide the entire cylinder group into small clusters, each containing just a few cylinders. These clusters were later assembled to build up the whole array. The hydrodynamic properties of the wave field were calculated at each level and subsequently added to produce the resultant solution. This procedure allowed Kashiwagi (2000) to analyse the scattering and radiation phenomena of an array of 5120 cylinders.

the elliptic coordinate system, for wave diffraction with an array of elliptical cylinders. Tao et al. (2009b) studied the wave field behaviour in the presence of two adjacent square caissons. Song et al. (2010) further extended the wave diffraction problem to multiple cylinders of arbitrary cross-sections.

Though intensive attention has been paid to wave interaction with vertical cylinders, the investigation of wave forces on horizontal cylinders is also of great significance in ocean engineering. Horizontal cylinders occur in pipelines, which are used for offshore oil and gas transportation, seawater extraction, wastewater disposal and seafloor communication cable protection (Roopsekhar and Sundar, 2004). They can also be structural elements for composite ocean installations. O'leary (1985) employed a numerical technique, based on multipole potentials, to investigate the radiation and diffraction of surface waves due to a group of parallelly-laid horizontal circular cylinders. Sibetheros et al. (2000) employed a third-order Volterra model to analyse the response of a horizontal cylinder exposed to random waves. Shen et al. (2007) studied the interaction of oblique waves with an array of horizontal cylinders, and provided analytical expressions for the diffracted and radiated velocity potentials by employing the multipole expansion method. Feng et al. (2008) used the same method to investigate wave diffraction and radiation due to an array of infinitely long horizontal circular cylinders in a two-layer fluid domain with infinite depth.

2.1.2 Structural analysis

Traditional structural analyses adopt attainable wave pressure formulations, such as the Morison equation, to explicitly calculate the wave force without elaborately solving the governing equation of the wave domain. Jain and Datta (1987) investigated the dynamic behaviour of an offshore tower under regular and random waves. The offshore tower was modelled as an assemblage of beam elements. The horizontal hydrodynamic force was estimated using the Morison equation, in which the water particle kinematics were evaluated using the linear wave theory, completely ignoring the fluctuation of the water surface. Similar theory was applied by Jain (1997) when evaluating the nonlinear response of a tension leg platform (TLP) subjected to regular wave forces. Jain (1997) calculated the wave force acting on the structural members of the TLP using a modified Morison equation, taking into account the relative velocity and acceleration between structure members and fluid particles. The water particle kinematics were calculated up to the actual wave surface level using the
modified Airy wave theory proposed by Chakrabarti (1971). The hydrodynamic force was determined at each time instant, and the nonlinear equation of motion of the structure was solved in the time domain by the Newmark’s beta integration technique.

Adrezin and Benaroya (1999) also investigated the structural behaviour of a TLP using the Morison equation to explicitly calculate the acting wave force. Chandrasekaran and Jain (2002) later employed Jain’s (1997) formulation to study the dynamic behaviour of square and triangular TLPs under regular wave loads. Agarwal and Jain (2003a; 2003b) redirected the analysis to the dynamic response of offshore spar platforms under regular waves.

Apart from the Morison equation, another available wave pressure formulation can be identified as Zhu’s (1993) formula, which describes the variation of the first-order short-crested wave pressure acting on a circular cylinder. It was used in Eicher et al. (2003) to evaluate the deformation and stress distribution of an offshore concrete pile. Li et al. (2010b) and Li et al. (2011) also employed this formula to investigate the response of an offshore monopile when subjected to short-crested incident waves.

Unfortunately, explicit wave expressions are not always available when structures are featured by irregular geometric configurations or multiple structures have complex combinations. Under these circumstances, both the wave field and the structural domain need to be formulated, taking into consideration the wave-structure interaction. In this regard, only a few studies are documented, which are reviewed as follows.

Wu et al. (1995) investigated the wave-induced response of an elastic floating plate. The wave domain study was conducted in a two-dimensional scope by solving the Laplace equation under the concept of linear potential flow. The structural analysis assumed that the amplitude of the plate’s motion is small compared with the plate’s dimension. The horizontal motion of the plate was restricted, and only the deflection was addressed using the elastic beam theory. Their results agreed well with experiments, however, the response of the plate was overestimated in the high-frequency region.

Lee and Wang (2000) addressed the dynamic behaviour of a TLP with net-cage system when exposed to ocean waves. Similar to Wu et al.’s (1995) work, the wave domain was governed by the Laplace equation, applying the small amplitude wave
theory, and the motion of the structure was assumed to be small. The solutions were derived in the form of infinite series summation, linearly and two-dimensionally.

Srisupattarawanit et al. (2006) presented a time-accurate computation of the interaction between elastic structures and random waves of a finite water depth. The formulation of wave behaviour was based on the potential flow theory and the velocity potential satisfied the Laplace equation at every time instant. The structure was modelled as a nonlinear beam, and its dynamic behaviour was evaluated using the Newmark time integration technique. The coupling procedure is described as:

- At the current time step, the wave domain was solved and the hydrodynamic force at the wave-structure interface was computed and passed to the structure;
- Subsequently, the structural analysis was performed integrating the wave force. The structural displacement and velocity were calculated and passed back to re-evaluate the boundary conditions of the wave analysis.

This process was iterated until the residuum between two successive calculations was negligible. The fourth-order Adams-Bashforth-Moulton (ABM) predictor-corrector scheme was used to perform the time integration of the coupling governing equations, in which the Laplace equation for each time instant needed to be solved first for prediction and again for correction. This requires substantial computational time and memory. In addition, in the wave analysis, the radiation condition was modelled using an absorbing zone, where all waves passing behind the structure were attenuated. The contribution of these waves was neglected when evaluating the structure’s response, which inevitably weakened the performance of the coupling scheme.

Recently, Ge et al. (2010) investigated the interaction between waves and a submerged floating tunnel (SFT). As was the case in previous studies, the linear potential flow theory was adopted to describe the wave field, which was governed by the Laplace equation. The SFT was modelled using shell elements. Ge et al. (2010) did not present an explicit discussion as to how to deal with the radiation condition when solving the wave domain. Furthermore, the elastodynamic behaviour of the tunnel was evaluated in the frequency domain, to avoid having to deal with the memory issue involved in the time-domain analysis.
Summarising the current study of wave-structure interaction problems, it is found that intensive effort has been directed towards wave analyses to investigate wave parameters due to the existence of structures, such as the free surface elevation, wave propagation velocity and the hydrodynamic pressure. These studies provide explicit wave data for subsequent structural analyses. However, they are normally formulated in certain situations for specific applications. The Morison equation is effective when the structure’s influence on the wave transformation is negligible. Zhu’s (1993) formula is only applicable to circular cylinders subjected to short-crested waves. In cases involving complex structural configurations, the link between wave analyses and structural analyses becomes challenging. For an effective evaluation of the structural behaviour, an investigation involving both waves and structures needs to be performed.

However, due to the dissimilar physical nature as found in waves and structures, the study in this aspect has not yet been adequately documented. Waves exist in an extensive domain, and their behaviour is governed by scalar equations, normally with the velocity potential as the unknown variable. Structures, on the other hand, are characterised by finite dimensions, and their behaviour is described by vector equations addressing displacements and stresses. These differences in the theoretical background between wave and structural analyses impede the compiling of a wave-structure interaction model. In this regard, the above reported studies covering both waves and structures set a precedent. Nevertheless, these studies were mostly conducted either in two dimensions or in the frequency domain, and the boundary conditions associated with the wave analysis were not treated properly. This is directly or indirectly related to the time consumption and memory requirements involved in three-dimensional or time-domain analyses. Moreover, the methodology employed in the analysis can also be a reason for this restriction. Therefore, the following section reviews the current methods, to identify the most appropriate one for an effective and efficient three-dimensional investigation of the wave field behaviour in the presence of structures and structural responses due to wave loads.

### 2.2 Current methods used in wave-structure interaction analysis

The current methods used in wave-structure interaction analyses are categorised as analytical, experimental and numerical, the first two of which are briefly addressed in
Sections 2.2.1 and 2.2.2. Numerical methods are detailed in Section 2.2.3 as they are more relevant to this PhD project.

2.2.1 Analytical method

Apart from the early studies (Morison et al., 1950; MacCamy and Fuchs, 1954; Spring and Monkmeyer, 1974) mentioned in Section 2.1.1, Rahman (1984) employed the linear diffraction theory to successfully formulate an exact second-order diffraction theory by incorporating the second-order effect of the Morison equation. Contrary to the formal perturbation technique, no quadratic velocity potential needed to be evaluated.

Following MacCamy and Fuchs’ work (1954) on linear plane wave diffraction around large vertical cylinders, Zhu (1993) further investigated the diffraction of short-crested waves by a circular cylinder. The expression of the scattered velocity potential was constructed as the superimposition of two groups of Hankel functions with unknown constants. This expression simultaneously satisfied the governing Laplace equation and the Sommerfeld radiation condition. The unknown constants were determined to be functions of the Bessel function and the Hankel function and their derivatives by the boundary condition at the wave-structure interface.

Yilmaz (1998) used Kagemoto and Yue’s (1986) exact algebraic method to obtain the analytical solution of plane wave diffraction and radiation by a group of truncated vertical cylinders. Shen et al. (2007) adopted the multipole expansion method to formulate the diffraction and radiation of liner oblique waves around multiple horizontal circular cylinders submerged in water of finite depth.

The review in this section leads to the conclusion that analytical solutions, often having rather complex formulations, are normally available for simple cases. They are generally derived to gain a preliminary understanding of the physical problem. For the complex interaction problem specified in this PhD study, it would be exceedingly difficult to formulate an analytical solution.

2.2.2 Experimental method

Laboratory and in-situ experiments are conducted to obtain qualitative information about the physical phenomenon when the analysis of the prototype is very complex. They can also be used to verify the basic assumptions inherent in an analytical
solution. Kagemoto et al. (2002) conducted an experiment to examine the wave decay phenomenon in the presence of a long array of cylinders. They used fifty vertical cylinders in a wave tank and measured the free surface elevation using capacitance-type wave probes. They learnt that the scale effect is an important factor in an experimental analysis of such multi-column structures.

Other experimental investigations involve Jung et al.’s (2004) observation of flow characteristics in a water tank with a fixed rectangular structure. They intended to understand the vorticity and turbulence phenomena in wave-structure interaction problems. Jung et al. (2005) used the same tank to examine flow characteristics in the presence of a free-rolling rectangular structure. Neelamani and Gayathri (2006) studied wave transmission and reflection by single surface plates and twin plate barriers.

Normally, experimental analyses are designed for specific purposes and are conducted under certain modelling laws. They are generally cost ineffective and labour intensive. Moreover, in processes of model design, environment simulation and result interpretation, it is difficult to satisfy the geometric, dynamic and kinetic similarities simultaneously. In addition, the accuracy of the results is apparatus-dependent and vulnerable to unpredictable changes in experiments.

2.2.3 Numerical method

Analytical and experimental investigations were predominant prior to the rapid development of computational capability and numerical techniques. However, as discussed, both analytical and experimental analyses have certain limitations and restrictions. For this reason, numerical algorithms are inevitably sought as effective alternatives. There is a wealth of literature reporting the investigations of wave-structure interaction problems using numerical algorithms, the most commonly used of which are reviewed in the following sections.

2.2.3.1 Finite element method

Eatock Taylor and Zietsman (1981) used FEM to investigate wave diffraction and radiation problems. They constructed two localised FEM schemes to deal with the unbounded wave domain. In both cases, the wave domain was divided into a localised finite element domain and an outer region. In the first scheme, the outer region was
represented by a series of eigenfunctions, whereas in the second scheme, an integral equation was satisfied at the interface between the two regions. Their analysis concluded that the second scheme is more suitable for a general two-dimensional wave diffraction and radiation analysis, and it may readily be applied to cases with infinite water depth.

A two-dimensional examination of the interaction between multiple floating structures and directional waves was performed by Sannasiraj et al. (2000) using FEM. The infinite wave domain was treated by incorporating plane boundary dampers at a radiation boundary, which was located at a certain distance from the structure. Sannasiraj et al. (2000) stated that the optimal distance should be large enough to avoid the local disturbance, and small enough to ensure an economical and accurate calculation. However, using plane boundary dampers to absorb outgoing waves necessitates an extensive computational domain. Therefore, the accuracy requirements and the computing capability should be considered when performing the FEM analysis.

Turnbull et al. (2003) studied two-dimensional wave interaction with submerged horizontal cylinders in a finite element numerical wave tank. A hybrid mesh consisting of a regular structured mesh and a Voronoi unstructured mesh was utilised to discretise the free surface. The hybrid mesh was adaptive and it tracked the vertical movement of the free surface nodes using a Lagrange-Eulerian scheme. In this numerical tank, no infinite domain was involved, and the Neumann boundary condition in terms of the velocity potential was enforced.

Wang and Wu (2007) conducted a time domain analysis of the second-order wave-structure interaction problem, taking advantage of FEM’s ability in handling non-linear problems. An unstructured two-dimensional finite element mesh was generated on the still water level, and then extended along the vertical direction to form a three-dimensional mesh. Small-size elements were used near the free surface to account for the intense free surface fluctuation. The two-dimensional mesh was truncated at a ‘control surface’ located some distance from the structure, on which the radiation condition was imposed. Unfortunately, there were no explicit discussions as to how to determine the location of the ‘control surface’ and how to enforce the radiation condition.
Wang and Wu (2010) also discussed the wave-structure interaction problem in a finite element numerical wave tank. They examined the nonlinear wave behaviour in the presence of a cylinder array in a three-dimensional scope. As was the case in Turnbull et al. (2003), a hybrid two-dimensional mesh combing structured and unstructured meshes was used to discretise the horizontal free surface. They adopted the same technique as in Wang and Wu (2007) to extend the two-dimensional mesh in the vertical direction. A ‘numerical damping zone’, together with the Sommerfeld condition, was employed to account for the radiation condition, and was formulated as an artificial viscous term into the free surface boundary condition. However, a constant adjusting of the effectiveness of the damping zone was introduced and a detailed discussion on the determination of the constant was not given.

Zhong and Wang (2009) employed the stabilised finite element approximation to investigate the nonlinear behaviour of shallow water waves in the presence of cylindrical structures. They adopted linear radiation conditions, as an approximation of open boundary conditions, to propagate waves out of the computational domain.

From the above discussions, a disadvantage is identified when using FEM to solve wave-structure interaction problems, which is associated with the treatment of unbounded wave domains. This has usually been achieved by truncating the remote region and imposing artificial boundary conditions. The artificial boundary should be introduced sufficiently far from the region under investigation to avoid erroneous results. However, this may incur excessive and impractical computational overheads, especially for three-dimensional problems.

2.2.3.2 Boundary element method

Au and Brebbia (1983) employed standard BEM to formulate wave diffraction by vertical cylinders of constant cross-section throughout the height. The boundary value problem was formulated in two dimensions by the Helmholtz equation \( \nabla^2 \phi + k^2 \phi = 0 \), \( \nabla \) is the Laplace operator; \( k \) is the wave number, governing the diffracted velocity potential \( \phi \), and boundary conditions at the cylinder surface and the infinity. The fundamental solution (denoted by \( G \)) of the two-dimensional Helmholtz equation was determined as:

\[
G = \frac{i}{4} H^{(1)}_0 (kr)
\] (2.2)
where \( r = \sqrt{(x-x_0)^2 + (y-y_0)^2} \) with \((x_0, y_0)\) and \((x, y)\) denoting the coordinates of a source point and a field point in two dimensions; \( i = \sqrt{-1} \) and \( H_0^{(1)} \) is the zero-order Hankel function of the first kind. Equation (2.2) naturally leads to the boundary condition at infinity to be satisfied. Thus only the cylinder surface needs to be discretised. Nodal values were obtained by solving a matrix-form equation. As the imaginary unit ‘\( i \)’ was introduced in the formulation of the fundamental solution, the matrix equation was in complex-number form. Wave diffractions with cylinders of circular, square and elliptical cross-sections were examined in this study, which laid a foundation for the application of BEM in wave-structure interaction analysis.

Chen and Mahrenholtz (1992) studied the wave interaction with floating twin cylinders using the boundary integral method. They detailed the discussion about irregular frequency, and claimed that the irregular frequencies for twin bodies are the same as those for a single body with identical geometry. In their study, the irregular frequencies were predicted prior to the calculation so that potential numerical difficulties can be avoided. However, they made it clear that in other situations when irregular frequencies are not known beforehand, careful consideration and specific techniques are required.

Bai and Eatock Taylor (2006) formulated a higher-order BEM solution to the wave interaction with oscillating vertical cylinders in a wave tank. The studied problem was symmetrical with respect to the \( xy \) plane and involved a horizontal seabed located at a level of \( z = -d \). Therefore, the Rankine source and its image were used as the Green’s function (the fundamental solution):

\[
G = \frac{1}{4\pi} \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \right) \tag{2.3}
\]

with

\[
\begin{align*}
R_1 &= \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} \\
R_2 &= \sqrt{(x-x_0)^2 + (y+y_0)^2 + (z-z_0)^2} \\
R_3 &= \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z+z_0+2d)^2} \\
R_4 &= \sqrt{(x-x_0)^2 + (y+y_0)^2 + (z+z_0+2d)^2}
\end{align*} \tag{2.4}
\]
In Equation (2.4), \((x_0, y_0, z_0)\) and \((x, y, z)\) are the coordinates of a source point and a field point in three dimensions, respectively. The boundary integral only needed to be evaluated over half of the domain boundary, which significantly reduced the computational burden. To further improve the efficient, they adopted the technique of domain decomposition (De Haas and Zandbergen, 1996), with continuity conditions enforced on interfaces between adjacent subdomains. A similar study was conducted investigating wave interaction with fixed and floating flared structures in Bai and Eatock Taylor (2009).

Kim et al. (2007) reviewed Au and Brebbia’s work (1983), and derived the wave diffraction by a vertical circular cylinder using BEM. Kim and Cao (2008) further extended this derivation to evaluate the wave force acting on two vertical cylinders.

Zheng et al. (2008) formulated oblique waves interaction with long prismatic structures. Contrary to convention, in which the fundamental solution satisfies the boundary conditions specified at the free surface, wave domain bottom and the radiation condition at infinity, Zheng et al. chose a fundamental solution not satisfying any boundary condition in the form of:

\[
G = \frac{K_0(kr \sin \theta)}{2\pi} \tag{2.5}
\]

with \(K_0\) being the zero-order modified Bessel function of the second kind; \(k\) the wave number; \(r\) the distance from the source to the field point and \(\theta\) the angle measured with respect to the positive \(x\) direction. All boundaries of the computational domain needed to be discretised, but the subsequent evaluation of the singular integral was very convenient. However, they adopted the approach presented by Bai and Yeung (1974) in the treatment of open boundaries. An accurate relation between the velocity potential and its corresponding derivatives was established on the open boundary, which was then placed as close to the structure as possible. Although this approach resulted in a minor study domain with moderate computational demand, the ‘accurate relation’ was rather complex.

Chen et al. (2009a) investigated the application of dual boundary element method to the wave interaction with vertical cylinders. They used \(G = -\frac{\pi i}{2} H_0^{(1)}(kr)\) (all
variables hold the same meaning as previously described) as the fundamental solution, and employed Burton and Miller’s (1971) method to avoid irregular frequencies.

From the above review, it is observed that BEM is an attractive alternative to FEM when modelling unbounded domains. BEM can rigorously satisfy the radiation condition at infinity if a proper fundamental solution is applied. Issues associated with domain truncations and artificial boundary conditions need not to be addressed, as would be the case in FEM. However, the complexity of the fundamental solution increases dramatically with the complexity of the physical problem. Furthermore, BEM needs to evaluate singular integrals and suffers from irregular frequencies, and the matrices involved in the BEM calculation are asymmetric and fully populated. These disadvantages weaken the merits of reducing one spatial dimension of the problem, and limit BEM’s application in the interaction problem specified in this PhD study.

2.2.3.3 Scaled boundary finite element method

Deeks and Cheng (2003) first introduced SBFEM to potential flow around streamlined and bluff structures in an infinite fluid domain. The two-dimensional Laplace equation ($\nabla^2 \phi (x, y) = 0$, $\phi$ is the velocity potential) governing the potential flow motion was addressed, and the weighted residual technique was employed to derive the corresponding scaled boundary finite element equation. This pioneering and introductory work marks a significant breakthrough in SBFEM’s developmental history and sets a precedent to its application in wave-structure interaction problems.

Subsequently, Li et al. (2005a) formulated SBFEM for problems with parallel side faces by modifying the scaled boundary transformation equation. Li et al. (2005b) applied the modified formulation to calculate wave diffraction by fixed structures and wave radiation excited by oscillation structures in water of finite depth. Li et al. (2004; 2006) extended SBFEM to solve the two-dimensional Helmholtz equation ($\nabla^2 \phi (x, y) + k^2 \phi (x, y) = 0$, $k$ is the wave number) which governs wave diffraction problems. Substructure techniques were employed to account for multiple structures, as shown in Figure 2-3. The entire wave domain was divided into several bounded domains $S_i$ ($i = 1, 2, \ldots, n$) and one unbounded domain $S_\infty$. The bounded domain formulation was similar to that presented in Song and Wolf (1998). In the unbounded domain formulation, the asymptotic expansion suggested by Wasow (1965) was employed.
The bounded and unbounded domains were assembled by matching boundary conditions on relevant interfaces. Li et al. (2004; 2006) found that SBFEM is free from irregular frequencies, a numerical difficulty that conventional Green’s function-based methods, such as BEM, often encounter. In addition, SBFEM does not suffer from numerical problems at sharp corners. On the contrary, it analytically resolves the velocity singularity by locating the scaling centre at sharp corners. Furthermore, SBFEM produces more efficient solutions than other numerical methods with far less number of DOFs.

Figure 2-3. SBFEM substructure configuration of domain division (Li et al., 2006)

Tao et al. (2007; 2009a; 2009b) also conducted a series of research regarding wave diffraction around structures using SBFEM. They commenced with the short-crested wave diffraction by a vertical cylinder (Tao et al., 2007). In this study, they took advantage of the properties of the Hankel function to satisfy the Sommerfeld radiation condition, thus avoiding the formulation of the Hamiltonian matrix and the subsequent matrix decomposition. The number of DOFs throughout the solution process remained the same as the original. Additional computational memory was not required, which however is the case in the standard SBFEM solution process. In the presented benchmark examples, Tao et al. (2007) used four elements to discretise half the cylinder circumference when \( ka < 3 \) (\( k \) is the wave number and \( a \) is the radius of the cylinder) and eight elements when \( ka = 5 \), to achieve the same accuracy as the analytical solution.

Following the above study, Tao and Song (2008) investigated the wave diffraction by an elliptic cylinder. They introduced a virtual circular cylinder to separate the entire infinite wave domain into one unbounded domain \( S_\infty \) and one bounded domain...
$S_b$, as shown in Figure 2-4. The bounded domain was further divided into four subdomains $S_i$ ($i = 1, 2, 3$ and $4$). The virtual cylinder had a circular cross-section so that the unbounded domain formulation followed that proposed in Tao et al. (2007). The bounded subdomains adopted similar formulation to that presented in Li et al. (2006). Subsequently, the analyses of wave diffraction by a square caisson (Song and Tao, 2008), two rectangular caissons (Tao et al., 2009b) and multiple cylinders of arbitrary shape (Song et al., 2010) were conducted in succession. At the same time, the wave diffraction by concentric porous cylindrical structures (Tao et al., 2009a) and non-uniform porous cylinders (Song and Tao, 2010b) were studied, which laid a foundation for the more recent study of wave diffraction by cylindrical structures with double-layered concentric perforated walls (Liu et al., 2012a).

![Figure 2-4. Two-dimensional illustration of subdomain division for wave diffraction by an elliptic cylinder (Tao and Song, 2008)](image)

With all the aforementioned effort focusing on solving linear partial differential equations to address wave diffraction problems, Li (2007) and Song and Tao (2010a) attempted the second-order wave diffraction problem using SBFEM. A series of pre-processing was required before SBFEM could be applicable.

In summary, it is encouraging to see that when analysing wave behaviour in the presence of structures, SBFEM successfully overcomes the difficulties that both FEM and BEM encounter. Contrary to FEM, domain truncations and artificial boundary conditions do not need to be treated; Unlike BEM, fundamental solutions and singular integral evaluations are not required, and numerical difficulties associated with irregular frequency and sharp corners do not need to be addressed. From the work
reviewed, SBFEM has demonstrated high efficiency and accuracy in solving wave domains when waves interact with structures. This, spontaneously but inspiringly, provokes the idea: Is it feasible to utilise SBFEM to solve the research problem identified in Section 2.1? Before answering this question, a thorough understanding of the theoretical and technical fundamentals of SBFEM as well as its advantages and limitations is indispensable, as detailed in Section 2.3. Section 2.2.3.4 continues the discussion about other numerical methods that have been used in wave-structure interaction analyses, for the purpose of integrity.

### 2.2.3.4 Other numerical methods

Other numerical methods used in the wave-structure interaction analysis include the Finite Difference Method (FDM), semi-analytical method, FEM-BEM coupling method, etc. Li and Lu (1987) employed FDM, in combination with the time stepping technique to calculate non-linear wave forces exerted on large coastal or offshore structures. Bingham and Zhang (2007) used a two-dimensional FDM-based solution scheme to solve a nonlinear potential flow problem, and suggested the combination of the fourth-order scheme with stretched vertical grid spacing in the finite difference discretisation. Ducroz et al. (2010) adopted this suggestion to investigate a three-dimensional nonlinear wave-structure interaction problem, in which a fine mesh was used in the vicinity of the structure, whereas a comparatively coarse mesh was responsible for the far region.

Chakrabarti (1978) presented the mechanism of multiple scattering among a vertical cylinder group. This work was further extended in Chakrabarti (2000; 2001) to address arbitrarily-shaped and floating structures. Eatock Taylor and Huang (1996) developed a method to obtain a particular solution for second-order regular wave diffraction by an axisymmetric structure. Huang and Eatock Taylor (1996) employed this method in the case of a truncated cylinder. Subsequently, Malenica et al. (1999) used the same method to address the second-order wave diffraction with multiple bottom-mounted cylinders.

The Trefftz method was used to calculate wave forces on offshore structures by Cheung et al. (1991). The complete and non-singular systems of Trefftz functions were employed to formulate the solution of the Helmholtz equation. Jiang and Wang (1995) studied the diffraction of cnoidal nonlinear shallow water waves around a
vertical cylinder using the Generalised Boussinesq (GB) numerical modal. Stojek et al. (2000) employed ‘frameless’ Trefftz-type finite elements to calculate diffraction loads on multiple vertical cylinders with rectangular cross sections. Li and Lin (2001) employed the large eddy simulation (LES) method to study the three-dimensional wave interaction with a vertical square cylinder.

Czygan and von Estorff (2002) were interested in coupling FEM and BEM to examine the wave-structure interaction problem. FEM was used for account for the material inhomogeneity of the structure, whereas the infinite extent of the wave domain was modelled by BEM. A fundamental solution associated with BEM needed to be sought before the volume discretisation of the structure and the boundary discretisation of the waves could be coupled. In addition, the computational efficiency of the coupling was not assessed if three-dimensional problems were involved.

Summarising the methodologies currently used in wave-structure interaction analyses, it is concluded that both analytical and experimental methods are inappropriate to the interaction problem specified in this PhD study. FEM, BEM and other numerical methods have certain drawbacks, and are not as effective and efficient as SBFEM, which has encouragingly demonstrated its potential applicability in three-dimensional interaction problems. For further validation, a systematic examination of SBFEM, in terms of its capability and efficiency, is carried out and presented in the next section.

### 2.3 The Scaled Boundary Finite Element Method

This section concentrates on SBFEM, with its concept origin and technical development presented in Section 2.3.1. Its applications in engineering fields are illustrated in Section 2.3.2, not including the application in coastal and offshore engineering which has already been reported in Section 2.2.3.3. Its limitations are introduced in Section 2.3.3. This section provides a clear picture of SBFEM for a rational and reasonable understanding of the method, to assist the construction of the framework of this PhD study.

#### 2.3.1 Origin and development

Many engineering problems can be formulated using partial differential equations (PDEs) in multiple independent variables, to explain the underlying physical mechanism. In the subject of dynamic unbounded medium-structure interaction, the
solution of PDEs is normally overwhelmingly difficult. If PDEs can be transformed to ordinary differential equations (ODEs) in a single variable, a solution may feasibly be sought.

Wolf and Song (1996), Song and Wolf (1996a; 1996b) employed the concepts of similarity and discretisation to successfully fulfil the above-mentioned transformation, thus introducing a new computational procedure, termed the Consistent Infinitesimal Finite Element Cell Method (Song and Wolf, 1996a; Song and Wolf, 1996b). It was later renamed as the Scaled Boundary Finite Element Method (SBFEM) when Song and Wolf (1997) reformulated the derivation procedure. A local dimensionless coordinate system \((\xi, \eta, \zeta)\) (see Figure 2-5) was introduced, consisting of a radial coordinate \(\xi\) pointing outwards from an origin \(O\), and two circumferential directions \(\eta, \zeta\) (for three-dimensional problems; and one circumferential direction \(\eta\) for two-dimensional problems) which are parallel to the domain boundary. This coordinate system is far more flexible than the commonly used Cartesian or cylindrical coordinate systems in that it does not require the orthogonality amongst the coordinates. The radial coordinate \(\xi\) assists the scaling of the boundary with respect to the origin \(O\). The other two coordinates, \(\eta\) and \(\zeta\), normally in line with the domain boundary, are locally defined for a convenient description of boundary discretisation. Therefore, the study domain is expressed by \(\xi, \eta\) and \(\zeta\) through a scaling of the discretised boundary according to the radial coordinate. The coordinate system \((\xi, \eta, \zeta)\) is called the scaled boundary coordinate system, and the origin \(O\) is called the scaling centre.

The solution concept of SBFEM originates from two robust numerical techniques, namely FEM and BEM, as can be read from its nomenclature. A number of early work (Song and Wolf, 1997; Song and Wolf, 1998; Wolf and Song, 1998; Song and Wolf, 1999; Song and Wolf, 2000; Wolf and Song, 2000; Wolf and Song, 2001) detailed the fundamental derivations and solution procedures of SBFEM, employing elastodynamics for illustration. Generally, SBFEM commences with a transformation of the domain geometry from the original coordinate system to the scaled boundary coordinate system, and performs a boundary discretisation afterwards. The Galerkin’s weighted residual method (or the virtual work principle (Deeks and Wolf, 2002b)) is applied in circumferential directions, reducing the governing PDEs to ODEs in the
The nodal displacement function \( \{u(\xi)\} \) with the radial coordinate \( \xi \) being an independent variable. The nodal displacement function \( \{u(\xi)\} \) describes the displacement variation along the radial coordinate, and is to be calculated analytically from ODEs. In the circumferential directions, the displacement is obtained by employing the interpolation concept in FEM using shape functions. Thus, SBFEM is accurate in the radial direction and is exact in the finite element sense in circumference directions. It is a semi-analytical fundamental solution-less BFM based on FEM. It possesses both the accuracy of analytical analysis and the flexibility of numerical analysis (Wolf, 2003). It combines the advantages of FEM and BEM, also has its own appealing features. A comparison of the three methods is summarised as shown in Table 2-1.

Since the emergence of SBFEM, pioneer researchers have been dedicated to solve technical difficulties and mathematical barriers ever encountered, extending the range of problems to which SBFEM can be applied. Song and Wolf (1999) derived the explicit expression for concentrated loads and loads varying as power functions in the radial coordinate by using the technique of variation of parameters. Doherty and Deeks (2003a; 2003b) proposed a derivation in which Fourier series was employed to formulate SBFEM in cylindrical coordinate system, which is well suited for problems involving axisymmetric geometry. Deeks (2004) detailed a technique allowing displacements to be prescribed along side faces. Song (2004) identified a potential numerical difficulty associated with the power series formulation of the nodal
Table 2-1. Comparisons between FEM, BEM and SBFEM (based on Song and Wolf, 1997)

<table>
<thead>
<tr>
<th>Comparison items</th>
<th>FEM</th>
<th>BEM</th>
<th>SBFEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reduction of the spatial dimension by one as only the boundary is discretised with surface finite elements, reducing the data preparation and computational efforts</td>
<td></td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>Analytical solution achieved inside the problem domain</td>
<td></td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>No fundamental solution required, expanding the scope of application and avoiding singular integrals</td>
<td>×</td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>Radiation condition at infinity satisfied exactly when modelling unbounded (infinite or semi-infinite) domain</td>
<td></td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>No discretisation of free and fixed boundaries or interfaces between different materials</td>
<td></td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>No approximation other than that of the surface finite elements on the boundary</td>
<td></td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>Free from numerical difficulties associated with singular integral, sharp corners and irregular frequency</td>
<td>×</td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>Symmetric dynamic-stiffness and unit-impulse response matrices for unbounded media</td>
<td></td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>Symmetric static-stiffness and mass matrices for bounded media</td>
<td></td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>Body loads processed without additional domain discretisation and thus additional approximation</td>
<td></td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>Straightforward calculation of stress concentrations and intensity factors based on their definition</td>
<td></td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>No fictitious eigenfrequencies for unbounded media</td>
<td></td>
<td></td>
<td>×</td>
</tr>
<tr>
<td>Straightforward coupling by standard assemblage of structure discretised with finite elements with unbounded medium</td>
<td></td>
<td></td>
<td>×</td>
</tr>
</tbody>
</table>

displacement foundation, and proposed a matrix function solution to overcome problems caused by parallel eigenvectors resulted from underlying logarithmic terms in the solution.
As SBFEM is being extended to large-scale problems, an improvement of its computational capacity is necessary. Song (2006) constructed a reduced set of block-orthogonal base functions, excluding the unnecessary high-order modes, to allow for the asymptotic expansion of the dynamic stiffness matrix being evaluated at a lower frequency, hence significantly reducing the computational time. In the meanwhile, a Padé series approximation, allowing the dynamic stiffness matrix at any specified frequency being calculated directly without performing numerical integral from a high frequency, was proposed by Song and Bazyar (2007). In the following year, they developed an algorithm to exploit the sparseness of the coefficient matrices of the scaled boundary finite element equation to identify the reduced set of base functions (Song and Bazyar, 2008). In combination with the technique of reduced set of base functions, the Padé series approximation demonstrates a high rate of convergence and a satisfactory level of accuracy.

Closely related to the Padé series approximation, a continued fraction solution of the dynamic stiffness matrix was determined by Bazyar and Song (2008) to formulate a high-order local transmitting boundary for wave propagation problems in unbounded domains. The coefficient matrices of the continued fraction are calculated recursively from the scaled boundary finite element equation. By introducing auxiliary variables corresponding to high-order terms, the number of degrees of freedom (DOF) of the resulting equation increases, however, the solution converges rather rapidly. The computational expense associated with convolution integrals is circumvented in the solution. The continued fraction can also be employed together with the technique of reduced set of base functions to further improve its numerical performance. Standard procedures, such as the Newmark’s integral method, are applied in a straightforward way when continued fraction technique is used for structural dynamic analyses (Song, 2009). Most recently, Birk et al. (2012) improved the continued fraction solution of the dynamic stiffness matrix for more robust calculation of large-scale three-dimensional problems.

By using the continued fraction technique, the nodal force-nodal displacement relationship, viz the equation of motion, is formulated as a local open boundary condition (or transmitting boundary condition) to model the unbounded domain. The open boundary condition is applicable to radiant fields to absorb wave propagation energy, and is singly asymptotic at high frequency limits. Prempramote et al. (2009)
developed high-order doubly asymptotic open boundaries for scalar wave propagation to address evanescent waves and low-frequency responses. Birk and Song (2010) applied the same boundary concept for transient diffusion problems in a semi-infinite homogeneous layer.

Through the contribution of many researchers, SBFEM is becoming more and more robust. To date, it has attracted the attention of several research groups all over the world. Some of their work is reviewed in the following section to illustrate the broad application of SBFEM.

2.3.2 Applications of SBFEM

SBFEM has been utilised in various engineering fields with rapid recognition and acknowledgment. Apart from the wave propagation problem within the framework of dynamic unbounded medium-structure interaction, from which the concept of SBFEM was originally derived, SBFEM has been employed to solve soil-structure interaction problems in geotechnical engineering; wave diffraction problems in ocean engineering; and waveguide eigen-problems in electromagnetic engineering. It has also been applied to fracture mechanics and hydraulics.

2.3.2.1 Geotechnical engineering

In dynamic soil-structure interaction (DSSI) analyses, the dynamic properties of the unbounded medium are represented by the interaction force \( \{R\} \)-displacement \( \{u\} \) relationship associated with the DOFs on the soil-structure interface (see Figure 2-6). Wolf and Song (1998; 2002) and Wolf (2002) formulated the response matrix \( [S^\infty(t)] \) of the unbounded medium to a unit impulse of displacement using SBFEM. The dynamic stiffness matrix \( [S^\infty(\omega)] \) at high frequency can be asymptotically expanded in a power series of \( i\omega \) in descending order as:

\[
[S^\infty(\omega)] \approx i\omega[C_\omega] + [K_\omega] + \sum_{j=1}^{m} \frac{1}{(i\omega)^j} [A_j]
\]  

(2.6)

Performing the inverse Fourier transformation of Equation (2.6) leads to the expression of \( [S^\infty(t)] \) as:

\[
[S^\infty(t)] \approx [C_\omega]\delta(t) + [K_\omega]\delta(t) + \sum_{j=1}^{m} \frac{1}{(j-1)!} [A_j] t^{j-1} H(t)
\]  

(2.7)
where \([C_\infty]\) and \([K_\infty]\) are the constant dashpot matrix and constant spring matrix; \([A_j]\) 
\((j = 1, \ldots, m)\) are the unknown constant coefficient matrices. They are determined from the coefficient matrices of the SBFEM equation. \(\delta(t)\) and \(H(t)\) are the Dirac delta function and the Heaviside step function, respectively.

Zhang et al. (1999) employed the above formulation to model the unbounded soil when analysing three-dimensional DSSI problems in the time domain. The original formulation is spatially and temporally global, and thus computationally expensive. In their study, an approximation scheme in time and space was posed, thereby considerably reducing the computational effort for calculating the unit-impulse response matrix and the interaction force. The response matrix was approximated by a few linear segments and assumed to be piece-wise constant over each time step. The soil-structure interface was divided into substructures and then assembled as in FEM.

In DSSI problems, the structure and near field region may exhibit non-linear behaviour, a hybrid method combining FEM and SBFEM was proposed so that both the nonlinearity and infinity can be addressed taking advantage of the two methods (Ekevid and Wiberg, 2002; Genes and Kocak, 2002; Yan et al., 2004; Doherty and Deeks, 2005; Lehmann, 2005; Wegner et al., 2005; Bazyar and Song, 2006). The conventional FEM is used to deal with the nonlinearity of the finite region and SBFEM is used to model the linear elastic soil region (see Figure 2-7). Thus, the equation of motion of the structure is written in the time domain as:
\[
\begin{bmatrix}
M_{ss} & M_{sb} \\
M_{bs} & M_{bb}
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_s(t) \\
\ddot{u}_b(t)
\end{bmatrix}
+ \begin{bmatrix}
C_{ss} & C_{sb} \\
C_{bs} & C_{bb}
\end{bmatrix}
\begin{bmatrix}
\dot{u}_s(t) \\
\dot{u}_b(t)
\end{bmatrix}
+ \begin{bmatrix}
K_{ss} & K_{sb} \\
K_{bs} & K_{bb}
\end{bmatrix}
\begin{bmatrix}
u_s(t) \\
u_b(t)
\end{bmatrix} = \begin{bmatrix}
P_b(t) \\
P_b(t) - R_b(t)
\end{bmatrix}
\]

(2.8)

where \([M], [C] and [K]\) are the mass, damping and stiffness matrices, respectively; \(\{u(t)\}, \{\dot{u}(t)\} and \{\ddot{u}(t)\}\) are displacement, velocity and acceleration vectors; \(\{P(t)\}\) is the external force vector; \(\{R(t)\}\) is the interaction force vector representing the contribution of the unbounded domain on the generalised soil-structure interface, and is expressed as:

\[
\{R(t)\} = \int_0^t G(t - \tau) \ddot{u}(\tau) d\tau
\]

(2.9)

\(G(t)\) is the acceleration unit-impulse response matrix of the unbounded soil and is determined by SBFEM. Subscripts \(b\) and \(s\) correspond to the generalised soil-structure interface and the structure, respectively.

Figure 2-7. Coupling FEM and SBFEM model for soil-structure interaction analysis
(reproduced from Genes and Kocak, 2005)

One issue in the coupling of FEM and SBFEM when analysing DSSI problems can be identified as the location of the generalised soil-structure interface, i.e. if it is not far enough, fictitious reflections may be generated on the interface and pollute the solution; if it is too far away, an unnecessarily excessive number of DOFs will be used to model the linear soil region, thus increasing the computational demand. Doherty and Deeks (2005) developed an adaptive technique to identify the distance. An iterative procedure is performed, in which an adaptive finite-element mesh is
extended if plasticity is detected in the outer band of elements. Similarity is enforced in the dynamic mesh extension procedure for the sake of SBFEM formulation.

### 2.3.2.2 Structural engineering

Lindemann and Becker (2000) employed SBFEM to calculate the inter-laminar stress and analysed the free-edge effect in composite laminates. They found that SBFEM demonstrated higher efficiency and accuracy in predicting the stress field near the free edge of the laminate compared with an equivalent finite element analysis. Song and Wolf (2002) introduced SBFEM to represent stress singularities at crack tips or at material interfaces in fracture mechanics (see Figure 2-8). The crack faces or the interfaces between different materials are not discretised, however boundary conditions on them are satisfied exactly. Enforcing the scaling centre to coincide with the crack tip, SBFEM permits an analytical representation of stress singularities in the radial direction. In circumferential directions where stress varies smoothly, the solution converges as in the weighted-residual formulation in FEM. This study identified one prominent advantage of SBFEM as modelling the stress singularity and paved the way for SBFEM to develop in fracture mechanics.

![Figure 2-8. Concrete bulk with a crack (reproduced from Yang and Deeks, 2007b) (a) physical model and (b) SBFEM model](image)

Taking this advantage, Chidgzey and Deeks (2005) continued Song and Wolf’s work and applied SBFEM to the calculation of linear elastic crack tip field. The coefficients in the Williams expansion were directly obtained from the SBFEM solution, allowing the evaluation of SIFs, T-stress and high-order coefficients. Case studies showed satisfactory performance of the SBFEM result obtained with relatively few DOFs compared with other numerical solutions. Yang (2006a) further discussed the application of SBFEM in static and dynamic fraction problems. For static
problems, the mixed-mode crack propagation in brittle or quasi-brittle materials was fully-automatically modelled by SBFEM in combination with a Linear Elastic Fracture Mechanics (LEFM)-based re-meshing algorithm. For dynamic problems, the newly-developed Frobenius solution (Yang et al., 2007a) to the scaled boundary finite element equation in the frequency domain was used to calculate stress intensify factors (SIFs). They concluded that SBFEM is capable of predicting static and dynamic SIFs, mixed-mode cracking paths and load-displacement relations accurately, efficiently and effectively. A further enhancement using SBFEM in transient dynamic fracture analysis was presented in Yang and Deeks (2007a) and Yang et al. (2007b). Yang and Deeks (2007b) later extended their work to cohesive crack growth problems through a FEM-SBFEM coupled method. Chidgzey et al. (2008) and Bird et al. (2010) highlighted the combination of BEM and SBFEM for accurate computations in fracture mechanics.

Apart from being employed to address fracture problems, SBFEM was also used to study the structural behaviour of coastal and offshore monopiles. It was extended by Li et al. (2010b) and Li et al. (2011) to the area of offshore renewable energy. They investigated the response of monopile foundations, used to support offshore wind turbines, to external wave loads. The studies demonstrated favourable capability of SBFEM, and at the same time contributed to a more economical and reliable design of monopile foundations.

2.3.2.3 Other engineering fields

Taking advantage of the well-established SBFEM formulation of the Laplace equation, which governs the potential flow problem, Liu et al. (2010) introduced SBFEM to electromagnetic engineering. They started from an electrostatic problem, which set a precedent for the subsequent investigation of waveguide eigenvalue problems using SBFEM (Lin et al., 2011). Subsequently, Liu and Lin (2011a; 2011b) and Liu et al. (2012b) detailed the waveguide eigenvalue problems of quadruple corner-cut ridged circular, elliptical and square wave guides. These studies demonstrated the high efficiency, accuracy and rate of convergence of SBFEM in solving electromagnetic problems.

As the development of SBFEM’s ability in solving DSSI problems, a number of researchers introduced SBFEM to hydraulic engineering to solve dam-reservoir-
foundation interaction problems. Early studies (Lin et al., 2007a; Lin et al., 2007b; Li et al., 2008) were confined to in the frequency domain before Li (2009) applied the diagonalisation procedure (Paronesso and Wolf, 1996; Yan et al., 2004) to reduce the computational burden in the time-domain analysis. Subsequently, Li (2012a; 2012b; 2011) conducted a few studies using this technique to solve time dependent dam-infinite reservoir interaction problems. More recently, Lin et al. (2012) evaluated the dam-reservoir interaction problem with advanced radiation boundary formulation. Gao et al. (2011) and Wang et al. (2011) used the high-order doubly asymptotic open boundary to accurately model the semi-infinite reservoir over the entire frequency range, which advanced SBFEM’s application in hydraulic engineering to a new level.

2.3.3 Limitations of SBFEM

A few circumstances where SBFEM shows certain shortfalls are examined:

- As discussed, SBFEM is currently only capable to solve problems formulated with linear governing equations, though it was applied to address nonlinear problems (Song and Tao, 2010a) by a linearisation of the physical problem prior to the SBFEM calculation.
- SBFEM is not competitive to FEM in solving problems with bounded geometric scope and when smooth stress variations present, but it outperforms FEM in cases where stress singularity appears even though with a bounded computational domain (Yang, 2006b).
- SBFEM is not flexible to deal with arbitrary material property distribution, though some specific distribution modes can be addressed (Doherty and Deeks, 2003b). In this regards, the substructing technique can be applied as a resort though it will introduce extra number of DOFs.

SBFEM has become well-established over the last decade, and is seeing many of its superiorities and attractive properties. It offers more than just combining the advantages of FEM and BEM, also exhibiting its own appealing features which are summarised as: (1) reducing the dimension of the problem by one and accordingly the computational effort; (2) achieving analytical solutions in the radial direction and rigorously satisfying the boundary condition at infinity; (3) capable of dealing with problems involving stress singularities and discontinuity; (4) free from numerical difficulties associated with irregular frequencies, sharp corners and singular integrals.
With limited number of shortcomings, SBFEM has been applied to many engineering fields and has great potential to further expand its applications.

2.4 Summary

This chapter reviews the wave-structure interaction study and existing research methodologies with special emphasis on SBFEM. Keeping in view the current knowledge of the physical problem, and based on the understanding of all available methods, several research questions arise, one of which specifically addresses the research question, namely, \textit{Is it feasible to advance SBFEM to analyse the complex interaction between waves and structures, and encompass both the wave motion in the presence of structures and the structural reactions subjected to wave forces?} This research proposal is motivated by its significance, from a theoretical angle, in updating the concept of wave-structure interaction, while simultaneously advancing the development of SBFEM; from an engineering viewpoint, in providing valuable information to assist structural design and safety assessment for ocean constructions.

Prior to a detailed study, a systematic plan is made for this PhD project in terms of: \textit{What theoretical assumptions are enforced? In which specific aspects and to what extent is the interaction problem investigated?} These questions are addressed in the following text in sequence:

\textit{What theoretical assumptions are enforced?}

Though quite a few studies have been directed to non-linear investigation of wave properties in the literature, the interaction problem specified in this study is restricted in the linear scope by applying the Airy wave theory in that: (1) the Airy wave theory offers a description of wave parameters with sufficient accuracy for many purposes (Goda, 2010; Dean and Dalrymple, 1991), such as the structural design and reliability evaluation from the engineering point of view. Moreover, this linear theory allows for a quick and preliminary evaluation of wave properties and their effects on structures; (2) nonlinear kinematic and dynamic wave parameters can be estimated from linear analysis results (Phillips, 1977); (3) as an initial trial to examine the feasibility of advancing SBFEM to the complex interaction problem, it is essentially conducive to start with a linear formulation. The effects of wave non-linearity can be the subject of further research once this study is accomplished. In these circumstances, the wave
field analysis is conducted under the assumptions that the fluid is inviscid and incompressible, and the flow motion is irrotational.

*In which specific aspects and to what extent is the interaction problem investigated?*

In this study, the structural entity is chosen to be represented by pile foundations, due to their extensive application in coastal and offshore constructions. They appear as monopile foundations in offshore wind farms; as pile group foundations supporting oil and gas exploitation platforms or airport platforms; as clusters of piers in ports and harbour areas for bridges and mooring purposes. Therefore, the wave interaction with both single pile foundations and group pile foundations are addressed. In addition, this study is carried out in a fully three-dimensional scope, and in both steady scope and time-dependent context.

The subsequent chapters present a thorough and systematic investigation according to the aforementioned aspects to fulfil the *Three-Dimensional Wave-Structure Interaction Modelling Using the Scaled Boundary Finite Element Method*. 
Chapter 3  SBFEM Development*

Unlike its counterparts, FEM or BEM for instance, SBFEM integrates intensive matrix calculations, such as matrix inversion and decomposition. The stability and accuracy of the SBFEM calculation are highly dependent on matrix properties as well as matrix manipulation techniques. This chapter investigates the numerical performance of the SBFEM calculation. Its fundamental derivations are briefly reviewed and outlined in Section 3.1. Subsequently, technical issues in terms of matrix manipulation techniques and matrix properties are addressed in Sections 3.2 and 3.3, respectively. Illustrative examples are employed to identify possible reasons of potential numerical difficulties. Corresponding solution schemes are also proposed to overcome these problems. This chapter aims to lay a solid foundation for credible investigations of the wave-structure interaction in the subsequent studies.

3.1 SBFEM fundamental derivations

The fundamental solution procedures of SBFEM are summarised following Wolf (2003), Wolf and Song (1996). First, a coordinate system transformation is performed to reformulate the partial differential equation (PDE), which governs the physical problem, into a matrix-form ordinary differential equation (ODE). Subsequently, the ODE is solved through a series of matrix manipulations to obtain the analytical nodal function. After the nodal function is acquired, interpolation using shape functions and specification of the scaled boundary coordinates lead to the semi-analytical solution of the entire domain. The following subsections will employ three-dimensional elasticity mechanics to present some key technical derivations of SBFEM for later references.

3.1.1 Scaled boundary coordinate system

The main framework of the SBFEM solution procedure begins with a coordinate transformation from a conventional coordinate system, the Cartesian coordinate system \((\hat{x}, \hat{y}, \hat{z})\) for example, to a scaled boundary coordinate system \((\xi, \eta, \zeta)\). This coordinate system, as shown in Figure 3-1, is locally constructed by a scaling centre \(O(x_0, y_0, z_0)\), a defining boundary \(S\) and a dimensionless radial coordinate \(\xi\). The radial coordinate originates from the scaling centre and is assigned as 0, 1 and \(\infty\) at the scaling centre, on the defining boundary \(S\) and at infinity, respectively. Thus, the bounded domain \(V_b\) can be obtained by scaling the defining boundary with respect to the radial coordinate for a range of \(0 < \xi < 1\), or alternatively, a scaling of \(S\) for \(1 < \xi < \infty\) will lead to an unbounded domain \(V_\infty\). Two curvilinear coordinates \(\eta, \zeta\) are introduced in the circumferential directions tangential to the defining boundary \(S\), with their magnitudes locally varying over an interval of \([-1, 1]\) for each discretised element on the defining boundary. The discretisation is only applicable to the defining boundary. Other boundaries, termed as side faces, do not need to be discretised.

![Figure 3-1. Scaled boundary coordinate system (reproduced from Wolf and Song, 1996)](image)

* As a convention in SBFEM, the coordinate in the Cartesian space is represented by \((\hat{x}, \hat{y}, \hat{z})\) as \((x, y, z)\) is reserved to denote the coordinates on the boundary. However, \(x, y\) and \(z\) are still used when indicating directions in the following discussions.
According to the notation defined in Figure 3-1, where \( \{x\}, \{y\} \text{ and } \{z\} \) represent the coordinates of the discretised nodes on the defining boundary, any position in the study domain can be definitely located by specifying \( \xi, \eta \text{ and } \zeta \) using the following transformation and the mapping function \([N(\eta, \zeta)]\):

\[
\hat{x}(\xi, \eta, \zeta) = \xi[N(\eta, \zeta)]\{x\} + x_0 \\
\hat{y}(\xi, \eta, \zeta) = \xi[N(\eta, \zeta)]\{y\} + y_0 \\
\hat{z}(\xi, \eta, \zeta) = \xi[N(\eta, \zeta)]\{z\} + z_0
\]

Equation (3.1) describes the geometrical interrelation between the Cartesian coordinate system and the scaled boundary coordinate system, and it is the core of the SBFEM concept. The SBFEM equation is derived from Equation (3.1), which will be shown in the next subsection.

### 3.1.2 Transformation to the SBFEM equation

A three-dimensional elasto-dynamic problem will be employed herein to illustrate the derivation of the SBFEM equation. The differential equation of motion is written in the Cartesian coordinate system in a vector form as:

\[
[L]^T \{\sigma\} + \omega^2 \rho \{u\} = 0
\]

with the partial differential operator \([L]\):

\[
[L] = \begin{bmatrix}
\frac{\partial}{\partial \hat{x}} & 0 & 0 & \frac{\partial}{\partial \hat{y}} & \frac{\partial}{\partial \hat{z}} \\
0 & \frac{\partial}{\partial \hat{y}} & 0 & \frac{\partial}{\partial \hat{z}} & \frac{\partial}{\partial \hat{x}} \\
0 & 0 & \frac{\partial}{\partial \hat{z}} & \frac{\partial}{\partial \hat{y}} & \frac{\partial}{\partial \hat{x}} & 0
\end{bmatrix}^T
\]

\(\omega\) is the excitation frequency; \(\rho\) is the mass density. The stress amplitude \(\{\sigma\}\) is related to the strain amplitude \(\{\varepsilon\}\) and the elastic matrix \([D]\) as:

\[
\{\sigma\} = [D]\{\varepsilon\}
\]

The strain amplitude \(\{\varepsilon\}\) and displacement amplitude \(\{u\}\) are related by \([L]\) in the form of:

\[
\{\varepsilon\} = [L]\{u\}
\]
Equations (3.2)-(3.5) form the governing equations of three-dimensional elasto-dynamics. With the geometric mapping shown in Equation (3.1), the differential operator \([L]\) is reformulated in the scaled boundary coordinate system using \(\xi, \eta \) and \(\zeta\) as:

\[
[L] = \left[ b^1(\eta, \zeta) \right] \frac{\partial}{\partial \xi} + \frac{1}{\xi} \left[ b^2(\eta, \zeta) \right] \frac{\partial}{\partial \eta} + \left[ b^3(\eta, \zeta) \right] \frac{\partial}{\partial \zeta} \tag{3.6}
\]

in which \([b^1(\eta, \zeta)], [b^2(\eta, \zeta)]\) and \([b^3(\eta, \zeta)]\) are determined by the scaled boundary discretisation on the defining boundary \(S\), and independent of the radial coordinate \(\xi\).

\[
\begin{bmatrix}
    b^1(\eta, \zeta)
\end{bmatrix} = \frac{1}{\bar{J}(\eta, \zeta)} \begin{bmatrix}
    y_\eta z_\xi - z_\eta y_\xi & 0 & 0 \\
    0 & z_\eta x_\xi - x_\eta z_\xi & 0 \\
    0 & 0 & x_\eta y_\xi - y_\eta x_\xi \\
    0 & x_\eta y_\xi - y_\eta x_\xi & 0 \\
    y_\eta z_\xi - y_\xi z_\eta & 0 & y_\eta z_\xi - z_\eta y_\xi \\
    z_\eta x_\xi - x_\eta z_\xi & y_\eta z_\xi - z_\eta y_\xi & 0 \\
\end{bmatrix} \tag{3.7}
\]

\[
\begin{bmatrix}
    b^2(\eta, \zeta)
\end{bmatrix} = \frac{1}{\bar{J}(\eta, \zeta)} \begin{bmatrix}
    z_\eta y_\xi - y_\eta z_\xi & 0 & 0 \\
    0 & x_\eta z_\xi - z_\eta x_\xi & 0 \\
    0 & 0 & y_\eta x_\xi - x_\eta y_\xi \\
    0 & y_\eta x_\xi - x_\eta y_\xi & 0 \\
    y_\eta x_\xi - x_\eta y_\xi & 0 & z_\eta y_\xi - y_\eta z_\xi \\
    x_\eta z_\xi - z_\eta x_\xi & z_\eta y_\xi - y_\eta z_\xi & 0 \\
\end{bmatrix} \tag{3.8}
\]

\[
\begin{bmatrix}
    b^3(\eta, \zeta)
\end{bmatrix} = \frac{1}{\bar{J}(\eta, \zeta)} \begin{bmatrix}
    y_\eta z_\eta - z_\eta y_\eta & 0 & 0 \\
    0 & z_\eta x_\eta - x_\eta z_\eta & 0 \\
    0 & 0 & x_\eta y_\eta - y_\eta x_\eta \\
    0 & x_\eta y_\eta - y_\eta x_\eta & 0 \\
    y_\eta x_\eta - x_\eta y_\eta & 0 & z_\eta y_\eta - y_\eta z_\eta \\
    x_\eta z_\eta - z_\eta x_\eta & z_\eta y_\eta - y_\eta z_\eta & 0 \\
\end{bmatrix} \tag{3.9}
\]

with \(\bar{J}(\eta, \zeta)\) denoting the determinant of the Jacobin matrix and expressed as:

\[
\bar{J}(\eta, \zeta) = x\left(y_\eta z_\xi - z_\eta y_\xi\right) + y\left(z_\eta x_\xi - x_\eta z_\xi\right) + z\left(x_\eta y_\xi - y_\eta x_\xi\right) \tag{3.10}
\]

Using the same shape function \([N(\eta, \zeta)]\) as for boundary discretisation, the displacement amplitude is expressed as:

\[
\{u(\xi, \eta, \zeta)\} = [N(\eta, \zeta)]\{u(\xi)\} \tag{3.11}
\]
whereby \( \{u(\xi)\} \) represents the nodal displacement function in the radial coordinate \( \xi \). This function is the basic unknown function in the upcoming scaled boundary finite element equation. Once it has been solved, the displacement field within the study domain can be obtained using Equation (3.11) with specified scaled boundary coordinates \( \zeta, \eta \), and subsequently the stress and strain fields can be calculated as:

\[
\{\varepsilon\} = \{\varepsilon(\xi, \eta, \zeta)\} = \left[ B^1 \right] \{u(\xi)\} + \frac{1}{\xi} \left[ B^2 \right] \{u(\xi)\}
\]

\[
\{\sigma\} = \{\sigma(\xi, \eta, \zeta)\} = \left[ D \right] \left( \left[ B^1 \right] \{u(\xi)\} + \frac{1}{\xi} \left[ B^2 \right] \{u(\xi)\} \right)
\]

(3.12)

with \( \left[ B^1 \right] \) and \( \left[ B^2 \right] \) formulated as:

\[
\left[ B^1 (\eta, \zeta) \right] = [b^1(\eta, \zeta)] [N(\eta, \zeta)]
\]

\[
\left[ B^2 (\eta, \zeta) \right] = [b^2(\eta, \zeta)] [N(\eta, \zeta)] + [b^3(\eta, \zeta)] [N(\eta, \zeta)]_{\zeta}
\]

(3.13)

After the geometric transformation, Equations (3.2)-(3.5) are weakened by employing either the weighted residual technique or the variational principle along the discretised circumferential directions. Through a series of mathematical manipulations, the governing PDEs (3.2)-(3.5) are transformed into the second-order matrix-form Euler-Cauchy ODEs in terms of the nodal displacement function \( \{u(\zeta)\} \):

\[
\left[ E^0 \right] \xi^2 \{u(\xi)\} + \left( 2[E^0]^T + [E^1]^T - [E^1] \right) \xi \{u(\xi)\} + \left( [E^1]^T - [E^2] \right) [u(\xi)] = 0
\]

(3.14)

with the internal nodal force \( \{q(\xi)\} \) being expressed as:

\[
\{q(\zeta)\} = \left[ E^0 \right] \xi^2 \{u(\xi)\} + \left[ E^1 \right]^T \xi \{u(\xi)\}
\]

(3.15)

Equation (3.14) is termed as the scaled boundary finite element equation. In Equation (3.14), only the radial coordinate \( \xi \) appears. The other two coordinates \( \eta \) and \( \zeta \) are incorporated in the coefficient matrices in the form of:
\[ [E^0] = \int_{\eta_1}^{\eta_1} \int_{\zeta_1}^{\zeta_1} [B^0(\eta, \zeta)]^T [D] [B^0(\eta, \zeta)] [J(\eta, \zeta)] \, d\eta d\zeta \]
\[ [E^1] = \int_{\eta_1}^{\eta_1} \int_{\zeta_1}^{\zeta_1} [B^1(\eta, \zeta)]^T [D] [B^1(\eta, \zeta)] [J(\eta, \zeta)] \, d\eta d\zeta \]
\[ [E^2] = \int_{\eta_1}^{\eta_1} \int_{\zeta_1}^{\zeta_1} [B^2(\eta, \zeta)]^T [D] [B^2(\eta, \zeta)] [J(\eta, \zeta)] \, d\eta d\zeta \]
\[ [M^0] = \int_{\eta_1}^{\eta_1} \int_{\zeta_1}^{\zeta_1} [N(\eta, \zeta)]^T [\rho N(\eta, \zeta)] [J(\eta, \zeta)] \, d\eta d\zeta \]  
(3.16)

The coefficient matrices \([E^0]\), \([E^1]\), \([E^2]\) and \([M^0]\) in Equation (3.16) are first formulated for each individual element discretised on the defining boundary \(S\) and then assembled in the same way as in FEM.

### 3.1.3 Solution of the SBFEM equation

Equation (3.14) is a linear second-order matrix-form ordinary differential equation, the solution \(\{u(\zeta)\}\) of which represents the analytical variation of the nodal displacement in the radial direction. For elasto-static problems with \(\omega = 0\), Equations (3.14) and (3.15) are formulated on the boundary, where the nodal force \(\{R\}\)-nodal displacement \(\{u\}\) relationship is introduced as:

\[ \{R\} = [K]\{u\} \]  
(3.17)

with \([K]\) representing the static stiffness matrix on the boundary. A new variable \(\{X(\zeta)\}\), incorporating the nodal displacement function \(\{u(\zeta)\}\) and the nodal force function \(\{q(\zeta)\}\), is introduced as:

\[ \{X(\zeta)\} = \begin{bmatrix} \zeta^{0.5} \{u(\zeta)\} \\ \zeta^{-0.5} \{q(\zeta)\} \end{bmatrix} \]  
(3.18)

By introducing \(\{X(\zeta)\}\) and employing a Hamiltonian matrix \([Z]\), Equation (3.14) is transformed into a first-order homogeneous linear ODE in \(\{X(\zeta)\}\) expressed by Equation (3.19). The number of DOFs associated with the physical problem is doubled. However the order of the matrix-form equation is reduced from two to one.

\[ \zeta \{X(\zeta)\}_{\zeta} = -[Z] \{X(\zeta)\} \]  
(3.19)

The Hamiltonian matrix \([Z]\) is calculated by the coefficient matrices \([E^0]\), \([E^1]\), \([E^2]\) and the identity matrix \([I]\) as:
\[
[Z] = \begin{bmatrix}
E^0 & -0.5\mathbb{I} & -E^0 \\
-E^2 + E^1 & E^0 & -E^1 \\
-E^1 & E^0 & E^1
\end{bmatrix}
\] (3.20)

For elasto-dynamic problems, the nodal displacement function \( \{u(\xi)\} \) represents the displacement variation in the radial direction, at the same time, it records the variation history with respect to time. The nodal force \( \{R\} \)-nodal displacement \( \{u\} \) relationship is introduced on the discretised boundary as:

\[ \{R\} = [S(\omega)]\{u\} \] (3.21)

with \([S(\omega)]\) representing the dynamic stiffness matrix. With \( \{R\} = \{q(\xi = 1)\} \), the scaled boundary finite element equation is rewritten in dynamic stiffness matrix as:

\[
\left([S(\omega)] - [E^1]\right) - \left([S(\omega)] - [E^1]^T\right) - \left([S(\omega)] - [E^0]\right) + \omega^2 [S(\omega)] = 0
\]

which is a non-linear first-order matrix-form ordinary differential equation. In this instance, the main objective is to solve the dynamic stiffness matrix \([S(\omega)]\) from Equation (3.22) and back substitute to Equation (3.21) to obtain the nodal history \( \{u\} \).

Either being formulated in terms of the nodal displacement function \( \{u(\xi)\} \) or the dynamic stiffness matrix \([S(\omega)]\), once the nodal degrees of freedom \( \{u\} \) is obtained, the solution of the entire domain can be calculated by specifying the scaled boundary coordinates \( \xi, \eta \) and \( \zeta \). The solution is exact in the radial direction and converges in the finite element sense in circumferential directions. The solution procedures described above can be illustrated by the flow chart shown in Figure 3-2.

One significant concern when assessing SBFEM’s practical applicability, which is the same as other numerical methods, lies in the performance of its solution, more specifically, the numerical stability and accuracy associated with SBFEM calculation. Most published literature so far has concentrated on the derivation of its conceptual framework (Wolf and Song, 2001; Wolf and Song, 2000; Song and Wolf, 2000; Song and Wolf, 1998; Song and Wolf, 1997), and the solution algorithms of the scaled boundary finite element equation (Birk et al., 2012; Song, 2009; Prempramote et al., 2009; Bazyar and Song, 2008; Song and Bazyar, 2007; Song, 2006; Song, 2004). Possessing the attributes of a numerical methodology, the credibility and capacity of SBFEM pose necessary discussions. The following two sections will systematically
investigate the numerical quality and property of SBFEM, identify possible technical difficulties, and propose corresponding solutions to overcome the outlined problems.

Figure 3-2. Solution procedure of SBFEM

3.2 Matrix decomposition technique*

The main techniques in solving the matrix-form scaled boundary finite element equation for both elasto-static and elasto-dynamic problems have been summarised in Section 3.1.3. A Hamiltonian matrix $[Z]$ is formulated using the coefficient matrices $[E^0]$, $[E^1]$ and $[E^2]$ of the scaled boundary finite element equation (3.14), whereby the nodal displacement function $\{u(\xi)\}$ is the basic unknown function. A new

intermediate function \( \{X(\xi)\} \) is introduced in relation to the nodal displacement function \( \{u(\xi)\} \) and the internal nodal force function \( \{q(\xi)\} \). This leads to a reduction from a second-order ordinary differential equation (3.14) to a first-order differential equation (3.19) at the expense of doubling the number of unknowns.

### 3.2.1 Eigenvalue decomposition and its inherent numerical problems

In the existing solution procedure for elasto-static problems, the displacement field is hypothesised in the form of the power series of the radial coordinate \( \xi \). Therefore, the solution of Equation (3.19) can be formulated using a finite power series as:

\[
\{X(\xi)\} = c_1\xi^{-\lambda_1}\{\phi_1\} + c_2\xi^{-\lambda_2}\{\phi_2\} + \cdots + c_n\xi^{-\lambda_n}\{\phi_n\} \tag{3.23}
\]

with \( n \) denoting the dimension of the Hamiltonian matrix \([Z]\). Substituting Equation (3.23) into Equation (3.19) leads to the eigenproblem:

\[
[Z]\{\phi\} = \lambda_i\{\phi\} \tag{3.24}
\]

where \( \lambda_i \) is the eigenvalue of \([Z]\) and \( \{\phi\} \) is the corresponding eigenvector. Equation (3.23) can be reformulated in a matrix form as:

\[
\{X(\xi)\} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} \xi^{\lambda_1} \\ \xi^{\lambda_2} \end{bmatrix} \begin{bmatrix} \{C_1\} \\ \{C_2\} \end{bmatrix} \tag{3.25}
\]

Note, if \( \lambda \) is the eigenvalue of \([Z]\), then \(-\lambda\), \(\bar{\lambda}\) (conjugate complex number) and \(-\bar{\lambda}\) are eigenvalues of \([Z]\). The eigenvalues \( \lambda_i \) of matrix \([Z]\) can be arranged in such a way that all the eigenvalues in \( \Lambda_1 \) have positive real parts, and all the eigenvalues in \( \Lambda_2 \) have negative real parts. According to Equation (3.25) and Equation (3.18), \( \{u(\xi)\} \) and \( \{q(\xi)\} \) can be expressed as:

\[
\{u(\xi)\} = \xi^{-0.5}\left(\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \xi^{\lambda_1}\{C_1\} + \Phi_{12} \xi^{\lambda_2}\{C_2\}\right) \tag{3.26}
\]

\[
\{q(\xi)\} = \xi^{-0.5}\left(\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \xi^{\lambda_1}\{C_1\} + \Phi_{22} \xi^{\lambda_2}\{C_2\}\right) \tag{3.27}
\]

where the integral constants \( \{C_1\} \) and \( \{C_2\} \) are to be determined according to the prescribed boundary conditions.
The displacement amplitude at the scaling centre where $\xi = 0$ for a bounded domain should be finite. Since the real parts of $\lambda_i$ in $\Lambda_2$ are negative, Equation (3.26) and Equation (3.27) are reduced to

$$\{ u(\xi) \} = \xi^{-0.5} [\Phi_{11}] [\xi^{H_i}] \{ C_1 \}$$

(3.28)

and

$$\{ q(\xi) \} = \xi^{-0.5} [\Phi_{21}] [\xi^{H_i}] \{ C_1 \}$$

(3.29)

Eliminating the constant vector $\{ C_1 \}$ from Equation (3.28) and Equation (3.29), and noticing that $\{ R \} = \{ q(\xi = 1) \}$ on the boundary, the following expression yields by a comparison with Equation (3.17):

$$[K] = [\Phi_{21}] [\Phi_{11}]^{-1}$$

(3.30)

Consequently, the nodal displacement vector $\{ u \}$ and the constant vector $\{ C_1 \}$ can be calculated from Equation (3.17) and Equation (3.28), successively.

After $\{ C_1 \}$ is determined, the nodal displacement function $\{ u(\xi) \}$ along the line defined by connecting the scaling centre and the corresponding node on the boundary is analytically obtained from Equation (3.28). Accordingly, the displacement amplitude at any point can be determined by interpolation using Equation (3.11). For unbounded domains, the displacement amplitude at $\xi = \infty$ must remain finite and $\{ R \} = -\{ q(\xi = 1) \}$ applies.

In real cases, however, the power series formulation may not provide a complete general solution, as logarithmic terms exist in problems involving particular geometric configurations, material composition and boundary conditions (Sinclair, 2000; Sinclair, 1999; Gadi et al., 2000; Chen, 1996). In this case, multiple eigenvalues or near-multiple eigenvalues of the Hamiltonian matrix $[Z]$ might be present, corresponding to parallel eigenvectors and indicating the existence of logarithmic terms in the solution. Consequently, matrices $[\Phi_{11}]$ and $[\Phi_{21}]$ in Equation (3.30) (or $[\Phi_{12}]$ and $[\Phi_{22}]$ for the case of an unbounded domain) formulated by parallel eigenvectors are rank-deficient and irreversible, which results in inaccurate solutions or even the failure of the eigenvalue decomposition when solving the scaled boundary finite element equation.
3.2.2 Real Schur decomposition

Deeks and Wolf (2002a; 2003) investigated a two-dimensional unbounded domain problem governed by the Laplace equation using SBFEM, in which case, the displacement amplitude is infinite in the near field. This infinite term is represented by an additional logarithmic mode, associated with the rigid body translation, to the power series formulation of the solution. Song (2004) proposed a matrix-function solution in combination with the real Schur decomposition to address this multiple-eigenvalue issue, in which terms in the series solution are not restricted to power function form. Unlike the work presented in Deeks and Wolf (2002a; 2003), Song’s (2004) matrix function method does not require prior knowledge of the presence of logarithmic terms, and copes well with the power functions, logarithmic functions and their transitions in the solution. Li et al. (2010a) further discussed the outperformance of the real Schur decomposition over the conventional eigenvalue decomposition technique.

The real Schur decomposition of the Hamiltonian matrix $[Z]$ can be expressed as:

$$[Z] = [V][S][V]^T$$  \hspace{1cm} (3.31)

where $[V]$ is an orthogonal matrix and $[S]$ is a block upper triangular matrix with 1-by-1 and 2-by-2 blocks on the diagonal. The eigenvalues are revealed by the diagonal elements and blocks of $[S]$, and the columns of $[V]$ constitute a basis offering superior numerical properties to a set of eigenvectors $\{\phi\}$ in Equation (3.24) (Paige and Vanlovan, 1981). Partitioning $[S]$ and $[V]$ into submatrices of equal size as:

$$[S] = \begin{bmatrix} S_n & \ast \\ 0 & S_p \end{bmatrix}, \quad \text{and} \quad [V] = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix},$$

where $\ast$ stands for a real matrix. The diagonal elements of matrix $[S_n]$ are negative and those of matrix $[S_p]$ are positive. Block-diagonalising $[S]$ using an upper-triangular matrix and using Equation (3.31) leads to:

$$[\Psi]^{-1}[Z][\Psi] = \begin{bmatrix} S_n & 0 \\ 0 & S_p \end{bmatrix}$$

Similar to the formulation in Equation (3.25), the general solution of Equation (3.19) using the real Schur decomposition is expressed as:
\( \{ X(\xi) \} = \begin{bmatrix} [\Psi_{u1}] & [\Psi_{u2}] \\ [\Psi_{q1}] & [\Psi_{q2}] \end{bmatrix} \begin{bmatrix} \xi^{-[s_1]} \\ \xi^{-[s_2]} \end{bmatrix} \{ \{ C_1 \} \} \)

Accordingly, \( \{ u(\xi) \} \) and \( \{ q(\xi) \} \) can be expressed as:

\[
\{ u(\xi) \} = \xi^{-0.5} \left( [\Psi_{u1}] \xi^{-[s_1]} \{ C_1 \} + [\Psi_{u2}] \xi^{-[s_2]} \{ C_2 \} \right)
\]

(3.32)

\[
\{ q(\xi) \} = \xi^{-0.5} \left( [\Psi_{q1}] \xi^{-[s_1]} \{ C_1 \} + [\Psi_{q2}] \xi^{-[s_2]} \{ C_2 \} \right)
\]

(3.33)

The following solution procedure is the same as described for the eigenvalue decomposition in Section 3.2.1. By performing the real Schur decomposition, the inverse of a possibly close-to-singular matrix \([ \Phi_{11} ]\) (or \([ \Phi_{12} ]\)) can be avoided by inverting only an upper triangular matrix \([\Psi_{u1} ]\) (or \([\Psi_{u2} ]\)). In addition, real Schur decomposition is more stable and suffers less from numerical difficulties than the eigenvalue decomposition. A three-dimensional cylindrical pile subjected to a uniformly distributed load is provided in the next subsection as a case study to demonstrate its efficiency.

### 3.2.3 Case study

The deformation of a cylindrical pile subjected to a uniformly distributed load is examined herein. The cylindrical pile with a radius of \( a \) and a height of \( h \) subject to uniformly distributed pressure \( p \) is shown in Figure 3-3. The bottom of the cylinder pile is fixed. The pile is assumed to be a plain concrete pile with Young’s modulus \( E \) and Poisson’s ratio \( \nu \), and exhibit pure elastic property. The geometric dimension and the material properties of the pile are listed in Table 3-1.

Table 3-1. Parameters of the cylindrical pile

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( h ) (m)</th>
<th>( a ) (m)</th>
<th>( E ) (Pa)</th>
<th>( \nu )</th>
<th>( p ) (Pa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnitudes</td>
<td>10</td>
<td>1</td>
<td>( 2.8 \times 10^{10} )</td>
<td>0.25</td>
<td>( 3 \times 10^{8} )</td>
</tr>
</tbody>
</table>

The scaling centre is chosen at the bottom centre of the pile. The circumferential boundary, as well as the top surface of the cylinder is discretised with quadratic eight-node quadrilateral isoparametric elements. A representative scaled boundary element
is shown in Figure 3-4, accompanied by corresponding shape function expressions. An example of the discretisation scheme is illustrated in Figure 3-5 (a).

$$N_1(\eta, \zeta) = \frac{-1}{4}(\eta - 1)(\zeta - 1)(\eta + \zeta + 1) \quad N_2(\eta, \zeta) = \frac{1}{2}(\eta^2 - 1)(\zeta - 1)$$

$$N_3(\eta, \zeta) = \frac{1}{4}(\zeta - 1)(-\eta^2 + \zeta\eta + \zeta + 1) \quad N_4(\eta, \zeta) = \frac{-1}{2}(\eta + 1)(\zeta^2 - 1)$$

$$N_5(\eta, \zeta) = \frac{1}{4}(\eta + 1)(\zeta + 1)(\eta + \zeta - 1) \quad N_6(\eta, \zeta) = \frac{-1}{2}(\eta^2 - 1)(\zeta + 1)$$

$$N_7(\eta, \zeta) = \frac{1}{4}(\eta - 1)(\zeta + 1)(\eta - \zeta + 1) \quad N_8(\eta, \zeta) = \frac{1}{2}(\eta - 1)(\zeta^2 - 1)$$

Figure 3-4. A typical scaled boundary element and the shape functions

The real Schur decomposition is employed in the calculation. The convergence test shows that 8 elements are needed around the pile circumference, 1 element along the radius and 16 elements along the height of the pile. The displacement of point A (see Figure 3-3) on the edge of the pile head converges to 8.0357 mm in the $x$ direction, and that in the $z$ direction is 52.344 mm.

An equivalent FEM analysis is carried out for comparison purposes. Three dimensional 20-node hexahedral solid elements are used in the FEM model, as shown in Figure 3-5 (b). A convergence test shows that 28 elements are required around the circumference, 5 elements along the radius and 50 elements for the height. The
displacement of point A in the x direction converges to 8.0357 mm, and that in the z direction reaches 52.345 mm.

Figure 3-5. Discretisation illustration of the pile foundation for (a) SBFEM model and (b) FEM model

Figure 3-6. Displacement comparison between SBFEM and FEM models
The displacement distributions of line $AB$ (see Figure 3-3) from both SBFEM and FEM models are plotted in Figure 3-6. The solid line, the dashed line and the broken line represent the FEM results. The dots, pentagons and crosses denote the SBFEM results. The comparison shows excellent performance of the real Schur decomposition in the SBFEM solution process, which ensures the credibility of the model for future analysis.

In order to demonstrate the superiority of the real Schur decomposition over the eigenvalue decomposition, the radial and vertical displacements of point $A$, calculated

\[
\begin{array}{c|c|c}
\text{Discretisation scheme} & \text{Eigenvalue} & \text{Real Schur} \\
9 \times 8 & 8.020 & 8.024 \\
9 \times 10 & 8.028 & 8.032 \\
9 \times 12 & 8.036 & 8.040 \\
9 \times 14 & 51.70 & 51.85 \\
9 \times 16 & 52.00 & 52.15 \\
\end{array}
\]

Figure 3-7. Comparison between the eigenvalue decomposition and the real Schur decomposition methods for: (a) radial displacement and (b) vertical displacement.
using the two matrix decomposition algorithms, are compared in Figure 3-7. The tick labels on the horizontal axis represent different discretisation schemes. For example, $6 \times 10 \times 1$ signifies the numbers of elements in the circumferential, vertical and radial directions are 6, 10 and 1, respectively. It is found that by using the real Schur decomposition, no prior knowledge of the potential multiple eigenvalues is required and no complex number operation is performed, as is necessary in the case of the eigenvalue decomposition. The inversion of rank-deficient matrices can be efficiently avoided. The real Schur decomposition tends to give more stable and reliable results compared to the eigenvalue decomposition, as shown in Figure 3-7.

In brief conclusion, the power series formulation of the solution, introducing the eigenvalue decomposition of the Hamiltonian matrix, leads to underlying multiple eigenvalues associated with possible logarithmic terms in the solution. The real Schur decomposition can be adopted as an alternative since it circumvents this problem and provides more stable and accurate solutions. Furthermore, no manipulations of complex numbers are required as in the eigenvalue decomposition. The analysis of a cylindrical pile subject to a uniformly distributed pressure serves as a benchmark example. It shows reassuring and encouraging behaviour of the results when compared to an equivalent FEM analysis. In addition, the outperformance of the real Schur decomposition over the eigenvalue decomposition is fully demonstrated.

### 3.3 SBFEM non-dimensionalisation

The original partial differential equation governing the physical problem, by conducting the scaled boundary coordinate transformation and applying the weighted residual technique, is rewritten as a matrix-form ordinary differential equation, i.e. the scaled boundary finite element equation. The term ‘matrix-form’ refers to the coefficients of the equation, which are in the form of matrices, and are calculated using the discretisation information on the domain boundary. They are used to formulate a Hamiltonian matrix \([Z]\), of which a matrix decomposition is performed for subsequent calculations. To ensure the validity of the solution, high accuracy of the decomposition should be guaranteed, which closely depends on the property of the Hamiltonian matrix. This section further investigates numerical issues in the SBFEM calculation. An illustrative example is used to identify reasons for the numerical difficulty. Subsequently, a solution scheme is suggested and integrated in SBFEM to
overcome the problem. Three benchmark examples, in terms of static analysis, modal analysis and transient analysis are provided to demonstrate the satisfactory performance of the proposed scheme.

3.3.1 Numerical difficulties associated with matrix properties

The case of a cylindrical pile subject to a uniformly distributed pressure, as illustrated in Figure 3-3, is also utilised in this section for a systematic investigation of the numerical credibility of the SBFEM calculation. The basic information is the same as that shown in Table 3-1. The displacement components of point A in the x, y and z directions are to be examined.

In the SBFEM model, the scaling centre O is chosen at the geometric centre of the pile at (0, 0, 5), and the entire surface is discretised into 184 eight-node quadratic quadrilateral elements. The solution procedure presented in Section 3.2.2 for elasto-statics is followed. The displacement components in the x, y and z directions of point A are calculated as 4.431×10³ mm, 1.421×10⁴ mm and 7.178×10⁴ mm, respectively. Apparently, the results differ considerably from the exact solution, which should hold amplitudes of 8.036 mm, 0 and 52.382 mm, as depicted in Figure 3-7.

A close examination of the Hamiltonian matrix [Z] reveals that its condition number \( \kappa = 2 \times 10^{24} \). This implies that the Hamiltonian matrix is ill-conditioned, and any succeeding manipulations either directly or indirectly related to this matrix may fail upon any rounding error fluctuation. The exactness of Equation (3.31) is checked by examining the norm of a residual matrix \([Res_1]\):

\[
[Res_1] = [Z] - [V] [S][V]^T
\] (3.34)

A zero norm of \([Res_1]\) from Equation (3.34) is theoretically expected. However, a norm of 0.1345 is observed, which is far beyond the acceptable accuracy tolerance (Goldberg, 1991).

Another touchstone can be associated with the static stiffness matrix \([K]\). It is understood that the static stiffness matrix \([K]\) obtained from Equation (3.30) should satisfy Equation (3.22), replacing \([S(\omega)]\) with \([K]\) and \(\omega\) with 0. Therefore, another residual matrix \([Res_2]\) is defined in Equation (3.35), the norm of which is checked as 9 \( \times 10^{21} \).
By examining the Hamiltonian matrix, it is found that the maximum magnitude of its entries is of $10^{10}$, resulting from the input parameter, the Young’s modulus which holds a magnitude of $10^8$ in the present case. The minimum magnitude, however, is 0. This significant magnitude difference among matrix entries leads to the ill-condition of the matrix. Therefore, it is necessary that input parameters are processed prior to calculations to improve the quality of matrices, thereby benefiting subsequent computations. It is motivated to introduce the non-dimensionalisation scheme into the SBFEM calculation, as it allows all quantities to have relatively similar order of magnitudes. The detailed procedure of non-dimensionalisation and its incorporation into the SBFEM formulation are presented in the next section.

### 3.3.2 Non-dimensionalisation implementation

Wolf and Song (1996) presented a dimensional analysis identifying independent variables to which the dynamic stiffness matrix is related. The non-dimensionalisation scheme proposed in this study follows their idea. Denoting the dimensionless length $r$, Young’s modulus $E$ and the mass density $\rho$ as $r^*$, $E^*$ and $\rho^*$, respectively (with superscript * representing dimensionless quantities), the reference variables, with respect to which all variables and matrices are non-dimensionalised, are defined in Table 3-2.

<table>
<thead>
<tr>
<th>Reference variables</th>
<th>Magnitudes</th>
<th>Dimensions</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_t$</td>
<td>$r/r^*$</td>
<td>$L$</td>
<td>m</td>
</tr>
<tr>
<td>$E_t$</td>
<td>$E/E^*$</td>
<td>$ML^{-1}T^2$</td>
<td>N/m²</td>
</tr>
<tr>
<td>$\rho_t$</td>
<td>$\rho/\rho^*$</td>
<td>$ML^3$</td>
<td>kg/m³</td>
</tr>
</tbody>
</table>

The dimensions of the dynamic stiffness matrix $[S(\omega)]$ and the independent variable frequency $\omega$ are expressed as ($'[]'$ is used to denote the dimension of a quantity):

$$[[S]] = L^{3}MT^{-2}$$  \hspace{1cm} (3.36)
\[
[\omega] = T^{-1}
\]  
(3.37)

with \( s \) representing the spatial dimension of the study domain (\( s = 2 \) for two-dimensional problems and \( s = 3 \) for three-dimensional problems). The following equality is introduced, with exponents \( n_i \) (\( i = 1, 2, 3, 4 \) and \( 5 \)), to formulate the dimensionless \([S^*(\omega)]\) and \(\omega^*\):

\[
\left[ [S]^{n_1} [r_r]^{n_2} [E_r]^{n_3} [\rho_r]^{n_4} [\omega]^{n_5} \right]^{\omega^*} = L^{(s-3)n_1+n_2-n_3-3n_4} M^{n_1+n_2+n_4} T^{-2n_1-2n_3-3n_4}
\]  
(3.38)

It is worth mentioning that Equation (3.38) is formulated using the reference variables, rather than the corresponding material parameters, as is the case in Wolf and Song (1996). This allows more flexibility in the non-dimensionalisation process.

By examining Equation (3.38), the following equations yield:

\[
(s-3)n_1 + n_2 - n_3 - 3n_4 = 0
\]
\[
n_1 + n_3 + n_4 = 0
\]
\[
-2n_1 - 2n_3 - n_5 = 0
\]

and are used to determine the five parameters \( n_i \) (\( i = 1, 2, 3, 4 \) and \( 5 \)). It is noticed that two of them are arbitrarily chosen. Given \( n_1 = 1 \) and \( n_5 = 0 \) yields the dimensionless dynamic stiffness matrix \([S^*(\omega)]\):

\[
[S^*(\omega)] = r_r^{2-s} E_r^{-1} [S(\omega)]
\]  
(3.40)

Or, if \( n_1 = 0 \) and \( n_5 = 1 \):

\[
\omega^* = \frac{\omega r_r}{\sqrt{E_r/\rho_r}}
\]  
(3.41)

Analogically, the following expressions hold for the static stiffness matrix \([K]\), the mass matrix \([M]\), and the damping matrix \([C]\):

\[
[K^*] = E_r^{-1} r_r^{-1} [K] \quad [M^*] = \rho_r^{-1} r_r^{-3} [M] \quad [C^*] = \frac{\sqrt{E_r/\rho_r}}{E_r r_r^2} [C]
\]  
(3.42)

The coefficient matrices \([E^0]\), \([E^1]\), \([E^2]\) and \([M^0]\) are normalised accordingly as:

\[
[E^0*] = E_r^{-1} r_r^{-2s} [E^0] \quad [E^1*] = E_r^{-1} r_r^{-s} [E^1] \quad [E^2*] = E_r^{-1} r_r^{-2s} [E^2] \quad [M^0*] = \rho_r^{-1} r_r^{-3} [M^0]
\]  
(3.43)
Another independent variable, the time $t$, needs to be reformulated in the time-domain analysis as:

$$ t^* = \frac{E_t/\rho_c}{r_c} t $$  \hspace{1cm} (3.44)

With all the above expressions, Equation (3.14) retains exactly its original form. For ease of presentation, all asterisks are removed from the mathematical expressions thereafter unless specified otherwise.

In order for the proposed parametric non-dimensionalisation scheme to be incorporated into the SBFEM calculation, a group of reference variables need to be specified beforehand so that, upon non-dimensionalisation, all relevant quantities engaged in the calculation are of similar magnitude. Precision has to be considered at various intermediate stages, such as the real Schur decomposition of the Hamiltonian matrix $[Z]$, the general eigenvalue decomposition of $[E^0]$ and $[M^0]$ and the static stiffness matrix $[K]$, the mass matrix $[M]$, and the damping matrix $[C]$ satisfying their corresponding algebraic equations.

It should be mentioned that Equations (3.40), (3.42) and (3.43) suggest how the matrices are non-dimensionalised with respect to the reference variables. They are not explicitly formulated in the solution procedure. Calculated results are dimensionless and require subsequent interpretation in order to be applicable to engineering practice. For example, a variable with dimension $L$ should be multiplied by the reference length $r_c$ to obtain the corresponding dimensional value. The following examples will clarify these procedures.

### 3.3.3 Numerical experiments

#### 3.3.3.1 Static analysis

The displacement of the cylindrical pile is reconsidered after the introduction of the non-dimensionalisation scheme. Four sets of reference variables are arranged, as shown in Table 3-3, to illustrate SBFEM’s performance under different situations. Considering the magnitude of the input parameters, four values of $E_t$ are selected as 1 Pa, $1 \times 10^3$ Pa, $1 \times 10^7$ Pa and $2.8 \times 10^{10}$ Pa to non-dimensionalise the Young’s modulus $E$ and the external pressure $p$, both of which have dimensions $ML^{-1}T^{-2}$. The reference length $r_c$ equals 1 m, as same SBFEM model is used for the four cases and the
dimension of the model is identical to the physical prototype of the pile. The mass density is irrelevant to this example and therefore is not discussed. For each of the four cases, the maximum difference $\Delta M_{\text{max}}$ in the magnitude of the entries of the Hamiltonian matrix and the condition number $\kappa$ of the Hamiltonian matrix, as well as the norms of two residual matrices $Res_1$ and $Res_2$ are examined. The displacement components $u_x$, $u_y$ and $u_z$ of point A are restored into the corresponding dimensional values and are listed in Table 3-3.

Table 3-3. Numerical performance illustration of SBFEM using a static analysis

<table>
<thead>
<tr>
<th>Parameters</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_r$ (Pa)</td>
<td>1</td>
<td>$1\times10^3$</td>
<td>$1\times10^7$</td>
<td>$2.8\times10^{10}$</td>
</tr>
<tr>
<td>$r_1$ (m)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\Delta M_{\text{max}}$</td>
<td>$4\times10^{10}$</td>
<td>$4\times10^7$</td>
<td>$4\times10^3$</td>
<td>138.74</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>$2\times10^{24}$</td>
<td>$8\times10^{16}$</td>
<td>$8\times10^8$</td>
<td>$2\times10^5$</td>
</tr>
<tr>
<td>$Res_1$</td>
<td>0.1345</td>
<td>$1\times10^{-4}$</td>
<td>$3\times10^{-8}$</td>
<td>$7\times10^{-11}$</td>
</tr>
<tr>
<td>$Res_2$</td>
<td>$9\times10^{21}$</td>
<td>$1\times10^9$</td>
<td>0.0011</td>
<td>$9\times10^{-12}$</td>
</tr>
<tr>
<td>$u_x$ (mm)</td>
<td>$4.431\times10^3$</td>
<td>11.507</td>
<td>8.0357</td>
<td>8.0357</td>
</tr>
<tr>
<td>$u_y$ (mm)</td>
<td>$1.4205\times10^4$</td>
<td>5.744</td>
<td>$4.625\times10^{-8}$</td>
<td>$1.394\times10^{-8}$</td>
</tr>
<tr>
<td>$u_z$ (mm)</td>
<td>$7.178\times10^4$</td>
<td>51.534</td>
<td>52.358</td>
<td>52.358</td>
</tr>
</tbody>
</table>

It is found that, by checking the four indices, $\Delta M_{\text{max}}$, $\kappa$, $Res_1$ and $Res_2$, as the reference parameter $E_r$ gradually increases, the numerical performance of the SBFEM calculation improves accordingly. The maximum magnitude difference among the element entries of matrix $[Z]$ decreases from a magnitude of $10^{10}$ to $10^2$. The condition number of $[Z]$ thus is calculated to decrease from $10^{24}$ to $10^5$. Consequently, the norms of the two residual matrices are found to converge to zero when $E_r$ reaches $2.8\times10^{10}$ Pa. The readings of the displacement components also indicate a trustworthy calculation when appropriate reference variables are employed. It is suggested that the reference parameters are defined in such way that all variables involved in the matrix calculation hold similar magnitude regardless of their dimensions. For the present
case, a combination of $E_r = 2.8 \times 10^{10}$ Pa and $r = 1$ m generates a magnitude of 1 for both the length dimension $L$ and the pressure dimension $ML^{-1}T^{-2}$.

### 3.3.3.2 Modal analysis

The previous case examines how the accuracy of SBFEM results is affected by the magnitude of input parameters in static analysis, thus highlighting the necessity of parametric non-dimensionalisation in the SBFEM calculation. As SBFEM is also planned to be employed in time-dependent analyses, it is necessary to perform the non-dimensionalisation procedure when solving elasto-dynamic problems. The modal analysis presented in this section and the transient analysis in the next section will examine the performance of the dimensionless SBFEM calculation in elasto-dynamics.

To enhance the structure and the content of the thesis, formulations of the time-dependent SBFEM model will be discussed in more detail in Section 6.2. The $L$-shaped panel presented in Song (2009) is re-examined herein with the same geometric configuration, however, the Young’s modulus $E$ and the mass density $\rho$ are assigned as $2.8 \times 10^{10}$ Pa and 2400 kg/m$^3$, respectively (the Poisson’s ratio remains as 1/3).

A sketch of the $L$-shaped panel is reproduced in Figure 3-8 (a), illustrating the geometric configuration ($b = 1$ m) and the boundary conditions: Line $EF$ is fully constrained in both $x$ and $y$ directions; $AB$ is fixed only in the $x$ direction. In the SBFEM model shown in Figure 3-8 (b), the $L$-shaped panel is divided into three

![Figure 3-8](image-url)
subdomains with the scaling centres located at the geometric centre of each subdomain. Therefore, all the boundaries as well as the two interfaces between the subdomains are discretised. This analysis can also be carried out by treating the L-shaped panel as a single domain and locating the scaling centre at point O, thus only those lines apart from OA and OF need to be discretised. Three-node quadratic elements are used for the boundary discretisation, which results in 196 DOFs for the problem. The continued fraction technique is employed and an order of 6 is used when formulating the global stiffness and mass matrices. The reference parameters are chosen as \( r_t = 1 \) m, \( E_r = 2.8 \times 10^{10} \) Pa and \( \rho_r = 2400 \) kg/m\(^3\).

The first 110 order dimensionless natural frequencies calculated from SBFEM are plotted in Figure 3-9 (a). The dimensional natural frequencies are thus obtained by reformulating Equation (3.41) as:

\[
\omega = \sqrt{\frac{E_r}{r_t}} \omega^* 
\]  

(3.45)

The dimensional result is compared with that of an equivalent FEM modal analysis in Figure 3-9 (b). The two curves agree extremely well. The same analysis using dimensional parameters in the SBFEM model is also attempted but fails due to the error accumulation from the real Schur decomposition onwards, which renders the subsequent calculation meaningless.

![Figure 3-9. Natural frequency of the L-shaped panel: (a) dimensionless natural frequency from SBFEM model and (b) comparison between FEM and SBFEM results](image)
3.3.3.3 Transient analysis

The L-shaped panel is also utilised in this section to illustrate the incorporation of the parametric non-dimensionalisation into the transient analysis. The prescribed force condition, similar to that described in Song (2009), is specified as: uniformly distributed along line BC (see Figure 3-8 (a)) with the magnitude varying in the time domain as depicted in Figure 3-10 (a). The values of $p_{\text{peak}}$, $t_{\text{total}}$, $t_{\text{peak}}$, $t_{\text{zero}}$ can be referred to in Table 3-4, in which both the dimensional and the dimensionless parameters employed in this analysis are listed.

Table 3-4. Parameters of the transient analysis of an L-shaped panel

<table>
<thead>
<tr>
<th>Material parameters</th>
<th>Dimensional</th>
<th>Dimensionless</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$ (Pa)</td>
<td>2.8×10^{10}</td>
<td>$E^*$</td>
</tr>
<tr>
<td>$\rho$ (kg/m$^3$)</td>
<td>2400</td>
<td>$\rho^*$</td>
</tr>
</tbody>
</table>

| Temporal variables | | |
|-------------------|-------------------|
| $t_{\text{total}}$ (s) | 0.022 | $t^*_{\text{total}}$ | 75 |
| $t_{\text{peak}}$ (s) | 1.4639×10^{-4} | $t^*_{\text{peak}}$ | 0.5 |
| $t_{\text{zero}}$ (s) | 2.9277×10^{-4} | $t^*_{\text{zero}}$ | 1 |
| $\Delta t$ (s) | 7.3193×10^{-6} | $\Delta t^*$ | 0.025 |

| Natural circular frequencies | | |
|-----------------------------|-------------------|
| $\omega_1$ (rad/s) | 1371.682 | $\omega_1^*$ | 0.4032 |
| $\omega_2$ (rad/s) | 2819.265 | $\omega_2^*$ | 0.8259 |

| External pressure | | |
|-------------------|-------------------|
| $p_{\text{peak}}$ (Pa) | 2.8×10^{7} | $p^*_{\text{peak}}$ | 1×10^{-3} |

The SBFEM analysis adopts the same discretisation model as that shown in Figure 3-8 (b), and the reference material properties, $E_r$ and $\rho_r$, are selected as 2.8×10^{10} Pa and 2400 kg/m$^3$, respectively. In the calculation, all the temporal variables associated with the time integration, in which the Newmark’s integral technique with $\alpha = 0.25$ and $\delta = 0.5$ (Clough and Penzien, 1975) being employed, are non-dimensionalised according to Equation (3.44). The Rayleigh material damping effect is taken into consideration with a material damping ratio of 0.05 (assuming the L-shaped panel is made of concrete). From the modal analysis in Section 3.3.3.2, $\omega_1 = 1371.682$ rad/s and $\omega_2 = 2819.265$ rad/s corresponding to two orthogonal modal shapes are selected.
However, it is their dimensionless counterparts are used for the formulation of the damping matrix. The magnitude of the external pressure at any time step is normalised in accordance with the Young’s modulus.

![Graph](image)

(a) (b)

Figure 3-10. SBFEM transient analysis of an L-shaped panel: (a) pressure variation with respect to time and (b) dimensionless displacement history in the y direction of point A

The dimensionless displacement history of point A (refer to Figure 3-8 (a)) in the y direction is shown in Figure 3-10 (b). With reference length $r_i = 1$ m, the dimensional displacement should hold the same amplitude as $u_y^*$, whereas the time variable should be calculated by reformulating Equation (3.44) in terms of $t$ to obtain its dimensional counterpart.

An equivalent FEM analysis is also carried out for comparison purposes. Excellent agreement is observed from Figure 3-11, which compares the displacement histories in the y direction of points A, B, C and D (refer to Figure 3-8 (a)) from both FEM and SBFEM calculations. An attempt of using original parameters in the SBFEM analysis does not produce any reasonable results.

To summarise, as SBFEM relies on intensive matrix computations, the property of all relevant matrices is of significant importance to the stability and accuracy of the result. A numerical problem has been identified to be associated with the magnitude of matrix entries calculated from input parameters. Therefore, it is proposed that a group of reference variables be pre-defined to non-dimensionalise input parameters, such as the geometric dimension, material properties and temporal variables. All
Figure 3-11. Comparison of vertical displacement history of the $L$-shaped panel between SBFEM and FEM models at: (a) Point $A$; (b) Point $B$; (c) Point $C$ and (d) Point $D$. 
relevant matrices thus present favourable properties to ensure the correctness of the calculation. The dimensionless output needs reinterpretation to be applicable in engineering practice. Numerical examples in terms of elasto-static, modal and transient analyses are shown to produce satisfactory comparisons with the corresponding FEM analyses, thus demonstrating the credibility of the proposed non-dimensionalisation scheme.

3.4 Summary

In this chapter, first, the main technical aspects are outlined in terms of the scaled boundary coordinate transformation, derivation of the scaled boundary finite element equation and the corresponding solutions. Subsequently, emphasis is redirected to the discussion of the numerical performance of SBFEM, which has not been systematically addressed in the literature. The discussion is carried out in two aspects, namely the matrix manipulation technique and the matrix property. For each aspect, the reasons for the potential problem are fully investigated. Corresponding solution schemes are proposed and verified using benchmark examples. This chapter allows for accurate investigation of the wave-structure interaction problem using SBFEM in the subsequent chapters.
Chapter 4  Wave Interaction with Monopile Foundation

From this chapter, SBFEM starts to explore its application in ocean engineering by addressing wave-structure interaction problems, with a view to providing valuable information for ocean structure design. As an introductory investigation, this chapter discusses a monopile foundation backed by one of the most commonly used structural concepts in offshore renewable energy. Generally embedded in shallow waters, offshore monopiles are characterised by relatively large geometric dimensions, compared with other pile foundations, to support the massive wind tower and transfer all the loads to the seabed. They are exposed to a harsh ocean environment and are subjected to detrimental wave forces throughout their entire lifespan. Therefore, the wave-induced structural stability and behaviour analysis of the monopile foundation are of the utmost significance to offshore wind farm projects.

The wave field behaviour in the presence of a single-standing pile with cylindrical configuration has been intensely investigated. Thus, the wave-induced pressure acting upon a cylindrical structure is analytically attainable. This chapter employs SBFEM and the wave pressure formulation presented in Zhu (1993) to study the structural behaviour of a monopile foundation. Parametric analysis in terms of the wave number, wave amplitude and water depth are concurrently carried out to gain a further insight into the monopile’s response to varying wave conditions. The work presented in this chapter is seen as an introductory study of advanced explorations of the wave-structure interaction problems using SBFEM.

---


4.1 Problem formulation

A monopile-supported wind turbine is associated with three physical aspects according to its surrounding media, i.e. aerodynamically with winds, hydrodynamically with sea water, and geotechnically with the seabed. The wind exerts aerodynamic forces on the turbine rotor during the wind turbine operation. There are also static axial loads transferred from the turbine tower and act on the monopile. These aspects, not being the main focus of this study, are not addressed in the current discussion. In addition, the monopile is assumed to be fixed at the seabed level, and the relative motion between the monopile and the seabed is neglected. These assumptions are also enforced for the problem formulations in Chapters 5 and 6. Therefore, a free-standing monopile foundation engaged in the ocean environment is concisely illustrated by its \( \textit{xz} \) plane view shown in Figure 4-1. The origin \( O \) of the Cartesian coordinate system is specified at the seabed level, and the \( z \) axis is pointing positively upwards. The physical designation of the parameters and the information about the monopile foundation and the wave condition are listed in Table 4-1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Notations</th>
<th>Magnitudes</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pile parameters</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pile radius</td>
<td>( a )</td>
<td>2.5</td>
<td>m</td>
</tr>
<tr>
<td>Pile height</td>
<td>( h )</td>
<td>40</td>
<td>m</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>( E )</td>
<td>( 2.8 \times 10^{10} )</td>
<td>Pa</td>
</tr>
<tr>
<td>Possion’s ratio</td>
<td>( \nu )</td>
<td>0.25</td>
<td>-</td>
</tr>
<tr>
<td>Wave parameters</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Water depth</td>
<td>( d )</td>
<td>30</td>
<td>m</td>
</tr>
<tr>
<td>Incident wave angle</td>
<td>( \alpha )</td>
<td>0</td>
<td>rad</td>
</tr>
<tr>
<td>Wave number</td>
<td>( k )</td>
<td>0.10</td>
<td>m(^{-1})</td>
</tr>
<tr>
<td>Water density</td>
<td>( \rho_w )</td>
<td>1000</td>
<td>kg/m(^3)</td>
</tr>
<tr>
<td>Wave amplitude</td>
<td>( A )</td>
<td>2.5</td>
<td>m</td>
</tr>
<tr>
<td>Gravitational acceleration</td>
<td>( g )</td>
<td>9.81</td>
<td>m/s(^2)</td>
</tr>
</tbody>
</table>
Considering a critical situation in which the monopile is subjected to a dynamic wave pressure with the maximum magnitude, the physical problem is identified as an elasto-static problem. Therefore, the monopile behaviour is governed by Equations (3.2) to (3.4) with $\omega$ being replaced by 0 as (Gould, 1994):

$$
\begin{align*}
[L]^T \{\sigma\} &= 0 \\
\{\sigma\} &= [D]\{\varepsilon\} \\
\{\varepsilon\} &= [L]\{u\} \\
\end{align*}
$$

Equation (4.1) describes the structural behaviour of any point within the monopile foundation. They are solved with the conditions specified at the monopile bottom, sea water-monopile interface and the boundary of the monopile above mean water level. Zero displacements are enforced at the seabed level where the monopile foundation is rested, i.e.

$$\{u\} = 0, \text{ at } z = 0$$

The wave pressure acting on the monopile foundation is the resultant pressure from the dynamic and the hydrostatic components. The analytical expression of the dynamic wave pressure acting upon a cylindrical pile proposed by Zhu (1993) is adopted, the magnitude of which is given in Equation (4.2) as:

$$p_m(a, \theta, z) = \frac{\rho g A}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon_n r^m Q_{mn}(r, \theta) \cosh k_d z \cosh k_d d$$

Figure 4-1. $xz$ plane view of a monopile foundation in ocean environment
with modification of the hyperbolic cosine function in terms of \( z' \), corresponding to the coordinate system constructed in Figure 4-1. \( z' \) is the stretched coordinate and is calculated according to Wheeler (1969) as:

\[
z' = \frac{zd}{d + \eta_\theta}
\]

The azimuth angle \( \theta \) is measured along the monopile circumference in the anti-clockwise direction. For ease of subsequent discussions, the orientation specifications of two angular variables, the azimuth angle \( \theta \) and the incident wave angle \( \alpha \), are defined in Figure 4-2.

\[
\eta_\theta = \frac{A}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon_m \varepsilon_n j^{m-n} Q_{mn} (r, \theta)
\]

In Equations (4.2) and (4.4), \( \varepsilon_m, \varepsilon_n \) and \( Q_{mn}(r, \theta) \) are defined as:

\[
\varepsilon_m, \varepsilon_n \begin{cases} 
1 & m, n = 0 \\
2 & m, n \neq 0 
\end{cases}
\]

\[
Q_{mn} (r, \theta) = \left[ J_m (k, r) J_{2n} (k, r) - A_m H_{m+2n} (kr) \right] \cos(m + 2n) \theta \\
+ \left[ J_m (k, r) J_{2n} (k, r) - B_m H_{m+2n} (kr) \right] \cos(m - 2n) \theta
\]
\[ A_{mn} = \frac{k_x J_m(k_xa) J_{2n}(k_ya) + k_y J_m(k_ya) J_{2n}(k_xa)}{k H_{m+2n}(ka)} \]

\[ B_{mn} = \frac{k_x J'_m(k_xa) J_{2n}(k_ya) + k_y J'_m(k_ya) J_{2n}(k_xa)}{k H_{m-2n}(ka)} \]

In Equations (4.5) to (4.7), \( J \) represents the Bessel function of the first kind and \( H \) represents the Hankel function. \( k_x \) and \( k_y \) explain the periodic property of the short-crested waves at the free-surface level in the \( x \) and \( y \) directions, respectively. They are related to the incident wave angle \( \alpha \) by:

\[ k_x = k \cos \alpha, \quad k_y = k \sin \alpha \]

Generally, the wave period \( T \) for wind waves in ocean environments ranges from 5 to 20 seconds. With this condition, the wave number \( k \) in Equation (4.2) can be evaluated using the wave dispersion equation

\[ (2\pi/T)^2 = gk \tanh(kd) \],

given the water depth \( d \).

The hydrostatic pressure \( p_h \) is calculated as:

\[ p_h = \rho_s g (d - z') \]

### 4.2 SBFEM model and verification

#### 4.2.1 SBFEM model

The monopile has a rather large aspect ratio of height to diameter as 8:1. It is suggested that the entire pile be divided into five subdomains, shown in Figure 4-3, with relatively well-proportioned length in all dimensions to avoid any potential numerical problems. Local scaled boundary coordinate systems are introduced in each subdomain, with the scaling centre located at the geometric centre of each subdomain. Therefore, the boundary of the monopile, as well as the interfaces between adjacent subdomains is discretised using the eight-node quadratic quadrilateral isoparametric elements shown in Figure 3-4.
Figure 4-3. SBFEM model of the monopile

For each subdomain, the local SBFEM formulation corresponding to Equation (4.1) is written as:

\[
[E^0]_{\xi \xi} [u(\xi)] + \left(2[E^0] + [E^1]^T - [E^2]\right)_{\xi \xi} [u(\xi)] + \left([E^1]^T - [E^2]\right)_{\xi \xi} [u(\xi)] = 0
\] (4.10)

Following the procedure presented in Section 3.2, the nodal force-nodal displacement relationship is obtained at the discretised domain boundary, which is subsequently assembled for the entire monopile. The nodal displacement \{u\} of the entire discretisation is then solved from the assembled equation by enforcing prescribed boundary conditions. Afterwards, it is extracted back into each subdomain to calculate the integral constant vector \{C_1\}, and accordingly the solution inside the domain by specifying \(\xi\), \(\eta\) and \(\zeta\).

The SBFEM model is non-dimensionalised following the procedure prescribed in Section 3.3, with the reference parameters listed in Table 4-2.

<table>
<thead>
<tr>
<th>Reference parameters</th>
<th>Descriptions</th>
<th>Magnitudes</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>Radius of monopile</td>
<td>2.5 m</td>
<td>(L)</td>
</tr>
<tr>
<td>(E)</td>
<td>Young’s modulus</td>
<td>2.8\times10^{10} Pa</td>
<td>(ML^1T^2)</td>
</tr>
</tbody>
</table>
4.2.2 Convergence test and model verification

To validate the proposed model and its numerical performance, a monopile foundation subjected to a hydrostatic pressure is studied as a benchmark example, employing the parameters listed in Table 4-1. The hydrostatic pressure is expressed as \( p_h = \rho_w g (d-z) \). To examine the convergence, the displacements in the \( z \) direction at the monopile head level are plotted in Figure 4-4 (a) for five discretisation schemes, which are explained in Table 4-3. For example, ‘Mesh1’ corresponds to ‘8x8x2’, meaning that 8 elements along the circumference, 8 elements along the height, and 2 elements for the radius. Figure 4-4 (a) shows a satisfactory convergence tendency of the proposed SBFEM model.

![Graph](image)

Figure 4-4. SBFEM and FEM model results versus discretisation scheme for (a) displacement convergence and (b) Number of DOFs

Table 4-3. Discretisation scheme description and result comparison

<table>
<thead>
<tr>
<th>Discretisation scheme</th>
<th>Mesh1</th>
<th>Mesh2</th>
<th>Mesh3</th>
<th>Mesh4</th>
<th>Mesh5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8x8x2</td>
<td>8x12x2</td>
<td>8x16x2</td>
<td>16x16x4</td>
<td>16x20x4</td>
</tr>
<tr>
<td>Number of DOFs</td>
<td>SBFEM</td>
<td>1059</td>
<td>1347</td>
<td>1635</td>
<td>4335</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>1623</td>
<td>2367</td>
<td>3111</td>
<td>10947</td>
</tr>
<tr>
<td>Displacement in ( z ) direction (x10^-5)</td>
<td>SBFEM</td>
<td>2.991</td>
<td>3.015</td>
<td>3.025</td>
<td>3.025</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>2.979</td>
<td>3.008</td>
<td>3.020</td>
<td>3.023</td>
</tr>
</tbody>
</table>
An equivalent finite element analysis is also carried out to verify the numerical credibility and competency of the SBFEM model. The \( z \) displacements of any point on the monopile head from the FEM model are plotted in Figure 4-4 (a) for comparison with those calculated from SBFEM. Only a 0.056\% discrepancy is observed for the converged displacement, which guarantees the accuracy of the results from SBFEM. With respect to the DOFs consumed by the two models, as compared in Figure 4-4 (b) and Table 4-3, it is easy to see that SBFEM requires a significantly smaller number of DOFs for the same mesh than that of an equivalent FEM model, but achieves equally satisfactory level of accuracy. This demonstrates the favourable numerical efficiency of the SBFEM model. Taking into consideration the number of DOFs involved and the level of accuracy achieved, ‘Mesh3’ will be adopted in the subsequent analyses.

4.3 Monopile behaviour analysis

With the proposed SBFEM model, the problem described in Section 4.1, viz the structural response of a monopile foundation to ocean wave loads is studied herein. For convenient interpretation of the results, displacements at two representative locations on the monopile foundation, shown in Figure 4-5, are specifically examined: one is line \( L-L' \) along the monopile height at \( \theta = 0 \); the other is arc \( R-R' \) around the monopile circumference at the monopile head level.

![Diagram](a)

![Diagram](b)

Figure 4-5. Two representative locations on monopile foundation: (a) \( L-L' \) along the monopile height and (b) \( R-R' \) around the monopile circumference
As the physical problem is symmetric with respect to the incident wave direction, the displacement in the \( y \) direction along \( L-L' \) is zero. Displacements in the \( z \) direction are less significant compared with the \( x \) counterparts and are not of the major concern. Therefore, the following discussion will mainly focus on the displacements in the \( x \) direction, which reflect the lateral deflection of the monopile when subjected to wave loads. It is also noted that the displacements are calculated from a dimensionless SBFEM model, with their magnitudes non-dimensionalised with respect to the radius of the monopile, as indicated in Table 4-2.

The non-dimensionalised lateral displacement along \( L-L' \) in Figure 4-6 (a) shows a maximum displacement of \( 0.8254 \times 10^{-3} \) at the monopile head level when the wave amplitude, wave number and the water depth are 1, 0.25 and 12, respectively. The displacement variation along monopile circumference at \( z = 16 \) is illustrated by the polar plot in Figure 4-6 (b). Note that numerics in the polar plot mark the scale of the radial axis, and this is the same for the following polar plots. It is noticed that the maximum displacements at the monopile head level are the same everywhere around the monopile circumference when \( \theta \) ranges from 0 to \( \pi \).

Figure 4-6. Lateral displacement of monopile foundation at \( A = 1, k = 0.25 \) and \( d = 12 \): (a) along \( L-L' \) and (b) around \( R-R' \).
To understand how each force component, i.e. the dynamic wave pressure and the hydrostatic pressure, contributes to the monopile behaviour, displacements caused by the two components are illustrated separately. It is observed from Figure 4-7 that the dimensionless displacement at the monopile head level caused by the hydrostatic component is greater than that from the dynamic part, with corresponding magnitudes being $0.4354 \times 10^{-3}$ and $0.3899 \times 10^{-3}$, respectively. Propagating in the positive $x$ direction with $A = 1$ and $k = 0.25$, the incident short-crested wave generates a total free-surface elevation $\eta_0$ as being depicted in Figure 4-8. It is noticed that $\eta_0$ at $\theta = \pi$ is greater than that at $\theta = 0$. This elevation distribution leads to a resultant hydrostatic force acting on the monopile in the incident wave direction and causes a displacement of $0.4354 \times 10^{-3}$. Prevailing throughout the entire vertical length from the free surface to the seabed, the hydrostatic pressure contributes more to the monopile deflection than the dynamic pressure, which predominates only around the free surface and decays rapidly into the water. The displacements around the monopile circumference are uniformly distributed for both the hydrostatic and the dynamic components, as read from Figure 4-7 (b).

Figure 4-7. Lateral displacement of monopile foundation subject to dynamic wave pressure and hydrostatic pressure separately with $A = 1$, $k = 0.25$ and $d = 12$: (a) along $L-L'$ and (b) around $R-R'$
4.4 Parametric study on wave number, amplitude and water depth

Key wave parameters such as wave number, wave amplitude and water depth are of great significance to the monopile behaviour. Considering the real situation where these parameters vary within a certain range, a study on the monopile response to the variation of these parameters is carried out to gain further insight into the functional performance of the monopile foundation. In this study, the analysis mainly focuses on how these parameters affect the wave load on the monopile and accordingly the monopile behaviour.

4.4.1 Effect of wave number, \( k \)

For a water depth of 30 m, and with the wave period ranging from 5 s to 20 s, the wave number varies approximately from \( 0.02 \text{ m}^{-1} \) to \( 0.18 \text{ m}^{-1} \). Therefore, the non-dimensionalised wave number \( k \) is chosen as 0.05, 0.15, 0.25, 0.35 and 0.45 to investigate how it affects the monopile behaviour. Other relevant parameters are listed in Table 4-1.

The wave number influences the wave pressure distribution on the monopile foundation in the vertical direction as well as the horizontal direction. Being dominated by the \( z \) component: \( \cosh k z' / \cosh kd \) of the pressure formulation Equation (4.2), the dynamic wave pressure shows a rather rapid decay with water depth when it is associated with a higher wave number. Superimposed with the hydrostatic pressure, the total wave pressure variation in the vertical direction along \( L-L' \) on the monopile foundation for varying \( k \) is plotted in Figure 4-9 (a). Horizontally, on the other hand, greater wave numbers indicate more frequent waves acting on the monopile. With
incident waves propagating in the positive $x$ direction, the wave pressure generated with a relatively small wave number is distributed axisymmetrically around the monopile circumference as shown in Figure 4-9 (b) for $k = 0.05$ and 0.15. When the wave number $k$ gradually increases to 0.45 by an increment of 0.10, the wave pressure acting on the upstream face of the monopile (corresponds to $\pi/2 < \theta < \pi$) increases, whereas that on the other side ($0 < \theta < \pi/2$) decreases. This, consequently, results in a substantial increase in the magnitude of the resultant force acting on the monopile in the incident wave direction.

![Figure 4-9](image)

Figure 4-9. Total wave pressure distribution at $A = 1$ and $d = 12$ for varying $k$: (a) along $L-L'$ and (b) around $R-R'$

Figure 4-10 (a) shows the lateral displacement of the monopile along $L-L'$ for varying wave numbers at $A = 1$ and $d = 12$. With increasing wave numbers from 0.05, 0.15, 0.25, 0.35 to 0.45, the maximum displacement at the monopile head level increases from $0.0076 \times 10^{3}$, $0.1878 \times 10^{3}$, $0.8254 \times 10^{3}$, $1.8805 \times 10^{3}$ to $3.3194 \times 10^{3}$. It is noticed from Figure 4-10 (b) that for each individual case, the displacement at the monopile head level are the same everywhere around the monopile circumference when $\theta$ ranges from 0 to $\pi$. Plotting the maximum displacements against the wave number $k$ in Figure 4-11 and examining the slope of the curve, it can be concluded
that as $k$ becomes greater, the increase in the maximum displacement becomes more noticeable.

Figure 4-10. Lateral displacement of monopile foundation at $A = 1$ and $d = 12$ for varying $k$:
(a) along $L-L'$ and (b) around $R-R'$

Figure 4-11. Maximum lateral displacement versus wave number
4.4.2 Effect of wave amplitude, $A$

The magnitude of wave amplitude reflects the kinetic energy associated with the wave motion. In this analysis, the dimensionless wave amplitude $A$ ranges from 0.5 to 2.0 at 0.5 increments, which corresponds to a wave height of 2.5 m, 5.0 m, 7.5 m and 10.0 m, respectively. The wave height of 10.0 m represents a wave condition which may serve as an extreme case for engineering design. The total wave pressure at the mean water level, shown in Figure 4-12, increases evenly as the wave amplitude $A$ increases.

![Figure 4-12. Total wave pressure distribution at $k = 0.25$ and $d = 12$ for varying $A$: (a) along $L-L'$ and (b) around $R-R'$](image)

The lateral displacement along $L-L'$ of the monopile for each case is plotted in Figure 4-13 (a). The corresponding polar plot, illustrating the variation in the lateral displacement with respect to the azimuth $\theta$, is shown in Figure 4-13 (b). Similarly, for each case with a certain wave amplitude, equal lateral displacement is examined around the monopile circumference although the pressure distribution, shown in Figure 4-12 (b), is not uniform when $\theta$ goes from 0 to $\pi$. With the wave amplitude increasing from 0.5 to 2.0, the maximum displacement increases from $0.4353 \times 10^{-3}$,
0.8254\times10^{-3}, 1.5024\times10^{-3} to 2.2067\times10^{-3} at the monopile head level. The dependence of the structural behaviour of the monopile on wave amplitude is presented in Figure 4-14. Physically, the greater the wave amplitude, the greater the energy associated with the wave motion, accordingly, the greater the displacement of the monopile foundation induced by the wave load.

Figure 4-13. Lateral displacement of monopile foundation at \( k = 0.25 \) and \( d = 12 \) for varying \( A \): (a) along \( L-L' \) and (b) around \( R-R' \)

Figure 4-14. Maximum lateral displacement versus wave amplitude
4.4.3 Effect of water depth, \( d \)

The variation of water depth inevitably affects the hydrostatic pressure, as well as the dynamic pressure according to Equations (4.2) and (4.9). Therefore, it is an important parameter when assessing monopile’s response to ocean wave loads. In this study, the dimensionless water depth in shallow water conditions varies from 9 to 13 with an increment of 1. Being linearly related to the water depth, the hydrostatic pressure increases with the water depth. The dynamic wave pressure, on the other hand, is related to the water depth by the hyperbolic cosine function \( \cosh kzd / \cosh kd \).

The resultant total wave pressures acting along \( L-L' \) for varying \( d \) at \( k = 0.25 \) and \( A = 1 \) are plotted in Figure 4-15 (a). Those acting upon the monopile foundation around \( R-R' \) at the mean water level are the same for different \( d \), and are overlapped as shown in Figure 4-15 (b).

![Figure 4-15. Total wave pressure distribution at \( k = 0.25 \) and \( A = 1 \) for varying \( d \): (a) along \( L-L' \) and (b) around \( R-R' \)](image)

As shown in Figure 4-16, the corresponding maximum lateral displacement for varying water depth at \( k = 0.25 \) and \( A = 1 \) is \( 0.4827 \times 10^{-3}, 0.5966 \times 10^{-3}, 0.7232 \times 10^{-3}, 0.8254 \times 10^{-3} \) and \( 0.8954 \times 10^{-3} \) when the water depth is 9, 10, 11, 12 and 13, respectively. Similar to Figure 4-11, the maximum displacement at the monopile head level is plotted against the water depth in Figure 4-17, which illustrates that the deeper
the water, the more significant the displacement becomes. Also, same lateral
displacement is discovered around the monopile circumference for each individual
water depth.

Figure 4-16. Lateral displacement of monopile foundation at \( k = 0.25 \) and \( A = 1 \) for varying \( d \);
(a) along \( L \)-\( L' \) and (b) around \( R \)-\( R' \)

Figure 4-17. Maximum lateral displacement versus wave depth
4.5 Summary

A three-dimensional SBFEM model is developed in this chapter to study the structural behaviour of a monopile foundation when subjected to varying ocean wave loads. By introducing a local scaled boundary coordinate system, the SBFEM model reduces the PDE governing the structural behaviour of the monopile foundation to matrix-form ODE in the radial direction. Only the DOFs associated with the discretised monopile boundaries are involved when formulating the coefficient matrices. Subsequently, the ODE is solved analytically for the nodal displacement function, representing the displacement variation in the radial direction. Adopting the same interpolation concept as in FEM, the SBFEM model explores the displacement field within the monopile foundation by specifying the radial coordinate in the nodal displacement function and the two circumferential coordinates in the shape functions. The model demonstrates analytical as well as numerical features in the solution process, and has displayed favourable applicability in modelling monopile behaviour through comparison with an equivalent FEM model in the validation process.

Structural behaviour of the monopile foundation in response to ocean wave loads are studied non-dimensionally, with the main findings summarised as follows:

1) The hydrostatic pressure is a more dominant factor contributing to the monopile deflection in the incident wave direction than its dynamic counterpart. The hydrostatic pressure prevails from the free water surface to the seabed, whereas the dynamic pressure only predominates at the sea surface level and demonstrates a rapid decay as it goes into the water.

2) The lateral displacement of the monopile increases when the wave number, wave amplitude and the water depth increase. For all cases, equal lateral displacement is observed around the monopile circumference at the pile head level regardless of the variation of the wave parameters.

The study presented in this chapter demonstrates favourable capability of SBFEM in modelling the structural behaviour of a monopile foundation, employing an accessible wave-induced pressure expression. It extends SBFEM to more advanced analyses in the subject of wave-structure interaction, presented in subsequent chapters, to derive a better understanding of the interaction nature.
Chapter 5  Wave Interaction with Pile Group Foundation

The previous chapter applies SBFEM to the investigation of wave interaction with a single pile foundation, and mainly focuses on the structural behaviour of the monopile with analytical wave pressure formulation. For constructions such as oil-drilling platforms and offshore airports, piles normally appear in clusters, i.e. a group of piles are arranged in specific layouts and separated by certain distances to provide enhanced functionality. This naturally leads to the question of wave interaction with pile group foundations, which differs from wave interaction with single pile foundations. The effect caused by multiple closely-spaced piles on the wave motion and the subsequent structural behaviour needs to be thoroughly understood and is of paramount importance when assessing the performance of pile group foundations.

Unlike the case with a single cylindrical pile foundation, an analytical solution to the wave field behaviour is currently unattainable when multiple piles with complex geometry are present. Therefore, an SBFEM formulation of the wave domain must be sought. Encouragingly, SBFEM has been extended to solve wave diffraction problems around piles, breakwaters and caissons (Tao et al., 2007; Li, 2007; Song et al., 2010; Song and Tao, 2008). This chapter enhances the development of the previous work, utilising the benefits offered by SBFEM to explore the wave field behaviour in the presence of multiple piles, and simultaneously investigate the structural behaviour of the pile group. Parametric analysis in terms of wave properties and structural layouts are concurrently performed, aiming to provide valuable information for the design and safety evaluation of pile group foundations in ocean installations.


5.1 Problem formulation

A group of pile foundations in an ocean environment with their bottoms fixed on the seabed and subjected to ocean wave forces are illustrated in Figure 5-1. The origin $O$ of the Cartesian coordinate system is located at the seabed surface. $x$ and $y$ denote two orthogonal horizontal directions, and $z$ positively points upwards. Here, $\Omega$ represents a three-dimensional domain with sea water. The designation of other variables remains the same as previously defined.

![Figure 5-1. A pile group foundation in ocean environment](image)

5.1.1 Wave behaviour

Due to the presence of pile foundations, the original wave motion is altered. Following the linear superposition principle (Illingworth, 1991), the resultant wave field can be considered as the superimposition of the original incident wave field and the scattered wave field. Using the velocity potential $\Phi$, the above statement can be formulated as (Zhu, 1993):

$$\Phi_T = \Phi_I + \Phi_S$$

(5.1)

where $\Phi_T$, $\Phi_I$, and $\Phi_S$ represent the velocity potential for the total wave, the incident wave and the scattered wave, respectively.
Under the assumption that the flow motion is irrotational, and that the fluid is
inviscid and incompressible, the velocity potential of the wave field $\Omega$ is determined by
the following Laplace’s equation (Mei, 1989):

$$\nabla^2 \Phi = 0 \text{ in } \Omega \quad (5.2)$$

The wave field is considered infinite at the horizontal scope but with a finite depth
from the free surface level down to the seabed. The boundary condition at the seabed
level indicating no flow across the seabed surface is specified as:

$$\Phi_{\tau} = 0 \text{ at } z = 0 \quad (5.3)$$

The condition at the free surface level is linearised as:

$$\Phi_{\tau} + g \Phi_{z} = 0 \text{ at free surface} \quad (5.4)$$

Due to the linear nature of Equation (5.2), the concept of separation variables is
employed to decompose the velocity potential into univariate functions in terms of
independent variables, i.e. the spatial variables $x, y,$ and $z,$ and the temporal variable $t:$

$$\Phi (x, y, z, t) = \phi (x, y) Z (z) e^{i \omega t} \quad (5.5)$$

In Equation (5.5), $Z(z)$ is formulated as:

$$Z (z) = \frac{\cosh kz'}{\cosh kd} \quad (5.6)$$

to satisfy Equation (5.3) with $z'$ being calculated according to Equation (4.3).
Equation (5.6) describes an attenuation of the magnitude of the velocity potential as it
goes deep into the sea. Equation (5.4) is satisfied by the following frequency
dispersion relation at the free surface:

$$\omega^2 = gk \tanh kh \quad (5.7)$$

Substituting Equation (5.5) into Equation (5.2) leads to a Helmholtz equation
governing the wave motion at the free surface level in two dimensions:

$$\nabla^2 \phi + k^2 \phi = 0 \quad (5.8)$$

It should be mentioned that due to the linear superimposition property reflected in
Equation (5.1), notations $\Phi$ and $\phi$ appearing in Equations (5.2) to (5.8) represent any
of the total velocity potential, the incident velocity potential or the scattered velocity
potential.
The Neumann boundary condition in terms of the total velocity potential and the Sommerfeld radiation condition (Sommerfeld, 1949) associated with the scattered velocity potential are expressed respectively as:

\[
\phi_{T,n} = 0 \quad \text{at the wetted structure surface} \tag{5.9}
\]

and

\[
\lim_{kr \to \infty} (kr)^{1/2} \left( \phi_S - ikr \phi_S \right) = 0 \tag{5.10}
\]

Therefore, by introducing Equations (5.5) and (5.6), a three-dimensional wave diffraction problem described by Equation (5.2) is spatially reduced to a two-dimensional one governed by Equation (5.8), from which the wave field behaviour is solved with Equations (5.9) and (5.10).

### 5.1.2 Structural response

The structural behaviour of each individual pile in the pile group is investigated three dimensionally by solving Equation (4.1) with specified boundary conditions that the piles are fixed on the seabed and are subjected to external forces from ocean waves.

### 5.2 SBFEM model and verification

#### 5.2.1 SBFEM model of wave domain

As discussed in Section 5.1.1, the three-dimensional wave diffraction due to the presence of structures governed by Equation (5.2) has been transformed to a problem addressed by Equation (5.8) two dimensionally at the free surface level. The xy plane view of an infinite wave domain with a group of pile foundations is illustrated in Figure 5-2, where the number of piles is arbitrarily chosen as three (denoted by P1, P2 and P3), and the piles are of circular cross-sections and are randomly positioned. An auxiliary circular envelope, represented by the dashed line, is introduced to divide the entire wave domain into two: one unbounded domain \( S_\infty \) extending from the circular envelope towards infinity, and one bounded domain \( S_b \) within the envelope enclosing the pile group. Before introducing the local scaled boundary coordinate system, a further division of \( S_b \) is performed to meet the requirement that for each subdivided domain, any position on the domain boundary can be visible from a specific location, namely the scaling centre. The subdomain division in SBFEM does not follow any...
particularly definite standard. It is subjected to the complexity of the boundaries and interfaces involved in the geometric model, i.e. the cross-section and the plane layout of the pile foundations in this study. Generally, subdomains with relatively uniform shapes are favoured. This is similar to the discretisation concept in FEM, where severely distorted polygons (elements) with rather sharp angles are avoided. The location of the scaling centre can theoretically be anywhere inside the domain, as long as the visibility of the domain boundary from the scaling centre is guaranteed. However, a location allowing for well-balanced distances from the scaling centre to the domain boundary is preferable. It is therefore suggested the scaling centre be positioned at the geometric centre of the corresponding subdomain, to eliminate any possible numerical inaccuracy also benefit the pre-process of the SBFEM calculation. Thus, illustrated in Figure 5-2, the entire wave field is discretised into eight subdomains, with seven bounded subdomains \( S_i \) \( (i = 1, 2, \ldots, 7) \) separated by solid lines inside the circular envelope and one outer unbounded subdomain \( S_\infty \) extending to infinity.

![Figure 5-2. SBFEM subdomain division](image)

A local scaled boundary coordinate system, taking subdomain \( S4 \) for example, is constructed in Figure 5-3 (a) by a scaling centre \( O (x_0, y_0) \) and a defining curve \( S \), i.e. the boundary of \( S4 \). Scaling the defining curve \( S \) according to a radial coordinate \( \zeta \) with respect to \( O \) leads to a bounded domain \( (S4 \text{ in this case}) \) when \( \zeta \) runs from \( \zeta_0 = 0 \) at the scaling centre \( O \) to \( \zeta_1 = 1 \) at the defining curve \( S \), or alternatively as shown in Figure 5-3 (b), the scaling leads to an unbounded domain \( S_\infty \) when \( \zeta \) goes from \( \zeta_0 = 1 \) at the defining curve \( S \) to \( \zeta_1 = \infty \) at infinity.
In the SBFEM discretisation, only the defining curve needs to be discretised. The local scaled boundary coordinate system \((\xi, s)\) is related to the Cartesian coordinate system \((\hat{x}, \hat{y})\) as:

\[
\begin{align*}
\hat{x}(\xi, s) &= \xi [N(s)] \{x\} + x_0 \\
\hat{y}(\xi, s) &= \xi [N(s)] \{y\} + y_0
\end{align*}
\]

(5.11)

where \(\{x\}\) and \(\{y\}\) represent the coordinates of the discretised nodes on \(S\); \([N(s)]\) is the geometric mapping function. With this geometric mapping, the gradient operator \(\nabla\) is reformulated in the scaled boundary coordinate system using \(\xi\) and \(s\) as:

\[
\nabla = \left[ b^1(s) \right] \frac{\partial}{\partial \xi} + \frac{1}{\xi} \left[ b^2(s) \right] \frac{\partial}{\partial s}
\]

(5.12)

in which, \([b^1(s)]\) and \([b^2(s)]\) are only related to the boundary discretisation on \(S\), and independent of the radial coordinate \(\xi\). Using the same shape function \([N(s)]\) as for the boundary discretisation, the velocity potential \(\phi\) is expressed as:

\[
\{\phi(\xi, s)\} = [N(s)] \{a(\xi)\}
\]

(5.13)

where \(\{a(\xi)\}\) represents the nodal velocity potential function varying in the radial direction \(\xi\). Accordingly, the velocity vector \(\{\upsilon(\xi, s)\}\) can be calculated as:
\[ \{ u(\xi, s) \} = \left[ B^1(s) \right] \{ a(\xi) \}_{\xi} + \frac{1}{\xi} \left[ B^2(s) \right] \{ a(\xi) \} \] (5.14)

with

\[ [B^1(s)] = [b^1(s)][N(s)] \]
\[ [B^2(s)] = [b^2(s)][N(s)] \] (5.15)

Denoting any boundary with prescribed velocity \( \bar{v}_n \) in the outward normal direction \( n \) as \( \Gamma_n \),

\[ \phi_{|n} = \bar{v}_n \text{ on } \Gamma_n, \] (5.16)

and applying the weighted residual technique with a weighting function \( w \) and Green's theorem, Equation (5.8) is translated into an integral equation using Equation (5.16) as:

\[ \int_{\Omega} \nabla^T w \nabla \phi d\Omega - \int_{\Omega} wk^2 \phi \theta d\Omega - \int_{\Gamma} w\bar{v}_n d\Gamma = 0 \] (5.17)

Formulating the weighting function \( w \) using the shape function \([N(s)]\) as:

\[ w(\xi, s) = [N(s)] w(\xi) = w(\xi)^T [N(s)]^T \] (5.18)

and through a series of mathematical manipulations, the following expression yields:

\[ \{ w(\xi_1) \}^T \left[ E^0 \right]_{\xi_1} \{ a(\xi_1) \}_{\xi} + \left[ E^1 \right]^T \{ a(\xi_1) \} - \int_{s} \left[ N(s) \right]^T \bar{v}_n(\xi_1, s) \xi_1 ds \]

\[ -\{ w(\xi_0) \}^T \left[ E^0 \right]_{\xi_0} \{ a(\xi_0) \}_{\xi} + \left[ E^1 \right]^T \{ a(\xi_0) \} + \int_{s} \left[ N(s) \right]^T \bar{v}_n(\xi_0, s) \xi_0 ds \] (5.19)

\[ -\int_{\Gamma} \{ w(\xi) \}^T \left[ E^0 \right]_{\xi} \{ a(\xi) \}_{\xi} + \left[ E^1 \right]^T \{ a(\xi) \} + \int_{s} \left[ N(s) \right]^T \bar{v}_n(\xi, s) \xi ds \]

Equation (5.19) is valid for any arbitrary \( \{ w(\xi) \} \), therefore, the coefficients of \( \{ w(\xi) \} \) should be zero, resulting in:

\[ \left[ E^0 \right]_{\xi_0} \{ a(\xi_0) \}_{\xi} + \left[ E^1 \right]^T \{ a(\xi_0) \} = -\int_{s} \left[ N(s) \right]^T \bar{v}_n(\xi_0, s) \xi_0 ds \] (5.20)

\[ \left[ E^0 \right]_{\xi_1} \{ a(\xi_1) \}_{\xi} + \left[ E^1 \right]^T \{ a(\xi_1) \} = \int_{s} \left[ N(s) \right]^T \bar{v}_n(\xi_1, s) \xi_1 ds \] (5.21)
Equations (5.20) and (5.21) explain the relationships between the nodal velocity potential \( \{ a(\xi) \} \) (appearing at the left-hand side of both equations) and the integral of the velocity along boundaries \( \xi_0 \) and \( \xi_1 \) (the right-hand side of the equations). By examining the left-hand sides of these two equations, the concept of nodal flow function, denoted as \( \{ q(\xi) \} \) and analogous to the internal nodal force function defined in Song and Wolf (1998), is introduced. The formulation of \( \{ q(\xi) \} \) is written as:

\[
\{ q(\xi) \} = [E^2] \xi \{ a(\xi) \}\left|_{\xi} \right. + [E^3]^T \{ a(\xi) \}
\]

(5.23)

Equation (5.22) is the scaled boundary finite element equation corresponding to Equation (5.8). It is a second-order matrix-form homogeneous ODE in terms of the nodal velocity potential function \( \{ a(\xi) \} \). Only the radial variable \( \xi \) appears. The other coordinate \( s \) is incorporated in the coefficient matrices in the form of:

\[
\begin{align*}
[E^0] &= \int_{\xi}^1 [B^1(s)]^T [B^1(s)] |J| \, ds \\
[E^1] &= \int_{\xi}^1 [B^2(s)]^T [B^1(s)] |J| \, ds \\
[E^2] &= \int_{\xi}^1 [B^2(s)]^T [B^2(s)] |J| \, ds \\
[M^0] &= \int_{\xi}^1 [N(s)]^T [N(s)] |J| \, ds
\end{align*}
\]

with \( |J| \) denoting the determinant of the Jacobian matrix calculated for the discretised curve \( S \).

The detailed solution procedure of the scaled boundary finite element equation formulated in the frequency domain for elasto-dynamic problems has been documented in Song and Wolf (1998) within the context of solid mechanics, and has been employed by Li et al. (2006) with appropriate modifications to solve wave diffraction problems. Song et al. (2010) adopted an analogous procedure to deal with the bounded domain, whereas used a special function, namely the Hankel function to account for the Sommerfeld radiation condition when formulating the solution of the unbounded domain. Key procedures of solving Equation (5.22) for the entire wave domain are represented in the following subsections.
5.2.1.1 Bounded domain

The following variable \(\{X(\xi)\}\), combining the nodal velocity potential function \(\{a(\xi)\}\) and the nodal flow function \(\{q(\xi)\}\), is introduced to reduce the order of Equation (5.22) from two to one, though at the expense of doubling the number of DOFs involved in the system:

\[
\{X(\xi)\} = \begin{bmatrix} \{a(\xi)\} \\ \{q(\xi)\} \end{bmatrix}
\]  

(5.25)

Consequently, Equation (5.22) is rewritten as a first-order matrix-form ODE:

\[
\ddot{\xi}\{X(\xi)\} = -[Z]\{X(\xi)\} - \ddot{\xi}^2 [M]\{X(\xi)\}
\]  

(5.26)

with a newly-defined variable \(\ddot{\xi} = ka\xi\), and the Hamiltonian matrix \([Z]\) formulated by the coefficient matrices of Equation (5.22) as:

\[
[Z] = \begin{bmatrix}
\left[ E^0 \right]^{-1} \left[ E^1 \right]^T & -\left[ E^0 \right]^{-1} \\
-\left[ E^2 \right] + \left[ E^0 \right] \left[ E^0 \right]^{-1} \left[ E^1 \right]^T & -\left[ E^1 \right] \left[ E^0 \right]^{-1}
\end{bmatrix}
\]  

(5.27)

Matrix \([M]\) in Equation (5.26) is calculated from the coefficient matrix \([M^0]\) as:

\[
[M] = \frac{1}{a^2} \begin{bmatrix}
0 & 0 \\
M^0 & 0
\end{bmatrix}
\]  

(5.28)

Solving Equation (5.26) involves a matrix decomposition of the Hamiltonian matrix \([Z]\). For two-dimensional problems, the Jordan’s decomposition is suggested (Song et al., 2010; Li et al., 2006):

\[
[Z][T] = [T][\Lambda]
\]  

(5.29)

in which \([T]\) is the invertible Jordan matrix; \([\Lambda]\) is constructed by the eigenvalues in the form of:

\[
[\Lambda] = \begin{bmatrix}
\lambda_j & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}
\]  

(5.30)
where \( j = 1, 2, \ldots, n-1; \) \( n \) is the number of DOFs in Equation (5.22); \( \text{Re}(\lambda_j) \geq 0. \)

Through a series of matrix manipulations, the solution of Equation (5.26) is sought as:

\[
\{ X(\xi) \} = \begin{bmatrix} T \end{bmatrix} \begin{bmatrix} R(\xi) \end{bmatrix} \begin{bmatrix} \xi^{(1)}[\xi] \end{bmatrix} \begin{bmatrix} C \end{bmatrix}
\]

(5.31)

with \( \begin{bmatrix} R(\xi) \end{bmatrix} \) being expressed as a power series in \( \xi \):

\[
\begin{bmatrix} R(\xi) \end{bmatrix} = \begin{bmatrix} I \end{bmatrix} + \xi^{1}[R_1] + \xi^{2}[R_2] + \cdots + \xi^{2m}[R_{m}] + \cdots
\]

(5.32)

The coefficient matrices \( \begin{bmatrix} R_i \end{bmatrix} \) \( (i = 1, 2, \ldots, m) \) can be obtained by solving the following equations:

\[
\begin{bmatrix} P_0^* \end{bmatrix} = \begin{bmatrix} \Lambda \end{bmatrix}
\]

\[
\begin{bmatrix} \Lambda \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix} - \begin{bmatrix} [\Lambda] \end{bmatrix} - 2 \begin{bmatrix} [R_1] \end{bmatrix} = \begin{bmatrix} P_1^* \end{bmatrix} + \begin{bmatrix} M \end{bmatrix}
\]

\[
\begin{bmatrix} \Lambda \end{bmatrix} \begin{bmatrix} R_m \end{bmatrix} - \begin{bmatrix} [\Lambda] \end{bmatrix} - 2m \begin{bmatrix} [R_m] \end{bmatrix} = \begin{bmatrix} P_m^* \end{bmatrix} + \begin{bmatrix} C_m \end{bmatrix}, \quad m = 2, 3, \ldots
\]

(5.33)

with

\[
\begin{bmatrix} C_m \end{bmatrix} = \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} R_{m-1} \end{bmatrix} + \sum_{i=1}^{m-1} \begin{bmatrix} R_i \end{bmatrix} \begin{bmatrix} P_{m-i}^* \end{bmatrix}
\]

(5.34)

The upper-triangular matrix \( \begin{bmatrix} U \end{bmatrix} \) with zero diagonal entries is written as:

\[
\begin{bmatrix} U \end{bmatrix} = \begin{bmatrix} 0 & P_{1}(\lambda_1-\lambda_2)/2 & \cdots & P_{N}(\lambda_1-\lambda_N)/2 \\ 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}
\]

(5.35)

\( (N = 2n) \)

Introducing \( \begin{bmatrix} Y(\xi) \end{bmatrix} = \xi^{(\nu)} \) and \( \begin{bmatrix} K(\xi) \end{bmatrix} = \begin{bmatrix} T \end{bmatrix} \begin{bmatrix} R(\xi) \end{bmatrix} \) for brevity, Equation (5.31) is partitioned as:

\[
\begin{bmatrix} X(\xi) \end{bmatrix} = \begin{bmatrix} K_{11}(\xi) \end{bmatrix} \begin{bmatrix} K_{12}(\xi) \end{bmatrix} \begin{bmatrix} \xi^{(1)} \end{bmatrix} + \begin{bmatrix} Y_{11}(\xi) \end{bmatrix} \begin{bmatrix} Y_{12}(\xi) \end{bmatrix} \begin{bmatrix} \{ C_1 \} \end{bmatrix}
\]

\[
\begin{bmatrix} K_{21}(\xi) \end{bmatrix} \begin{bmatrix} K_{22}(\xi) \end{bmatrix} \begin{bmatrix} \xi^{(1)} \end{bmatrix} + \begin{bmatrix} Y_{21}(\xi) \end{bmatrix} \begin{bmatrix} Y_{22}(\xi) \end{bmatrix} \begin{bmatrix} \{ C_2 \} \end{bmatrix}
\]

(5.36)
For a bounded domain, the solution at the scaling centre where \( \tilde{\xi} = k a \tilde{\xi} = 0 \) must be finite, resulting in \( \{ C_2 \} = 0 \). Comparing Equation (5.36) with Equation (5.25) leads to:

\[
\begin{align*}
\{ a(\tilde{\xi}) \} &= [A(\tilde{\xi})] \{ C_1 \} \\
\{ q(\tilde{\xi}) \} &= [Q(\tilde{\xi})] \{ C_1 \} 
\end{align*}
\] (5.37)

with

\[
\begin{align*}
[A(\tilde{\xi})] &= \left[ K_{11}(\tilde{\xi}) \right] \tilde{\xi}^{i_1} \left[ Y_{i1}(\tilde{\xi}) \right] \\
[Q(\tilde{\xi})] &= \left[ K_{21}(\tilde{\xi}) \right] \tilde{\xi}^{i_1} \left[ Y_{i1}(\tilde{\xi}) \right] 
\end{align*}
\] (5.38)

Eliminating the constant vector \( \{ C_1 \} \) from Equation (5.37) yields the algebraic equation, viz the nodal flow function \( \{ q(\tilde{\xi}) \} \) and the nodal velocity potential function \( \{ a(\tilde{\xi}) \} \) relationship:

\[
\{ q(\tilde{\xi}) \} = [H(\tilde{\xi})] \{ a(\tilde{\xi}) \}
\] (5.39)

with

\[
[H(\tilde{\xi})] = [Q(\tilde{\xi})][A(\tilde{\xi})]^{-1} - [K_{21}(\tilde{\xi})][K_{11}(\tilde{\xi})]^{-1}
\] (5.40)

Equation (5.39) is formulated individually for each bounded subdomain \( S_i \) (i = 1, 2, ..., 7) based on a local scaled boundary coordinate system. It is later assembled for the solution of the entire wave domain, following the same assemblage concept in FEM, which is discussed in Section 5.2.1.3.

### 5.2.1.2 Unbounded domain

The unbounded domain can be represented by scaling the circular envelope towards infinity as depicted in Figure 5-3 (b). The radial coordinate \( \tilde{\xi} \) equals 1 on the circular envelope and \( \infty \) at infinity. Taking advantage of the geometric property of the circular envelope, Equation (5.11) can be reformulated as:

\[
\begin{align*}
\hat{x}(\tilde{\xi}, s) &= \tilde{\xi} R \cos(s/R) + x_0 \\
\hat{y}(\tilde{\xi}, s) &= \tilde{\xi} R \sin(s/R) + y_0
\end{align*}
\] (5.41)

with \( R \) representing the radius of the circular envelope.
Accordingly, the following relationships for the coefficient matrices in Equation (5.24) hold:

\[
\begin{align*}
\begin{bmatrix} E^0 \end{bmatrix} &= \frac{1}{R} \int_{-1}^{1} [N(s)]^T [N(s)] k ds \\
\begin{bmatrix} E^1 \end{bmatrix} &= 0 \cdot [I] \\
\begin{bmatrix} E^0 \end{bmatrix}^{-1} \begin{bmatrix} M^0 \end{bmatrix} &= R^2 [I]
\end{align*}
\]

Equation (5.22) is thus simplified as a matrix-form Bessel’s differential equation:

\[
\overline{\xi}^2 \{a(\overline{\xi})\} - \overline{\xi} \{a(\overline{\xi})\} - \begin{bmatrix} E^0 \end{bmatrix}^{-1} \begin{bmatrix} E^2 \end{bmatrix} \{a(\overline{\xi})\} + \overline{\xi} \{a(\overline{\xi})\} = 0
\]

by redefining \( \overline{\xi} \) for the unbounded domain as: \( \overline{\xi} = kR\overline{\xi} \).

Combining the two linearly independent solutions of the Bessel’s differential equation, i.e. the Bessel functions of the first kind and the Bessel functions of the second kind, the Hankel functions of the first kind are chosen as the base functions to formulate the series solution of Equation (5.43):

\[
\{a(\overline{\xi})\} = \sum_{j=1}^{M} c_j H_{\alpha_j} (\overline{\xi}) T_j
\]

where \( T_j \) are the eigenvectors of \( [E^0]^{-1}[E^2] \) resulting from an eigenvalue problem

\[
\left( \begin{bmatrix} E^0 \end{bmatrix}^{-1} \begin{bmatrix} E^2 \end{bmatrix} - r_j^2 [I] \right) T_j = 0
\]

formulated by substituting Equation (5.44) into Equation (5.43); \( r_j^2 \) are the corresponding eigenvalues. Hence, the boundary condition at infinity Equation (5.21) i.e. the Sommerfeld radiation condition Equation (5.10) is satisfied automatically.

Similar to Equation (5.39) for the bounded domains, the nodal flow function-nodal velocity potential function relationship for the unbounded domain is formulated by substituting Equation (5.44) into Equation (5.23), and noticing \( [E^1] = 0 \cdot [I] \), leads to:

\[
\{q(\overline{\xi})\} = \begin{bmatrix} E^0 \end{bmatrix} \overline{\xi} \{a(\overline{\xi})\} = \begin{bmatrix} E^0 \end{bmatrix} \overline{\xi} \sum_{j=1}^{M} c_j H_{\alpha_j} (\overline{\xi}) T_j
\]

Using Equation (5.44) again, Equation (5.46) is rewritten as:

\[
\{q^*_{\alpha} (\overline{\xi})\} = \begin{bmatrix} H^* (\overline{\xi}) \end{bmatrix} \{a^*_{\alpha} (\overline{\xi})\}
\]

where
\[
\begin{bmatrix}
H^s(\xi)
\end{bmatrix}
= 
\left[ E^0 \right] \overline{\xi}[T][H_{bb}][T]^t
\]

with \([T]= [T_1 \ T_2 \ \ldots \ T_m] \)

and \([H_{bb}] = \text{diag}\left[ H_{s1}(\xi)/H_{s1}(\xi) \ H_{s2}(\xi)/H_{s2}(\xi) \ \ldots \ H_{sn}(\xi)/H_{sn}(\xi) \right]\).

It should be made clear that the Sommerfeld radiation condition is only associated with the scattered waves. Therefore, a subscript ‘s’ is introduced in Equation (5.47) to signify that the nodal flow function and the nodal velocity potential function are in terms of the scattered waves only. A superscript ‘\(\infty\)’ is used for the unbounded domain.

5.2.1.3 Wave domain solution

As indicated in Sections 5.2.1.1 and 5.2.1.2, each subdomain (bounded or unbounded) has an individual SBFEM formulation associated with a particular scaled boundary coordinate system. These formulations are independent, and are only effective for their own-defined domain. To solve the entire wave field, these formulations need to be assembled according to the relationships in terms of the nodal variables at the discretised interfaces between adjacent subdomains. These nodal values are then solved from the assembled equation subject to boundary conditions and the incident wave information. Subsequently, they are extracted back into each subdomain to calculate the integral constants (\(\{C_1\}\) in Equation (5.37) for bounded domains and \(c_1\) in Equation (5.44) for the unbounded domain). Afterwards, analytical nodal functions are formulated according to Equation (5.37) for bounded domains or Equation (5.44) for the unbounded domain. Finally, the solution of the entire wave domain can be obtained by specifying the scaled boundary coordinates \(\xi\) and \(s\). The subdomain assemblage process and the solution procedure are detailed as follows.

Equation (5.39) is formulated on the discretised boundary for bounded subdomains \(S_i (i = 1, 2, .., 7)\) as:

\[
\{q^h\} = \left[ H^h \right] \{d^h\}
\]

and then is assembled for the entire bounded subdomain \(S_b\) according to the conditions that the velocity potentials (\(\phi_r\), \(\phi_i\) and \(\phi_s\)) are continuous (Equation
(5.50)) at the subdomain interfaces $\Gamma_{\text{inf}}$ (refer to Figure 5-2); the flows ($q_T$, $q_L$ and $q_S$), however, hold the same magnitude but opposite signs (Equation (5.51)).

\[
\phi = \phi^{\text{adj}} \text{ on } \Gamma_{\text{inf}} \tag{5.50}
\]

\[
q = -q^{\text{adj}} \text{ on } \Gamma_{\text{inf}} \tag{5.51}
\]

$\phi$ and $q$ with superscript ‘$\text{adj}$’ refer to the velocity potential and the flow from an adjacent subdomain. Expressing the total nodal flow and the total nodal velocity potential by the sum of corresponding incident and scattered components, the assembled nodal flow-nodal velocity potential relationship for the entire bounded domain $S_b$ is written as:

\[
\{q^b_T\} + \{q^b_S\} = [H^b]([a^b_T] + [a^b_S]) \tag{5.52}
\]

In the subsequent assemblage of $S_b$ and $S_\infty$, the equalities on the $\Gamma_{\text{auxi}}$ are addressed. Again, the velocity potentials ($\phi_T$, $\phi_I$ and $\phi_S$) on $\Gamma_{\text{auxi}}$ are continuous (Equation (5.53)), and the flows ($q_T$, $q_L$ and $q_S$) are equal in magnitude but opposite in sign (Equation (5.54)).

\[
\phi^{\infty} = \phi^{\text{adj}} \text{ on } \Gamma_{\text{auxi}} \tag{5.53}
\]

\[
q^{\infty} = -q^{\text{adj}} \text{ on } \Gamma_{\text{auxi}} \tag{5.54}
\]

As the outward normal directions of the bounded domain and the unbounded domain on the auxiliary circular envelope are opposite to each other, Equation (5.47) is reformulated in conformity with Equation (5.52) for assemblage purposes as:

\[
[H^\infty]-\{a^\infty_S\}=-\{q^\infty_S\} \tag{5.55}
\]

and is added by $[H^\infty]\{a^\infty_T\}$ to both sides, leading to:

\[
[H^\infty]\{a^\infty_T\}=-\{q^\infty_S\}+H^\infty\{a^\infty_T\}\tag{5.56}
\]

By rearranging Equation (5.52) as:

\[
[H^b]\{a^b_T\}=\{q^b_T\}=-\{q^\infty_S\}-\{q^\infty_I\} \tag{5.57}
\]

and combining Equations (5.56) and (5.57), noticing $\{a^\infty_T\}=\{a^\infty_I\}$ on $\Gamma_{\text{auxi}}$ yields:
\[
\left[ \begin{bmatrix} H_b^t \\ \end{bmatrix} \right] - \left[ \begin{bmatrix} H^x \\ \end{bmatrix} \right] \{a_i^p\} = - \{q_i^w\} - \left[ \begin{bmatrix} H^x \\ \end{bmatrix} \right] \{a_i^x\} 
\]

Equation (5.58) represents the assembled nodal flow-nodal velocity potential relationship for the entire discretisation. Its right-hand side is in terms of the incident wave information, which enables the total velocity potential \( \{a_i^p\} \) of all discretised nodes to be obtained by simply solving a linear algebraic equation. Once the nodal total velocity potential \( \{a_i^p\} \) for the entire discretisation is solved, they are extracted for each individual subdomain according to the degrees of freedom involved. For each bounded subdomain \( S_i \) \((i = 1, 2, \ldots, 7)\), the constant vector \( \{C_1\} \) in Equation (5.37) is first determined according to the nodal value on the discretised boundary. Subsequently, \( \{a(\tilde{z})\} \) is calculated for any specific \( \tilde{\xi} \) (ranging from 0 to 1), and \( \{\phi(\tilde{\xi},s)\} \) within the subdomain can be obtained by specifying the circumferential coordinate \( s \). For the unbounded domain outside the auxiliary circular envelope, as the series expression in Equation (5.44) is only associated with the scattered waves, the nodal scattered velocity potential is extracted first by subtracting the incident component from the total nodal velocity potential. Afterwards, the constant \( c_j \) in Equation (5.44) is obtained and the total velocity potential for the unbounded domain is retrieved by adding the incident component to the scattered counterpart at any \( \tilde{\xi} \) (ranging from 1 to \( \infty \)) and \( s \). Thus the velocity potential field and the velocity field for the entire wave domain are obtained. Subsequently, other derivative physical quantities, such as the free surface elevation \( \eta_0 \) and the dynamic wave pressure \( p_d \) acting upon pile foundations can be computed as:

\[
\eta_0 = \frac{i \omega}{g} \phi 
\]

\[
p_d = -\rho_w \Phi_{st} 
\]

### 5.2.2 SBFEM model of structural domain

The structural behaviour is formulated the same way as illustrated in Section 4.2 for each pile in the pile group.
5.2.3 Solution procedure

The SBFEM formulation and solution procedure are outlined in Section 5.2.1 for the wave field behaviour, and in Section 5.2.2 for the pile foundation response, respectively. The wave field behaviour is solved first in the presence of pile foundations so that wave-induced forces can be applied for the subsequent structural behaviour investigations. As discussed, the SBFEM formulation of the wave field corresponds to the two-dimensional Helmholtz equation at the free surface level, whereas that of the pile foundations is addressed in three dimensions. Therefore, the wave field solutions need to be reinterpreted into three dimensions in order to be applicable to the structural response investigations. Referring back to the transformation from Equation (5.2) to Equation (5.8), it is consequently reasonable to combine the analytical expression Equation (5.6) with the SBFEM solution to gain the three-dimensional wave field solution.

During the numerical implementation, conformity is required between the discretisation on $\Gamma_p$ (see Figure 5-2) for the wave field analysis and on pile circumferences for structural analysis. Therefore, the physical quantities calculated from the two-dimensional mesh on $\Gamma_p$ (denoted by $\bullet$ in Figure 5-2) are matched correspondingly onto the three-dimensional pile mesh according to the nodal $x$ and $y$ coordinates. Equation (5.6) is programmed into the two-dimensional SBFEM solution of the wave domain based on the nodal $z$ coordinates to restore the analytical variation of the wave field solution in the $z$ direction. Upon formulation into three dimensions, the wave field solution can then be applied to the subsequent structural behaviour investigations. A flowchart, illustrating the whole procedure from solving the wave field to analysing the structural behaviour, is shown in Figure 5-4. Its feasibility and performance are demonstrated by the subsequent validation process.

It should be made clear that although the SBFEM model is formulated using a group of three cylindrical piles as an illustration, it is capable of addressing wave interaction with structures in a more general sense regardless of the attributes associated with the structures, such as the geometric configuration or the quantity and the spatial layout when multiple structures are involved. The proposed model can always be utilised following the procedure detailed in Section 5.2. Separating the entire infinite wave domain into one unbounded domain and one bounded domain, the auxiliary circular envelope is recommended to be large enough to enclose all the
structures. The unbounded domain outside the circular envelope is addressed in terms of the scattered wave field. The bounded domain, in which a further division into several subdomains should be considered, is formulated with respect to the total wave field. The subdivision of the bounded domain $S_b$ behaves such that the scaling centre of each subdomain can be efficiently defined to make any position on the subdomain boundary visible from it. In this study, the subdomain division follows a well-balanced pattern rather than a completely arbitrary one, thereby each subdomain holding relatively even distances between the scaling centre and the domain boundary. The scaling centre of the unbounded domain is positioned at the centre of the circular envelope. These can be easily implemented in the pre-process to improve the efficiency of the SBFEM model.

5.2.4 Model verification

The structural behaviour of a single pile foundation subject to ocean wave loads as presented in Chapter 4, in which an explicit wave pressure expression is used, can be employed herein to validate the solution scheme proposed for the wave-pile group interaction. In this validation example, plane waves are selected as incident waves, with the incident velocity potential $\Phi_i$ being expressed as (Mei, 1989):
\[ \Phi_I = -\frac{igA}{\omega} Z(z) e^{i(k_x x + k_y y - \omega t)} \] (5.61)

In Equation (5.61), \( \omega \) represents the angular frequency of the incident plane waves; \( i = \sqrt{-1} \). The pile radius \( a \) is chosen as 1 m; the pile height \( h = 10 \) m; water depth \( d = 7.5 \) m, and the wave amplitude \( A = 0.5 \) m. Other relevant parameters remain the same as listed in Table 4-1.

For the wave domain solution, a two-dimensional SBFEM model is established as shown in Figure 5-5. A radius \( R \) twice that of the pile radius \( a \) is chosen for the auxiliary circular envelope. The bounded domain within the envelope, i.e. the annulus, is further divided into four subdomains, each having a scaling centre located at the geometric centre of the subdomain. All solid lines in Figure 5-5 (a) are discretised using three-node quadratic elements as displayed in Figure 5-5 (b).

![Figure 5-5. Two-dimensional SBFEM model for wave domain solution: (a) subdomain division and (b) scaled boundary element](image)

\( \eta_s(s) = s(s-1)/2 \)
\( \eta_s(s) = s(s+1)/2 \)
\( \eta_s(s) = (1+s)(1-s) \)

The normalised free surface elevation \( |\eta_s|/A \), defined as the ratio of the free surface elevation \( |\eta_s| \) to the wave amplitude \( A \), is calculated under the prescribed wave conditions and compared with the analytical solution (Zhu, 1993) in Figure 5-6 for a finite region and in Figure 5-7 around the pile circumference, respectively. The two figures demonstrate excellent performance of the wave field solution.
Figure 5-6. $|\eta|/A$ of a finite region of the wave field for waves ($k = 0.10$ m$^{-1}$ and $\alpha = 0$) interaction with a cylindrical pile from (a) analytical expression and (b) numerical solution.

Subsequently, a three-dimensional model of the pile foundation is prepared for the structural behaviour analysis. Five subdomains with well-proportioned geometric dimensions are designed with the scaling centre of each subdomain coincident with the geometric centre. The surface of the pile, as well as the interfaces between adjacent subdomains is discretised using eight-node quadratic quadrilateral elements. In order for the nodal physical quantity from the wave field solution to be matched onto the pile model, a consistent discretisation scheme along the pile circumference between the two meshes is required. Following the procedure outlined in Section 4.2,
the displacement components in the $x$, $y$ and $z$ directions of the pile foundation are calculated and those along $L-L'$ (defined in Figure 4-5), labelled by the suffix ‘-Numerical’, are plotted in Figure 5-8. They are compared with the displacements calculated from the analytical wave pressure, denoted by the suffix ‘-Analytical’. As can be seen, the results agree well between the two calculations.

![Graph showing displacement comparison](image)

Figure 5-8. Displacement comparison of the validation example

In order to justify the applicability of the proposed SBFEM model in engineering practice and demonstrate the validity of the linear wave theory in the specified study, the model is compared against the experiment conducted by Chakrabarti and Tam (1975), in which dynamic effects due to waves on a large vertical circular cylinder were measured in a wave tank in the form of pressures, forces and moments. Detailed information about the experimental model, apparatus and the test procedure can be found in Chakrabarti and Tam (1975). A plot of the effective inertia coefficient $C_M$ over the ratio of cylinder diameter to wave length $ka/\pi$ from the experimental data is used to compare the corresponding values calculated from the SBFEM model, as shown in Figure 5-9. The satisfactory agreement, together with those achieved in Figure 5-6 to Figure 5-8, demonstrates the credibility of the proposed SBFEM model in solving wave-structure interaction problems. This model will be employed in the investigation of wave interaction with pile group foundations, which is discussed in the subsequent sections.
5.3 Wave interaction with two piles

The verified SBFEM model is employed in the following analyses to investigate wave interaction with a group of two piles, with the intention to provide enlightening information when more piles or complex structures are involved. The geometric sketch illustrating the physical problem is shown by an $xz$ plane view and an $xy$ plane view in Figure 5-10. The two piles, denoted by P1 and P2, are of identical radius $a = 0.5$ m. They are seated on the $x$ axis and are placed symmetrically with respect to the $y$ axis. The net distance between the two piles $d_{Net}$ equals three times the pile radius $a$. Other parameters hold the same magnitudes as listed in Table 4-1.

Figure 5-10. Geometric model of a group of two piles in ocean environment: (a) $xz$ plane view and (b) $xy$ plane view
The SBFEM model for the wave field is constructed in the \(xy\) plane by introducing an unbounded domain \(S_\infty\) and a bounded domain \(S_b\) using an auxiliary circular envelope with the radius \(R = 8a\). \(S_b\) is further divided into six bounded subdomains \(S_i\) \((i = 1, 2, \ldots, 6)\) as illustrated in Figure 5-11. The structural analysis of the pile foundations employs the same SBFEM model as that presented in Figure 4-3.

![SBFEM Model of Wave Field Analysis for Wave Interaction with Two Piles](image)

Figure 5-11. SBFEM model of wave field analysis for wave interaction with two piles

With the specified wave parameters, the wave field behaviour is calculated and the normalised free surface elevation \(|\eta_0|/A\) is plotted in Figure 5-12 for a finite region of the infinite wave domain. An obvious curvature of the wave crests and troughs is perceived, indicating a change in the propagation direction as waves approach the piles. The local enlargement of \(|\eta_0|/A\) provides information that \(|\eta_0|/A\) in the vicinity of \(P1\) is generally greater than that of \(P2\) under the prescribed wave conditions.

![Wave Field Behavior](image)

Figure 5-12. \(|\eta_0|/A\) of a finite region with a local magnification for wave diffraction with two piles when \(k = 0.10\) m\(^{-1}\) and \(\alpha = 0\)
A clearer examination of \(|\eta_0|/A\) distribution around pile circumferences is obtained from Figure 5-13. A symmetric pattern of the distribution with respect to the \(x\) axis is observed due to the symmetric definition of the problem. It is noticed that, with other conditions being identical, \(|\eta_0|/A\) around P1 circumference is overall greater than that around P2. At the same time, the difference of \(|\eta_0|/A\) at the two sides of the piles, the upstream side where \(-\pi \leq \theta \leq -\pi/2 \cup \pi/2 \leq \theta \leq \pi\) and the lee side where \(-\pi/2 \leq \theta \leq 0 \cup 0 \leq \theta \leq \pi/2\), is significantly mitigated after the wave diffraction with P1 occurs.

Figure 5-13. \(|\eta_0|/A\) around pile circumferences when \(k = 0.10\) m\(^{-1}\) and \(\alpha = 0\)

Demonstrated by Equations (5.59) and (5.60), the quantitative relationship between the dynamic wave pressure \(p\) and the free surface elevation \(\eta_0\) at a certain horizontal level is \(p = \rho g \eta_0\). Therefore, it is reasonable to infer from Figure 5-13 that the resultant force acting in the \(x\) direction on P1 is greater than the corresponding resultant force on P2, and that in the \(y\) direction on both piles is zero. This will consequently result in a greater displacement in the \(x\) direction for P1 than that for P2, and zero displacement in the \(y\) direction for both piles, as shown by the solid and dotted lines in Figure 5-14 (The dash dotted lines in Figure 5-14 and Figure 5-15 depict displacement components of P, a single pile standing in isolation and subject to the same wave conditions as P1 and P2. The structural behaviour of P will be addressed later when comparing pile group displacements and single pile displacements). Negative displacements at the bottom of both piles in Figure 5-14 (a)
are found to be associated with the hydrostatic pressure, which reaches its maximum at the pile bottom level when linearly increases with the water depth.

![Graph showing displacement components along pile height at θ = 0 with k = 0.10 m⁻¹ and α = 0](image)

(a) in the x direction and (b) in the y direction

Figure 5-14. Displacement components along pile height at θ = 0 with k = 0.10 m⁻¹ and α = 0

Figure 5-15 shows a polar plot of the displacement variation around pile circumferences at the pile head level. Represented by the solid and the dotted lines for P1 and P2 respectively, the x displacements are uniformly distributed around the pile circumferences, with magnitudes of 0.008 mm and 4.765x10⁻⁴ mm. Due to symmetry, the y displacement component is zero all around the pile circumferences.

Figure 5-14 and Figure 5-15 also show the displacement components (denoted by the dash dotted lines) of a single pile P, calculated under the same wave condition, for comparison purposes. It is found that the displacement amplitude of P is smaller than that of P1, and slightly greater than that of P2. The presence of P2 affects the wave field behaviour around P1, accordingly the structural response of P1, and vice versa. This mutual influence between individual piles in the pile group results in their structural behaviour different from that of piles standing in isolation. The displacement comparison implies that the leading pile, to which waves approach first,
experiences a greater resultant wave force than the other pile, and the force is also greater than what the pile would experience in an isolated situation. The other pile, however, with which waves interact afterwards, is subjected to a relatively mild external force compared with the leading pile. It also experiences a milder wave force than it would do if it were standing alone. This contributes to the outperformance of pile groups over single pile foundations in the respect that with randomly generated incident waves from different directions, each pile in the pile group alternately acts against the wave force first, while at the same time keeping other piles sheltered. Therefore, no particular pile will have to consistently withstand extreme wave forces, as is the case with a single pile foundation. This positively prolongs the lifespan and ensures reliable service of pile group foundations.

This section presents the wave field behaviour in the presence of two piles and the corresponding response of the two piles, with a view to providing fundamental as well as informative understanding of the interaction phenomenon in other situations, such as with pile groups having more piles with circular or irregular cross-sections. The detailed analysis process follows that proposed in Section 5.2 with a slight difference lying in the subdomain division when constructing the SBFEM model. Illustrations of the subdomain division for wave interaction with three piles located at the vertices of
an equilateral triangle, and four piles positioned at the vertices of a square are shown in Figure 5-16 (a) and (b), respectively. Analogous division pattern can be arranged for more complex structural configurations. As discussed, the proposed SBFEM-based solution process is applicable to very general wave-structure interaction problems.

![Subdomain division for wave interaction with (a) three piles and (b) four piles](image)

Figure 5-16. Subdomain division for wave interaction with (a) three piles and (b) four piles

### 5.4 Parametric analysis under varying incident wave angle

The parametric analysis performed in Chapter 4 provides a clear picture regarding the influence of the parametric variation, in terms of the wave number, wave amplitude and the water depth, on the structural behaviour of a single pile foundation. For wave interaction with pile groups, other parameters, such as the incident wave angle and the dimension ratio of the pile distance to the pile radius are of more significance. Considering the geometric layout of the two piles, the incident wave angle varies within a range of \([0, \pi/2]\) and is specified as 0, \(\pi/4\) and \(\pi/2\), corresponding to three situations in which the waves incident from a direction in parallel, oblique and orthogonal to the connecting line between the two piles, as illustrated in Figure 5-17. The ratio of the pile distance to the pile radius, abbreviated as the dimension ratio in the following text for brevity and denoted as \(e\), is arranged to vary from 1 to 5 with an increment of 1. The wave number is still considered as an important factor affecting both the wave field behaviour and the subsequent structural response. From relevant wave information and the dispersion equation (Equation
(5.7)), the wave number is calculated to change from 0.05 m\(^{-1}\) to 0.15 m\(^{-1}\) by an increment of 0.025 m\(^{-1}\). Table 5-1 lists the variation of the three parameters, which will be employed in the subsequent discussions.

![Diagram of wave direction and parameters](image)

*Figure 5-17. Variation of the incident wave direction*

**Table 5-1. Parameter variation for parametric analysis**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Magnitudes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wave number (k) (m(^{-1}))</td>
<td>0.05      0.075  0.10  0.125  0.15</td>
</tr>
<tr>
<td>Dimension ratio (e = d_{Net}/a)</td>
<td>1           2       3       4     5</td>
</tr>
<tr>
<td>Incident wave direction (\alpha) (rad)</td>
<td>0       (\pi/4) (\pi/2) -       -</td>
</tr>
</tbody>
</table>

The parametric analysis is sectioned into three parts according to the prescription of the incident wave angle shown in Table 5-1. In each case, discussions are focused on how the variation of individual parameters, i.e. the wave number \(k\) and the dimension ratio \(e\), and the combination of the two parameters affect the wave field behaviour and the structural response. The parametric analysis is expected to provide valuable information, which will shed light on the subject of wave-structure interaction.

### 5.4.1 Incident wave angle \(\alpha = 0\)

Figure 5-18 depicts the variation of \(|\eta_0|/A\) around pile circumferences for varying wave number \(k\) when \(e = 3\) and \(\alpha = 0\). In each of the subplots, the solid lines depict the
\(|\eta_0|/A| distribution for \(k = 0.05 \text{ m}^{-1}\); the dotted lines for \(k = 0.075 \text{ m}^{-1}\); the dash dotted lines for \(k = 0.10 \text{ m}^{-1}\); the dashed lines for \(k = 0.125 \text{ m}^{-1}\) and the solid lines with dotted markers for \(k = 0.15 \text{ m}^{-1}\). With \(\alpha = 0\), the distribution of \(|\eta_0|/A| around pile circumferences present a symmetric pattern with respect to the \(x\) axis. Generally, \(|\eta_0|/A| around P1 circumference is greater than that of P2 for corresponding cases. A relatively uniform distribution of \(|\eta_0|/A| around pile circumferences is noticed for both piles when the wave number is small. As the wave number increases, \(|\eta_0|/A| increases considerably on the upstream side of P1, whereas that on the lee side increases with a small magnitude. Subsequently, an elongated shape is developed with its longer axis in alignment with the incident wave direction as the wave number becomes larger. For P1, the maximum \(|\eta_0|/A| always appears at \(\theta = \pi\) where the pile first confronts against the incoming waves. The location of the minimum \(|\eta_0|/A|, however, is visible at where \(\theta\) is slightly less than \(\pi/2\) with mild shifts when \(k\) changes. For P2, the maximum \(|\eta_0|/A| is found at \(\theta = 0\), the lee side of the pile, for relatively small \(k\). It switches to \(\theta = \pi\), the upstream side of the pile, when the wave number becomes significant. The minimum \(|\eta_0|/A| on P2 circumference varies roughly around \(\theta = \pi/2\). It can be inferred that the maximum \(|\eta_0|/A| on the pile circumference always appears in the incident wave direction, and the minimum \(|\eta_0|/A| can be approximately found in its orthogonal direction.

![Figure 5-18. \(|\eta_0|/A| around pile circumferences for varying \(k\) when \(e = 3\) and \(\alpha = 0\) for: (a) P1 and (b) P2](image)
Following the discussion of the location of $|\eta_0|/A$ extrema around pile circumferences, a corresponding examination of the magnitude variation is carried out. The magnitude variation of $|\eta_0|/A$ is not solely related to any single parameter, i.e. the wave number $k$ or the dimension ratio $e$. It is a multivariate function of both parameters. Thus, Figure 5-19 is drawn to identify the interrelationship between the combination of the two parameters and the extrema. As shown in subplot (a), the maximum $|\eta_0|/A$ on P1 circumference increases and the increase becomes more noticeable as the value of $k$ rises, especially with a greater $e$. Examining the subplot

![Graph (a)](image)

![Graph (b)](image)

![Graph (c)](image)

![Graph (d)](image)

Figure 5-19. Variation of $|\eta_0|/A$ extrema with $k$ for varying $e$ when $\alpha = 0$ for: (a) maximum for P1; (b) maximum for P2; (c) minimum for P1 and (d) minimum for P2
for each individual $k$, denoted by the same markers on different lines (‘+’ for $k = 0.05$ m$^{-1}$, ‘o’ for $k = 0.075$ m$^{-1}$, ‘V’ for $k = 0.10$ m$^{-1}$, ‘□’ for $k = 0.125$ m$^{-1}$ and ‘0’ for $k = 0.15$m$^{-1}$), it is found that for any specific $k$, the maximum increases when $e$ goes up. However, this increase is rather mild for small $k$. It becomes remarkable when $k$ becomes greater but with a slight downgrade in the increase with respect to $e$ as shown by ‘0’ in the plot.

Figure 5-19 (b) and (c) provide similar variation pattern of the maximum $|\eta_0|/A$ for P2 and the minimum $|\eta_0|/A$ for P1. The extrema decrease first when $e$ is no less than 2 followed by a subsequent rise as $k$ increases, or show a constant decrease when $e = 1$. Analogically, by viewing the subplots for any definite $k$, an increase in the maximum $|\eta_0|/A$ for P2 or the minimum $|\eta_0|/A$ for P1 is observed as the value of $e$ rises.

The minimum $|\eta_0|/A$ around P2 illustrated in Figure 5-19 (d) shows a tendency of decrease with $k$ for any particular $e$ although it increases as the two piles are further away from each other for any certain $k$.

An overview of Figure 5-19 reveals that a greater wave number corresponds to more significant wave diffraction phenomenon in the presence of pile foundations. The variation of the wave number has a considerable influence on the $|\eta_0|/A$ extrema for both piles regardless of the variation of $e$. The effect of the variation of $e$, however, is largely related to the magnitude of the wave number. For any value of $k$, all subplots show an increase in $|\eta_0|/A$ extrema, however the increase becomes less prominent as $e$ increases. This tendency is far more obvious with large wave numbers than that with small ones.

The corresponding pile behaviour in response to the wave forces is studied, with the displacement component of the centre of the pile head plotted in Figure 5-20. As the physical problem is symmetric with respect to the $x$ axis when $\alpha = 0$, the displacement component in the $y$ direction is zero, and thus is not presented. Overall, the $x$ displacement component of P1 is greater than that of P2 for corresponding cases. This is due to the fact that the dynamic wave energy associated with the wave motion has been mitigated after the wave diffraction with P1 happens. The $x$ displacement of P1 shows substantial increase with $k$ for any $e$, but it is noticed that for a specific $k$, it decreases when $e$ increases. The $x$ displacement of P2, on the other hand, demonstrates a different variation pattern. The lower part of subplot (b) shows
negative displacements, indicating an opposite displacement direction to that of P1 for a combination of small $k$ and $e$. The magnitude first increases and then goes down when $k$ increases, however, it shows a monotonic decrease as the value of $e$ rises. When $e$ and $k$ become greater at the upper corner, positive displacements are observed and the magnitudes increase with respect to both $k$ and $e$. Once again, the influence of the variation in dimension ratio $e$ on the pile displacement only becomes prominent when the wave number is large, and the degree of influence is steadily degraded as $e$ keeps increasing.

![Figure 5-20. Variation of displacement with $k$ for varying $e$ when $\alpha = 0$ for: (a) P1 and (b) P2](image)

5.4.2 Incidence wave angle $\alpha = \pi/4$

When the incidence wave direction is $\pi/4$, small wave number still generates uniform $|\eta_0|/A$ distribution around the pile circumferences. As the wave number increases, the magnitudes of $|\eta_0|/A$ become greater around where the incident waves first approach the pile. Due to the presence of P2, the location of the maximum $|\eta_0|/A$ on P1 circumference moves around $\theta = -3\pi/4$ with a slight shift to the right. The corresponding minimum appears at around $\theta = 2\pi/3$. As far as P2 is concerned, the maximum $|\eta_0|/A$ occurs at the left of $\theta = \pi/4$ for relatively small $e$ and $k$, however, when the net distance between the two piles becomes substantial, the influence of P1’s presence on the wave behaviour around P2 is weakened to an extent that the maximum $|\eta_0|/A$ switches to $\theta = -3\pi/4$ for $e > 3$ and $k = 0.15$ m$^{-1}$. The minimum $|\eta_0|/A$
Figure 5-21. $|\eta_\theta/A$ distribution around pile circumferences for varying $k$ when $e = 3$ and $\alpha = \pi/4$ for: (a) P1 and (b) P2

Figure 5-22. Variation of $|\eta_\theta/A$ extrema with respect to $k$ and $e$ when $\alpha = \pi/4$ for: (a) maximum for P1; (b) maximum for P2; (c) minimum for P1 and (d) minimum for P2
remains at around $\theta = -\pi/4$ for any combination of $e$ and $k$. It can be readily concluded that, for both piles, the maximum $|\eta_0|/A$ always locates approximately in alignment with the incident wave direction. The minimum, however, is roughly in the orthogonal direction. Figure 5-21 shows the $|\eta_0|/A$ variation around pile circumferences of the case with $e = 3$ and $\alpha = \pi/4$ for illustration purposes. The variations of the maximum $|\eta_0|/A$ on P1 circumference and the minimum $|\eta_0|/A$ on P2 circumference in Figure 5-22 show the same tendency as their counterparts when $\alpha = 0$ (see Figure 5-19). The

Figure 5-23. Variation of displacement extrema with respect to $k$ and $e$ when $\alpha = \pi/4$ for: (a) maximum $u_x$ for P1; (b) maximum $u_x$ for P2; (c) maximum $u_y$ for P1 and (d) maximum $u_y$ for P2

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minimum $|\eta_0|/A$ on P1 circumference decreases as $k$ increases, but shows a tendency of potential increase with $k$ when $e$ is large enough. The maximum $|\eta_0|/A$ on P2 circumference reduces when the wave number $k$ increases, and it is not affected much by the dimension ratio $e$ except when its location shifts abruptly from around $\theta = \pi/4$ to $\theta = -3\pi/4$ when $k = 0.15 \text{ m}^{-1}$ and $e > 3$.

The displacement components in the $x$ direction for both piles demonstrate similar variation pattern with respect to $e$ and $k$ when compared with the corresponding plots in Figure 5-20 for $\alpha = 0$, with a slight difference in the magnitudes. As the physical problem is no longer symmetric when $\alpha = \pi/4$, significant displacement in the $y$ direction for both piles are observed and they present similar variation pattern to their $x$ counterparts. Both $x$ and $y$ displacement components of P1 are greater than those of P2, which signifies that the leading pile generates greater displacements. The other pile, however, experiences a relatively gentle wave impact due to the presence of the preceding pile, and consequently presents comparatively moderate displacement.

### 5.4.3 Incident wave angle $\alpha = \pi/2$

When the incident wave angle shifts to $\pi/2$, the incident wave direction is orthogonal to the connecting lining between the two piles. The physical problem is symmetric with respect to the $y$ axis, and the two piles are equivalent to each other. Accordingly, the wave field behaviour demonstrates a symmetric pattern. The $|\eta_0|/A$ distribution around P1 circumference mirrors that of P2 with respect to the $y$ axis, and

![Figure 5-24](image_url)

Figure 5-24. $|\eta_0|/A$ distribution around pile circumferences for varying $k$ when $e = 3$ and $\alpha = \pi/2$ for: (a) P1 and (b) P2
vice versa. The maximum $|\eta_0|/A$ on both pile circumferences appears at around $\theta = -\pi/2$. The minimum $|\eta_0|/A$ can be visible at corresponding symmetric positions on P1 and P2 circumferences, i.e. around $\theta = 0$ for P1 and $\theta = \pi$ for P2.

The maximum and minimum $|\eta_0|/A$ on P1 and P2 circumferences show identical variation patterns with respect to the two parameters, and only the plots for P1 are presented in Figure 5-25. The maximum $|\eta_0|/A$ first decreases as $k$ increases to 0.075 m$^{-1}$, followed by a subsequent significant rise when $k$ increases to 0.15 m$^{-1}$. Unlike the plots in Figure 5-19 and Figure 5-22, in which the maximum either increases or decreases consistently as the value of $e$ rises for certain $k$, the maximum $|\eta_0|/A$ shown in Figure 5-25 first decreases when $e$ goes up from 1 to 3 and then increases when $e$ continues to go up to 5. This indicates that when the incident wave direction is orthogonal to the connecting line of the two piles, the maximum $|\eta_0|/A$ on the pile circumference is not a monotonic function of the pile distance, there is a dimension ratio $e$ when the maximum $|\eta_0|/A$ falls to a minimum, which equals 3 in the present case. The minimum $|\eta_0|/A$, however, displays a simple declining tendency with $k$, and monotonically increases with a dropping slope when $e$ becomes larger for a certain $k$.

![Figure 5-25. Variation of $|\eta_0|/A$ extrema with respect to $k$ and $e$ when $\alpha = \pi/2$: (a) maximum and (b) minimum](image)

The variation of the displacement components in both $x$ and $y$ directions are plotted in Figure 5-26 to examine the corresponding structural behaviour. Though Figure 5-25 presents a non-sequential variation pattern of the maximum $|\eta_0|/A$, the
displacement components show a sequential variation pattern. Both $x$ and $y$
displacement components increase with $k$ for any definite $e$, and descend with a
reducing slope as $e$ increases for any particular $k$. Another noteworthy phenomenon is
that the displacement component in the $y$ direction is larger than its $x$ counterpart as
indicated from the $|\eta_0|/A$ distribution in Figure 5-24, the resultant force acting on the
pile in the $y$ direction is greater than that in the $x$ direction. P2 holds the same
magnitude of the displacement in the $x$ direction as P1 but with an opposite sign. The
displacement in the $y$ direction is precisely the same as that of P1 in both magnitude
and direction.

![Figure 5-26. Variation of displacement with respect to $k$ and $e$ when $\alpha = \pi/2$ for P1: (a) maximum and (b) minimum](image)

5.5 Summary

The study presented in this chapter advances SBFEM’s application in wave-
structure interaction by addressing two contemporary debatable issues within the
context of ocean engineering: the wave field behaviour in the presence of multiple
cylindrical structures and the corresponding structural response when subjected to the
resultant wave forces. Prior to the SBFEM formulation of the wave field, the Laplace
equation governing the entire wave field is processed into the Helmholtz equation by
separating the vertical-direction component from the velocity potential expression.
This leads to a two-dimensional SBFEM investigation of the wave field at the free
surface level. Well-planned subdomain division of the two-dimensional wave domain into one unbounded and several bounded subdomains is required so that: (1) the Sommerfeld radiation condition at infinity can be precisely satisfied by employing the Hankel function in the formulation of the unbounded domain; (2) irregular geometric configuration or complex spatial arrangement of piles can be easily dealt with. The solution concept, employing SBFEM to address the wave diffraction phenomenon developed in the previous work (Li et al., 2006; Song et al., 2010), is referred to in the discussion of the wave field solution.

To incorporate the wave field results into the subsequent structural analysis, conformity of the discretisation scheme around pile circumference is required when constructing the SBFEM model. The wave field outputs are reinterpreted into three dimensions by programming the analytical attenuation function in the $z$ direction into the two-dimensional results. The formulation of the subsequent structural behaviour analyses follows that presented in Chapter 4. The proposed model is verified using wave interaction with one single pile with satisfactory performance: (1) accurately captures the boundary condition at infinity for the unbounded wave domain; (2) effectively addresses the wave-pile group interaction problem in three dimensions however with released computational burden; (3) successfully incorporates a scalar field in which the wave field behaviour is explored, and a vector field where the structural response is investigated.

The SBFEM model, though employing a group of three cylindrical piles to present its derivation, is applicable to wave-structure interaction problems more generally without any specifications on the configuration, quantity or spatial layout of the structures. It is used to analyse wave interaction with a group of two piles as an illustrative case in this chapter, with the intention of providing enlightening and essential information for more complex wave-structure interaction analyses. The main findings are summarised as follows:

1) Symmetric problem configuration leads to symmetric wave field behaviour and accordingly, symmetric structural response of the piles.

2) Small wave numbers correspond to relatively uniform $|\eta_0|/A$ distribution around pile circumferences, indicating insignificant resultant wave forces acting on piles, thus moderate pile displacements.

3) Greater wave numbers render the $|\eta_0|/A$ distribution around pile
circumferences an elongated shape with its longer axis approximately oriented in the incident wave direction, where the maximum $|\eta_0|/A$ is located. The minimum, however, is roughly visible at the perpendicular direction.

4) The difference of $|\eta_0|/A$ at the upstream side and the lee side is greater for the leading pile, to which waves approach first, implying greater resultant wave force and consequently larger displacement of the leading pile than those of succeeding piles. With randomly generated wave forces, each pile in the pile group serves alternately as the leading pile, dissipating a portion of the mechanical energy associated with the wave motion, therefore contributing to a longer lifespan of the pile group.

5) The wave number has a noteworthy influence on $|\eta_0|/A$ distribution and also the pile behaviour no matter what the pile distance is. The effect of the dimension ratio $e$ on the wave filed and the pile behaviour, however, largely depends on the magnitude of the wave number. The physical quantity ($|\eta_0|/A$ or displacement), either increasing or decreasing with respect to $e$, displays more obvious variation tendency with a large wave number than that with a small one.

All the above discussions demonstrate the reliability of SBFEM in addressing complex wave-structure interaction problems. Further investigation, focusing on the time-dependent analysis, will be presented in Chapter 6 for more profound exploration of SBFEM’s potential as well as additional insight into the nature of the physical problem.
Chapter 6  Time-Dependent Wave Interaction with Pile Foundation*

In Chapter 4, SBFEM is employed to examine wave interaction with a monopile foundation, in which the wave field behaviour is analytically available. In Chapter 5, SBFEM is advanced to address the issue of wave interaction with multiple piles, combining both the wave domain and the structural domain to compensate the inaccessibility of explicit wave pressure expression. In both cases, the time-dependent term is excluded from the wave field solution and the structural response is analysed in a static scope. In the real world, nevertheless, wave properties vary constantly with respect to time. Therefore, research on the wave field behaviour, and the subsequent structural response, in the context of wave-structure interaction, should be conducted in the time domain for a more realistic representation of the ocean situation. In this chapter, SBFEM is going to further its adventure in the application in wave-pile foundation interaction by taking into consideration the time-dependent nature of the physical problem. Upon reformulation in the time domain, SBFEM is used to investigate the wave interaction with one single pile and a group of two piles. Parametric analysis is also presented for insightful comprehension of the time-dependent wave structure interaction.

6.1 Problem formulation

The physical problem can be represented by Figure 5-1, in which a group of pile foundations are engaged in ocean environment and are subject to ocean wave loads, which are redefined as time-dependent. The wave field behaviour possesses the same mathematical formulation as shown in Section 5.1.1, activating the time-dependent term $e^{i\omega t}$. The structural behaviour of the pile foundations follows the elasto-dynamic

differential equation presented in Section 3.1.2. It is duplicated here for ease of reference.

\[
\begin{align*}
[L]^T \{\sigma\} + \omega^2 \rho_p \{u\} &= 0 \\
\{\sigma\} &= [D]\{\varepsilon\} \\
\{\varepsilon\} &= [L]\{u\}
\end{align*}
\]

(6.1)

with \(\rho_p\) specifically denoting the mass density of pile foundations. Equation (6.1) is solved with the boundary condition that the displacement of piles is zero at the seabed level, and the piles are subjected to time-dependent wave forces.

### 6.2 SBFEM model and verification

#### 6.2.1 Time-dependent SBFEM model

The SBFEM formulation of the wave field follows that detailed in Section 5.2.1. As indicated in Equation (5.5), the time-dependent property can be readily incorporated into the wave field behaviour by analytically attaching the time-dependent term \(e^{i\omega t}\) to the wave field solution. The formulation of the dynamic structural analysis and the corresponding solution routine differ from those of the static analysis, and require more advanced techniques.

The SBFEM equation expressed in nodal displacement \(\{u(\xi)\}\), corresponding to Equation (6.1) is written as:

\[
\begin{align*}
[E^0]\xi^2\{u(\xi)\}_{xx} + \left(2[E^0]+[E^1]^T-[E^1]\right)\xi\{u(\xi)\}_x + \left([E^1]^T-[E^2]\right)\{u(\xi)\} \\
+\omega^2\left[M^0\right]\xi^2\{u(\xi)\} = 0
\end{align*}
\]

(6.2)

It is transformed into:

\[
\begin{align*}
\left([S(\omega)]-\left[E^1\right]\right)\left[E^0\right]^{-1}\left([S(\omega)]-\left[E^1\right]^T\right)-\left[E^2\right]+[S(\omega)] \\
+\omega[S(\omega)]_{\omega} + \omega^2[M^0] = 0
\end{align*}
\]

(6.3)

from which the dynamic stiffness matrix \([S(\omega)]\) is solved, and substituted into the nodal force \(\{R\}\)-nodal displacement \(\{u\}\) relationship formulated on the discretised boundary:

\[
\{R\}=[S(\omega)]\{u\}
\]

(6.4)
to obtain nodal DOFs \{u\}. Equation (6.3) is a non-linear first-order matrix-form Riccati differential equation. Solving the dynamic stiffness matrix \([S(\omega)]\) from it is not straightforward. One method is to postulate the formation of \([S(\omega)]\) using the stiffness matrix \([K]\) and the mass matrix \([M]\) as:

\[
[S(\omega)] \approx [K] - \omega^2 [M] + [\Theta]
\]  

(6.5)

A series of unknown coefficient matrices in \([\Theta]\) are to be determined according to the methodology adopted for the approximation (Bazyar and Song, 2008; Song, 2009; Song and Bazyar, 2007; Prempramote et al., 2009; Birk et al., 2011). Song (2009) employed the continued fraction technique and formulated the dynamic stiffness matrix as:

\[
[S(x)] = [K] + x[M] - x^2 [S^{(1)}(x)]^{-1}
\]  

(6.6)

with \(x = -\omega^2\), and \(M_{ij}\) represents the order of continued fraction.

In order to determine the static stiffness matrix \([K]\), the mass matrix \([M]\) and the high-order term \([S^{(1)}(x)]\), Equation (6.6) is substituted into Equation (6.3), leading to:

\[
\]  

\[
+ x^2 \left( [M] [E^0]^{-1} [M] - ([K] - [E^1]) [E^0]^{-1} [S^{(1)}(x)]^{-1} - [S^{(1)}(x)]^{-1} [E^0]^{-1} ([K] - [E^1]^T) \right)
\]  

\[
+ 2x \left( [S^{(1)}(x)]^{-1} \right) + x^2 \left( [S^{(1)}(x)]^{-1} [E^0]^{-1} [S^{(1)}(x)]^{-1} \right)
\]  

\[
= 0
\]  

(6.7)

The terms in Equation (6.7) are arranged in an ascending order of the power of \(x\), which results in a constant term, a linear term and a high-order term. The coefficient matrix of each term should be zero so that Equation (6.7) can be satisfied for any arbitrary \(x\). The constant term leads to an algebraic Riccati equation:
(6.8)\
\begin{align*}
\begin{bmatrix} K & -E^1 \\ -E^1 & 0 & 1 & 2 & \end{bmatrix} \begin{bmatrix} E^0 \\ E^1 \end{bmatrix} + \begin{bmatrix} K & -E^1 \\ -E^1 & 0 & 1 & 2 & \end{bmatrix} \begin{bmatrix} E^0 \\ E^1 \end{bmatrix} + K = 0
\end{align*}

from which the static stiffness matrix $[K]$ is determined as:

$$[K] = [V_{21}][V_{11}]^{-1}$$  

(6.9)

with $[V_{21}]$ and $[V_{11}]$ formulated from the real Schur decomposition of a Hamiltonian matrix:

$$\begin{bmatrix} E^0 & -0.5I & -E^0 \\ -E^2 & +E^1 & E^0 & -E^1 & E^0 & +0.5I \\ \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \\ V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} S_{11} \\ S_{12} \\ 0 \\ S_{22} \end{bmatrix}$$  

(6.10)

The linear term in Equation (6.7) yields a Lyapunov equation from which the mass matrix $[M]$ can be obtained:

$$\begin{bmatrix} K & -E^1 \\ -E^1 & 0 & 1 & 2 & \end{bmatrix} \begin{bmatrix} M \\ E^0 \end{bmatrix} + \begin{bmatrix} K & -E^1 \\ -E^1 & 0 & 1 & 2 & \end{bmatrix} \begin{bmatrix} M \\ E^0 \end{bmatrix} + 3[M] - [M^0] = 0$$  

(6.11)

In Equation (6.6), $[S^{(i)}(x)]$ is generalised for the $i$th order continued fraction as:

$$\begin{bmatrix} S^{(i)}(x) \end{bmatrix} = \begin{bmatrix} S_0^{(i)} \\ x[ S_1^{(i)} ] - x^2 \begin{bmatrix} S^{(i+1)}(x) \end{bmatrix}^{-1} \end{bmatrix}$$  

(6.12)

in which the coefficient matrices $[S_0^{(i)}]$ and $[S_1^{(i)}]$ can be determined from the high-order term in Equation (6.7) following the subsequent recursive procedure. First, a group of coefficient matrices are initialized, as shown in Equation (6.13), after performing the real Schur decomposition (Equation (6.10)) and obtaining the static stiffness matrix $[K]$ and the mass matrix $[M]$.

$$\begin{bmatrix} a^{(i)} \end{bmatrix} = [E^0]^{-1}$$

$$\begin{bmatrix} V^{(i)} \end{bmatrix} = [V_{11}]$$

$$\begin{bmatrix} U^{(i)} \end{bmatrix} = 2[I] - [S_{11}]$$

$$\begin{bmatrix} b_0^{(i)} \end{bmatrix} = [V^{(i)}][U^{(i)}][V^{(i)}]^{-1}$$

$$\begin{bmatrix} b_1^{(i)} \end{bmatrix} = [E^0]^{-1}[M]$$

$$\begin{bmatrix} c^{(i)} \end{bmatrix} = [M][E^0]^{-1}[M]$$

(6.13)
Subsequently, they are employed to calculate the intermediate coefficient matrix \( Y_0^{(i)} \) from a Lyapunov equation:

\[
\begin{bmatrix} Y_0^{(i)} \end{bmatrix} \begin{bmatrix} U^{(i)} \end{bmatrix} + \begin{bmatrix} U^{(i)} \end{bmatrix}^T \begin{bmatrix} Y_0^{(i)} \end{bmatrix} = \begin{bmatrix} V^{(i)} \end{bmatrix}^T \begin{bmatrix} c^{(i)} \end{bmatrix} \begin{bmatrix} V^{(i)} \end{bmatrix}
\]

(6.14)

before \( \begin{bmatrix} S_0^{(i)} \end{bmatrix} \) is solved from:

\[
\begin{bmatrix} S_0^{(i)} \end{bmatrix}^{-1} = \begin{bmatrix} V^{(i)} \end{bmatrix}^T \begin{bmatrix} Y_0^{(i)} \end{bmatrix} \begin{bmatrix} V^{(i)} \end{bmatrix}^{-1}
\]

(6.15)

Correspondingly, \( \begin{bmatrix} Y_1^{(i)} \end{bmatrix} \) is calculated from Equation (6.16) to formulate \( \begin{bmatrix} S_1^{(i)} \end{bmatrix} \) in Equation (6.17):

\[
\left( \begin{bmatrix} I \end{bmatrix} + \begin{bmatrix} U^{(i)} \end{bmatrix}^T \right) \begin{bmatrix} Y_1^{(i)} \end{bmatrix} + \begin{bmatrix} Y_1^{(i)} \end{bmatrix} \left( \begin{bmatrix} I \end{bmatrix} + \begin{bmatrix} U^{(i)} \end{bmatrix} \right) = \begin{bmatrix} V^{(i+1)} \end{bmatrix}^T \left( \begin{bmatrix} b_0^{(i)} \end{bmatrix} \begin{bmatrix} S_0^{(i)} \end{bmatrix} + \begin{bmatrix} S_0^{(i)} \end{bmatrix} \begin{bmatrix} b_1^{(i)} \end{bmatrix} \right) \begin{bmatrix} V^{(i+1)} \end{bmatrix}
\]

\[
\begin{bmatrix} S_1^{(i)} \end{bmatrix} = \begin{bmatrix} V^{(i+1)} \end{bmatrix}^T \begin{bmatrix} Y_1^{(i)} \end{bmatrix} \begin{bmatrix} V^{(i+1)} \end{bmatrix}^{-1}
\]

(6.16)

(6.17)

The coefficient matrices in Equation (6.13) are then updated for \( i > 1 \) using the following recursive relationships.

\[
\begin{bmatrix} a^{(i+1)} \end{bmatrix} = \begin{bmatrix} c^{(i)} \end{bmatrix}
\]

\[
\begin{bmatrix} b_0^{(i+1)} \end{bmatrix} = \begin{bmatrix} V^{(i+1)} \end{bmatrix} \begin{bmatrix} U^{(i+1)} \end{bmatrix} \begin{bmatrix} V^{(i+1)} \end{bmatrix}^{-1}
\]

\[
\begin{bmatrix} b_1^{(i+1)} \end{bmatrix} = -\begin{bmatrix} b_1^{(i)} \end{bmatrix} + \begin{bmatrix} c^{(i)} \end{bmatrix} \begin{bmatrix} S_1^{(i)} \end{bmatrix}
\]

\[
\begin{bmatrix} c^{(i+1)} \end{bmatrix} = \begin{bmatrix} a^{(i)} \end{bmatrix} - \begin{bmatrix} b_1^{(i)} \end{bmatrix} \begin{bmatrix} S_1^{(i)} \end{bmatrix} - \begin{bmatrix} S_1^{(i)} \end{bmatrix} \begin{bmatrix} b_1^{(i)} \end{bmatrix} + \begin{bmatrix} S_1^{(i)} \end{bmatrix} \begin{bmatrix} c^{(i)} \end{bmatrix} \begin{bmatrix} S_1^{(i)} \end{bmatrix}
\]

(6.18)

Equations (6.14) - (6.18) will be processed repeatedly until the specified continued fraction order \( M_{cf} \) is achieved.

Substituting Equation (6.6) into Equation (6.4) yields the following expression:

\[
\{ R \} = [K]\{ u \} + x[M]\{ u \} - x\{ u^{(1)} \}
\]

(6.19)

with
\[ x^{(i)} \{ u \} = x^2 \left[ S^{(i)}(x) \right]^{-1} \{ u \} \tag{6.20} \]

which can be reformulated as:

\[ x \{ u \} = \left[ S^{(i)}(x) \right] \{ u^{(i)} \} \tag{6.21} \]

and generalised for the \( i \)th order continued fraction as:

\[ x^{(i)} \{ u^{(i)} \} = \left[ S^{(i)}(x) \right] \{ u^{(i)} \} \quad (i \geq 1) \tag{6.22} \]

where \( \{ u^{(i)} \} \ (i \geq 1) \) are the auxiliary displacements introduced by the order of the continued fraction. Substituting Equation (6.12) into Equation (6.22), and using Equation (6.20) lead to:

\[ x^{(i)} \{ u^{(i)} \} = \left[ S^{(i)}_0 \right] \{ u^{(i)} \} + x \left[ S^{(i)}_1 \right] \{ u^{(i)} \} - x \{ u^{(i+1)} \} \quad (i \geq 1) \tag{6.23} \]

Rearrange Equation (6.19) and Equation (6.23) into similar form as:

\[ [K] \{ u \} + x[M] \{ u \} - x[I] \{ u^{(i)} \} = \{ R \} \]
\[ \left[ S^{(i)}_0 \right] \{ u^{(i)} \} - x[I] \{ u^{(i-1)} \} + x \left[ S^{(i)}_1 \right] \{ u^{(i)} \} - x[I] \{ u^{(i+1)} \} = 0 \tag{6.24} \]

Notice that the first equation in Equation (6.24) corresponds to \( i = 0 \) and \( \{ u \} \) represents the structural DOFs. The second equation corresponds to \( i \geq 1 \) and \( \{ u^{(i)} \} \) stands for the introduced auxiliary DOFs from the continued fraction. These two equations can be combined and written in matrix form algebraic equations as:

\[ \left( [K] - \omega^2 [M] \right) \{ y \} = \{ f \} \tag{6.25} \]

with

\[ [K] = \text{diag} \left( [K] \left[ S^{(1)}_0 \right] [S^{(2)}_0] \ldots [S^{(M)}_0] \right) \]
\[ [M] = \left[ \begin{array}{cccc}
[M] & -[I] & 0 & \ldots & 0 \\
-I \left[ S^{(1)}_0 \right] & -[I] & \ldots & 0 \\
0 & -[I] \left[ S^{(2)}_0 \right] & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & [S^{(M)}_0] \\
\end{array} \right] \tag{6.26} \]

\[ [M_s] = \left[ \begin{array}{cccc}
[M] & -[I] & 0 & \ldots & 0 \\
-I \left[ S^{(1)}_1 \right] & -[I] & \ldots & 0 \\
0 & -[I] \left[ S^{(2)}_1 \right] & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & [S^{(M)}_1] \\
\end{array} \right] \tag{6.27} \]
\[
\{y\} = \{u\} \quad \{u^{(1)}\} \quad \{u^{(2)}\} \quad \ldots \quad \{u^{(M)}\}^T
\]  
(6.28)

and

\[
\{f\} = \{R\} \quad 0 \quad 0 \quad \ldots \quad 0^T
\]  
(6.29)

By the inverse Fourier transformation, Equation (6.25) is rewritten in the time domain as:

\[
[K_s]\{y\} + [M_s]\{y\} = \{f\}
\]  
(6.30)

In cases where subdomain division is applicable, the coefficient matrices \([K_s]\) and \([M_s]\) listed in Equations (6.26) and (6.27) are calculated for each subdomain. They are assembled into global matrices \([K_G]\) and \([M_G]\) in such a way that the entries corresponding to structural DOFs follow the assemblage concept in FEM; those corresponding to the auxiliary DOFs are allocated afterwards into the global matrices in sequential order. It is worth mentioning that the number of DOFs involved in the global equation of motion Equation (6.31) is larger than the number of structural DOFs due to the employment of the continued fraction technique, and the higher the order of continued fraction, the greater the number of DOFs, accordingly, the more intensive the computational demand. Therefore, an appropriate continued fraction order is suggested to avoid the computational memory issue.

\[
[K_G]\{y\} + [M_G]\{\ddot{y}\} = \{f\}
\]  
(6.31)

Equation (6.31) represents an undamped forced vibration. It can be employed to examine the natural frequencies of the pile foundation if the external excitation \(\{f\}\) equals 0. For the structural behaviour analysis defined in this study, the material damping effect has to be taken into consideration. This can be implemented by adding the damping force into Equation (6.31) as (the subscript ‘G’ is dropped for conciseness):

\[
[M]\{\ddot{y}\} + [C]\{\dot{y}\} + [K]\{y\} = \{f\}
\]  
(6.32)

using the damping matrix \([C]\), which can be formulated by the linear combination of the stiffness matrix \([K]\) and the mass matrix \([M]\) as:

\[
[C] = \alpha_d [M] + \beta_d [K]
\]  
(6.33)
\( \alpha_d \) and \( \beta_d \) are two frequency-independent coefficients. They are calculated as:

\[
\alpha_d = 2r_{md}\omega_1\omega_2/(\omega_1 + \omega_2) \\
\beta_d = 2r_{md}/(\omega_1 + \omega_2)
\]

(6.34)

where \( r_{md} \) is the material damping ratio. \( \omega_1 \) and \( \omega_2 \) are natural circular frequencies corresponding to two orthogonal modal shapes in the modal analysis of the pile foundation.

It should be emphasised that the calculation is based on the non-dimensionalised model. Utilising the reference variables listed in Table 3-2, the displacement-related variables and the external force vector in Equation (6.32) are non-dimensionalised as:

\[
\{y\}^* = \{y\}/r_\text{E} \\
\{\dot{y}\}^* = \{\dot{y}\}/\sqrt{E_r/\rho_r} \\
\{\ddot{y}\}^* = \{\ddot{y}\}/\rho_r r_\text{E} \\
\{f\}^* = \{f\}/E_r r_\text{E}^2
\]

(6.35)

Rearrange Equation (6.35) and Equation (3.37) in terms of the dimensional variables and substitute into Equation (6.32), a constant term \( E_r r_\text{E}^2 \) appears for each term throughout the equation and can be cancelled out. Therefore, the dimensionless form of Equation (6.32) stays the original format. It holds the traditional format of the dynamic equation of motion as a matrix-form second-order ODE in terms of the nodal displacement history. It is solved in the time domain using the Newmark integral technique. Once the time history of the nodal displacement is obtained, the time-dependent behaviour of the pile foundation is sought.

### 6.2.2 Model verification

The solution process combining the wave formulation and the structural formulation follows the details provided in Section 5.2.3, the procedure of which has been verified in Section 5.2.4. The transient analysis using an \( L \)-shaped panel illustrated in Section 3.3.3.3 serves as a qualified benchmark of the SBFEM model for dynamic structural analysis. The following discussions are diverted to the investigation of wave-pile foundation interaction using the time-dependent model.
6.3 Wave interaction with monopile foundation

The time-dependent wave interaction with a monopile foundation is examined first. Figure 6-1 describes the geometric model, and relevant parameters employed in the analysis are listed in Table 6-1.

![Figure 6-1](image)

Figure 6-1. Illustration of wave interaction with a monopile foundation (a) $xz$ plane view and (b) $xy$ plane view of the pile head

A representative plot of the time-dependent analysis result is illustrated in Figure 6-2 in a relatively defined time frame, showing the transient response of point $O'$ (see Figure 6-1 (b)), located at the centre of the monopile head, to the dynamic wave loads. The time variable $t$ is assumed to be zero at the instant the pile starts to experience wave loads. 50 periods of the pile motion are calculated and plotted. Employing the parameters listed in Table 6-1, it takes approximately ten periods for the mechanical energy associated with the pile’s free motion to be dissipated. The pile thereafter arrives at a steady state vibration, and moves at the same frequency as waves. The following discussions are concentrated on the steady state vibration considering the real ocean situation.
Table 6-1. Parameters of time-dependent wave interaction with a monopile foundation

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Notations</th>
<th>Magnitudes</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pile radius</td>
<td>$a$</td>
<td>0.5</td>
<td>m</td>
</tr>
<tr>
<td>Pile height</td>
<td>$h$</td>
<td>10</td>
<td>m</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>$E$</td>
<td>$2.8 \times 10^{10}$</td>
<td>Pa</td>
</tr>
<tr>
<td>Mass density</td>
<td>$\rho_p$</td>
<td>2400</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>Damping ratio</td>
<td>$r_{md}$</td>
<td>0.05</td>
<td>-</td>
</tr>
<tr>
<td>Possion’s ratio</td>
<td>$\nu$</td>
<td>0.25</td>
<td>-</td>
</tr>
</tbody>
</table>

Pile parameters

| Water depth                         | $d$       | 7.5           | m     |
| Incident wave angle                 | $\alpha$  | 0             | rad   |
| Wave number                         | $k$       | 0.10          | m$^{-1}$ |
| Water density                       | $\rho_w$  | 1000          | kg/m$^3$ |
| Wave amplitude                      | $A$       | 0.5           | m     |

Wave parameters

| Time steps                          | $N$       | 800           | -     |
| Time interval                       | $\Delta t$| 0.4977        | s     |

Temporal variables

| Natural circular frequencies        | $\omega_1$| 29.966        | rad/s |
|                                    | $\omega_2$| 29.966        | rad/s |

| Gravitational acceleration         | $g$       | 9.81          | m/s$^2$ |

Figure 6-3 (a) plots the displacement of two periods for five observation points located at different height levels along $L-L'$ where $z = 2$ m, 4 m, 6 m, 8 m and 10 m (see Figure 6-1 (a)). It is observed that the five points move at the same frequency with different amplitudes. The further away the point is from the seabed level where the pile is fixed, the greater the amplitude of the movement. This feature can be more visually recognised in Figure 6-3 (b), in which the motion of the pile for one complete period, represented by the movement of $L-L'$ is depicted. The four curves labelled as 1, 2, 3 and 4 match the four time instants marked by corresponding numbers in Figure 6-3 (a). The pile starts off from its equilibrium position, denoted by curve 1, moves towards positive $x$ direction and reaches its maximum coverage marked by curve 2 at $t$.
= T/4 (T denotes the period of the pile motion), then moves back to curve 3 and
continues to swing to curve 4 to achieve the maximum on the other side, and
eventually returns to its initial status upon completion of a cycle when t = T. The pile
swings back and forth between curve 2 and curve 4 under the action of the wave loads.
Curve 1 or 3 denotes the equilibrium position of the pile in its cyclic movement. It
does not coincide exactly with the pile’s neutral still status due to the effect of the
hydrostatic pressure. The hydrostatic pressure is theoretically time-independent, yet it
does vary with time in this situation when the sea surface fluctuates, rendering

![Graph](image)

(a)

![Graph](image)

(b)

Figure 6-2. Transient response of point O' to the dynamic wave loads (a) in the x direction
and (b) in the y direction
the water depth changes constantly. The hydrostatic pressure also accounts for the negative displacement appearing at the bottom of the pile in that it acts normally to the pile circumference and reaches its maximum at the seabed level when it linearly increases with the water depth.

Figure 6-3. Dynamic behaviour of the monopile foundation (a) Displacement versus time for representative locations along $L-L'$ and (b) Monopile motion description for one period

Figure 6-4. Displacement versus time for representative locations on the monopile head
Figure 6-4 reflects the displacement of five specific locations at the pile head level (shown in Figure 6-1 (b)). Four of them are located on the pile circumference at $\theta = 0, \pi/2, \pi$ and $-\pi/2$, and the fifth is selected at the centre of the pile head. They present exactly the same motion pattern with identical frequency and amplitude. Considering Figure 6-3 (b) and Figure 6-4, the time-dependent response of pile foundations will be addressed by examining the displacement of point $O'$ for ease of discussion.

### 6.4 Wave interaction with pile group foundation

A group of two piles is considered in this section to explore the time-dependent wave interaction with pile group foundations. Unlike that with a single pile foundation, wave interaction with pile groups demonstrates different behaviour under different incident wave angles. The relative orientation between the incident wave direction and the structural layout of the piles significantly affects $|\eta_0|/A$ distribution around pile circumferences, and the behaviour of the piles accordingly. Thus, employing the incident wave angles specified in Table 5-1, this section is organised into three parts, corresponding to $\alpha = 0, \pi/4$ and $\pi/2$, respectively. Each case represents a situation in which the incident wave direction is in alignment with, oblique with, and orthogonal to the connecting line of the two piles. Parametric analyses are concurrently carried out in each section to present the varying time-dependent response of the piles when subject to changing wave conditions. Moreover, the variation of one parameter affects the influence of the variation of other parameters on the transient response of the pile foundations is also worth exploring.

#### 6.4.1 Incident wave angle $\alpha = 0$

The incident wave angle $\alpha = 0$ leads to the symmetry of the piles’ behaviour with respect to the $x$ axis, and accordingly zero displacement components in the $y$ direction. Therefore, only the displacements in the $x$ direction are plotted in Figure 6-5 for both piles. They correspond to situations with different $e$ varying from 1 to 5 by an increment of 1 when the wave number $k = 0.10$ m$^{-1}$. It is noticed that P1 and P2 have different response to wave loads when the pile distance between them varies. Generally, as discussed in Chapter 5, the displacement of P1 is comparatively greater than that of P2 for identical $e$. For P1, the maximum displacement in the positive $x$ direction is 0.0070 mm, 0.0075 mm, 0.0080 mm, 0.0087 mm and 0.0097 mm when $e$ increases from 1 to 5 at an increment of 1. At the same time instant, the displacement
for P2 is read as -0.0018 mm, -3.13 x 10^{-4} mm, 4.77 x 10^{-4} mm, 9.65 x 10^{-4} mm and 0.0013 mm, with the negative sign indicating the negative x direction.

With \( \epsilon \) no more than 3, P1 and P2 move in opposite directions, i.e. using Figure 6-3 (b), when P1 follows the trajectory of 1→2→3→4→1, P2 moves in accordance with 1→4→3→2→1. This is related to the spatial proximity between the two piles. When the pile distance is relatively small, the resultant force acting on P2 is in the opposite direction to that on P1, and the two piles are moving one moment towards each other and the next moment away from each other. As the net distance becomes larger, the effect of P1’s presence on the behaviour of the waves around P2 is not as significant, and the resultant force acting on P2 is in the same direction as that on P1. Consequently, both piles move in the same direction.

Figure 6-5. Displacement versus time when \( k = 0.10 \text{ m}^{-1} \) and \( \alpha = 0 \) for varying \( \epsilon \): (a) P1 and (b) P2

Figure 6-6 is used to explore the variation of the time-dependent response of the two piles with respect to the wave number when \( \epsilon = 3 \). The two subplots resemble each other except for the displacement magnitudes. Apparently, both piles maintain consistent movement period as the external waves. Hence, the period of the pile motion changes from 14.994 s, 10.264 s, 7.964 s, 6.6259 s to 5.761 s as the wave number varies from 0.05 m^{-1} to 0.15 m^{-1}. Once again, the displacement amplitude of
P1 is greater than that of P2 at the same time instant. The displacement amplitude is read as 0.0015 mm, 0.0039 mm, 0.0080 mm, 0.0137 mm, and 0.0210 mm for P1 and -2.96×10⁻⁴ mm, -1.73×10⁻⁴ mm, 4.7699×10⁻⁴ mm, 0.0018 mm and 0.0041 mm for P2 as $k$ increases from 0.05 m⁻¹ to 0.15 m⁻¹ at a 0.025 m⁻¹ increment.

![Graphs showing displacement versus time](image)

Figure 6-6. Displacement versus time when $e = 3$ and $\alpha = 0$ for varying $k$: (a) P1 and (b) P2

Similar to that discussed in Chapter 5, the time dependent behaviour of the piles is not solely related to either $e$ or $k$. On the contrary, the piles have different performance for different $e$ and $k$ combinations. More specifically, for any $k$ except 0.10 m⁻¹, Figure 6-5 will present a dissimilar pattern of the displacement variation with respect to $e$. Analogously, when $e$ is not equal to 3, Figure 6-6 will show another relationship between the transient response of the pile and the wave number. In order to provide a clear overview of this phenomenon, Figure 6-7 is drawn in terms of the variation of the maximum displacement in the positive $x$ direction with respect to the two parameters. This actually exhibits the same variation pattern as that presented in Figure 5-18, in which the maximum wave pressure is adopted to calculate the pile displacement, and the maximum displacement variation is drawn with respect to $k$ for varying $e$. Instead, the displacement variation is plotted versus $e$ for any certain $k$ in Figure 6-7. It is found that P1 and P2 present rather different variation pattern: (1) the displacement amplitude of P1 is bigger in comparison with that of P2 for same cases; (2) displacement decreases with $e$ for any fixed $k$ for P1. As for P2, by reading the lower part of Figure 6-7 (b), it is found that in situations with small wave numbers or
when P2 is closely located to P1, an opposite movement direction to that of P1 is observed, and the magnitude decreases as the value of $e$ goes up. The upper part of the plot shows an increasing displacement amplitude with respect to $e$, also the same motion direction as that of P1; (3) viewing any particular symbol to examine how the displacement amplitude changes with respect to $k$ for a certain $e$, it can be concluded that the displacement of P1 shows a consistent increase with $k$. As far as P2 is concerned, the displacement amplitude first rises and later falls with increasing $k$ for rather small $e$. For relatively large $e$, it increases steadily when the value of $k$ becomes larger. Whether the displacement amplitude is increasing or decreasing when $e$ varies, both subplots present a rather moderate slope of variation for cases with small wave numbers, and relatively prominent slope when the wave number is large.

![Figure 6-7. Displacement versus $e$ for varying $k$ when $\alpha = 0$ for: (a) P1 and (b) P2](image)

### 6.4.2 Incident wave angle $\alpha = \pi/4$

When the incident wave direction is oblique to the connecting line between the two piles at an angle of $\pi/4$, the pile’s behaviour is no longer symmetric. Therefore, displacements in both $x$ and $y$ directions are examined. Figure 6-8 offers general knowledge as to how piles’ transient behaviour vary with $e$ for $k = 0.10$ m$^{-1}$. The displacements in the $x$ direction of the two piles resemble those for $\alpha = 0$ in Figure 6-5, other than the magnitude. For $\alpha = \pi/4$ and $k = 0.10$ m$^{-1}$, the maximum displacement of P1 in the positive $x$ direction decreases from 0.0076 mm, 0.0070 mm, 0.0064 mm, 0.0060 mm to 0.0057 mm when $e$ increases. The corresponding readings
of P2 are -0.0019 mm, -9.71×10^{-4} mm, -3.28×10^{-4} mm, 1.0919×10^{-4} mm and 4.2057×10^{-4} mm, respectively. The negative signs associated with small \( e \) indicate opposite motion directions of P2 to P1. The accompanying plots of the \( y \) displacement show opposite directions of motion of the two piles for any \( e \). The displacement magnitudes are found to be 0.0131 mm, 0.0110 mm, 0.0097 mm, 0.0087 mm and 0.0080 mm for P1 and -0.0050 mm, -0.0037 mm, -0.0027 mm, -0.0019 mm and -0.0014 mm for P2 at the same time instant. Regardless of the direction of motion, the displacement amplitude of both piles decreases as \( e \) increases.

Figure 6-8. Displacement versus time when \( k = 0.10 \text{ m}^{-1} \) and \( \alpha = \pi/4 \) for varying \( e \): (a) \( u_x \) of P1; (b) \( u_x \) of P2; (c) \( u_y \) of P1 and (d) \( u_y \) of P2
Figure 6-9 suggests the fluctuation of displacement with respect to time for various $k$ when $\epsilon = 3$. For P1, the displacement components in the $x$ and $y$ directions are similar to each other, apart from the amplitude, which are 0.0012 mm, 0.0032 mm, 0.0064 mm, 0.0110 mm and 0.0168 mm in the $x$ direction and 0.0020 mm, 0.0050 mm, 0.0097 mm, 0.0161 mm and 0.0244 mm in the $y$ direction. They consistently increase as the value of $k$ rises. The response of P2, shown in Figure 6-9 (b) and (d), presents a slightly complex varying pattern as $k$ changes. The pile first moves towards the negative $x$ direction when $k$ is $0.05 \text{ m}^{-1}$, $0.075 \text{ m}^{-1}$ and $0.10 \text{ m}^{-1}$. When $k$ is greater than $0.10 \text{ m}^{-1}$, P2 moves in the same direction as P1, with relatively smaller amplitude.

Figure 6-9. Displacement versus time when $\epsilon = 3$ and $\alpha = \pi/4$ for varying $k$: (a) $u_x$ of P1; (b) $u_x$ of P2; (c) $u_y$ of P1 and (d) $u_y$ of P2
Examining the displacement in the $y$ direction, it is found that P2 moves in the opposite direction to P1 regardless of the wave number. As for the amplitude, it is read as 0.0011 mm, 0.0020 mm, 0.0027 mm, 0.0030 mm and 0.0029 mm for $k = 0.05$ m$^{-1}$, 0.075 m$^{-1}$, 0.10 m$^{-1}$, 0.125 m$^{-1}$ and 0.15 m$^{-1}$ respectively.

Figure 6-10 is drawn to assist the understanding of the displacement variation with respect to the two parameters. P1 presents a rather simple reduction in the amplitude as $e$ increases for each individual $k$ in both $x$ and $y$ directions (Figure 6-10 (a) and (c)). The reduction is more obvious when the wave number is greater than that of smaller ones. The amplitudes in both $x$ and $y$ directions increase with $k$ for any specific $e$, which is recognised by viewing the same markers in the subplots, and the increase becomes more remarkable as $k$ becomes greater. The displacement variation of P2 is slightly complex. First of all, Figure 6-10 (b) provides similar information to the corresponding plot in Figure 6-7 (b) for $\alpha = 0$. The pile moves oppositely to its counterpart P1 when the wave number is small or when the two piles are rather closely spaced. When P2 is located further away from P1 and under the condition of great wave numbers, it tends to move in company with P1 in the same direction, however, with a smaller amplitude. The plot in the $y$ direction (Figure 6-10 (d)), on the other hand, shows that whatever the combination of $e$ and $k$ is, P2 always has opposite motion direction to that of P1. The amplitude of the motion decreases with respect to $e$ for any specific $k$, and the decrease becomes more significant when $k$ is large. By fixing $e$ and examining the variation of the displacement magnitude with respect to $k$, it is shown that when $e$ is less than 3, the displacement magnitude monotonically increases with $k$ and the increase becomes less significant as the value of $k$ rises. When $e$ continues to increase to 3, 4 and 5, the displacement magnitude increases first and then decreases. The complex variation feature of P2’s displacement once again proves that both wave properties and the spatial layout of pile members affect pile group’s response, which should be taken into consideration in the foundation design and safety evaluation processes. In addition, the leading pile, which interacts with external waves first, normally experiences much greater resultant wave force, and accordingly shows greater displacement amplitude than subsequent piles. The behaviour of subsequent piles is more vulnerable to parameter variations.
Figure 6-10. Displacement versus $\epsilon$ for varying $k$ when $\alpha = \pi/4$ for: (a) maximum $u_x$ for P1; (b) maximum $u_x$ for P2; (c) maximum $u_y$ for P1 and (d) maximum $u_y$ for P2

6.4.3 Incident wave angle $\alpha = \pi/2$

When the incident wave direction is orthogonal to the connecting line between the two piles, the physical configuration is symmetrical with respect to the $y$ axis. Therefore, both piles possess the same displacement amplitude but opposite motion directions along the $x$ axis. In the $y$ direction, they hold identical motion pattern in both magnitudes and directions. This is demonstrated in Figure 6-11, which shows the displacements of both piles when $k = 0.10 \text{ m}^{-1}$ and $\alpha = \pi/2$ for varying $\epsilon$. 
Figure 6-11. Displacement versus time when $k = 0.10 \text{ m}^{-1}$ and $\alpha = \pi/2$ for varying $e$: (a) $u_x$ of P1; (b) $u_x$ of P2; (c) $u_y$ of P1 and (d) $u_y$ of P2.

Figure 6-12 depicts the displacement history in both $x$ and $y$ directions of P1 for varying $k$ when $e = 3$. Subplots (a) and (b) present similar variation pattern of the displacement with the amplitudes in the $x$ direction being $7.4447 \times 10^{-4}$ mm, 0.0017 mm, 0.0029 mm, 0.0045 mm and 0.0065 mm, and the corresponding readings in the $y$ direction are $6.66 \times 10^{-4}$ mm, 0.0022 mm, 0.0050 mm, 0.0095 mm and 0.0159 mm for every increasing $k$. 
Figure 6-12. Displacement versus time when \( e = 3 \) and \( \alpha = \pi/2 \) for P1 for varying \( k \): (a) in the \( x \) direction and (b) in the \( y \) direction.

Figure 6-13 shows the variation of displacement amplitude with respect to \( e \) for varying \( k \). Both \( x \) and \( y \) displacement amplitudes decrease when \( e \) increases from 1 to 5 with an increment of 1, and the decrease is more distinct when the wave number is greater. In addition, it is presented that both displacement amplitudes increase with \( k \), and the smaller the net distance between the two piles, the quicker the displacement amplitude increases.

Figure 6-13. Displacement versus \( e \) of P1 for varying \( k \) when \( \alpha = \pi/2 \): (a) in the \( x \) direction and (b) in the \( y \) direction.
6.5 Summary

Forwarding the study presented in Chapter 5, this chapter promotes the investigation of wave-structure interaction problems to the time-dependent context using SBFEM. The wave field analysis remains the same formulation as that detailed in Chapter 5, and is straightforwardly attached an analytical time-related expression to incorporate the time-dependent nature. The structural analysis, however, requires additional consideration, and starts over from the elasto-dynamic differential equation. The transformed scaled boundary finite element equation, in terms of the dynamic stiffness matrix, is a non-linear first-order matrix-form Riccati differential equation, the solving of which is not analytically feasible. The continued fraction formulation is adopted to approximate the dynamic stiffness matrix by employing an optimally chosen continued fraction order. Substituting the dynamic stiffness matrix into the nodal force history-nodal displacement history relationship on the boundary, and taking the material damping effect into consideration, the scaled boundary dynamic equation of motion holds the traditional format, through which the nodal displacement history is solved using Newmark integral technique. Non-dimensionalisation technique is employed to ensure credible performance of the time-dependent model.

The model is successfully used to investigate the time-dependent behaviour of monopile foundations and pile group foundations. It is found that:

1) With the monopile foundation:
   a) Shown in a relatively defined time frame, under the effect of material damping, the pile comes to a steady state vibration shortly after the action of the external wave pressure, and moves coordinately with the waves at the same frequency. This is because the mechanical energy associated with structures’ free vibration is irreversibly transformed into the internal energy.
   b) The pile moves cyclically with respect to a central status. Different locations along the pile height demonstrate different displacement amplitudes, but the same frequency. The displacements of the five points at the monopile head exhibit both the same magnitude and frequency.

2) The parametric analyses carried out for a group of two piles under different incident wave angles show that:
   a) Symmetric problem configurations lead to symmetric pattern of the transient behaviour of the two piles, illustrated by cases when the incident wave
direction is in alignment with \((\alpha = 0)\) or orthogonal to \((\alpha = \pi/2)\) the connecting line of the two piles.

b) Different wave numbers correspond to different excitation frequency from the waves. Therefore, the piles exhibit different motion periods as the wave number varies. The bigger the wave number, the shorter the period.

c) The structural behaviour is a multivariable function of the wave properties and the geometric layout of the piles. The piles show different response to different parametric combinations. Emphasis is placed on two predominant parameters: the wave number \(k\) and the ratio \(e\) of the pile distance to the pile radius. The displacement amplitude of the leading pile presents a rather moderate decrease with \(e\) when the wave number is small, but the decrease is quite significant when the value of \(k\) is large. The displacement increases with \(k\) for any definite \(e\), and the increase is slightly more obvious in cases with a lower \(e\) than that with a higher one.

d) The displacement amplitude of the succeeding pile is not a monotonic function. It is susceptible to parametric variations, easily demonstrating complex variation patterns. Analogous to the results shown in Chapter 5, the succeeding pile always presents a smaller displacement amplitude than that of the leading pile. When the wave effect is moderate, the succeeding pile displaces mildly with respect to \(e\) for small wave numbers and shows substantial responses when the wave number is large.

e) In situations when the wave number is small, or when the two piles are closely spaced, the two piles move in opposite directions. Other situations result in the same motion direction of the two piles.

The above discussions provide a clear picture as to how individual piles in the pile group behave under the action of time-dependent wave pressure. At the same time, they offer illuminative information on the transient response of pile group foundations to varying wave conditions. The study presented in this chapter brings the advancement of SBFEM to wave-structure interaction problems, specified in this PhD project, to a satisfactory completion. The next chapter presents a systematic conclusion of the entire study, highlighting its practical and theoretical significance. In the meantime, a future research proposal regarding sustainable continuation of the current study is outlined.
Chapter 7 Conclusions and Future Research Plan

Upon completion of all the research objectives set in Section 1.2, this chapter highlights significant contributions of the study carried out in this PhD project in respect of: (1) the advancement of SBFEM in the investigation of the three-dimensional wave-structure interaction problem; and (2) the interaction mechanism between ocean waves and structures. In addition, an initial proposal regarding the extension of SBFEM in other recognised topics is outlined as a continuation of the current study. They are illustrated in the following three sections.

7.1 Research achievements

The research has advanced SBFEM in solving three-dimensional wave-structure interaction problems by successfully addressing the following items in succession:

1) Careful examinations have been placed on the numerical stability and accuracy of the SBFEM calculation. It is identified that the numerical instability and inaccuracy inherent in the calculation are related to matrix manipulation techniques as well as matrix properties. The real Schur decomposition and parametric non-dimensionalisation schemes are employed to overcome these difficulties, with a view to strengthening SBFEM’s credibility in subsequent studies.

2) Wave interaction with a monopile foundation is examined first as a convenient introduction to the investigation. Accessible wave parameters are employed, therefore, only the structural behaviour needs to be formulated in the SBFEM model.

3) Advancing with increasing complexity, the wave interaction with pile group foundations is discussed. The SBFEM model incorporates both the wave field and the structural domain, as no wave field solution is attainable to be directly applied to the structural analyses.
4) The investigation is further promoted to dynamic context by interpreting the time-dependent nature of the interaction problem. SBFEM assesses the wave field behaviour by attaching a time-dependent term to the corresponding steady state solution, yet reformulates a dynamic model for the structural analysis.

7.2 Significant contributions

7.2.1 Advancement of SBFEM

By addressing the above items, this project convincingly advances SBFEM to deal with wave-structure interaction problems in that:

1) It solves ordinary instead of partial differential equations, by performing the scaled boundary transformation, and therefore obtains analytical formulations of the nodal function in the radial direction, while keeping the solution in circumferential directions numerically accurate.

2) It investigates the interaction problem in three dimensions, however with released computational burden in terms of the memory requirement by only discretising the domain boundary. This significantly reduces the number of degrees of freedom associated with the three-dimensional problem, especially when an infinite wave domain is involved.

3) It combines, geometrically, an unbounded domain and a bounded domain, and physically, a scalar field with a vector field.

4) Though pile foundations with a circular cross-section are used throughout this study, the developed methodology is applicable to wave-structure interaction problems in a more general sense regardless of the structure’s configuration, quantity or layout, etc. With well-designed subdomain division, which is considered essential: (a) the Sommerfeld radiation condition at infinity of the wave field is exactly satisfied by incorporating the Hankel function in the solution formulation of the unbounded domain; (b) complex configurations and irregular boundaries of the geometric model of the interaction problem can be efficiently dealt with; (c) balanced geometric dimension of subdomains avoids possible numerical failure in the SBFEM solution.
5) It explores time-dependent behaviour of the structure through the traditional format of the dynamic equation of motion, in which standard algorithms, such as the Newmark integral technique is directly applicable to the solution process.

7.2.2 Exploration of the interaction mechanism

Throughout this study, the wave field behaviour is discussed by examining the normalised free surface elevation $|\eta_0|/A$, and the structural response is represented by lateral displacement components of pile foundations. The physical mechanism of the wave-structure interaction problem is found to involve:

1) The symmetry of the problem formulation, resulting from the spatial layout of piles and the relative orientation of incident waves and piles, affects the symmetry of the wave field behaviour and subsequently the structural response.

2) $|\eta_0|/A$ distribution reflects the wave pressure distribution. Small wave numbers, leading to uniform distribution of $|\eta_0|/A$, signify little pressure difference around pile circumferences. Therefore, the resultant wave force acting on piles is insignificant, and the pile displacement is moderate. Relatively great wave numbers introduce obvious $|\eta_0|/A$ difference on the upstream side and the lee side of the pile. Therefore, an elongated shape of $|\eta_0|/A$ distribution is formed in the incident wave direction, around which the maximum $|\eta_0|/A$ is normally located. The minimum $|\eta_0|/A$ is always found in the perpendicular direction. This $|\eta_0|/A$ distribution pattern results in significant pile displacement in the incident wave direction when compared to cases with small wave numbers.

3) By examining the displacement of different locations along the pile height, negative displacement is found on the lee side at the bottom of the pile. This is related to the hydrostatic pressure, which is perpendicular to the pile circumference and reaches its maximum at the pile bottom as it increases with the depth. The hydrostatic pressure also contributes more to the pile displacement than its dynamic counterpart, as it prevails from the free water surface to the seabed level, whereas the dynamic wave pressure only predominates at the sea surface, and decays quite rapidly towards the seabed. Equal displacement is obtained at the pile head level. Specifically, for time-
dependent analysis, locations on the pile head present the same variation pattern with respect to time both in magnitude and frequency.

4) As far as a single pile is concerned, the pile displacement increases when wave number, wave amplitude or water depth increases with other parameters remaining constant. For pile group foundations, the spatial arrangement of the piles also affects the behaviour of piles, and it is recognised that the pile displacement is not a univariate function of any particular parameter. It is subject to the variation of parametric combinations. The wave number has direct influence on piles’ behaviour regardless of the pile distance. The effect of the pile distance, on the other hand, depends on the wave number. The higher the wave number, the greater the influence.

5) The leading pile in pile group foundations, to which waves approach first, experiences greater resultant wave force, and thus exhibits greater displacement than those of other piles. Fortunately, when the incident wave direction randomly changes, piles in pile groups take turns acting as the leading pile and share the workload of protecting other piles, which improves the performance of pile groups in terms of longer lifespan.

6) The displacement amplitude of the leading pile in pile groups decreases with respect to the dimension ratio $e$ in such way that the decrease is rather mild when $k$ is small and quite significant for large $k$ values. It increases with $k$, and the increase is slightly more noticeable in cases with smaller $e$ values. The displacement variation of the succeeding pile, illustrated by the wave interaction with two piles, is more vulnerable to the parametric variation than that of the leading pile. In cases with small wave numbers or when piles are closely located to each other, the succeeding pile tends to have an opposite displacement direction to that of the leading pile. The displacement magnitude does not monotonically increase or decrease as is the case with the leading pile. Instead, it demonstrates a rather complex variation pattern, which should be made clear in the pile group foundation design process.

7) Particularly applicable to the time-dependent analysis, the steady state motion of piles holds the same motion frequency as the external waves, taking the
material damping effect into consideration, and the greater the wave number, the greater the frequency.

This PhD project advances SBFEM in the subject of wave-structure interaction, and simultaneously clarifies the nature of the physical problem. It has immediate engineering significance in that it provides useful information for offshore and coastal structural design. At the same time, it promotes the theoretical development and practical application of SBFEM to an advanced level, and is expected to introduce SBFEM to topics lying outside the scope of this study, such as wave-seabed interaction, seabed-structure interaction, even wave-structure-seabed interaction. On the other hand, there are certain aspects associated with this PhD project that need improvement. The problem formulation for both static and time-dependent wave-structure interaction corresponds to linear waves throughout this study, which provides a satisfactory estimate of the wave loads in moderate ocean conditions. In situations with steep incoming waves, a number of nonlinear phenomena, such as wave run up and ringing are prominent, and the interactions of such nonlinear waves with structures can be significantly different from those evaluated by using the linear theory. The major difficulties associated with nonlinear analyses lie in the nonlinear boundary conditions, both kinematic and dynamic, that have to be satisfied at the instantaneous free surface. From the viewpoint of numerical calculation, various types of numerical instabilities and the substantial computational time also pose substantial challenges. At this stage, due to the inherent limitations of SBFEM, a number of linearisation procedures need to be performed before SBFEM can be considered applicable for the analysis and the solution of nonlinear problems.

In addition, the present study intensively focuses on the wave-structure interaction, external loads from wind actions acting on wind farm monopiles or superstructures acting on pile groups are not considered. To guide engineering design when a specific project is involved, these loads need to be applied in the SBFEM model, following the same way as in other analysis techniques. Another simplification is in relation to the boundary condition at the seabed level, where pile foundations are seated and their displacements are assumed as zero. The relative motion between piles and the seabed are beyond the scope of the present study. A proposal regarding the incorporation of the seabed into the wave-structure interaction model is detailed in the next section as a continuation of the current study.
7.3 Future research proposal

The wave-structure interaction is neither independent of nor isolated from its relevant surroundings. It is closely related to another important element in the ocean environment, namely, the seabed. As waves propagate, the permeability of the seabed causes a dampening effect on the wave motion, thereby, altering wave properties. At the same time, the seabed, where coastal structures and fixed offshore structures are seated, experiences scour or liquefaction under the cyclic wave motion in shallow water conditions. This will consequently weaken the stability of the seabed and the functional ability of structures. Therefore, an extension of the present research to wave-structure-seabed interaction is consequentially meaningful from the engineering design and safety point of view. The following illustration, Figure 7-1, provides comprehensive knowledge to the interaction problem.

The main challenge of the wave-structure-seabed interaction problem is associated with one component, the seabed, which exhibits inhomogeneity and demonstrates non-linear mechanical behaviour in the vicinity of structures. In this situation, SBFEM is not suitable for the near-field seabed behaviour formulation, due to its inherent limitations. Considering the performance of FEM in handling materials with
inhomogeneity and complex mechanical behaviour, it is suggested that FEM be used in the near-field to address the non-linear behaviour of the bounded region, while SBFEM is applied at the far-field (in contrast to near-field) where linear nature and homogeneity can be assumed. The geometric extension of the near-field requires careful consideration. The interface between the near-field and the far-field should be at an appropriate distance, not too far away to cause any computational memory problems, nor too close to violate the non-linear performance of the seabed. The discretisation associated with SBFEM is only applied on the interface, and the boundary condition at infinity is exactly satisfied. By combining SBFEM and FEM, both the unbounded geometric scope and the non-linear material property will be successfully addressed, which cannot otherwise be simultaneously explored by either of the two methods.

The investigation of wave-structure-seabed interaction can be broken down into manageable tasks in terms of wave-structure interaction, which is well-based upon the present study, of wave-seabed interaction and structure-seabed interaction. This can be achieved by addressing the six aspects illustrated in Figure 7-1. The three interactions can be finally incorporated into one according to the associated boundary conditions prescribed at relevant interfaces between any two of the three media. Parametric analyses in relation to wave, structure and seabed properties should also be arranged to gain more in-depth understanding of the wave-structure-seabed interaction mechanism.
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