Tomography of binary quantum detectors *

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Abstract— **A tomography method for binary detectors is developed. In this method, different input states are employed and the measurement data are then collected. First a primary estimation of the detector is obtained through least squares estimation, without considering the restriction on the eigenvalues of the detector. Then this possibly nonphysical estimation is projected onto the physical subspace to obtain a final estimation. We analyze the computational complexity of this algorithm, and present a theoretical error upper bound. Numerical simulation on a two-qubit example validates the effectiveness of the algorithm.**

I. Introduction

Quantum science and technology is a hot research topic in the recent decades, which consists of many branches like quantum computation [1], quantum communication [2] , quantum sensing [3], etc. Quantum computation utilizes the superposition of quantum states to perform certain computation tasks with an efficiency potentially much higher than classical (non-quantum) computers. Quantum communication encodes information in quantum states to realize theoretically secure exchange of key information. Quantum sensing employs quantum properties to perform measurements with high sensitivity or precision. These achievements, together with many other developing branches, are promising candidates for next-generation technologies in many information-related subjects.

One of the most notable features of quantum objects is that the measurement on them usually changes their states, which is in sharp contrast to the classical world. Therefore, measurement not only extracts the information encoded in quantum systems, but also can serve as an effective way to affect or alter the system. This highlights the significance of measurement in investigating and controlling a quantum system [4, 5]. To name a few examples, a sequence of appropriately designed measurements can perform quantum computation tasks [6]. Quantum key distribution [7], as a significant branch of quantum communication, requires measurement as a vital necessity. In quantum metrology, the Heisenberg limit can be achieved through adaptive measurement in phase estimation [8]. Therefore, quantum measurement is also important quantum resource, and it is desirable to develop methods to characterize a partially or fully unknown measurement devices (called *detectors*). The task to characterize quantum measurement devices [9, 10] is thus called *quantum detector tomography*, which is fundamental for other estimation tasks like quantum state tomography [11]- [15], Hamiltonian identification [16]- [21] and quantum process tomography [22]- [25].

To the best of our knowledge, specific quantum detector tomography protocols were first proposed in [26], where the Maximum Likelihood Estimation (MLE) method was employed to estimate an unknown POVM detector. POVM is short for positive operator valued measure, a measure with values being non-negative self-adjoint operators on the underlying Hilbert space and the integral being the identity operator. The matrix-version definition of POVM will be presented in Sec. II. MLE method has been widely employed in quantum tomography research [11, 27]. It can preserve the positivity and completeness of the detector, but it is difficult to characterize the error and computational complexity from a theoretical perspective. One common classification of quantum detectors is to divide them as phase-insensitive ones and phase-sensitive ones. The former ones correspond to diagonal matrices in the photon number basis and are thus relatively straightforward to be reconstructed. The latter ones mean that nondiagonal elements exist in the detector matrices in the photon number basis. The reconstruction of phase-insensitive detectors is modelled as a linear-regression problem in [28], and a least squares solution is obtained. In [29, 30], phase-insensitive detector tomography was established as a convex quadratic optimization problem and an efficient numerical solution was derived. This method was then experimentally tested in [31, 32], and was further developed in [33] and [34] to model phase-sensitive detector tomography as a recursive constrained convex optimization problem, where the unknown parameters are recursively estimated. For phase-insensitive detectors with a large linear loss, an extension of detector tomography method is introduced in [35] and tested on a superconducting multiphoton nanodetector.

In this paper, we focus on phase-sensitive detectors with only two POVM matrices (often called *binary detectors* or *on-off detectors*), which is a case relatively easy to tackle compared with the cases of three or more POVM matrices. By showing that we can equivalently focus on only one POVM matrix for binary detectors, we develop a binary detector tomography protocol. First J number of different types of states (called *probe states*) are employed and the measurement data are obtained. Then we employ least squares regression to obtain a primary estimation, which can be non-physical due to possible eigenvalues larger than 1 or being negative. Finally we analytically locate the projection

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of this primary estimation onto a physical space, in the sense of Frobenius norm. Since our algorithm has closed-form formula, it is amenable to theoretical analysis. We prove that the mean squared error of the final estimation scales as $O(1/N)$ with N the resource number of each input quantum state. We also demonstrate that our algorithm has computational complexity $O(d^2 J)$ for any d-dimensional binary detectors. We perform numerical simulation on a two-qubit phase-sensitive binary detector to showcase the effectiveness of our algorithm and illustrate the theoretical error analysis.

The organization of this paper is as follows. Section II introduces some preliminary knowledge and formulates the tomography problem for binary detectors. Section III presents the procedure of our algorithm and analyzes its computational complexity. Error analysis in theory is provided in Section IV. A numerical example is given in Section V to showcase the effectiveness of our algorithm and to illustrate the result of theoretical error analysis. Section VI concludes this paper.

II. Preliminaries and Problem Formulation

The state of a d-dimensional closed quantum system is characterized by a unit complex vector (denoted as $|\phi\rangle$) in the d-dimensional Hilbert space $\mathcal{H}$. Its dynamics is described by the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\phi(t)\rangle = H|\phi(t)\rangle, \tag{1}$$

where i is the imaginary unit, H is the system Hamiltonian and we set $\hbar = 1$ by using atomic units in this paper. For the case of open systems or mixed states, a density matrix ρ is employed to describe the system. ρ should always satisfy $\mathrm{Tr}(\rho) = 1$ and $\rho \geq 0$ (positive semidefinite). Measurement on a quantum system has various types, where one of the most common types is the positive operator valued measure (POVM). Mathematically, a POVM is associated with a set of positive semidefinite operators $\{A_i\}_{i=1}^n$ such that

$$\sum_{i=1}^{n} A_i = I,$$

where I is the identity matrix. The probability p_i for the i-th outcome to happen is given by Born's rule

$$p_i = \mathrm{Tr}(\rho A_i). \tag{2}$$

Assume that N copies of a state are employed in an experiment and the occurrence of the i-th outcome is N_i. We thus use $\hat{p}_i = N_i/N$ as the estimate of p_i, and throughout this paper we use $\hat{x}$ to denote the estimate of x. Quantum detectors are physical devices to perform a measurement, and we call $\{A_i\}_{i=1}^n$ a detector in this paper.

When the quantum states employed for measurement (called probe states) are known and $\hat{p}_i$ are obtained, the technique to deduce the unknown $\{A_i\}_{i=1}^n$ is called quantum detector tomography. The type of detectors with $n = 2$ is fundamental in quantum optics, and this type is usually called *binary detectors*, which is the main focus of this paper.

Suppose the k-th measurement result using the j-th quantum state ρ_j $(1 \leq j \leq J)$ is estimated as $\hat{p}_{jk}$, where $k = 1, 2$. We thus consider the following tomography problem for binary detector $\{A_1, A_2\}$.

Problem 1: Given states ρ_j and data $\hat{p}_{jk}$, estimate $\{\hat{A}_1, \hat{A}_2\}$ such that $\hat{A}_1 \geq 0$, $\hat{A}_2 \geq 0$ and $\hat{A}_1 + \hat{A}_2 = I$.

We first note the following two conditions are equivalent:
(i) $\hat{A}_1 \geq 0$ and $I - \hat{A}_1 \geq 0$;
(ii) $\forall 1 \leq i \leq d$, $0 \leq \lambda_i(\hat{A}_1) \leq 1$, where $\lambda_i(\hat{A}_1)$ is the i-th eigenvalue of $\hat{A}_1$.

Hence, we can transform Problem 1 into the following equivalent form:

Problem 2: Given states ρ_j and data $\hat{p}_{jk}$, estimate $\hat{A}_1$ such that $0 \leq \lambda_i(\hat{A}_1) \leq 1$. ($\hat{A}_2$ will be obtained from $\hat{A}_2 = I - \hat{A}_1$)

Remark 1: This equivalent transformation is a key difference that binary detectors distinguish from other types of detectors. When $n \geq 3$, we can no longer focus only on A_1, and thus the algorithm to be developed in Sec. III does not directly work.

III. Tomography algorithm

A. Least squares estimation

We first ignore the eigenvalue requirement

$$0 \leq \lambda_i(\hat{A}_1) \leq 1 \tag{3}$$

and seek for a preliminary estimation through least squares regression.

We parameterize the detector and the probe states. Let $\{Z_i\}_{i=1}^{d^2}$ be a complete Hermitian basis set of $\mathcal{C}_{d\times d}$ (the set of all d-dimensional Hermitian matrices), satisfying

$$\mathrm{Tr}(Z_i^\dagger Z_j) = \delta_{ij}$$

where δ_{ij} is the Kronecker delta function. We thus parameterize the detector as

$$A_1 = \sum_{i=1}^{d^2} a_i Z_i,$$

where $a_i \triangleq \mathrm{Tr}(A_1 Z_i)$. Also, the probe states are

$$\rho_j = \sum_{i=1}^{d^2} b_i^{(j)} Z_i,$$

where $b_i^{(j)} \triangleq \mathrm{Tr}(\rho_j Z_i)$. Let

$$a = (a_1, a_2, ..., a_{d^2})^T$$

and

$$b^{(j)} = (b_1^{(j)}, b_2^{(j)}, ..., b_{d^2}^{(j)})^T.$$

Thus we have

$$p_{j1} = \mathrm{Tr}(\rho_j A_1) = {b^{(j)}}^T a.$$

We further collect

$$p = (p_{11}, p_{21}, ..., p_{J1})^T,$$
$$Q = (b^{(1)}, b^{(2)}, ..., b^{(J)})^T.$$

We thus have

$$p = Qa.$$

Assume that the probe states have enough variety; i.e., $J \geq d^2$ and $(Q^TQ)^{-1}$ exists. Due to uncertainty or noise, $\hat{p}$ inevitably deviates from p in practice. We thus present a least squares estimation as

$$\hat{a} = (Q^TQ)^{-1}Q^T\hat{p}, \tag{4}$$

and the primary estimation is

$$\hat{B}_1 = \sum_{i=1}^{d^2} \hat{a}_i Z_i. \tag{5}$$

In (4) and (5) we use the notation $\hat{B}_1$ instead of $\hat{A}_1$ because at this stage we do not consider the eigenvalue constraint (3). Hence, $\hat{B}_1$ can be nonphysical, although it is always Hermitian. If $\hat{B}_1 \geq 0$ and $I - \hat{B}_1 \geq 0$, then we output the final estimation as $\hat{A}_1 = \hat{B}_1$ and our algorithm terminates here. Otherwise, we call $\hat{B}_1$ nonphysical and we need to further adjust $\hat{B}_1$ to obtain a final physical estimation. A natural idea is to project $\hat{B}_1$ onto the subspace consisting of all the physical estimations. Specifically, we aim to obtain the final estimation $\hat{A}_1$ through

$$\hat{A}_1 = \text{argmin}_{X\geq 0 \text{ and } I-X\geq 0}||X - \hat{B}_1||, \tag{6}$$

where $||\cdot||$ denotes the Frobenius norm throughout this paper.

B. Diagonalization

Suppose the spectral decomposition of $\hat{B}_1$ is

$$\hat{B}_1 = UEU^\dagger \tag{7}$$

where U is unitary and E is diagonal. Then we propose the following lemma to characterize the optimal solution $\hat{A}_1$ to (6).

Lemma 1: $U^\dagger \hat{A}_1 U$ is diagonal.

Proof: We prove by contradiction. First, from $\hat{A}_1 \geq 0$ and $I - \hat{A}_1 \geq 0$, we know

$$U^\dagger \hat{A}_1 U \geq 0 \tag{8}$$

and

$$I - U^\dagger \hat{A}_1 U = U^\dagger (I - \hat{A}_1) U \geq 0. \tag{9}$$

Eqs. (8) and (9) indicate that all the diagonal elements of $U^\dagger \hat{A}_1 U$ are in the region $[0, 1]$. Construct the diagonal matrix

$$F = \text{diag}[(U^\dagger \hat{A}_1 U)_{11}, (U^\dagger \hat{A}_1 U)_{22}, ..., (U^\dagger \hat{A}_1 U)_{dd}].$$

We thus know $F \geq 0$ and $I - F \geq 0$, which indicates

$$UFU^\dagger \geq 0 \tag{10}$$

and

$$I - UFU^\dagger \geq 0. \tag{11}$$

Eqs. (10) and (11) mean that $UFU^\dagger$ is a physical estimation.

Second, we assume that $U^\dagger \hat{A}_1 U$ has some nonzero non-diagonal elements. Then we know

$$\begin{aligned} ||UFU^\dagger - \hat{B}_1|| &= ||F - E|| \\ &< ||U^\dagger \hat{A}_1 U - E|| \\ &= ||\hat{A}_1 - \hat{B}_1||, \end{aligned}$$

which indicates that $UFU^\dagger$ is a better solution to (6) than $\hat{A}_1$, which contradicts the fact that $\hat{A}_1$ is the optimal solution. Hence, we reach the conclusion that $U^\dagger \hat{A}_1 U$ must be diagonal. ■

C. Eigenvalue correction

Now we investigate what values the diagonal elements in $U^\dagger \hat{A}_1 U$ should take. From the proof of Lemma 1 we know all the diagonal elements of $U^\dagger \hat{A}_1 U$ are in the region $[0, 1]$. Furthermore,

$$\begin{aligned} ||\hat{A}_1 - \hat{B}_1||^2 &= ||U^\dagger \hat{A}_1 U - E||^2 \\ &= \sum_{i=1}^d [(U^\dagger \hat{A}_1 U)_{ii} - E_{ii}]^2. \end{aligned} \tag{12}$$

Hence, to minimize $||\hat{A}_1 - \hat{B}_1||$ is equivalent to minimizing the diagonal elements one by one. For each $1 \leq i \leq d$, if $0 \leq E_{ii} \leq 1$, we should take $(U^\dagger \hat{A}_1 U)_{ii} = E_{ii}$; if $E_{ii} > 1$, we should take $(U^\dagger \hat{A}_1 U)_{ii} = 1$; otherwise if $E_{ii} < 0$, we should take $(U^\dagger \hat{A}_1 U)_{ii} = 0$.

D. General procedure and computational complexity

The general procedure and computational complexity of our algorithm are outlined as follows.

Algorithm 1: Step 1. Determine the basis set $\{Z_i\}_{i=1}^{d^2}$ and parameterize the detector and probe states. Using least squares estimation (4) and (5) to obtain a primary (possibly nonphysical) estimation $\hat{B}_1$. In (4), $(Q^TQ)^{-1}Q^T$ can be computed offline prior to the experiment. Hence, this step has online computational complexity $O(d^2J)$.

Step 2. Perform spectral decomposition (7), which has complexity $O(d^3)$ [36]. If $E \geq 0$ and $I - E \geq 0$, output the final estimation as $\hat{A}_1 = \hat{B}_1$ and terminate the algorithm. Otherwise, turn to Step 3.

Step 3. Define diagonal matrix F. For each $1 \leq i \leq d$,

$$F_{ii} = \begin{cases} 0 & : \; E_{ii} < 0; \\ E_{ii} & : \; 0 \leq E_{ii} \leq 1; \\ 1 & : \; E_{ii} > 1. \end{cases} \tag{13}$$

The final estimate is $\hat{A}_i = UFU^\dagger$. This step has complexity $O(d^3)$.

Since $J \geq d^2$, the total computational complexity of our algorithm is $O(d^2J)$. In practice we often have $J = O(d^2)$, and our computational complexity is thus reduced to $O(d^4)$.

IV. Error analysis

Theorem 1: Assume that the total number of copies of probe states employed for the detector tomography is N, evenly distributed to each state. The estimation error of our algorithm $\text{E}||\hat{A}_1 - A_1||$ scales as $O(\frac{1}{\sqrt{N}})$, where $\text{E}(\cdot)$ denotes the expectation on all possible measurement outcomes.

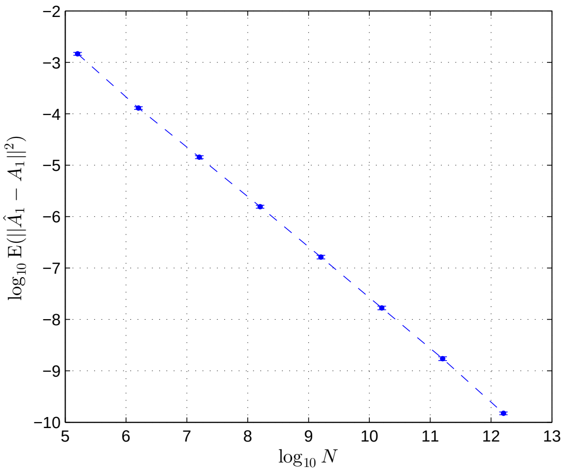

Fig. 1. Estimation errors versus resource numbers.

Proof: From the analysis in [12, 18], we know

$$\mathrm{E}(||\hat{B}_1 - B_1||^2) = \mathrm{E}(||\hat{a} - a||^2) \sim O(\frac{1}{N}). \tag{14}$$

If $\hat{B}_1$ is physical, we know

$$\mathrm{E}(||\hat{A}_1 - A_1||^2) = \mathrm{E}(||\hat{B}_1 - B_1||^2) \sim O(\frac{1}{N}),$$

which is the conclusion we expect to prove. Otherwise $\hat{B}_1$ is nonphysical, and from (6) we know

$$||\hat{A}_1 - \hat{B}_1|| \leq ||B_1 - \hat{B}_1||. \tag{15}$$

Hence, we have

$$\begin{aligned} ||\hat{A}_1 - A_1|| &\leq ||\hat{A}_1 - \hat{B}_1|| + ||\hat{B}_1 - B_1|| \\ &\leq 2||\hat{B}_1 - B_1||. \end{aligned} \tag{16}$$

Combining (14) and (16), we know

$$\mathrm{E}||\hat{A}_1 - A_1|| \sim O(\frac{1}{\sqrt{N}}),$$

which completes the proof. ■

V. Numerical results

We perform numerical simulation to showcase the performance of our algorithm. We choose a two-qubit detector

$$A_1 = e^{\mathrm{i}L} \begin{pmatrix} 0.9 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-\mathrm{i}L}$$

where

$$L = \begin{pmatrix} 1 & 2 & 2\mathrm{i} & 5 \\ 2 & 0 & -3\mathrm{i} & 0.7 \\ -2\mathrm{i} & 3\mathrm{i} & 4.4 & 2-0.1\mathrm{i} \\ 5 & 0.7 & 2+0.1\mathrm{i} & -1 \end{pmatrix}.$$

The pure probe states we used are $|m\rangle$, $\frac{1}{\sqrt{2}}(|j\rangle + |k\rangle)$ and $\frac{1}{\sqrt{2}}(|j\rangle + \mathrm{i}|k\rangle)$ for all $1 \leq m \leq d$ and $1 \leq j \neq k \leq d$. The results to estimate A_1 using our algorithm with different resources are shown in Fig. 1, where the horizontal axis is the logarithm of the number of copies for each probe state, $\log_{10} N$, and the vertical axis is the logarithmic estimation error $\log_{10} \mathrm{E}(||\hat{A}_1 - A_1||^2)$. Each point is the average of 50 repetitive runs. We see the simulation results match the result in Theorem 1 well.

VI. Conclusion

We have presented a tomography method for binary phase-sensitive detectors. Our algorithm has closed-form estimation formula, from which we deduce a theoretical error upper bound and computational complexity. We perform numerical simulation for a two-qubit binary detector to validate the effectiveness of our algorithm. The generalization of the method in this paper to the cases of $n \geq 3$ is a challenging task, because matrices even with the same dimensions lack a total order, and it is difficult to simultaneously diagonalize $n \geq 3$ matrices. We thus leave this extension to possible future work.

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