

Transport Time Scales in Soil Erosion Modelling

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26 **Core Ideas**

- 27 • Erosion time scales inherent in the Hairsine-Rose soil erosion are exposed
- 28 • Both fast and slow time scales are isolated, and can be estimated a priori
- 29 • The maximum sediment settling rate controls the possible range of timescales
- 30 • In practice, the full range of erosion time scales are not seen in flume experiments

31 **Abstract**

32 Unlike sediment transport in rivers, erosion of agricultural soil must overcome its cohesive
33 strength to move soil particles into suspension. Soil particle size variability also leads to fall
34 velocities covering many orders of magnitude, and hence to different suspended travel distances
35 in overland flow. Consequently, there is a large range of inherent time scales involved in
36 transport of eroded soil. For conditions where there is a constant rainfall rate and detachment is
37 the dominant erosion mechanism, we use the Hairsine-Rose (HR) model to analyze these
38 timescales, to determine their magnitude (bounds) and to provide simple approximations for
39 them. We show that each particle size produces both fast and slow timescales. The fast timescale
40 controls the rapid adjustment away from experimental initial conditions – this happens so
41 quickly that it cannot be measured in practice. The slow time scales control the subsequent
42 transition to steady state and are so large that true steady state is rarely achieved in laboratory
43 experiments. Both the fastest and slowest time scales are governed by the largest particle size
44 class. Physically, these correspond to the rate of vertical movement between suspension and the
45 soil bed, and the time to achieve steady state, respectively. For typical distributions of size
46 classes, we also find that there is often a single dominant time scale that governs the growth in
47 the total mass of sediment in the non-cohesive deposited layer. This finding allows a
48 considerable simplification of the HR model leading to analytical expressions for the evolution
49 of suspended and deposited layer concentrations.

50 **Keywords:** Erosion, transport, timescales, multi-size, detachment

51 **1. Introduction**

52 Human-induced soil erosion is a worldwide problem with significant economic and
53 environmental costs. Loss of surface soil leads to a reduction in soil fertility, structure and
54 resilience, an ultimately leads to non-productive land and desertification (Lal, 2001). Sediment is
55 a pollutant in its own right. It reduces light penetration and damages freshwater ecosystems. In
56 addition, it is a carrier of pollutants such as pesticides, phosphorus and bacteria, which promote
57 eutrophication and microbial contamination of surface water bodies. The growth of hypoxic
58 zones in coastal waters is related directly to river discharges containing high levels of sediment-
59 sorbed nutrients originating from agricultural runoff. Such zones occur in the Baltic, Black and
60 East China Seas, and in the Gulf of Mexico (Boesch et al., 2009; Diaz and Rosenberg, 2008). As
61 contaminants bind preferentially to clay and silt particles, predicting their transported loads also
62 requires the ability to predict the particle size distribution of the eroded sediment.

63 Depending on the spatial scale of sediment transport, there is a range of timescales involved that
64 determine transport behavior at that spatial scale. There is an associated advective timescale for
65 transport in suspension, a morphological timescale associated with bedform evolution (Fowler,
66 2011; McGuire et al., 2013), and a timescale for sediment to move through and exit a catchment.
67 These different timescales depend on the soil's particle size or settling velocity distribution since
68 this influences how sediment moves down a laboratory flume or through a landscape. In
69 addition, the size distribution of deposited sediment at the beginning of an erosion event affects
70 transported sediment fluxes for the different particle sizes (Cheraghi et al., 2016; Kim et al.,
71 2013; Sander et al., 2011). From simulations using the Hairsine-Rose (HR) model (Hairsine and
72 Rose, 1991, 1992b), Sander et al. (2011) confirmed that the particle size distribution and the
73 initial surface conditions of a soil determine not only the formation but also the shape of

74 hysteretic loops for suspended sediment concentration-versus-volumetric flow rate, as seen in
75 experimental data (Eder et al., 2010; Oeurng et al., 2010; Seeger et al., 2004; Williams, 1989).
76 Clockwise, anti-clockwise and figure eight (both flow orientations) hysteresis loops are
77 straightforward to obtain using the HR model. Physical explanations of the formation of the
78 different hysteresis loops are based on the availability of easily erodible sources of sediment and
79 its spatial distribution at the start of an erosion event (Oeurng et al., 2010; Smith and Dragovich,
80 2009). These sediment sources correspond to the readily erodible finer sediments as well as
81 material in the low-cohesion deposited layer of the HR model. The model's prediction of
82 different hysteretic curves arises from its specification of the initial size class distribution of this
83 layer along with its evolution, and that of the suspended sediment.

84 Recently, Cheraghi et al. (2016) tested the performance of the HR model against a series of
85 hysteretic experiments and found that it captured the behavior of all particle sizes. While
86 hysteresis was clearly shown to occur for the smaller particles, there was very little, if any,
87 hysteresis behavior for the larger particles. Sander et al. (2011) and Cheraghi et al. (2016)
88 demonstrated that a significant factor determining the size, shape and orientation of hysteresis
89 loops is the difference between the supply limit of fine sediment and transport limit of coarse
90 sediment, along with spatial variability in the state of the initial soil surface. This distinction is an
91 important attribute of any erosion model (Kirby, 2010). Kim et al. (2013) used a two-
92 dimensional numerical solution of the HR model and St Venant equations to analyses sediment
93 transport through the Lucky Hills watershed in Walnut Gulch. They also showed the importance
94 of watershed geometry and morphological evolution on the supply and transport-limited
95 movement of sediment sizes throughout the watershed.

96 With the growth of computational power along with the development of accurate, reliable and
97 efficient numerical schemes, landscape and catchment scale soil erosion modelling using the HR
98 formulation is possible (Fiener et al., 2008, Van Oost et al., 2004). For example, Le et al. (2015)
99 developed a two-dimensional scheme for which the stability criteria for time stepping is solely
100 governed by the Courant-Friedrichs-Lewy condition for the St Venant equations. This is a
101 significant advance over the schemes of Heng et al. (2009, 2011) and Kim et al. (2013), where
102 the controlling stability criterion was determined by the fall velocity of the largest size class.
103 Kim and Ivanov (2014) used a combined multi-dimensional HR, St Venant and morphological
104 model to study catchment-scale movement of eroded sediment, the scale dependence of erosion
105 rates and the associated contaminant and nutrient fluxes.

106 Kim and Ivanov (2014) noted that a controlling factor determining non-uniqueness of sediment
107 yield is the two timescales controlling the rapid rise to the peak concentration and the slow decay
108 to steady state. These two timescales were previously noted and discussed by Sander et al.
109 (1996) and Parlange et al. (1999), who developed an approximate analytical expression for the
110 HR model. The solution of Parlange et al. (1999) shows the importance of the largest size class
111 in determining the time for steady state to be achieved. However, there remains the question of
112 how the underlying soil properties determine these two transport time scales. Kim and Ivanov
113 (2014) showed there is a relationship with the dimensionless Shields parameter. However, the
114 more fundamental connection with soil properties, sediment size distribution, rainfall rate, and
115 erodibility of both the original and deposited soil was not considered.

116 Below, we show that due to the distribution of sediment sizes in a given soil, there is a wide
117 range of associated time scales that occur under rainfall detachment-controlled soil erosion. Not
118 only do we determine precise expressions for these, we show how these timescales combine to

119 control the overall behavior of the rapid rise in suspended sediment concentration and the slow
120 decline to steady state. In addition, we examine these time scales in terms of (i) what can be
121 realistically measured in the laboratory, and (ii) how they result in a rapid movement to a quasi-
122 equilibrium state between the deposited layer and the suspended sediment. In order to make our
123 analysis more tractable, a number of simplifying assumptions are invoked. These are that (i)
124 there is a constant rainfall rate, (ii) rainfall detachment is the dominant erosion mechanism and
125 that shear-driven entrainment processes can be neglected, (iii) only net erosion conditions occur
126 and (iv) the breakdown of aggregates (which change the soil's settling velocity distribution) is
127 not considered.

128 We note that this is the first time where such an analysis has been performed that relates erosion
129 timescales to both soil and hydraulic properties, for a multi-size class soil. There is a need to
130 understand the intrinsic behaviour of the models that are built, rather than just curve fitting or
131 calibrating them to data as a means of demonstrating their validity. Many complex models have
132 been developed without investigating their mathematical properties, other than a sensitivity
133 analysis to parameters. This does not inform users as to whether the functional dependence of
134 the model output to these parameters is physically sensible, except for the very small sensitivity
135 range that was tested. In our analysis, we are able to determine simple formulas that elucidate
136 the effect on the solution behaviour of the HR model for all physically relevant values of the soil
137 and hydraulic parameters. Consequently we can explain and interpret what these formulas imply
138 both physically and mathematically, and therefore gain further scientific understanding of
139 erosion modelling.

140

141 **2. HR model and solutions**

142 Under the just-given assumptions, the one-dimensional HR model for mass conservation of
143 water and eroded sediment is given by the following system of equations (Hairsine and Rose,
144 1991, 1992b),

$$\frac{\partial(Dc_i)}{\partial t} + \frac{\partial(qc_i)}{\partial x} = e_i + e_{di} - d_i, \quad i = 1, \dots, I, \quad (1)$$

$$\frac{\partial m_i}{\partial t} = d_i - e_{di}, \quad i = 1, \dots, I, \quad (2)$$

$$\frac{\partial D}{\partial t} + \frac{\partial q}{\partial x} = R, \quad (3)$$

145 where t is time (s), x is downstream distance (m), D is flow depth (m), q is the water flux per
146 cross-sectional width ($\text{m}^2 \text{s}^{-1}$), c_i is the suspended sediment concentration in size class i (kg m^{-3}),
147 m_i is the mass per unit area of deposited sediment of size class i (kg m^{-2}), and I is the total
148 number of sediment size classes. Eq. (3) is the kinematic approximation to the Saint-Venant
149 equations (Wooding, 1965). The excess rainfall rate, R (m s^{-1}), is the difference between the
150 rainfall rate, P , and the infiltration rate through the soil.

151 The conceptual layout of the HR model is shown in Fig. 1. The source terms on the right side of
152 Eqs. (1) and (2) represent the processes of raindrop detachment of original uneroded cohesive
153 soil, e_i , and the non-cohesive deposited layer, e_{di} , respectively ($\text{kg m}^{-2} \text{s}^{-1}$), and deposition of
154 suspended sediment due to gravity, d_i ($\text{kg m}^{-2} \text{s}^{-1}$). Note that Eq. (2) states that there is no flux
155 component moving sediment within the deposited layer, and that changes in its mass are due to
156 differences in erosion and deposition rates.

157 Expressions for the rainfall detachment and deposition rates are (Hairsine and Rose, 1991,
 158 1992b):

$$e_i = ap_i P(1-H), \quad e_{di} = a_d PH \frac{m_i}{m}, \quad d_i = \mathcal{G}_i c_i, \quad (4)$$

159 and following Sander et al. (1996), the HR model can be written as:

$$D \frac{\partial c_i}{\partial t} + q \frac{\partial c_i}{\partial x} = ap_i P(1-H) + a_d PH \frac{m_i}{m} - \mathcal{G}_i c_i - Rc_i, \quad i = 1, \dots, I, \quad (5)$$

$$\frac{\partial m_i}{\partial t} = \mathcal{G}_i c_i - a_d PH \frac{m_i}{m}, \quad i = 1, \dots, I. \quad (6)$$

160 The remaining parameters in Eq. (5) are the detachability, a (kg m^{-3}), of the original soil, the
 161 redetachability, a_d (kg m^{-3}), of the deposited soil, settling velocities, \mathcal{G}_i (m s^{-1}), and proportion of
 162 mass in each size class, p_i (with $\sum p_i = 1$). The total mass of soil in the deposited layer is $m =$
 163 $\sum_{i=1}^I m_i$, with H ($0 \leq H \leq 1$) determining the level of protection provided by the deposited layer
 164 to the original underlying soil:

$$H = \min\left(1, \frac{m}{m^*}\right). \quad (7)$$

165 The parameter m^* (kg m^{-2}) is the total mass required for complete protection by the deposited
 166 layer (i.e., $H = 1$).

167 Physically, Eq. (4) means that the detachment or redetachment rates, respectively, of a particle
 168 size are proportional to the rainfall rate, availability through p_i or m_i/m , and accessibility of the
 169 particles through $1 - H$ or H , respectively. The detachability, a , and redetachability, a_d , are
 170 decreasing functions of both the soil's cohesive strength and the overland flow depth, and since
 171 the deposited layer is non-cohesive, $a_d \gg a$.

172 The underlying time scales are found with the simplifications of the HR model used by Sander et
 173 al. (1996). These are (i) that temporal changes in c_i and m_i dominate over spatial gradients and
 174 (ii) that q and D can both be replaced by average (constant) values. This approximation was used
 175 to analyze effluent flume data under a variety of experimental conditions (Hogarth et al., 2004b;
 176 Jomaa et al., 2010, 2012b; Sander et al., 1996). Laboratory erosion experiments are typically
 177 conducted in flumes using an impervious base with a saturated soil and/or with high precipitation
 178 rates. In either case, infiltration can be neglected and $R = P$. Since D , a and a_d are constants, we
 179 define the following dimensionless variables and parameters:

$$\tau = \frac{Pt}{D}, \quad C_i = \frac{Dc_i}{m^*}, \quad M_i = \frac{m_i}{m^*}, \quad v_i = \frac{g_i}{P}, \quad \alpha = \frac{a_d D}{m^*}, \quad \beta = \frac{aD}{m^*}. \quad (8)$$

180 Eqs. (5)-(7) then reduce to the following linear system of $2I$ ordinary differential equations:

$$\frac{dC_i}{d\tau} = \beta(1-H)p_i + \alpha M_i - (1+v_i)C_i, \quad i = 1, \dots, I, \quad (9)$$

$$\frac{dM_i}{d\tau} = v_i C_i - \alpha M_i, \quad i = 1, \dots, I, \quad (10)$$

181 since under net erosion conditions $m < m^*$ and Eq. (7) then becomes $H = m/m^*$. In Eqs. (9) and
 182 (10), β and α are non-dimensional detachability and redetachability coefficients, respectively,
 183 with $\alpha > \beta > 0$, and $M = \Sigma M_i = H$.

184 Each size class has a characteristic non-dimensional settling velocity, v_i . We consider the case of
 185 an initially uneroded soil, and solve Eqs. (9) and (10) subject to zero initial concentrations of all
 186 size classes in the water and deposited layer, i.e., $C_i(0) = M_i(0) = 0$. Note that this problem was
 187 solved by Sander et al. (1996) in terms of the system's eigenvalues. Rather than using the
 188 method outlined in their paper, the problem is solved here using Laplace transforms as it leads to
 189 (i) approximate expressions for the eigenvalues (timescales), and (ii) additional physical insight

190 to the underlying erosion processes. The connection between the two solution methods will then
 191 be briefly discussed.

192 For notational convenience, we introduce $h(\tau) = 1 - H(\tau)$. When $H(\tau) = 1$, the original soil is
 193 completely shielded from erosion by the deposited soil and when $H(\tau) = 0$, the original soil is
 194 completely exposed. In Laplace space (denoted by overbars with Laplace variable s), the solution
 195 to Eqs. (9) and (10) is:

$$\bar{C}_i(s) = \frac{s + \alpha}{v_i} \beta p_i \bar{K}_i(s) \bar{h}(s), \quad (11)$$

$$\bar{M}_i(s) = \beta p_i \bar{K}_i(s) \bar{h}(s), \quad (12)$$

196 where

$$\bar{h}(s) = \frac{1}{s} - \bar{H}(s) = \frac{1}{s} - \sum_{i=1}^I \bar{M}_i(s) \quad (13)$$

197 and

$$\bar{K}_i(s) = \frac{v_i}{(s+1)(s+\alpha) + v_i s}. \quad (14)$$

198 While solutions to Eqs. (9) and (10) can be expressed as convolution integrals, for the present we
 199 consider aspects of the Laplace domain solution, which depend on inverting \bar{K}_i and \bar{h} . Note that
 200 the central role played by h (or H) in the solutions to Eqs. (9) and (10) is evident in Eqs. (11) and
 201 (12).

202 The inversion of \bar{K}_i is straightforward. For \bar{h} , we sum Eq. (12) over i , and use the definition of
 203 $h(\tau)$ to obtain:

$$\bar{h}(s) = \frac{s^{-1}}{1 + \beta \bar{K}(s)}, \quad (15)$$

204 where

$$\bar{K}(s) = \sum_{i=1}^I p_i \bar{K}_i(s) = \sum_{i=1}^I \frac{v_i p_i}{(s+1)(s+\alpha) + v_i s}. \quad (16)$$

205 From Eq. (15), the steady-state value of h , denoted h_∞ , is obtained by inverting the leading order
 206 term for $s \rightarrow 0$ as (Parlange et al., 1999):

$$h_\infty = h(\tau \rightarrow \infty) = \left(1 + \frac{\beta}{\alpha} \sum_{i=1}^I p_i v_i \right)^{-1} = \frac{\alpha}{\alpha + \beta v_{av}}, \quad (17)$$

207 where $v_{av} = \sum_{i=1}^I p_i v_i$ is the average settling velocity.

208 The inversion of Eqs. (11) and (12) to recover C_i and M_i depends on the singularities of $\bar{h}(s)$ in
 209 Eq. (15). There is a simple pole at $s = 0$, the residue of which gives the steady-state value of
 210 $h(\tau)$, i.e., Eq. (17). Otherwise, residues for s satisfying

$$\beta \bar{K}(s) = -1, \quad (18)$$

211 are needed. Since each \bar{K}_i in Eq. (16) has at most two distinct singularities, $\beta \bar{K}(s) = -1$ has at
 212 most $2I$ roots. We show in the Supplementary Material that there are indeed exactly $2I$ roots,
 213 which are all real and negative.

214 Equation (15) can be expressed as a rational function $\bar{h}(s) = \bar{p}(s) / \bar{q}(s)$, where $\bar{p}(s)$ is a
 215 polynomial in s and:

$$\bar{q}(s) = s \prod_{j=1}^{2I} (s - \lambda_j). \quad (19)$$

216 In this equation, the λ_j s are the roots of $\beta\bar{K}(s) = -1$, which in general must be found

217 numerically. Then, $\bar{h}(s)$ is expressed as:

$$\bar{h}(s) = \frac{A_0}{s} + \sum_{j=1}^{2I} \frac{A_j}{s - \lambda_j}, \quad (20)$$

218 where, from the steady solution to Eq. (15), $A_0 = \alpha(\alpha + \beta v_{av})^{-1}$, and values for the other A_j s can

219 be derived from the Heaviside expansion formula. The inversion of Eq. (20) is then:

$$h(\tau) = \frac{\alpha}{\alpha + \beta v_{av}} + \sum_{j=1}^{2I} A_j \exp(\lambda_j \tau). \quad (21)$$

220 We see in Eq. (21) that the λ_j s define the different time scales affecting the behavior of $h(\tau)$, as

221 well as $\bar{C}_i(s)$ and $\bar{M}_i(s)$, from Eqs. (11) and (12), respectively.

222 2.1 Solution as Convolutions

223 Since $h(\tau)$ is known explicitly from Eq. (21) – albeit in general it involves finding the roots of

224 Eq. (18) numerically – the inversion of Eqs. (11) and (12) can be expressed as convolutions. Size

225 class masses in the deposited layer are given by:

$$M_i(\tau) = p_i \beta \int_0^\tau K_i(\tau - y) h(y) dy, \quad (22)$$

226 where $K_i(\tau)$ is obtained by inverting $\bar{K}_i(s)$ from Eq. (14):

$$K_i(\tau) = \frac{V_i}{r_i - R_i} [\exp(r_i \tau) - \exp(R_i \tau)]. \quad (23)$$

227 With Eq. (23), inversion of Eq. (11) yields:

$$C_i(\tau) = p_i \beta \int_0^\tau L_i(\tau - y) h(y) dy, \quad (24)$$

228 where

$$L_i(\tau) = \frac{1}{r_i - R_i} \left[(r_i + \alpha) \exp(r_i \tau) - (R_i + \alpha) \exp(R_i \tau) \right]. \quad (25)$$

229 By summing Eq. (22), H takes the form of an integral equation:

$$H(\tau) = 1 - h(\tau) = \beta \int_0^\tau K(\tau - y) h(y) dy, \quad (26)$$

230 where $K = \sum_{i=1}^I p_i K_i$.

231 The constants R_i and r_i in Eqs. (23) and (25) are the roots of the quadratic in the denominator of

232 Eq. (14), i.e., for each particle size class, i ,

$$\begin{bmatrix} r_i \\ R_i \end{bmatrix} = -\frac{\nu_i + \alpha + 1}{2} \begin{bmatrix} 1 - \sqrt{1 - \frac{4\alpha}{(\nu_i + \alpha + 1)^2}} \\ + \sqrt{1 - \frac{4\alpha}{(\nu_i + \alpha + 1)^2}} \end{bmatrix}. \quad (27)$$

233 Since $\alpha > 0$ and $\nu_i > 0$, r_i and R_i are always real and negative. Eq. (27) also allows $\bar{K}_i(s)$ from Eq.

234 (14) to be written as:

$$\bar{K}_i(s) = \frac{\nu_i}{r_i - R_i} \left(\frac{1}{s - r_i} - \frac{1}{s - R_i} \right). \quad (28)$$

235 **2.2 Connection with the Solution of Sander et al. (1996)**

236 It is useful to show the connection with the solution of Sander et al. (1996). To relate the two

237 approaches, we briefly reproduce their result more directly. The general solution of Eqs. (9) and

238 (10) is given by the steady-state component (superscript “steady”):

$$C_i^{steady} = \frac{\alpha \beta p_i}{\alpha + \beta \nu_{av}}, \quad M_i^{steady} = \frac{\beta \nu_i p_i}{\alpha + \beta \nu_{av}}, \quad H^{steady} = \frac{\beta \nu_{av}}{\alpha + \beta \nu_{av}}, \quad (29)$$

239 plus the general solution of the homogeneous equation. Substituting $C_i(\tau) = C_i^{steady} + \gamma_i \exp(\lambda \tau)$
 240 and $M_{di}(\tau) = M_i^{steady} + \mu_i \exp(\lambda \tau)$ into Eqs. (9) and (10) and assuming $2I$ distinct eigenvalues λ_j
 241 yields:

$$C_i(\tau) = \frac{\alpha \beta p_i}{\alpha + \beta \bar{v}} + \sum_{j=1}^{2I} A_j \gamma_{ij} \exp(\lambda_j \tau), \quad i = 1, \dots, I, \quad (30)$$

$$M_i(\tau) = \frac{\beta v_i p_i}{\alpha + \beta \bar{v}} + \sum_{j=1}^{2I} A_j \mu_{ij} \exp(\lambda_j \tau), \quad i = 1, \dots, I, \quad (31)$$

242 where γ_{ij} and μ_{ij} are the i^{th} component of the eigenvectors associated with the j^{th} eigenvalue λ_j ,
 243 and are given by:

$$\gamma_{ij} = \frac{-\beta(\lambda_j + \alpha) p_i}{(\lambda_j + 1)(\lambda_j + \alpha) + \lambda_j v_i}, \quad (32)$$

$$\mu_{ij} = \frac{-\beta v_i p_i}{(\lambda_j + 1)(\lambda_j + \alpha) + \lambda_j v_i}. \quad (33)$$

244 By summing Eq. (31) over the size classes and noting that $\sum_{i=1}^I \mu_{ij} = -\beta \bar{K}(\lambda_j) = 1$, then:

$$H(\tau) = H^{steady} + \sum_{j=1}^{2I} A_j \exp(\lambda_j \tau), \quad (34)$$

245 in agreement with Eq. (21). The coefficients A_j are found by matching the initial conditions $C_i(0)$
 246 $= 0$, $M_i(0) = 0$, and in general must be found numerically.

247 The characteristic equation defining the eigenvalues in Eqs. (30) and (31) is $\beta \bar{K}(\lambda) = -1$, which,
 248 not surprisingly, also appears in the Laplace transform solution through Eq. (18). The
 249 singularities arising in the inversion of \bar{h} are the eigenvalues in Eqs. (30) and (31) that control
 250 the erosion timescales inherent in the HR model. Note that carrying out the integrations in Eqs.
 251 (22) and (24) – with Eq. (21) – results in Eqs. (30) and (31), respectively. The different forms of

252 the solution allow different insights and interpretations of the erosion processes to be obtained.
253 The temporal time scales appearing in the solutions of the HR model, and hence the effect of the
254 soil's particle size distribution on erosion timescales, is governed by the *distribution and size of*
255 *the eigenvalues*, which in general are calculated numerically. It is clear that on physical grounds
256 we would expect that all λ_j s in Eqs. (30) and (31) are negative; otherwise the solutions would
257 diverge at large times. Consequently, it is the magnitude of the λ_j s that determine the timescale
258 over which the separate contributions through $\exp(\lambda_j\tau) \rightarrow 0$, i.e., the system approaches steady
259 state. In the next section, we obtain simple approximations for the eigenvalues as functions of
260 erosion parameters and the settling velocity distribution.

261 **3. Time scale bounds**

262 In the Supplementary Material, several results describing the behavior of the λ_j s are derived
263 formally. These results are now used to interpret time scales in the HR model physically.

264 Differences between soils and experimental conditions are expressed through different values of
265 the dimensionless parameters ν_i , α , and β . While the HR model imposes the physical condition
266 $\alpha > \beta > 0$, in the Supplementary Material it is shown that α , β greater than or less than one also
267 plays an important role in the analysis of the eigenvalues, as might be expected from the
268 denominators of Eqs. (32) and (33). We examine in detail the case of $\alpha > \beta > 1$ as it occurs often
269 in practice (Sander et al., 1996), and consider the slight modifications for the other two cases, α
270 $> 1 > \beta$ and $1 > \alpha > \beta$, in the Discussion.

271

272

273 3.1 Example soil

274 To illustrate the features of the solution and how the bounds on the eigenvalues are obtained,
275 consider a soil composed of $I = 3$ particle sizes with fall velocities of (0.00018, 0.0033, 0.0125)
276 m s^{-1} subject to a constant rainfall rate of 56 mm h^{-1} . This results in dimensionless fall velocities
277 v_1, v_2, v_3 of 11.57, 212.1 and 803.6, respectively. Taking $\alpha = 25, \beta = 20$ and $p_i = 1/3$ results in the
278 solution curves from (9) and (10) as shown in Fig. 2. This figure shows that the total suspended
279 sediment concentration undergoes a rapid early rise to the peak concentration, followed by an
280 apparent exponential decline to steady state. The smallest size class makes the greatest
281 contribution to the peak due to its lowest settling velocity and therefore tends to remain in
282 suspension relative to the larger sediment sizes. This initial flush of fine sediment is regularly
283 seen in experimental data and is primarily responsible for the eutrophication and pollution of
284 surface water bodies through the additional transport of sorbed fertilizers and pesticides. The
285 larger size classes quickly fall out of suspension and make the greatest contribution to the growth
286 of the deposited layer and the magnitude of H . It is the rate of growth of H that determines the
287 time of the peak concentration and for the subsequent decline in C through the reduction in
288 access to small particle sizes. The smallest size class contributes little to H (and so to the
289 deposited layer). Hence, the only significant source of this size class to the suspended sediment
290 load is from the original uneroded soil. Due to the increase of H , the detachment process (i.e.,
291 raindrop-induced erosion) is unable replace the small particles that are transported downstream
292 and so C rapidly drops off from its peak. The form of the solution curves shown in Fig. 2
293 remains the same for any α or β when $\alpha > \beta$. Changes in their magnitude simply change the
294 position, magnitude and rate of decline from the peak concentration.

295 Returning to $\bar{K}(s)$, the form of this function for $\alpha = 25$ and $\beta = 20$ is shown in Fig. 3, where we
 296 observe that the roots R_i and r_i (labeled according to their magnitude such that $|R_i| > \alpha$ and
 297 $|r_i| < 1$) from Eq. (27) separate the eigenvalues into discrete intervals. This arises because $\bar{K}(s)$
 298 is made up from the sum of the I separate $\bar{K}_i(s)$ functions with each one approaching $+\infty$ or $-\infty$
 299 depending whether s approaches R_i or r_i from above or below. Of the $2I$ (six in this example)
 300 eigenvalues, $I - 1$ can be found between R_1 and R_I and $I - 1$ can be found within r_1 and r_I . The
 301 remaining two eigenvalues are located in the region between R_1 and r_1 , which can be further
 302 isolated into having one each in $(R_1, -\alpha)$ and $(-\alpha, -1)$. This distribution of the eigenvalues holds
 303 for any I when $\alpha > \beta > 1$ (Supplementary Material). Thus, increasing the number of size classes
 304 between v_1 and v_3 merely adds more intervals between both $-\infty$ and R_3 , and r_3 and 0. Note that
 305 from Eq. (27), both R_i and r_i depend only on the i^{th} settling velocity, v_i , and redetachability, α ,
 306 and that for $v_i \gg \alpha$, $R_i \rightarrow -v_i$ and $r_i \rightarrow 0$.

307 The analysis presented in the Supplementary Material, which generalizes the results shown in
 308 Fig. 3, can be summarized by the following four properties. For a soil that is composed of any
 309 number of particle size classes I , then for $\alpha > \beta > 1$:

- 310 (i) All the eigenvalues λ are real, simple and negative;
- 311 (ii) There are I eigenvalues in the interval $(-\infty, -\alpha)$;
- 312 (iii) There are $I - 1$ eigenvalues in the interval $(-1, 0)$;
- 313 (iv) There is 1 eigenvalue in the interval $(-\alpha, -1)$.

314 From (i), the solution will decay towards steady state without oscillations. Further, there are no
 315 solutions having terms of the form $\tau \exp(\lambda \tau)$. Since $\alpha > 1$, the eigenvalues in (ii), (iii) and (iv)

316 can be classified as ‘fast’, ‘slow’ and ‘intermediate’, respectively, as they represent the rate at
 317 which their individual contributions to the solution become negligible as τ increases, according
 318 to the decay rates $\exp(\lambda_j\tau)$.

319 **3.2 Eigenvalue approximations for a Black Earth soil**

320 Sander et al. (1996) solved the system of equations given by Eqs. (9) and (10), and successfully
 321 applied the solution to the experimental data of Proffitt et al. (1991) for two different soils, Black
 322 Earth (vertisol) and Solonchak (aridisol). The experimental conditions are consistent with the
 323 assumptions given in the Introduction. As both soils behave similarly, we will present results
 324 only for the Black Earth. The experiment using the Black Earth soil had a precipitation rate of P
 325 $= 56 \text{ mm h}^{-1}$ and an overland flow depth of $D = 2 \text{ mm}$, which results in $\alpha \approx 100$, $\beta \approx 50$ along
 326 with dimensionless settling velocities for 10 size classes as given in Table 1. Note the wide range
 327 in the dimensionless settling velocities ($10^{-1} - 10^5$).

328 In Table 2, the roots satisfying $\beta \bar{K}(s) = -1$ are presented along with their bounds as described in
 329 Theorems 1 and 2 in the Supplementary Material. It is straightforward to derive estimates for the
 330 fast eigenvalues, which lie in the interval $(-\infty, -\alpha)$, as they all sit very close to the corresponding

331 R_i (Fig. 3). Thus, in a given interval i , the dominant contribution from $-\beta \sum p_i \bar{K}_i(s) = 1$ comes
 332 from the i^{th} term due to $(s - R_i)^{-1}$ in Eq. (28), and so the summation can be simplified to a single
 333 term to give $-\beta p_i K_i(s) \approx 1$ for $i = 1, 2, \dots, I$, or $-\beta p_i v_i / R_i \approx s - R_i$ from Eq. (28) since $\lambda \gg r_i$. We
 334 therefore approximate the i^{th} fast eigenvalue as:

$$s_i^f = R_i - \frac{\beta v_i p_i}{R_i}, \quad (35)$$

335 which shows the weak (second-order) dependence of s^f on β . Noting that for real soils usually α
 336 $+v_i \gg 1$, then by combining with Eq. (27) and ignoring the second-order correction, Eq. (35)
 337 simplifies to:

$$s_i^f = -(\alpha + v_i). \quad (36)$$

338 Unlike the fast eigenvalues, the values of the slow eigenvalues in the interval $(-1, 0)$ wander
 339 between the bounds r_i , so reliable expressions corresponding to Eqs. (35) and (36) are not
 340 available. The closest estimate to each slow eigenvalue is then given by the bounds r_i , which
 341 from Eq. (27) with $\alpha + v_i \gg 1$ gives:

$$s_i^s \approx r_i \approx -\frac{\alpha}{\alpha + v_i}, \quad i = 2, 3, \dots, I. \quad (37)$$

342 Interestingly, Parlange et al. (1999) derived an approximate analytical solution to c_i and m_i based
 343 on an approach that did not consider the underlying eigenvalues. They obtained large time
 344 exponential decay terms of the form $\exp[-\alpha\tau(\alpha + v_i)^{-1}]$, which correspond to the timescales in Eq.
 345 (37). This helps explain the favorable comparison of their approximation with the exact
 346 analytical solution. While in general Eq. (36) is a good estimate of the fast eigenvalues as they
 347 always sit very close to R_i , Eq. (37) is less accurate for the slow eigenvalues as they can move
 348 within the bounds r_i and r_{i+1} as the soil properties change. This is the source of the small
 349 discrepancy between the approximate and exact solutions presented by Parlange et al. (1999).
 350 For instance, for the soil and parameter values used in Table 2, the best estimate for the slow
 351 eigenvalues is mostly given by the lower bound r_{i-1} rather than r_i .

352 For large α with $\alpha > \beta > 1$, the interval $(-\alpha, -1)$ containing the intermediate eigenvalue is large and
 353 a tighter bound would be preferred. From Theorems 1 and 2 (Supplementary Material), for the
 354 more common case of $\alpha > \beta > 0$, this interval can be considerably reduced to (s_L, s_U) , where:

$$s_L > \max \left(-\alpha, -1 - \frac{\beta \sum \frac{v_i p_i}{r_i - R_i}}{1 - \beta \sum \frac{v_i p_i}{(r_i - R_i)(-\alpha - R_i)}} \right), \quad (38)$$

355 and

$$s_U < \min \left(-1, r_1 - \frac{\beta \sum \frac{v_i p_i}{r_i - R_i}}{1 - \beta \sum \frac{v_i p_i}{(r_i - R_i)(-1 - R_i)}} \right). \quad (39)$$

356 For the Black Earth soil, the value of the intermediate eigenvalue is -38.88 (Table 2), with Eqs.
 357 (38) and (39) giving the bounds of $s_L = -43.86$ and $s_U = -38.64$. Other than for $\beta = 1$ when $s = -1$
 358 (see Remark 6.1 in the Supplementary Material), our extensive numerical simulations show that
 359 the upper bound s_U generally provides the closest estimate to the intermediate eigenvalue, as
 360 indeed it does for the Black Earth soil.

361 Equations (22), (24) and (26) show that, if h is known, then concentrations in suspension and the
 362 deposited layer are known explicitly. Although exact results rely on numerical calculation of the
 363 roots of $\beta \bar{K}(s) = -1$ (needed to determine h), we can estimate h by estimating $\bar{K}(s)$ in Eq. (15).

364 From Theorem 3 (Supplementary Material), we have $\bar{K}(s) < -B/s$, where $B =$

365 $\sum v_i p_i / (\alpha + v_i)$. Substituting this estimate for $\bar{K}(s)$ into Eq. (15), inverting and forcing the

366 approximation to reach the correct steady-state value, gives the following approximation for h or

367 $H = 1 - h$:

$$h(\tau) \approx (1 - h_\infty) \exp(-\beta B \tau) + h_\infty, \text{ or } H(\tau) = H^{steady} [1 - \exp(-\beta B \tau)], \quad (40)$$

368 where h_∞ is given by Eq. (17) and H^{steady} by Eq. (29). Figure 4 shows that Eq. (40) is potentially
 369 a useful approximation for h . This approximation is additionally valuable since it leads directly
 370 to analytical approximations for the complete solution to the HR model using the results in §2.1.

371 We have carried out simulations across a wide range of values for α and β where $\alpha > \beta > 1$,
 372 $\alpha > 1 > \beta$, $1 > \alpha > \beta$, with $\alpha / \beta = 1000, 100, 10$ and 2 for the particle size distributions of the
 373 three different soils of Proffitt et al. (1991), Polyakov and Nearing (2003) and Jomaa et al.
 374 (2010). All these simulations showed Eq. (40) to be a good approximation for $h(\tau)$, which
 375 improved as α/β decreased. Inspection of the simulation results showed that, independently of
 376 α , β or soil type, there is usually one and occasionally two or three of the coefficients A_j in Eqs.
 377 (30) and (31) that are at least an order of magnitude greater than the rest, and so isolate the key
 378 timescale controlling h . In addition, where there are two or three, they always occur for
 379 consecutive j s. By comparing the corresponding λ_j values with the values of βB , it was found
 380 that βB not only tracks these eigenvalues, it represents some averaged measure of them. The
 381 approximation Eq. (40) works well because so very few of the eigenvalue timescales contribute
 382 significantly to the summation term in Eq. (34) to H . Consequently, they can all be approximated
 383 by a single timescale and therefore a single exponential term of the form $\exp(-\beta B \tau)$.

384 **4. Discussion**

385 **4.1 Physical interpretation of the convolution integral solution**

386 The convolution integrals in §2.1 draw attention to the motion of a specific parcel of soil
 387 detached from the parent medium at a time $\tau = y$. The state at time τ of a soil parcel detached

388 at an earlier time y is specified by the response functions $K_i(\tau)$, $L_i(\tau)$, given, respectively, by
389 Eqs. (23) and (25). These functions represent the masses of this previously detached soil in the
390 deposited layer and in suspension, respectively. At the earlier time y , a fraction $h(y)$ of the soil
391 was exposed and the resulting detachment rate of a given size class was therefore $p_i\beta h(y)$, as
392 detachment is not size class selective (Hairsine and Rose, 1991). These parcels then propagate
393 through to time τ by the response functions. Thus, $C_i(\tau)$ and $M_i(\tau)$ are the integrals of detachment
394 over all earlier times, i.e., the convolutions of Eqs. (23) and (25). The total deposited mass, $1 -$
395 $h(\tau)$, is therefore an integral over its source at earlier times y , as given by Eq. (26). That is, Eq.
396 (26) balances the present mass of sediment in the deposited layer against the mass of detached
397 soil particles from earlier times y .

398 Figure 5 shows the response curves and h for the Black Earth soil for all ten grain size classes.
399 Both K_i and L_i display a rapid initial transient and by comparison, a slow decay, however, the
400 magnitude of the initial effect differs greatly with particle size. For a given v_i , the fast
401 eigenvalues, λ_j^{fast} , define the timescales of the initial transients in K_i and L_i while the slow
402 eigenvalues, λ_j^{slow} , control the decay to steady state. We also note that the majority of the
403 $L_i(\lambda_j^{slow})$ values are far smaller than the corresponding $K_i(\lambda_j^{slow})$ values. This indicates that
404 while suspended sediment concentrations and h can appear to be at steady state, the sediment
405 size class distribution within the deposited layer is *still undergoing considerable adjustment*.
406 This behavior is evident in Figs. 2 (measured and predicted total concentrations), 5 (c_i) and 6 (m_i)
407 of Sander et al. (1996), which show that the suspended sediment concentrations are essentially at
408 steady state, but those in the deposited layer are not. The largest particle size is also seen to

409 provide the timescale controlling the transition to steady state (Figs. 5 and 6 of Sander et al.,
410 1996).

411 **4.2 Interpretation of rate processes**

412 We saw above that the characteristic rates for the decoupled pairs have one fast rate $R_i < -\alpha$ and
413 one slow rate $-1 < r_i < 0$ and that the values of R_i and r_i depend only on the i^{th} settling velocities,
414 v_i , and redetachability, α . Moreover, as v_i increases (heavier sediment), the fast rate R_i gets faster,
415 and the slow rate r_i gets slower. However, with increasing detachability, β , the fast rates reduce
416 slightly, and the slow rates increase slightly. This is suggested in Fig. 3 through shifting of the
417 horizontal line $-\beta^{-1}$ upwards and noting the corresponding changes in the position of the circled
418 points. Since the eigenvalue bounds R_i and r_i depend only on α and the corresponding v_i , the
419 eigenvalues cannot vary strongly with β . This is more noticeable as the number of size classes
420 increase. The bounds R_i and r_i then crowd more densely on the intervals $(-\infty, -\alpha)$ and $(-1, 0)$,
421 giving the fast and slow eigenvalues less freedom to wander, and packing them tighter and
422 tighter together in these intervals.

423 Concerning the different rates as described by the eigenvalues of the HR model, several
424 observations can be made. These are that

425 (i) Fast and slow rates are associated primarily with uncoupled processes (deposition,
426 redetachment) as they depend primarily on α and one or two settling velocities.

427 Detachability, β , soil composition, p_i , and other settling velocities, v_i , have only minor
428 effects on the fast and slow eigenvalues;

429 (ii) When $\alpha > \beta > 1$, the only eigenvalue whose location is genuinely a result of the coupled
430 detachment process is the ‘intermediate’ eigenvalue, which is primarily determined by the

431 detachability, β (e.g., Fig. 3). This eigenvalue is a good estimate of the dominant timescale
 432 governing the evolution of h permitting an accurate explicit approximation for $h(\tau)$ to be
 433 obtained, Eq. (40). As mentioned above, with h known (approximately), C_i and M_i can be
 434 estimated through their convolution integrals (§2.1).

435 (iii) The fastest and slowest rates are largely determined by the maximum settling velocity,
 436 v_{max} , and are thus associated with movement of the heaviest sediment;

437 Intuitively, we might expect that the fast and slow processes are associated with fast and slow
 438 settling soil particles, but *this is not the case*. Both the fastest and slowest rates are determined
 439 primarily by the *maximum* settling velocity, v_{max} . Good approximations for the fast and slow
 440 eigenvalues are given by $\lambda_i^{fast} \approx -(\alpha + v_i)$ and $\lambda_i^{slow} \approx -\alpha(\alpha + v_i)^{-1}$, respectively, assuming
 441 $\alpha > \beta > 1$. Thus, the shortest timescale (largest λ^{fast}) process is approximated by

442 $O(-(\lambda_i^{fast})^{-1}) \approx O((\alpha + v_{max})^{-1})$ and is therefore associated with settling of the heaviest particles.

443 The longest timescale (smallest λ^{slow}) process is $O(-(\lambda_i^{slow})^{-1}) = O(1 + v_{max} / \alpha)$ and is associated
 444 with downslope movement of these *same* particles. Note that while the spatial sediment gradient
 445 is neglected in Eq. (9), the effect of advection is still present through the $-C_i$ term on the right
 446 side of Eq. (9). The possible range of timescales is of order v_{max}^2 if $v_{max} \gg \alpha$, as is generally
 447 expected in practice. In a real soil, the fastest processes (timescale 0.01 s for Black Earth)
 448 manifest themselves as an instantaneous initial jump, and cannot be resolved experimentally.
 449 Even the ‘intermediate’ rate process (timescale 3.4 s) is too fast to be measured for the Black
 450 Earth. The slow processes (timescale 5 min or more) are the ones that are observed in a
 451 laboratory experiment. However, the slowest processes (timescale 50 h for Black Earth) are
 452 sufficiently slow so that in any reasonable length experiment or rainfall event where raindrop

453 detachment dominates, they will not have run to completion. Thus, although values of C_i and M_i
454 may be varying slowly as measured in an ongoing laboratory experiment, usually steady state
455 values of C_i and M_i will not be attained.

456 The eigenvalue spectrum for the Black Earth soil is shown in Table 2, where it can be seen how
457 well the intermediate eigenvalue -38.88 is separated from the rest of the spectrum. Doubling the
458 number of size classes to $I = 20$ has a very small impact on this eigenvalue. Thus, it is very stable
459 to ν being discretized in various ways and is therefore a property of the soil and experimental
460 conditions. This occurs because the range of settling velocities is fixed for any given soil and
461 therefore, the range of time scales is also fixed. For this reason, the number of size classes
462 selected for a given soil does not have a great effect on the overall results.

463 The eigenvalues cover the complete possible range of rates by distributing themselves along
464 portions of the real axis, while their specific locations depend on how the soil is divided into size
465 classes. For instance, the fast eigenvalues are $\lambda_i \approx -(\nu_i + \alpha)$, so changing the number of size
466 classes of ν would give different eigenvalues. The particular values of the fast and slow rates
467 depend as much on the discretization of soil data, through ν_i , as on soil and experiment
468 conditions (given through P , D , m^* and a_d). However, the fast eigenvalues *collectively*, and the
469 slow eigenvalues *collectively* are soil and experiment properties and give the possible *range of*
470 *timescales*.

471 The differences between classes of eigenvalues are further emphasized by the behavior of the
472 associated eigenvectors. Below, we consider the eigenvectors associated with the fast,
473 intermediate and slow eigenvalues.

474 **Fast** By replacing λ_j in Eqs. (32) and (33) with the approximation $-(v_j + \alpha)$, then the
 475 components of the fast eigenvectors are approximated by:

$$\gamma_{ij} \approx \frac{\beta v_j p_i}{(v_j + \alpha)(v_j - v_i) - v_j} \approx -\mu_{ij}. \quad (41)$$

476 The suspended sediment components of γ_{ij} , are approximately the same magnitude but opposite
 477 in sign to those of the deposited sediment components, μ_{ij} . Consequently, the ‘fast’ eigenvectors
 478 represent predominantly a rapid exchange of material between suspension and the deposited
 479 layer. Note, in addition, that for $i \neq j$ all the eigenvector components are small compared to that
 480 for $i = j$, hence exchange between the suspended and deposited material of a given size class
 481 depends little on the concentrations of other size classes. This highlights the weak coupling
 482 between the size classes.

483 **Intermediate** For the intermediate eigenvalue $\lambda + \alpha > 0$ and hence Eqs. (32) and (33) show that
 484 the eigenvector components are of the same sign. All size classes now participate with the
 485 heavier size classes being more active in the deposited layer since as v_i increases in Eq. (33) so
 486 does μ_{ij} . At the same time, the lighter classes are more active in the suspension since γ_{ij}
 487 increases as v_i decreases in Eq. (32).

488 **Slow** These processes are associated with *resorting* of the deposited layer. From Eq. (32), $v_i \gamma_{ij}$
 489 is approximated by:

$$v_i \gamma_{ij} \approx \frac{-\alpha \beta v_i p_i}{(\lambda_j + 1)\alpha + \lambda_j v_i} = \alpha \mu_{ij}, \quad (42)$$

490 since for the slow eigenvalues, $\alpha \gg -\lambda_j$. The approximation Eq. (42), shows that the slow
 491 eigenvalues and associated eigenvectors correspond to the condition where $v_i \gamma_{ij} - \alpha \mu_{ij} \approx 0$, or

492 $v_i C_i - \alpha M_i \approx 0$. Since $dM_i / d\tau = v_i C_i - \alpha M_i$ and $H = \sum M_i$, this means that the deposited
 493 layer quickly obtains a state of quasi-equilibrium where $M_i \approx v_i C_i / \alpha$, which is then followed by
 494 a slow resorting of the actual contributions of each size class as they approach their steady state
 495 values over a long timescale. It was the recognition of this quasi-equilibrium state that was
 496 exploited by Parlange et al. (1999) to develop simple analytical expressions for $H(\tau)$, $M_i(\tau)$ and
 497 $C_i(\tau)$ that provided a good approximation to the solution given by Eqs. (30) and (31).

498 Short time processes occur on the timescale for vertical motion of soil particles and are related to
 499 exchange of material between the suspension and the deposited layer. At all times, there is a
 500 strong mass exchange between the soil bed and the suspension. The *net* mass exchange may, of
 501 course, be very small; at steady state there is indeed an exact balance. Any perturbation from
 502 steady state that leads to an imbalance between deposition and redetachment rates would rapidly
 503 be corrected. In practice, this happens so quickly it appears to be instantaneous, and in practical
 504 terms the soil bed is always in a state where $v_i C_i \approx \alpha M_i$.

505 **4.3 Timescale dependence on detachability parameters for cases where α or $\beta < 1$**

506 There are two further parameter cases that need to be considered, these being $\alpha > 1 > \beta$ and $1 > \alpha$
 507 $> \beta$. Remember that on physical grounds $\alpha > \beta$ resulting in $I - 1$ eigenvalues $< R_1$, $I - 1$
 508 eigenvalues $> r_1$, and two in the region (R_1, r_1) . Changes in the magnitudes of α and β simply
 509 reposition the two eigenvalues in (R_1, r_1) into the following two intervals (Lemma 6,
 510 Supplementary Material):

- 511 (i) $\alpha > \beta > 1$; $(R_1, -\alpha)$ and $(-\alpha, -1)$;
- 512 (ii) $\alpha > 1 > \beta > 0$; $(R_1, -\alpha)$ and $(-1, r_1)$;

513 (iii) $1 > \alpha > \beta > 0$; $(R_1, -1)$ and $(-\alpha, r_1)$.

514 While all three cases have I fast ($|\lambda| > 1$) eigenvalues, for $\beta < 1$ the intermediate eigenvalue is
 515 also less than unity, giving a total of I slow ($|\lambda| < 1$) eigenvalues. The special cases of $\beta = 1$ and α
 516 $= \beta$ result in $\lambda = -1$ and $\lambda = -\alpha$, respectively; however, it is only the former case that has any
 517 physical significance.

518 For $\beta < 1$, the bounds on the intermediate eigenvalue given in Eqs. (38) and (39) are modified to
 519 (Theorem 2, Supplementary Material):

$$s_L > \max \left(s_{\min}, s_{\max} - \frac{\beta \sum \frac{v_i p_i}{r_i - R_i}}{1 - \beta \sum \frac{v_i p_i}{(s_{\min} - R_i)(r_i - R_i)}} \right), \quad (43)$$

520 for the lower bound and

$$s_U < \min \left(s_{\max}, r_1 - \frac{\beta \sum \frac{v_i p_i}{r_i - R_i}}{1 - \beta \sum \frac{v_i p_i}{(s_{\max} - R_i)(r_i - R_i)}} \right), \quad (44)$$

521 for the upper bound. In the above equations (s_{\min}, s_{\max}) is given by $(-\alpha, -1)$, $(-1, r_1)$ or $(-\alpha, r_1)$ for
 522 the above-listed cases (i), (ii) and (iii), respectively.

523 **4.4 Spatial dependence**

524 The quantity $\alpha / (\alpha + v_i)$ not only controls the slow timescales and hence the time to reach
 525 steady state for $x > qt/D$, but it also determines the advective transport velocity of the different
 526 sediment size classes. We show this by first defining the additional dimensionless space variable
 527 $z = Px/q$, then along with Eqs. (8) and (10), we rewrite Eq. (5) as:

$$\frac{\partial C_i}{\partial \tau} + \frac{\partial M_i}{\partial \tau} + \frac{\partial C_i}{\partial z} = \beta(1-H)p_i - C_i, \quad i = 1, \dots, I. \quad (45)$$

528 As discussed in §4.2, the deposited layer rapidly adjusts itself so that deposition and
 529 redetachment are always in balance, except for very short times. Hence, rearranging Eq. (10) to:

$$M_i = \frac{v_i}{\alpha} C_i - \frac{1}{\alpha} \frac{\partial M_i}{\partial \tau}, \quad (46)$$

530 shows that $\alpha^{-1} \partial M_i / \partial \tau$ can be interpreted as the leading order correction to this balance.

531 Differentiating Eq. (46) with respect to τ , neglecting the second-order derivative correction, and
 532 substituting into Eq. (45) gives the following approximation to Eq. (5) (Hogarth et al., 2004a):

$$\frac{\partial C_i}{\partial \tau} + \frac{\alpha}{\alpha + v_i} \frac{\partial C_i}{\partial z} = \frac{\alpha}{\alpha + v_i} [\beta(1-H)p_i - C_i], \quad i = 1, \dots, I. \quad (47)$$

533 Equation (47) shows that disturbances in the individual particle concentrations will propagate
 534 down the slope with a characteristic speed of $\alpha / (\alpha + v_i)$, a quantity that appeared earlier as an
 535 estimate of the slow eigenvalues as given by Eq. (37). For the small particles, $\alpha \gg v_i$ and so
 536 $\alpha / (\alpha + v_i) \approx 1$. Thus, these particles travel at close to the water velocity, q/D . However, large
 537 particles with $v_i \gg \alpha$ travel downstream at a dimensionless speed of α / v_i with the longest travel
 538 time therefore given by the largest particle.

539 Since Eq. (5) is hyperbolic, the method of characteristics shows that for a constant initial
 540 condition, solutions for $x > qt/D$, found by solving Eqs. (9) and (10), depend only on time.
 541 However solutions in the region $x < qt/D$ can depend on both x and t . For an imposed boundary
 542 condition that will result in significant spatial effects for $x < qt/D$, then our analysis will still
 543 apply to measured effluent concentrations until $t = DL/q$, for a flume of length L . However, as
 544 zero concentration boundary and initial conditions are commonly used in flume experiments on

545 rainfall-driven erosion (e.g., Jomaa et al., 2010; Proffitt et al., 1991), then neglecting the spatial
546 derivative will still result in a good approximation to $C_i(\tau)$ at the end of the flume even for $t >$
547 DL/q , provided DL/q is greater than or equal to the time of the peak total concentration in C , as
548 determined from Eqs. (9) and (10).

549 **5 Conclusions**

550 The approximate solution of Sander et al. (1996) to the Hairsine-Rose model is a useful means to
551 analyze the range of timescales (denoted by λ) inherent in rainfall detachment erosion and
552 transport of soils. The HR model divides the soil into I different size classes. There are $2I$
553 timescales, two for each individual particle size. The timescales are characterized as ‘fast’,
554 ‘intermediate’ or ‘slow’. For $\beta < 1$, each of the I size classes has a fast ($|\lambda| > 1$) and a slow (
555 $|\lambda| < 1$) timescale, while for $\alpha > \beta > 1$ this total changes slightly to $I + 1$ fast and $I - 1$ slow
556 timescales. The fast timescales govern rapid transient adjustments from the initial conditions to a
557 state where the mass of sediment in suspension and the deposited layer are in quasi-equilibrium.
558 In practice, this happens so quickly (less than seconds) that they are not resolved in a flume
559 experiment. The slow timescales that govern the subsequent slow transition to steady state are
560 predominantly controlled by the resorting of size classes in the deposited layer. There is also an
561 additional timescale approximated by $(\beta B)^{-1}$ that provides a good estimate for determining the
562 rate of growth of the total mass of sediment in the deposited layer. This time scale appears in
563 analytical approximations for the suspended and deposited layer concentrations obtained in this
564 work.

565 The fastest and slowest timescales are both controlled by the largest settling velocity, v_l . As v_l
566 increases, these two timescales become faster and slower, respectively. These are interpreted as
567 the vertical movement (deposition) and downslope travel time of this particle size class, and
568 provide bounds that can be used, for example, to design laboratory experiment durations
569 appropriately.

570 Compared to a soil with large particles, soils made up of *smaller* size classes will therefore have
571 smaller λ^{fast} timescales and larger λ^{slow} timescales such that steady state occurs sooner. Tight
572 bounds on all the individual eigenvalues were obtained. These are independent of the mass
573 proportions p_i in each size class and the detachability of the original soil β . Thus, p_i and β can
574 affect the characteristic rates to only a very limited extent and the primary determinants of the
575 erosion timescales are the settling velocities, v_i , and redetachability (of the deposited sediment),
576 α .

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627 **Figure captions**

628 **Figure 1.** Conceptual layout for the Hairsine-Rose model (Hairsine and Rose, 1991, 1992a,b).

629 **Figure 2.** Dimensionless total and particle size class suspended sediment concentrations (top
630 plot), dimensionless deposited size classes masses and H (bottom plot) as a function of τ from
631 Eqs. (9) and (10). Labels 1, 2 and 3 correspond to particles sizes 1, 2 and 3, respectively.

632 **Figure 3.** Plot of $\bar{K}(s)$ and $-1/\beta$ (solid lines) showing how the solutions of $\bar{K}(s) = -1/\beta$
633 (circled) sit in well-defined intervals defined by R_i and r_i (dashes) for $i = 1, 2, 3$. These are found
634 from Eq. (27) and correspond to roots of the quadratics in the denominator of Eq. (16).

635 **Figure 4.** Comparison of exact $H(\tau) = \sum M_i = 1 - h(\tau)$ from Eq. (31) (solid line) and the
636 approximation for H from Eq. (40) (dashed-dotted line) for the Black earth soil (parameter
637 values given in Table 2).

638 **Figure 5.** Response functions K_i , (deposition, left plot) and L_i (suspension, right plot) defined by
639 Eqs. (23) and (25), respectively, for the Black Earth soil for $\alpha = 100$, $\beta = 50$ and v_i from Table 1.
640 Each plot also shows h (dashed line) obtained from (26), which appears in the convolution
641 integrals of Eqs. (22) and (24). The circles (two for each curve) correspond to K_i and L_i
642 calculated at both eigenvalues corresponding to v_i . The plots show the different possible
643 timescales for the different sediment size classes. Size class 1 ($v_i \ll \alpha$) contains the finest
644 particles, transitional size classes correspond to $i = 2, 3$ ($v_i \approx \alpha$) and heavy sediment size class
645 to $i \geq 4$ ($v_i \gg \alpha$).

646 **Table 1.** Dimensionless Black Earth particle size distribution ($I = 10$ size classes) for a rainfall
 647 rate of $P = 56 \text{ mm h}^{-1}$, $p_i = 0.1$, $i = 1, 2, \dots, 10$.

Size class i	1	2	3	4	5
v_i	0.225	11.57	212.1	803.6	1414
Size class i	6	7	8	9	10
v_i	2507	3535	5142	8357	19286

648

Table 2. Eigenvalues (left column) for Black Earth with 10 size classes, divided as equal intervals of $\log v$. Parameter values are $\alpha = 100, \beta = 50$. The three sections in the table are the ‘fast’, ‘intermediate’ and ‘slow’ eigenvalues (i.e., time scales), with the lists of Estimates and Bounds in the heading referring to these sections, respectively. S_L and S_U are given by Eqs. (A6) and (A7), respectively, and r_i and R_i by Eq. (A2). Note how close the ‘fast’ values are to the estimates (middle column) of $(v_i + \alpha)$ and the ‘slow’ values are to either of the bounds (right column) r_i or r_{i-1} .

Eigenvalues (Numerical)	Estimates	Bounds
	$-(v_i + \alpha)$	R_i
	S_U	$S_U \quad S_L$
	$-\alpha/(v_{i-1} + \alpha)$	r_i
		-19387
-19382	-19386	-8458
-8453	-8457	-5244
-5239	-5243	-3637
-3632	-3636	-2608
-2603	-2607	-1515
-1510	-1514	-904
-899.8	-904	-313
-308.9	-312	-112
-110.9	-111.6	-100.23
-100.21	-100.22	-43.86
-38.88	-38.64	-38.64
		-0.9977
-0.9975	-0.9978	-0.8955
-0.8838	-0.896	-0.3197
-0.2940	-0.320	-0.1106
-0.1003	-0.111	-0.0660
-0.05847	-0.0660	-0.03834
-0.03434	-0.0384	-0.02750
-0.02364	-0.0275	-0.01907
-0.01513	-0.0191	-0.01182
-0.007519	-0.0118	-0.005158