The topic of geometry includes developing students’ knowledge about aspects such as two-dimensional shapes and three-dimensional objects, symmetry, tessellations, transformation and geometric reasoning. Geometry also includes learning about position, location and arrangement in relation to maps, plans, scaled drawings and grid systems. Spatial knowledge extends from learning the simple language of position and arrangement in the early years—inside, outside, near, far, etc.—to using a map to determine distances in the later years. Also important in the topic of geometry are developing students’ spatial visualisation and spatial reasoning abilities, as well as making links to natural and built environments. Learning activities in this topic can involve exploration and investigation of buildings, bridges, consumer products, shapes and symmetry in plants and animals, and the construction of items from plans, involving spatial reasoning. The geometry topic also links to many occupations—architecture, landscape gardening, golf course designing, civil engineering, surveying, town planning, painting, crafts, art, building and dress designing. It is important to make connections from space to measurement (Battista 2010; Trafton & Hartman, 1997) and number topics, and to other curriculum areas such as art, craft, physical education, science and social studies.

A large body of research continues to highlight the link between early spatial reasoning and later performance in mathematics (Verdine, Golinkoff, Hirsch-Pasek, & Newcombe, 2017). This is an important consideration given the emphasis, perhaps over-emphasis on number in many school curricula. Spatial reasoning is considered critical for everyday tasks and helps us to understand, appreciate and interpret our geometric world (National Council of Teachers of Mathematics [NCTM], 2000). Mix and Cheng (2012) suggest that the link between spatial ability and mathematics is so well established that it no longer makes sense to ask whether developing spatial reasoning is related to mathematical learning. Given this, it is encouraging to note that school mathematical curricula are becoming increasingly spatial and graphic in nature and are moving away from predominantly word-based tasks to include a range of quantitative information (Lowrie, Logan, & Ramful, 2017). In Australia, spatial reasoning is identified as a core component in the Numeracy general capability within the Australian Curriculum and thus is to be incorporated into all subjects across the curriculum. The good news for teachers is that a large body of evidence suggests that most spatial skills are malleable and that these skills can improve as a result of spatial experiences, both more generally and also as a result of specific, intensive training (Stieff & Uttal, 2015).

The geometry strand is one of the richest in terms of language. Developing a range of strategies to support learning of the spatial language is important. Creating posters, books including big books for early years learners, word walls, flash cards, dictionaries, word games, and so on are useful and practical strategies for supporting learners to better understand the nuanced language of this strand. Words in this strand have specific meanings in relation to spatial concepts that are different to other strands of the mathematics curriculum as well as the non-mathematics meanings. A good example of this...
is the term “base” which is the geometry strand may refer to the bottom edge of a triangle or the bottom face of a prism whereas in the number strand “base 10” refers to a particular counting system. In the non-mathematics register, the term base can have many meanings such as ‘base camp’ or someone being based (located) at a particular site.

Teaching Tip
Create a word bank for the topic area you are covering. These should be created to suit the students’ needs
e.g. shapes – triangle, square, rectangle, circle, sides, angles, line, parallel, slope,
e.g. quadrilateral, rhombus, obtuse, acute, right angle, parallel, converging, gradient, convex, concave, regular, irregular
There are many dynamic geometric programs available on computers and hand-held devices. The potential of these tools to create new understandings appears to be more significant than the traditional pencil-and-paper approaches of the past (Chorney & Sinclair, 2018). Using dynamic geometry software has become an accepted part of contemporary mathematics education and so needs to be incorporated in teaching (Jones & Tzekaki, 2016). As educators (Larkin & Milford, 2018) caution in their work with apps – not all software is the panacea to good teaching so a critical analysis of the packages available is essential.

Topics in the geometry strand
Topics within the geometry strand include location (position, direction, coordinates), networks (maps, scale drawing, topology), three-dimensional objects (surface, corner, edge), two-dimensional shapes (sides, angles, polygons), symmetry (line symmetry, rotational symmetry), tessellations (tiling, design, art applications), dissections (puzzles), similarity (shape, perspective) and transformations (flips, slides, turns). Many of these topics are interrelated, and thus the geometry strand can be organised in different ways. Here we have organised the topics under shape and structure; transformation and symmetry; and location and arrangement.

Visualisation
Being able to visualise images is an important foundation for problem solving in mathematics (Lowrie et al., 2017). Developing a strong facility for visualisation will result in an increased capacity to solve problems. The development of visualisation is fostered through the study of geometry, so developing spatial sense is very important.

The study of geometry assists the development of visual processing—the capacity to create, manipulate and transform spatial images in the mind. The exercise in the diagram requires students to be able to visualise the rectangular image and rotate it in order to respond to the question. Visualisation also assists in interpretation of information represented in diagrams, graphs and maps. (Lowrie et al., 2017) have found that graphical literacy—the capacity to read diagrams, maps and graphs—is an important component of the geometry strand but it is a skill that needs to be developed, particularly from around the middle years of schooling—that is, from Grade 4 onwards. Their 2017 research has indicated marked gains in mathematics achievement following an intervention that developed students’ abilities in mental rotation, spatial orientation and spatial visualisation (Lowrie et al., 2017). Lowrie and Diezmann (2009) report that in the middle years, there are gender differences in graphical literacy of which teachers also need to be cognisant, and it is important to develop strategies to enhance these skills and redress differences.

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Visual processing is a key skill in mathematical problem-solving across the strands. It is recognised that the teaching of spatial concepts and processes is the key to students developing visualisation and spatial sense (Pegg & Davey, 1989). The most effective problem solvers have a strong visual sense as well as verbal, symbolic and abstract thinking. Throughout their study of geometry, students should continually be challenged so that they do not form inflexible images. An example of an inflexible image is when
students correctly identify a square when it is aligned horizontally but insist it is a diamond when it is tilted on one
corner. Students holding inflexible images often cannot identify forms other than an equilateral triangle as triangles. In a similar vein, triangles with their bases uppermost may be regarded as ‘wrong’ or ‘upside down’. Thus, in order to enhance visualisation skills, early shape-recognition activities must provide students with a range of forms and a variety of orientations for each particular shape. Teachers need to be able to recognize and articulate the properties of shapes and solids, including examples and non-examples (Tsamir, Tirosh, Levenson, Barkai, & Tabach, 2015).

The van Hiele levels of geometric thinking

The most cited and used theory underpinning the teaching of geometry is that proposed by Dutch researchers van Hiele and van Hiele who note that students’ geometric thinking typically can be represented by key characteristic styles, which they refer to as ‘levels’. According to this theory, students’ progression across the levels occurs in response to instruction, emphasising the impact of teaching on students’ thinking (the levels are not intended in the sense of broad developmental or biological progression.) The theory provides a useful model for planning teaching (Howse & Howse, 2015). The model has five levels, but for the primary school teacher the first three are most relevant. It is useful when teaching geometric thinking, particularly in the primary school years, that students have access to manipulatives, such as attribute blocks (Howse & Howse, 2015).

Level 1: Visualisation

At Level 1, students’ thinking about shapes relates solely to what they see. Students react to figures as wholes—a square is a square because it is a square. Similarly, an angle is an angle because it is an angle. Students do not think about the components of a shape or its properties. They might regard a square that has been rotated through 45° as a diamond—that is, they no longer see it as a square. Students at this level justify a circle being a circle because “it looks like a circle” (Ferguson, 2017).

Level 2: Analysis

At level 2, students begin to focus on the properties of shapes and to informally analyse the components of shapes—a square is a square because it has four equal sides and four right angles. However, they do not focus on relationships between properties.

Level 3: Abstraction or informal deduction

At level 3, students logically order properties, form abstract definitions and identify necessary and sufficient conditions associated with properties. For example, a square has some properties related to other shapes—a square is both a special rectangle and a special rhombus; if the opposite angles of a quadrilateral are equal, then the opposite sides are parallel; if I know that a parallelogram has one right angle, then it must be a rectangle; a triangle with three equal sides must have three equal angles.

Level 4: Deduction

Level 4 involves formal reasoning, where axioms, terms, underlying logical systems, definitions and theorems become the focus. This level is commonly associated with high school geometry, and includes developing formal arguments and proving theorems.

Level 5: Rigour

Level 5 involves comparing systems based on different axioms—for example, Euclidean geometry versus spherical geometry. Typically, the study of geometry associated with this level of thinking is undertaken beyond the high school years.

The importance of the van Hiele model

Central to the van Hiele model is that progression in geometric thinking is due to effective teaching, and is not necessarily associated with increasing age or maturation—effective teaching is the essential element. By focusing on the different ways students think geometrically, the van Hiele model allows teachers to identify how students are conceptualising geometric ideas and then how best to plan and organise effective learning so that the student can progress to more advanced thinking (Alex & Mammen, 2016). For example, a student working at level 2 may not be able to see that a square is a special kind of a rectangle. By recognising that the student is not seeing interconnections between properties, the teacher is able to organise appropriate learning.
experiences that have the goal of enabling the student to see connections between the properties of rectangles and of squares. In this way, students can become aware that a square is a special kind of a rectangle—a rectangle with the additional property that its adjoining sides (as well as its opposite sides) are equal.

For students in the primary years, the first three levels of the van Hiele model are generally very applicable. Using the model to carefully plan teaching can lead to enhanced understandings of geometry by students. However, as with any model, it should not be used to constrain learning and learners. If students are working beyond the first three levels, then teaching should scaffold students beyond these levels.

Teaching notes
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The research of van Hiele and van Hiele has informed much of the current teaching of geometry. Central to their work is the important role of teaching of geometric ideas. They argue that the complexity in students’ geometric thinking arises from the experiences provided by teachers. Geometry activities should link to and draw from the everyday world and build on students’ experiences with spatial aspects of their environments. Geometry activities should be multisensory, informal before formal, and provide ample opportunities for the development of visualisation.

Appropriate geometric activities should use and develop important skills such as:

- Visualising—manipulating and transforming spatial images in the mind; interpreting figural information; interpreting and reading diagrams or maps
- Communicating—developing accurate and precise geometric language, both spoken and written (start with children’s natural language and gradually develop more formal terminology)
- Drawing and modelling—sketching and modelling with appropriate accuracy
- Thinking and reasoning—asking students to give reasons for their answers: Why did you say that? How do you know that? What evidence is there for that?
- Applying geometric concepts and knowledge—drawing attention to real-life applications; deriving problems from real-life situations; considering relationships between shape and function.

Spatial concepts: Early activities

Play activities in the early years are important for developing spatial concepts. Rather than being random, play activities should be planned carefully and their purpose should be clear. Early activities include:

- Sorting—according to students’ criteria or criteria suggested by the teacher; for example, rolling or sliding, height, pointiness; games of identifying which one is different or which one is the same
- Building—Lego, blocks, junk; making the tallest or widest structure; making a symmetrical building; making an enclosure for this object
- Packing—placing materials neatly into storage boxes
- Modelling—with plasticine, clay, straws and string, cardboard and tape or rubber bands, toothpicks
- Touching—using touch to identify a shape (feely bag); describing the object to others
- Matching—shapes to diagrams; matching a picture and a model; building a three-dimensional structure from its picture; matching containers with models
- Drawing—solids from different viewpoints; identifying and drawing shapes after walking through the park, the playground, the garden; discussing reasons particular shapes have been used for various structures
- Visualising—replicating structures behind a screen that have been shown for a few seconds.

Shape and structure

Two-dimensional shapes

The study of two-dimensional shapes relates to topics such as polygons, lines and angles. Polygons are closed, two-dimensional shapes with three or more straight sides. The word polygon comes from two words: poly, meaning many, and gon, meaning sides. Hence a polygon is a many-sided, closed figure.
Polygons are a type of
plane shapes as they lie on the two-dimensional plane. The name of a specific polygon includes a prefix determined by its number of sides (see table).

**Names of some polygons**

<table>
<thead>
<tr>
<th>No. of sides</th>
<th>Prefix</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>tri-</td>
<td>triangle</td>
</tr>
<tr>
<td>4</td>
<td>quad-</td>
<td>quadrilateral</td>
</tr>
<tr>
<td>5</td>
<td>penta-</td>
<td>pentagon</td>
</tr>
<tr>
<td>6</td>
<td>hexa-</td>
<td>hexagon</td>
</tr>
<tr>
<td>7</td>
<td>hepta-</td>
<td>heptagon</td>
</tr>
<tr>
<td>8</td>
<td>octa-</td>
<td>octagon</td>
</tr>
<tr>
<td>9</td>
<td>nona-</td>
<td>nonagon</td>
</tr>
<tr>
<td>10</td>
<td>deca-</td>
<td>decagon</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>many</td>
<td>poly</td>
<td>polygon</td>
</tr>
</tbody>
</table>

Polygons are classified as regular or irregular. Regular polygons have all sides equal and therefore all angles equal. An irregular polygon has either some or all of its sides of different lengths, or some or all of its angles of different sizes, or both sides and angles differing. In some cases, students might be presented with polygons that are almost exclusively regular. Limiting students’ experiences in this way can result in their having an impoverished notion of polygons. They might have difficulty in naming certain polygons (Ozkan & Bal, 2017). Thus it is important that students engage in activities with both regular and irregular polygons.

In the early years, the focus is on assisting students to identify common shapes such as squares, triangles and circles (NB: a circle is not a polygon as a polygon must have only straight sides). Richer conceptual understanding is promoted through activities that direct students to consider properties of shapes, and by providing a variety of examples of particular polygons—both regular and irregular—in various spatial orientations (Seah, 2018). Sorting activities are useful for preventing the development of inflexible conceptualisations of particular shapes.

When teaching basic geometric shapes, students need to become familiar with shapes presented in a range of orientations. If students’ experience of two-dimensional shapes is mainly limited to shapes in an upright orientation (i.e. where the base line is on the horizontal orientation), this can result in misconceptions. For example, students may regard only one of the two squares in the diagram as a square. Therefore it is important that students are presented with examples of basic 2D shapes in a variety of orientations.

Tangrams are inexpensive and useful in the teaching of two-dimensional shape. Tangram activities not only support the learning of the names of shapes but also help students to develop spatial visualisation. Various tangram puzzles are available but a seven-piece tangram is most commonly used (van Hiele, 1999). The tangram allows students to make many shapes, both prescribed and free form. It also allows them to explore aspects of transformational geometry as they create or copy shapes.

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**Teaching Idea**

**Geoboard activities**

**Exploring lines**

- Make the shortest line segment possible on your geoboard.
- Make the longest line segment possible.
- Make three line segments of different lengths.
Exploring angles

- Make a narrow angle.
- Make a wide angle.
- Make an angle like the corner of a square.
- Make an angle with 2 nails between the rays.
- Make an angle with 0 nails between the rays.

Exploring regions

- Construct a region that has just one nail inside it.
- Construct a region that has 0 nails inside it.
- Construct a region that has 1 nail outside it.
- Construct a region that has 1 nail outside and 3 nails inside it.
- Construct a region that has 3 nails on the boundary.

Line and angle

Learning about line and angle requires students to be engaged in constructing a variety of lines and angles and discussing them (Host, Baynham, & McMaster, 2014). Geoboards are a valuable resource for investigating lines and angle. Geoboards are wooden (or plastic) boards with nails protruding in a grid pattern. Frequently they are in a square array, but can be in circular patterns as well. A variety of two-dimensional shapes can readily be created on a geoboard using rubber bands, and language associated with shapes, lines and angles can be fostered (Britton & Stump, 2001).

The geoboard is also a valuable resource for constructing and learning about angles. Many students develop misconceptions about angles (Mitchelmore, 2000), particularly in relation to the size of the arms of an angle versus the size of an angle. The notion of angle relates to measuring an ‘amount of turn’. A simple way to introduce students to the idea of angle is to ask them to stand with their arms outstretched in front of them and then open one arm to the side through an angle of 90°. This can be linked to turning all the way around, which is a turn through 360°, and turning halfway around, which is a turn of 180°. Outstretched arms can be dropped to the side of the body and turning can continue in order to emphasise that the notion of angle relates to an amount of turn rather than the length of the arms or the length of the lines making an angle.

Teaching Idea

Sorting shapes

Which of these shapes are triangles? Put a cross on the shapes that are triangles.

Cut out the shapes and sort the triangles into one group. Explain why you sorted the shapes out in that way.

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To focus further on angle (e.g. the angles made by the hands of an analogue clock) an angle tester can be created. This is simply a circle of paper folded into four quarters. The corner created is a right angle and can be used to find angles that are exactly 90°, less than 90° and greater than 90°. An angle wheel (see diagram) can be constructed using two circles of equal size cut from different coloured pieces of paper. By cutting a split to the mid-point of both circles, fitting the circles together and turning them, parts of each circle can be seen, from either side, forming angles of different sizes. Angles can initially be classified as acute (sharp), right angled (like the corner of a square), obtuse (blunt), and straight (flat line). An interactive website for developing 2-dimensional understanding of shapes and angles can be found here (<http://www.mathopenref.com/planegeometry.html>). The site is interactive, for example, students can create various angles, or create various polygons using a drag and drop action.

Three-dimensional objects

The topic of three-dimensional (3D) objects focuses on learning names and properties of solids (3D
objects), relationships among solids and visualisation skills associated with this topic. Learning about solids can include learning about 3D objects that are important in geometry (e.g. cubes, spheres, pyramids, prisms, cylinders and cones), and linking this knowledge to objects that occur in the natural and built environments. For example, a block of wood might have the shape of a prism, a basketball has the shape of a sphere and a funnel has the shape of a cone. Activities involving the construction of solids can support students’ development of spatial knowledge (Ambrose & Falkner, 2002).

Many syllabuses list the topic of 3D objects before the topic of 2D shapes. This is because young students can think more easily about solids than they can about planar (two-dimensional) shapes. Solids can be regarded as more experientially real for young students because they are essentially objects that can easily be handled whereas the notion of a 2D shape is more abstract. Indeed, students are often confused when teachers pick up an object and call it a shape.

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### Naming and classifying solids

It is important to understand the nomenclature of solids. There are parallels between the nomenclature of solids and that of plane figures, and teachers should be aware of the links in these two naming systems. The term ‘solid’ is used in an inclusive sense—any 3D object, whether it is ‘hollow’ or not, is a solid. Objects in the sub-group of solid figures that have flat surfaces (faces) only, are referred to as ‘polyhedra’ (singular ‘polyhedron’) as the word polyhedra comes from two Greek words: *poly*, meaning many, and *hedra*, meaning faces. Just as circles and semi-circles are 2D shapes that are not polygons, so spheres, cylinders and cones are examples of solids that are not polyhedra (because of their curved surfaces). Examples of polyhedra are cubes, rectangular prisms and pyramids.

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### Prisms and pyramids

Look at the diagram of solids above. Compare the prisms with the pyramids. What do you notice? The first thing you will likely notice is that the pyramids form a point. Pyramids are named by the two-dimensional shape of their base. All other sides of the shape are triangles. Prisms have at least one pair of opposite faces of particular polygons (e.g., rectangle, triangle, pentagon, hexagon) and the remaining four faces are rectangles.

**Teaching Idea**

Which of these solids roll and which of these solids don’t roll?

### Nets of solids

The notion of a ‘net’ is the basis of an important activity in geometry that helps to connect ideas in 2D and 3D space. The net of a solid is a plane shape that can be folded up to make a hollow representation of the solid. Early learning activities for nets commence when students take a cardboard box (such as a biscuit box) and unfold it. They can see how the 3D biscuit box was constructed from a flat 2D shape. The specific shapes that were used to construct the 3D box can readily be seen. Students can then be challenged to create their own net to create a new box. More advanced activities can involve students drawing nets for a range of solids, then using them to construct the solids. Keep in mind that nets can be made for some solid figures that are not polyhedra (e.g. cylinders and cones). It is important for students to realise that a net is a flat, 2D shape that is folded to create the 3D object. Even though the net can be seen to consist of several 2D shapes, they are connected together in one flat shape ready for folding. A very useful app, based on research, which can support understanding of nets of a cube is Click the Cube (<https://itunes.apple.com/au/app/click-the-cube/id1157365733?mt=8>). When using the app students can manipulate the cube to “see” the net being formed.

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Solid, hollow and skeletal forms

Using a wide array of resources is invaluable in terms of creating opportunities for students to construct understandings of solids and their properties. Seeing the same solid in three different forms (solid,
hollow and skeletal) is a key teaching strategy to develop deeper understandings of 3D shape properties. Teaching aids for 3D space typically include a set of solids made of wood. Hollow objects—that is, objects made from nets or from connecting plane figures—enable students to focus on the number of flat surfaces of the shape—that is, its faces. Skeletal figures can readily be made by using straws or toothpicks connected with marshmallows or Blu Tack. These shapes promote familiarity of language for 3D objects of vertices, faces and edges. When making hollow objects, students are working with each separate flat surface of the object—that is, its faces. When making skeletal models, the ‘skeleton bones’ clearly highlight the edges of the object and the connectors (marshmallows or Blu Tack) emphasise the vertices.

Euler’s formula

To expand students’ understanding of 3D solids, the exploration of Euler’s formula provides an interesting investigation. Euler (pronounced Oiler) was a mathematician of the eighteenth century. Provide students with a range of 3D polyhedra (remember, 3D solids with curved surfaces are not polyhedra so don’t include cones and cylinders) and have students create a data table of the number of vertices, faces and edges of each of the shapes. After they have completed their table, ask them whether they can see a relationship between the number of vertices, faces and edges. They will see that, for each shape, the number of vertices plus the number of faces is equal to the number of edges minus 2 (V + F = E - 2).

<table>
<thead>
<tr>
<th>Shape</th>
<th>Vertices</th>
<th>Faces</th>
<th>Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube</td>
<td>8</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>Rectangular prism</td>
<td>8</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>Square-based pyramid</td>
<td>5</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>Triangular-based pyramid</td>
<td>4</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Triangular-based prism</td>
<td>6</td>
<td>5</td>
<td>9</td>
</tr>
</tbody>
</table>

Transformation and symmetry

Transformation and symmetry are interrelated topics. Transformation, as the name implies, is about altering a shape in some way. This could be altering its position in space (congruent transformation), changing its size (projective transformation) or changing its features (topological transformation). The terms ‘object’ and ‘image’ are often used in association with transformation. The figure prior to the transformation is referred to as the object; after the transformation it is referred to as the image. Symmetry includes line symmetry and rotational symmetry in 2D space, and plane symmetry and rotational symmetry in 3D space. The topic of tessellations (or tiling) is an aspect of 2D space that draws on notions of both transformation and symmetry and can also reinforce conceptual understanding of area.

Congruent transformations

A shape transformation that leaves the original shape unchanged, but alters its position, is called a rigid or congruent transformation. Congruency is often a source of confusion of learners due to the way it has been taught (Plaksin & Patkin, 2016). Providing experiences with various shapes (triangles and quadrilaterals in the first instance) assists in building understandings of congruence. A rigid transformation occurs when a shape is reflected, translated or rotated from its original position. In some curriculums these transformations are referred to respectively as flips, slides and turns. In congruent transformations, the image is always congruent with the object. The transformations of flip, slide and turn relate closely to the topics of symmetry and tessellations (see below).
Projective transformations

Projective transformations are enlargements (dilations) or reductions of the original object. An enlargement (dilation) is a transformation where the object is made larger while maintaining the original shape. A reduction is a transformation where the object is made smaller while maintaining the same shape. Thus enlargements and reductions preserve shape, but not size. They also preserve directions of lines and orientation. Projecting shapes on to a wall and tracing the image, and then comparing it with the original object, provide a great way to introduce students to this type of shape transformation. Shadows can be linked to explorations of projective transformations.

Topological transformations

Topological transformations involve stretching and bending without breaking or tearing. Properties such as lengths of lines, sizes of angles and straightness are ignored. An object and its image under a topological transformation are said to be topologically equivalent. Thus a quadrilateral, for example, is topologically equivalent to any other polygon. A soccer ball that is not inflated is close to a flat shape. When inflated, it takes on the (spherical) shape of a soccer ball. The flatish shape and the air-filled shape are said to be topologically equivalent.

A lump of plasticine is an ideal starting point for exploring topological geometry. Rolling the plasticine into a sphere, then changing its shape into a cube demonstrates that these two shapes are topologically equivalent. Other shapes made by manipulating the plasticine—a bowl, a plate—are also equivalent. However, if a hole is put in the plasticine, the shape is no longer topologically equivalent. Thus, for example, a sphere and a doughnut shape are not topologically equivalent (see 'Teaching idea—Topological shapes').

Teaching Idea

Topological shapes

Shapes can be classified topologically by the number of holes in their surface.

- Start with a lump of play-dough with no holes in its surface. Make it into a ball shape. Now make it into a cube shape.
  1. Did the amount of play-dough change from what you started with?
  2. Are the shapes topologically the same?
- Start with a lump of play-dough and put a hole right through it (use a pencil). Make a doughnut and then the number 9 out of play-dough.
  1. Does each shape have just one hole?
  2. Did the amount of play-dough you started with change?
  3. Are the shapes topologically the same?
- With your play-dough, make the digits 0 to 9. Classify each one topologically.
- Find out if all the letters in your name belong to one particular topological category.
- Is it possible to have a given name in which all the letters are from one topological category?

The Möbius strip can be the basis of a rich topological investigation. Constructing a Möbius strip involves taking a long strip of paper, turning one end through one half-turn, and taping the two ends together. If a line is drawn down the middle of the Möbius strip, it eventually meets its beginning point. Thus a Möbius strip is considered to have one side only. Interesting activities can begin with making a longitudinal cut about one-third in from the edge of the strip (see 'Teaching idea—the Möbius strip'). Making cuts at other distances in from the edge of the strip yields different results. These activities can lead to conjectures, hypotheses, discussion and questioning.
Teaching Idea

The Möbius strip

1. Make a Möbius strip by cutting out a flat strip of paper about 50 cm long. Twist the paper (once) then join the two ends to make a closed ring.
2. Try colouring one side of the strip red and the other side green (or any two different colours). What do you notice?
3. Try to draw a line along the centre of the strip, continuing until you come back to the same point from which you started. Are you convinced that this strip has only one side?
4. Predict what you think will happen if you cut along the middle of the strip of paper. Try it. Were you surprised by the result?
5. Make another Möbius strip. This time draw a line along the strip one-third in from the edge. Continue to draw the line the same distance from the edge until you return to the point from which you started. Cut along the line you have just drawn. What happened? Did you expect that would occur?
6. Make a Möbius strip which is twisted twice before the ends are joined. Repeat activities 3, 4 and 5.
7. Make Möbius strips with three or four twists and repeat activities 3, 4 and 5. Can you discover a pattern emerging?

Symmetry

Symmetry is a geometrical property that can be identified in everyday objects in both the natural and built environments as well as in some geometric figures (2D shapes and 3D objects).

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Line or lateral symmetry

Symmetry can arise in several ways. The most easily recognisable kind of symmetry is that associated with 2D space and the idea of a mirror image. This kind of symmetry can be seen in natural objects such as leaves and in geometry in plane figures (2D figures) (Raisor & Hudson, 2018). As seen in the examples below, symmetry of this kind is associated with a line of symmetry. This kind of symmetry is referred to as line symmetry, lateral symmetry or mirror-image symmetry. An important activity for students is to use a mirror to reflect images of objects back on themselves in order to determine lines of symmetry. Another excellent resource is the Mira (a hard piece of translucent plastic) that can be placed perpendicular to a shape so that students can see both the original shape and the mirror shape. Activities using the Mira can be found here (<https://www.mrlondonsclass.com/mira.html>). Activities using mirrors or Miras can be applied to everyday flat objects (leaves), to pictures (a person’s face), to symbols such as many upper-case letters, and to 2D shapes. Place the mirror on the centre line of an object and see whether the reflected half of the image fits exactly on the actual object. Line symmetry can also be investigated through paper-folding along a line. To test whether a line of symmetry exists, cut the shape out of paper, and fold it in half back on itself along the predicted line of symmetry. If it folds back exactly, that is the line of symmetry.

Try to categorise the 26 upper-case letters according to the number of lines of symmetry. Students have opportunities to determine all possible lines of symmetry for a range of 2D shapes—various triangles and quadrilaterals (square, rhombus, kite, etc.), and various pentagons, hexagons, etc. There is also a correspondence between line symmetry and the congruent transformation of a shape by flip or reflection (see above). When a shape is flipped, it becomes a mirror image of the original object. A line is drawn to indicate the exact point where the flip occurred to result in the mirror image. This is the line of symmetry for an object under a flip transformation. The figure on one side of the line of symmetry is the image of the object on the other side.

Rotational symmetry in 2D space

Symmetry in 2D space can arise in two ways—line symmetry and rotational symmetry. Rotational symmetry is associated with turning the 2D shape through a unit fraction (1/2, 1/3, 1/4, etc.) of a full circle. To test for rotational symmetry, draw an outline of the shape on another piece of paper. Place the shape on its outline. Mark the ‘top point’ of the shape with a dot. Turn the shape until it completely matches its...
outline. Count the number of times it fits its outline until it reaches the original position. An object or
figure is said to have rotational symmetry of order two if, when turned through one-half of a circle (180°), the image corresponds exactly to the object in terms of the position it occupies. Rotational symmetry of order two is the simplest case of rotational symmetry and is referred to as point symmetry. In a similar vein, an object or figure is said to have rotational symmetry of order three if, when turned through one-third of a circle (120°), the image corresponds exactly to the object in terms of the position it occupies. Rotational symmetries of orders four, five, six, etc. are defined similarly.

As in the case of line symmetry, activities relating to rotational symmetry can involve flat objects, symbols (upper-case letters) and 2D shapes. Try to categorise the 26 upper-case letters in terms of whether or not they have rotational symmetry and, if so, of what order. A final point to keep in mind is the general separateness of line symmetry and rotational symmetry. Although an equilateral triangle has three lines of symmetry and rotational symmetry of order three, and a square has four lines of symmetry and rotational symmetry of order four, line symmetry and rotational symmetry are not always associated in this way. Try to draw figures that have rotational symmetry of various orders and no lines of symmetry, or figures with one or more lines of symmetry but no rotational symmetry.

Symmetry in 3D space
Symmetry can also be considered in 3D objects. Imagine cutting an object in half. Do you have two identical objects? If yes, then the cut that yielded the two objects that mirror each other is called a plane of symmetry. Consider a solid such as a cylinder—how many planes of symmetry does it have? Since circles have an infinite number of lines of symmetry, cylinders have an infinite number of planes of symmetry. Rotational symmetry can also be considered in 3D space. Rotational symmetry about a point in 2D space corresponds with rotational symmetry about an axis in 3D space. As in the case of planes of symmetry, students could investigate axes of symmetry, and the order corresponding to each axis, associated with 3D mathematical objects and other everyday solids.

Tessellations
The topic of tessellations is associated with 2D shape and links to the everyday notion of tiling and area. A 2D shape is said to tessellate if it can cover an area without gaps or overlaps. Below are some examples of tessellating shapes. They extend in all directions without gaps or overlaps.

The following examples do not tessellate as there are gaps when the shapes are placed together.
regular pentagon, regular hexagon, etc.
• Consider a scalene triangle, e.g. with sides of 6 cm, 8 cm and 13 cm. Will this triangle tessellate?
• Consider a quadrilateral with sides 4 cm, 7 cm, 9 cm and 12 cm. Will this figure tessellate?
• Try to find a pentagonal shape that will tessellate.

One of the appealing aspects of the topic of tessellation is its potential for integrating 2D geometry and art. Use of tessellation in artistic and decorative contexts has a long history. This topic lends itself well to investigations of library sources and websites for artistic examples of tessellations as they occur in a range of cultural and historical contexts (Paznokas, 2003). Investigations could include finding information about, and examples of, the work of Martin Escher, who embraced both art and geometry. His work included unusual and picturesque examples of tessellation (see ‘Teaching idea—Escher tessellations’). Students can use tessellation as a basis for their own artwork.

Teaching idea

**Escher tessellations**

Take a regular tessellating shape such as a rectangle.

Make a cut from one end and then slide this to the opposite end and glue. Do the same to the corresponding edges.

This odd shape may not appear as if it will tessellate. Students should be allowed to predict whether it will tessellate or not. Provide opportunities for students to create their own tessellations and share with the class.

To create Escher-type artworks, begin with a shape that tessellates. Change the shape of one or more sides—for example, by introducing curves and using congruence transformations—flips, slides or turns—as the basis for additional and compensating changes that preserve the tiling aspect of the shape. In this way, the topic of tessellation is seen to link closely to the topics of transformations and symmetry (Whitaker, 2001).

**Technology and geometry**

Several computer software programs have been developed to support teaching and learning in geometry. LOGO®, for example, was developed more than 20 years ago, and together with its recent derivations (e.g. Microworlds®) supports learning of geometry, and in particular the sub-topic of location. LOGO® has useful applications across the primary school years and is also useful for teaching coding. More recently developed programs for geometry are Cabri®, Geometer's Sketchpad® and Geogebra® - commercial programs suitable for the upper years of primary schooling and into secondary schooling. The usefulness of these programs relates to the ways in which they enable students to engage in dynamic geometry activities (Battista, 2002).

**Location and arrangement**

The topic of location focuses on ways to characterise position within a space. In curriculum documents, this sub-strand has a range of labels—location, arrangement or position—and includes learning about coordinate systems. At more advanced levels, this includes developing notions associated with map reading—for example, coordinates and scales—and using a globe to find locations and learn about topics such as latitude and longitude.

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The language of location and movement

Locational or positional language is a key component of early location work. Students come to school with varying experiences of language relating to the notion of location. Some students will have a well-established location language whereas others may not. Teachers need to spend time developing this aspect of location. The language of location includes words to describe objects in relation to others—the chair is behind the table; the vase is above the cabinet. Other language can be related to movement—move forward one step; jump backward three squares.

Following is a list of terms associated with the language of location:

<table>
<thead>
<tr>
<th>above</th>
<th>top</th>
</tr>
</thead>
<tbody>
<tr>
<td>below</td>
<td>bottom</td>
</tr>
<tr>
<td>right</td>
<td>forward</td>
</tr>
<tr>
<td>left</td>
<td>backward</td>
</tr>
<tr>
<td>in front of</td>
<td>beside</td>
</tr>
<tr>
<td>behind</td>
<td>around</td>
</tr>
</tbody>
</table>

Early location experiences

Students can begin to learn about coordinate systems in the early years (Dobler & Klein, 2002). Early experiences focus on students learning to read the grid references corresponding to a region where items appear. Typically, these involve an alpha-numerical pairing—that is, letters are used on one axis and numerals are used on the other. Each square on the map is characterised by a distinct alpha-numerical pair. When using a map such as this, by convention, students would work with letters on the x-axis (horizontal) and numerals on the y-axis (vertical). Using the alpha-numerical scale supports students’ learning to read the x-axis first and y-axis second. This is important for both the topic of location and further work with graphs and functions. Students need to experience two aspects of location. First, the coordinates are given and the student identifies the site—what is located at B2? Second, the site is given and the student determines the coordinates—what are the coordinates for the bag of money?

As students develop knowledge of alpha-numerical scales, they can be introduced to the use of ordered numerical pairs. Thus activities involving reading the value on the x-axis, then reading the value on the y-axis, serve to develop knowledge of the more abstract notion of an ordered pair. The term ‘ordered’ is used in the sense that, by mathematical convention, the first value of the pair corresponds to the value of the x-axis, and the second corresponds to the value on the y-axis. Notation such as (3, 5) is used for an ordered pair.

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Later experiences of location

Once students have learnt to read ordered pairs, teaching can focus on associating each ordered pair with a point rather than a region. As part of the progression to associating each ordered pair with a point, each numeral on the axes corresponds with a grid line rather than an interval on the axis. This can be seen on the map above, where the numerals are aligned with the grid lines.

Initial learning experiences may involve the items being placed on intersecting grid lines. As students’ fraction knowledge develops, items to be located can be placed at points where students can readily identify the point—this can involve simple fractions such as halves and quarters initially (1.5, 2–2.5), and later tenths (1.3, 1.8). This approach enables students to see the usefulness of finer gradations on the axes—points can be characterised more precisely.

Create problems for the students whereby they come to ‘discover’ the need for gradations in the pairings.

Applications

Once students have developed knowledge of ordered pairs, they can engage in games such as Battleships, where the objective is to locate and sink your partner’s ships (see ‘Teaching idea—
Battleships’). More realistic applications involve the use of maps (analogue and digital) to locate sites. Initially, this can involve local street directories, which typically use alpha-numerical scales and regions. Later experiences can focus on using maps in an atlas or online. An important point to keep in mind when working with maps in atlases is that coordinates are specified and read as follows—first latitude (North or South), then longitude (East or West). Thus the order of reading these map coordinates is at odds with the order of reading coordinates on a grid system—first horizontal (x-axis), then vertical (y-axis). Students may struggle with this at first, in part due to the order of reading the pairs, and also because of the use of four quadrants (NE, NW, SE, SW), but it is a life skill that is important for students to understand.

**Teaching idea**

**Battleships**

Using grid paper, students put a number of ships on the paper. In pairs, students list off coordinates to identify and sink their partner’s ships. The winner is the one who sinks all the ships first.

**Moving through space**

A common teaching practice is for students to be provided with a map and then given a series of directions—such as move forward two units, turn right, then move forward x units. Such activities often use names of streets to incorporate a reality element to the task. Commonly, this activity may involve starting at a point, such as a railway station, and then navigating through streets, turns, distances, and so on to arrive at a particular place, which the student must name. This requires the coordination of a number of variables and is quite complex. It is a useful activity as it resembles out-of-school experiences of using maps to find sites or moving through new territories. Diezmann and Lowrie (2010) report that in this activity students often make errors by not coordinating the various elements of the task, such as overlooking one element—ordinal language (the third street) or positional language (at the school or into Lambert Rd) or positional language (turn right). They report that the development of this skill appears to plateau in the middle years around Grade 6 or 7.

**Location and technology**

Research is beginning to emerge about students’ map-reading skills. We have shown the typical teaching process, where elements are built into the curriculum progressively so that students learn the required mapping skills. This approach seems to limit what students can do when they use digital environments. Typically the experiences in school mathematics have been contained to two-dimensional space—such as maps on paper. However, in the ICT and games environments, young children are exploring 3D representations where objects are moved along the two dimensions of the screen but also appear to ‘fall’ into three-dimensional spaces as well. Lowrie (2003) has shown how young children (eight-year-olds) have been found to construct complex images of the maps beyond the immediate visual cues and to construct three-dimensional images. In his work in this area, Lowrie has shown how young children not only create three-dimensional mental maps of the worlds in which they play (such as Mario Brothers or Pokemon or Minecraft) but are capable of creating mental images of the maps outside the screen area. The impact of this has not yet formally been recognised in curriculum documents, but it challenges the static views of map reading that are currently presented. Furthermore, middle school students are often involved in complex computer games where they are required to navigate through worlds to solve problems. The challenges they confront in these worlds are not only useful in terms of problem-solving but also help to develop visualisation, as they create the mental maps of the worlds they will encounter as they move from one screen to the next. This information is not available to them but must be created before they are able to move successfully into the next screen. The impact of these games will have a profound effect on how map reading is to be conceptualised in the emerging theorising of curriculum and knowing. Clearly, these technologies will create new learning opportunities for spatial thinking and visualisation that were inconceivable in the traditional pencil-and-paper format of previous times.

A further advance in modern society is the increasing availability of Global Positioning Systems (GPS). Various tools are now available that allow students to undertake location activities. The GPS tools allow for
activities such as orienteering or linking to other curriculum areas (such as science, to plot the location of species or to track animal migrations). While some of these tools may be expensive, others—tools such as car navigation tools—are more readily accessible and offer new representations of space and movement through space. Similarly, these tools are widely available on mobile phones and can readily be used to support learning activities. As mobile devices become increasingly available, access to such resources will enable new modes of teaching and learning to be made available to students and teachers.

**Direction**

Direction is another aspect of the geometry topic of location, and involves learning about compass points. Initially, students learn about the four points north, south, east and west. Depending on syllabus requirements, this is extended to eight points in all by the inclusion of north-east, north-west, south-east and south-west, or to 16 points by also including NNE, ENE, NNW, WNW, SSE, ESE, SSW and WSW. 

In students’ initial learning of the four major points, north and south are usually learnt relatively easily, but some confusion can arise when learning east and west. Mnemonic devices—such as Never Eat Soggy Weetbix—can be used for learning these as the students rotate clockwise through the points. A common instructional activity involves students finding a particular point by following directions on a grid map—‘two paces north, three paces west’, and so on. Orienteering activities involving the use of a compass are also useful for this aspect of location.

The cultural relativity of direction is an interesting area of application. For most students growing up in Western societies, the concept of direction is relative—that is, it is relative to where you are located. Terms such as left and right are commonly used for finding or giving directions. In some cultures, however, different applications are used. Some Indigenous Australian cultural groups use compass points in their everyday discussions (Harris, 1990). In other contexts, different methods are developed. For example, on Hawaii, relative (but significant) points are used as the terms of reference (Diamond Head, the mountains) so that compass points are not used when giving directions.

**Teaching geometry in the middle years**

The van Hiele levels of geometric thinking, summarised at the beginning of this chapter, are the foundation for instruction in the primary curriculum. Levels 4 and 5 would be the focus of activities in the primary years only informally. However, the middle years should be laying the foundation for students to achieve at level 4 in particular, where students develop the capacity to argue the case for why certain geometric situations are mathematically sound. This is the notion of proof, which is something that is not frequently used in the discourse of the middle years mathematics curriculum. However, calls have been made for proof to be an integral part of the middle years curriculum, in relation to providing convincing arguments in mathematics, and to show that “what is offered as a convincing argument by one person must be accepted by others” (Harel & Sowder, 2007, p. 808). Harel and Sowder discuss the notion of a proof scheme, rather than proof in the formal sense, where members of a (classroom) community determine the parameters for ascertaining and persuading the community about the ‘truth’ of an assertion. In such a community, students would be given tasks where they must persuade others of the legitimacy of their thinking. Their argument is based on examples and generalisations rather than reference to an authoritative source, such as the teacher, the internet or the textbook. To emphasise the importance of engaging students in discussions about geometric relationships and notions, Harel and Sowder provide the following assessment item used to determine Year 11 students’ understanding of mathematical argumentation and proof: ‘Jim says, “If a 4-sided figure has all equal sides, it is a square.” Which of the following figures might be used to prove that Jim is wrong?”

The number of students selecting each figure was 15, 31, 24 and 31 per cent respectively. The point is that middle years teachers need to move students beyond level 3 of the van Hiele model. Proof schemes require students to build mathematical arguments to convince others of their solutions, necessitating revisiting knowledge and understanding of properties of shapes and relationships.
To support students’ movement between the van Hiele levels, extensive opportunities to explore and experience spatial activities are required. In the primary years, hands-on activities that include building, designing, constructing and representing are advocated for space. This must continue in the middle years, with tasks becoming increasingly sophisticated. For example, constructing shapes from 2 cm cubes, then drawing these designs on isometric paper, promotes visualisation and arrangement that supports reading and interpretation of the architectural drawings students will most likely encounter in their lives in the future.

**Review questions**

15.1 What common observations might you expect from your students when teaching basic geometric plane (2D) shapes that would confirm the van Hiele theory?

15.2 Explain the difference between nets and skeletal forms, using examples to illustrate the difference/s.

15.3 Describe three introductory activities that would be appropriate for the topic of tessellation.

15.4 Students need to have considerable experiences in language when working in the geometry strand. Give examples of early language that should be developed and of how you would go about providing learning experiences to develop this language.

15.5 Outline learning activities suitable for the topic of symmetry in the middle primary years.

15.6 In a multi-age classroom, there is considerable diversity in student learning. How would you use one stimulus map to cater for a diverse range of learning when teaching mapping skills to students?

15.7 Find out how other cultural groups use directions in their everyday lives. Compare and evaluate this with standard curriculum expectations.

**Further reading**


**References**


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