Full Length Article

MHD wedge flow of nanofluids with an analytic solution to an especial case by Lambert W-function and Homotopy Perturbation Method

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A B S T R A C T

In the present work, an analytical investigation on the wedge flow of magnetic nanofluids is performed. A relatively new empirical model is used in an Eulerian framework to study the effectiveness of nanofluids against the pure base fluids. This new Eulerian model is considered mainly due to two significant reasons. First is to indicate some anomalies within the nanofluids system which are overlooked on using the popular classic models (e.g. Maxwell-Garnett relation for thermal conductivity and Brinkman/Einstein model for viscosity). The anomalies simply refer to the effects imposed due to different nanoparticles sizes as well as various thermal boundary conditions (here is to be the direction of heat transfer). The second reason behind the implementation of such a model is to follow some important notes recently revealed via a critical paper. In this regard, the nanoparticles migration due to either Brownian motion or thermophoresis effect is totally disregarded as the corresponding factors can be shown to be negligible for many actual nanofluids. Therefore, further parametric studies may be depreciated subject to a more rigorous physical ground. Before proceeding with nanofluids analysis, an especial case was found to be interesting. This case is referred to the MHD flow around a 90 degree corner where the magnetic parameter is equal to unity. The case is treated analytically firstly in an implicit manner using the inverse operator of Lambert W-function and secondly in an explicit manner using Homotopy Perturbation Method (HPM). Upon reading the existing literature, it may be confirmed that this case has not been noted so far. An insight to the classic convective heat transfer coefficient in Blasius flow is also additionally provided in this paper where this factor is simply obtained purely independent of any solution to energy similarity equation seemingly for the first time in the state of art.

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1. Introduction

In the past decades, nanofluids have received many theoretical/-analytical, numerical and experimental investigations. This happened since the flowing nanoparticles inside the pure base fluids showed potential for many pertinent applications. One of the applications of nanofluids is with regard to heat transfer augmentation. The widespread aspect towards the impact of nanofluids on heat transfer phenomenon positively supports the promising role of the dispersed nanoparticles into the base fluids in heat transfer augmentation (it is yet to be proven!). The most challenging part in studying nanofluids is with respect to the manipulation of thermo-physical properties. In this regard, there are many experimental and theoretical contributions in the state of art (see [1–6] as examples). The dispersed nanoparticles inside the pure base fluids can be studied subject to two different frameworks. First is the Eulerian one-phase framework which deals with the manipulation of thermo-physical properties of pure fluids in the presence of nanoparticles regardless of the dynamic effects imposed by the slip mechanisms. In other words, heat diffusion due to either Brownian motion or thermophoresis effect is ignored in this framework. Second is the Lagrangian two-phase framework which simply includes these two significant slip mechanisms. Using order of magnitude analysis, Buongiorno [7] first proved that among the seven known slip mechanisms, two of them are significant and hence should be taken into account. But as pointed out by him, mainly the empirical
factor regarding the thermophoresis effect of nanofluids is still missing in the literature. In the other hand, Corcione [1], by bringing into account many empirical data of different nanofluids proved that the classic Eulerian models (e.g. taking into account Maxwell relation for thermal conductivity and Einstein/Brinkman relations for viscosity) fail considerably to describe the behavior of thermophysical properties of nanofluids. Therefore, in the present work, the effect of nanofluids is included applying the new relations proposed in [1]. Here, it is really worth mentioning this fact that in [8], it has been argued that many similarity transformations regarding two-phase modeling of nanofluids are not complete. The aforementioned article is a recent review paper criticizing many analytical contributions in the literature respecting nanofluids. Two outcomes seem to be significant in this review article; first is that the effectiveness of nanofluids against the pure base fluids has been exaggerated over the past decade since rigorous experimental results indicate that many actual nanofluids may not be really promising for heat transfer purposes; second is that for the so far practical nanofluids (including Cu/Water nanofluids which is the target of the present work), Schmidt number is very large. Besides, the corresponding thermophoresis and Brownian numbers are incredibly small (see [8] for more information). Such outcomes simply inspire this fact that no further parametric solutions towards nanofluids may be encouraged. Therefore, following these notes, one can conclude that for nanofluids such as Cu/Water, one-phase modeling seems to be convincing enough. Here, nanofluids are considered to be electrically conductive. The new applied one-phase framework particularly reveals that the type of application (referring to different thermal boundary conditions) affects the behavior of nanofluids in such a way as now the structure of energy equation alters in the context of this new Eulerian framework. Similarity transformation method is used to convert the primary partial differential conservative equations into ODEs (Ordinary Differential Equations). It is worth mentioning that similarity transformation method stands as the most powerful analytical tool to tackle the system of PDEs (Partial Differential Equations) mainly in the field of fluid mechanics; therefore the method has received many contributions so far. Regarding similarity solutions of nanofluids, Jafarimoghaddam et al. [9] revealed interesting correlations for predicting convective heat transfer coefficient and also skin friction coefficient arise respecting the classical Blasius flow by considering the fact that some nanofluids may behave on the basis of non-Newtonian power law fluids with the power law index close to unity (this is most applicable to nanofluids with relatively high volumetric fraction of nanoparticles). Although the similarity transformation method used in [9] may not be so realistic regarding the energy equation, but at least the results for skin friction coefficient were significant as shown only a slight deviation in power law index results in considerable deviation in skin friction coefficient. Jafarimoghaddam [10] analytically solved the wall jet flow of nanofluids subject to a two-phase modeling framework. He was able to derive closed form analytic solutions to heat and mass transfer characteristics of a general exponentially decaying Glauber type wall jet flow of nanofluids where the topological representation of heat diffusion due to Brownian motions was specifically studied. In [11] an analytical investigation has been carried out on wall jet flow of magnetic nanofluids taking into account the classic Eulerian framework. As the main result of that work, Water/AA 7075 nanofluids have been reported to have a greater heat transfer rate compared to Water/AA 7072 nanofluids. There are many studies in the literature regarding wedge flow problem with or without MHD and nanofluids considerations (e.g. [12–21]). For example, in [12], flow past a wedge moving in the context of nanofluids has been studied; the effectiveness of the engaged factors has been then examined through a two-phase modeling framework. In [13], a two-phase modeling of magnetic nanofluids flow over a stretching wedge with thermophoresis effect and convective boundary condition has been reported. In [14], MHD flow over a wedge embedded in porous medium has been studied. In that study, the effects of magnetic parameter, porosity factor, injection/suction and pressure gradient have been visualized mainly by adapting the available solution for the flow around a 90 degree corner to MHD/porous cases by the means of Taylor series analysis. Regarding the numerical analysis on the MHD flows, we refer to some

### Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>$u$, $v$</td>
<td>velocity components along x and y directions</td>
</tr>
<tr>
<td>$U_e$</td>
<td>free stream velocity</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density</td>
</tr>
<tr>
<td>$\rho_{bf}$</td>
<td>density of base fluid at room temperature</td>
</tr>
<tr>
<td>$\rho_{nf}$</td>
<td>density of nanoparticle</td>
</tr>
<tr>
<td>$\rho_{nf1}$</td>
<td>density of nanofluids in classic model</td>
</tr>
<tr>
<td>$\rho_{nf2}$</td>
<td>density of nanofluids in new model</td>
</tr>
<tr>
<td>$\mu_{nf}$</td>
<td>dynamic viscosity of nanofluids in classic model</td>
</tr>
<tr>
<td>$\mu_{nf1}$</td>
<td>dynamic viscosity of nanofluids in new model</td>
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<tr>
<td>$\nu$</td>
<td>kinematic viscosity</td>
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<tr>
<td>$\nu_{nf}$</td>
<td>kinematic viscosity of nanofluids in classic model</td>
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<tr>
<td>$\nu_{nf1}$</td>
<td>kinematic viscosity of nanofluids in new model</td>
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<tr>
<td>$B_0$</td>
<td>magnetic constant</td>
</tr>
<tr>
<td>$B(x)$</td>
<td>magnetic field</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>electrical conductivity</td>
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<tr>
<td>$\sigma_{nf}$</td>
<td>electrical conductivity of nanofluids</td>
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<td>$\sigma_{nf2}$</td>
<td>electrical conductivity of nanofluids in new model</td>
</tr>
<tr>
<td>$\sigma_{bf}$</td>
<td>electrical conductivity of base fluids</td>
</tr>
<tr>
<td>$C_{np}$</td>
<td>specific heat capacity of nanoparticles</td>
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<tr>
<td>$C_{nf}$</td>
<td>specific heat capacity of nanofluids in new model</td>
</tr>
<tr>
<td>$C_{nf1}$</td>
<td>specific heat capacity of nanofluids in classic model</td>
</tr>
<tr>
<td>$\beta$</td>
<td>wedge angle</td>
</tr>
<tr>
<td>$M_{bf}$</td>
<td>magnetic parameter of base fluid</td>
</tr>
<tr>
<td>$M$</td>
<td>magnetic parameter</td>
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<tr>
<td>$T_{\infty}$</td>
<td>temperature of free stream</td>
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<tr>
<td>$T_w$</td>
<td>surface temperature</td>
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<tr>
<td>$T_f$</td>
<td>freezing point temperature of base fluid</td>
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<tr>
<td>$d_{np}$</td>
<td>nanoparticle diameter</td>
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<tr>
<td>$d_{nf}$</td>
<td>molecular diameter of base fluid</td>
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<tr>
<td>$\phi$</td>
<td>volume concentration</td>
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<td>$m$</td>
<td>wedge index</td>
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<tr>
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<td>free stream scaling parameter</td>
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<tr>
<td>$Re$</td>
<td>nanoparticles Reynolds number</td>
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<td>$\alpha$</td>
<td>thermal diffusivity factor</td>
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<td>$T$</td>
<td>temperature</td>
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<tr>
<td>$k_{np}$</td>
<td>thermal conductivity of nanoparticles</td>
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<tr>
<td>$k_{nf2}$</td>
<td>thermal conductivity of nanofluids in new model</td>
</tr>
<tr>
<td>$k_{bo}$</td>
<td>Boltzmann constant</td>
</tr>
<tr>
<td>$h(x)$</td>
<td>convective heat transfer coefficient</td>
</tr>
<tr>
<td>$Nu(x)$</td>
<td>Nusselt number</td>
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<tr>
<td>$Pr_{nf}$</td>
<td>Prandtl number of nanofluids in classic model</td>
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<tr>
<td>$Pr_{nf1}$</td>
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It is worth mentioning that similarity transformation method stands as the most powerful analytical tool to tackle the system of PDEs (Partial Differential Equations) mainly in the field of fluid mechanics; therefore the method has received many contributions so far. Regarding similarity solutions of nanofluids, Jafarimoghaddam et al. [9] revealed interesting correlations for predicting convective heat transfer coefficient and also skin friction coefficient arise respecting the classical Blasius flow by considering the fact that some nanofluids may behave on the basis of non-Newtonian power law fluids with the power law index close to unity (this is most applicable to nanofluids with relatively high volumetric fraction of nanoparticles). Although the similarity transformation method used in [9] may not be so realistic regarding the energy equation, but at least the results for skin friction coefficient were significant as shown only a slight deviation in power law index results in considerable deviation in skin friction coefficient. Jafarimoghaddam [10] analytically solved the wall jet flow of nanofluids subject to a two-phase modeling framework. He was able to derive closed form analytic solutions to heat and mass transfer characteristics of a general exponentially decaying Glauber type wall jet flow of nanofluids where the topological representation of heat diffusion due to Brownian motions was specifically studied. In [11] an analytical investigation has been carried out on wall jet flow of magnetic nanofluids taking into account the classic Eulerian framework. As the main result of that work, Water/AA 7075 nanofluids have been reported to have a greater heat transfer rate compared to Water/AA 7072 nanofluids. There are many studies in the literature regarding wedge flow problem with or without MHD and nanofluids considerations (e.g. [12–21]). For example, in [12], flow past a wedge moving in the context of nanofluids has been studied; the effectiveness of the engaged factors has been then examined through a two-phase modeling framework. In [13], a two-phase modeling of magnetic nanofluids flow over a stretching wedge with thermophoresis effect and convective boundary condition has been reported. In [14], MHD flow over a wedge embedded in porous medium has been studied. In that study, the effects of magnetic parameter, porosity factor, injection/suction and pressure gradient have been visualized mainly by adapting the available solution for the flow around a 90 degree corner to MHD/porous cases by the means of Taylor series analysis. Regarding the numerical analysis on the MHD flows, we refer to some
recent contributions in this field (see [22–24] as examples). In [22,23], authors have investigated the effect of MHD on natural convection in a vertical annulus filled with molten gallium and potassium by numerical means. In [24], the effects of radii and aspect ratio, Hartmann and Prandtl number on Nusselt number have been studied regarding the natural convection in the presence of a magnetic field inside a cylindrical annulus. Nonlinear dispersive Rayleigh–Taylor instabilities in Magnetohydrodynamic (MHD) flows have been investigated in [25]. In that work, the nonlinear evolution of Rayleigh–Taylor instabilities (RTI) have been examined in 2 + 1 dimensions in the context of MHD. In [26] the nonlinear evolution of the kink instability of a plasma with an elliptic magnetic stagnation line is studied by means of an amplitude expansion of the ideal magnetohydrodynamic equations.

Upon reading the existing literature, it was found that the application of this new Eulerian model has not yet been carefully examined in the context of nanofluids which is to be scrutinized here in a detailed manner. In addition, as pointed out in [8], it seems that many investigations on the applicability of nanofluids in different problems, have not been carried out rigorously and therefore, frankly speaking, there are many researches in this regard that both engaged equations and the associating parameters fail to completely describe many actual nanofluids. The MHD wedge flow of Cu/Water nanofluids subject to a more rigorous physical ground is to be studied in here (a schematic is drawn in Fig. 1).

In the present work, the derived ODEs for both energy and momentum equations are solved numerically using R-K algorithm which has been proved for its accuracy over decades. Effectiveness of the magnetic parameter is also visualized in the context of this new model. Although the main target of the present work is to reveal an insight into the flowing nanofluids (here is to be Cu/Water nanofluids) following the comments made in [8] and further subject to a realistic Eulerian model, but before proceeding with this, an especial case was found to be interesting; seemingly not treated before analytically in the literature. The case refers to the flow around a 90 degree corner where the magnetic parameter is equal to unity. As it will be shown, it is possible to explicitly deduce normalized skin friction coefficient for this case by simply reducing the order of derivatives. Moreover, it is found that for such an especial case, implicit parametric solution is possible. The parametric implicit solution is simplified further using Lambert W-function. As mentioned before, this especial case allows one to explicitly deduce the normalized skin friction coefficient. Therefore, the resulting second order nonlinear equation could be immediately tackled in a purely analytic manner by series methods. Among the existing series methods, Homotopy Perturbation Method (HPM) is used to obtain explicit analytic solutions for both hydrodynamics and thermal characteristics in this case. It should be mentioned that an interesting insight to the classic Blasius flow was also noted through the present work. In this regard, we could derive a semi-analytical relation expressing the convective heat transfer coefficient for the Blasius case. The relation is similar to that of already available known as Pohlhausen relation. The classic relation was obtained chiefly on the basis of numerical solutions of similarity energy equation. The former relation has been argued to be applicable for Prandtl numbers greater than 0.5 and for below this point an asymptotic relation was recommended. But here, we show that a similar relation could be established regardless of any numerical solution to the similarity energy equation. Therefore, it may be possible to conclude that the present relation is applicable to any arbitrary Prandtl number and hence may be considered as a replacement for the classic relations. The details on this matter will be also given in the present work.

2. Structure of magnetic fluids in brief

In the presence of a magnetic field, electrically conducting fluids undergo a body force imposed by the applied magnetic field. This Lorentz force can be obtained by the following relation:

\[ \mathbf{F}_b = \mathbf{j} \times \mathbf{B} \]

Besides:

\[ \mathbf{j} = \sigma (\mathbf{E} + \mathbf{V} \times \mathbf{B}) \]

Therefore, in the absence of an external electrical field, where the induced magnetic field is ignored with respect to the assumption of small magnetic Reynolds number (conductivity is not large in the absence of an external electrical field), the induced body force for a normal magnetic field can be obtained as:

\[ \mathbf{F}_b = -\sigma \mathbf{B}^2 \mathbf{V} \]

3. Similarity and scaling analysis

Including the body force introduced in the previous section, boundary layer equations may be approximated as:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \] (4)

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U_e \frac{\partial U_e}{\partial x} + \frac{\partial^2 u}{\partial y^2} + B(x)^2 \frac{\sigma}{\rho} (U_e - u) \] (5)

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial y^2} \] (to be studied for the classic nanofluids modeling) (6)

\[ \rho C_p \left( \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) \] (To be studied for the contemporary nanofluids modeling) (7)

As it is clear from Eq. (5), viscous dissipation and the induced magnetic field are ignored in the present work. In addition, variable magnetic field of \( B(x) \) will be restricted to a unique form through the similarity solution. Please also note that the flow is steady, incompressible and two dimensional.

Generally, defining \( \eta = ax^b \) (parameters of ‘a’ and ‘b’ will be specified later) and \( \frac{\partial}{\partial y} = f'(\eta) \) which are in accordance with the physics of flow of interest and further can be simply proven via group theory analysis, Eqs. (4) and (5) reduce to:

![Fig. 1. Schematic of the problem.](image-url)
Fluids and nanofluids can be simply performed regardless of the pure base fluids. The characteristics of nanofluids will appear in the derived ODE. Generally, in Eq. (8), by defining \( a \) as to be that of for the base fluid \((a_{bf})\), for nanofluids (which the volumetric fraction is assumed to be constant in the whole spatial domain and further properties are independent of the engaged variables) one reaches to:

\[
e_{bf}\frac{df}{d\eta} + f(\eta)f'(\eta) + \beta(1 - f^2(\eta)) = 0
\]

(9)

Which \( e_{bf} \) is a factor including the impact of nanofluids against the pure base fluids. In Eq. (8) with \( M = 0 \), changing \( f(\eta) \rightarrow \eta f(\eta) \), assuming \( ab = 1 \) (to certify equivalent boundary conditions at the infinity for the velocity function in similarity terms) and then dividing by \( a^2 b^2 \) we obtain:

\[
\frac{b}{a} \frac{df}{d\eta} + f(\eta)f'(\eta) + (1 - f^2(\eta)) = 0
\]

(10)

Now we have:

\[
\begin{align*}
\frac{b}{a} &= e_{bf} \\
ab &= 1 \Rightarrow a = \frac{1}{\sqrt{e_{bf}}}, \quad b = \sqrt{e_{bf}}
\end{align*}
\]

(11)

Therefore, it is obvious that if \( f(\eta) \) is a solution to Eq. (8) with \( M = 0 \), \( \sqrt{e_{bf}} f(\frac{1}{\sqrt{e_{bf}}}) \eta \) would be the solution to Eq. (9). This reveals that the solution for nanofluids (hydrodynamic characteristics) is already available and is exactly the same as that of for pure base fluids but now with a different scaling factor. Motivation for presenting this seemingly plain discussion was to insist that in the case of \( M = 0 \), the structure of momentum similarity equation is not interfered and this cannot be counted as an additional problem. Therefore, for this case, comparison between the pure base fluids and nanofluids can be simply performed regardless of the solution of Eq. (9). Besides, Now consider Eq. (8) where \( M \neq 0 \). Again, constructing this equation for nanofluids with \( a_{bf} \), we reach:

\[
e_{bf}\frac{df}{d\eta} + f(\eta)f'(\eta) + \beta(1 - f^2(\eta)) + M_{bf} e_{bf}(1 - f^2(\eta)) = 0
\]

(12)

In Eq. (12), \( M_{bf} \) is the magnetic parameter on the basis of base fluids properties. It is obvious that by transferring \( f(\eta) \rightarrow \eta f(\eta) \) in Eq. (8) \((M \rightarrow M_{bf})\), we cannot recover Eq. (12). This means that now, Eq. (8) should be considered as an additional problem for nanofluids if we would like to compare nanofluids to the base fluids on the basis of certain values of \( \beta \) and \( M \) (which can be \( M_{bf} \)).

In a general form (subject to two existing constant temperatures within the problem), by continuing similarity analysis for energy equation (Eq. (6)) by the use of \( \theta(\eta) = \frac{\theta - T_{\infty}}{T_{\infty} - T_{w}} \), one reaches the following set of ODEs (subject to constant thermo-physical properties and regardless of the fluid type):

\[
\begin{align*}
f''(\eta) + f(\eta)f'(\eta) + \beta(1 - f^2(\eta)) + M(1 - f'(\eta)) &= 0 \\
\theta'(\eta) + Pr \theta f(\eta)/f'(\eta) &= 0 \\
f(0) &= f'(0) = 0, \quad f'(\infty) = 1 \\
\theta(0) &= 1, \quad \theta(\infty) = 0
\end{align*}
\]

(13)

4. An immediate interesting insight

Here, before proceeding with the major scopes of the present work, an interesting insight to (13) is revealed. For \( \beta = 0 \) and \( M = 0 \), and having considered the well-known following simplification:

\[
\theta(\eta) = \int_0^\eta \frac{\theta'(0)e^{-Prtf(\eta)}d\eta}{e_1} + 1
\]

(14)

where:

\[
\theta'(0) = -\frac{1}{\int_0^\infty e^{-Prtf(\eta)/f'(\eta)}d\eta}
\]

(15)

By rearranging momentum equation as \( f'(\eta) = -\frac{f''(\eta)}{f'(\eta)} \) and reforming integrations within Eqs. (14) and (15), it is simple to obtain:

\[
\theta'(\eta)/\theta'(0) = \int_0^\infty f'(\eta)f'(\eta)e^{-prt}d\eta
\]

(16)

And consequently:

\[
\theta'(0) = -\frac{-f''(0)e^{-prt}}{\int_0^\infty f'(\eta)f'(\eta)e^{-prt}d\eta}
\]

(17)

Eq. (17) reveals that for \( Pr = 1 \), \( \theta'(0) = -f''(0) \) (respecting the boundary condition), and then Eq. (16) shows \( \theta'(\eta) = -f''(\eta) \) or with integrating \( \theta'(\eta) = 1 - f'(\eta) \) (obviously, these conclusions are of classic ones).

Interestingly, this analysis may go further. For this type of flow (which is the Blasius case with normal boundary conditions), it is easy to formulate convective heat transfer coefficient accurately and for all the arbitrary Prandtl numbers without any analysis on the similarity energy equation! For this, we note that \( f''(0) \approx 0.4696 \) [27]. Therefore, we only need to formulate \( f'(\eta) \). Because, Eq. (17) deals with the integration rather than differentiation, very slight deviations in formulating \( f'(\eta) \), will not likely affect the resulting value from the integration noticeably. Moreover, based on the boundary conditions which yield \( f''(0) = 0 \) and further noting the trends of the engaged functions, it is reasonable to formulate \( f'(\eta) \) as \( f'(\eta) = f'(0)e^{-a\eta} \) where \( a \) and \( b \) are positive factors and will be identified using Least Square Technique. Before that, we substitute the abovementioned formulation in Eq. (17) to get:

\[
\theta'(0) = -\frac{-f''(0)e^{-a\eta}}{\int_0^\infty e^{-a\eta}f'(\eta)f'(\eta)e^{-prt}d\eta}
\]

(18)

Relating the stretched exponential function to Gamma function by:

\[
\langle \tau^a \rangle \equiv \int_0^\infty e^{-\lambda \tau^a}d\tau = \frac{\Gamma\left(\frac{n}{a}\right)}{\Gamma\left(\frac{1}{a}\right)}
\]

(19)

We have:

\[
\int_0^\infty e^{-At\tau}d\tau = \frac{1}{B}A^{-\frac{1}{A}}\Gamma\left(\frac{1}{B}\right)
\]

(20)

And hence:

\[
\theta'(0) = -PrB^\frac{1}{A} \frac{b^a}{\Gamma\left(\frac{1}{a}\right)}
\]

(21)

Now, considering that for \( Pr = 1 \), we previously had:

\[
\theta'(0) = -f''(0)
\]

(22)

Therefore, we may conclude:

\[
b^a\Gamma\left(\frac{1}{a}\right) = f''(0)
\]

(23)
According to Eq. (23), one may conclude that there is now only one factor which should be optimized through the Least Square Technique. Applying the numerical analysis, we found $b = 2.907$ and based on Eq. (23) $a = 0.07972$.

A comparison between the formulation proposed for $f''(\eta)$ and the numerical result is shown in Fig. 2. Please note that R-Square for this formulation is reasonably equal to 1.

Therefore, one is able to conclude at this stage that the former version of formulation for convective heat transfer coefficient which was mainly on the basis of correlation to many numerical results from the similarity energy equation (the former formulation has been proposed chiefly for the band of $0.5 \leq Pr \leq 1000$) is now updated for any arbitrary Prandtl number as:

$$h(x) = \frac{0.4696}{\sqrt{2}} \frac{k}{\sqrt{Re, Pr^{m}}}$$ (25)

Please note that for $Pr << 1 (Pr \to 0)$, we previously had [28]:

$$\theta'(0) = \frac{1}{\sqrt{2}} (\frac{\eta}{x})^{\frac{1}{2}}.$$  

Practically, this relation is of the mathematical interest where Prandtl is negligibly small tending zero in value; therefore, the asymptotic formulation for $Pr \to 0$ may have actually no meaning in real applications. Summing up this section, one may simply note that Eq. (25) seems to be more compatible since the power of Prandtl is between that of for classic relation and the asymptotic one. Although based on the scale analysis [28] one may expect the asymptotic relation for very small Prandtl numbers, but we again insist that the asymptotic formulation cannot be practically used for any certain Prandtl values. Therefore, the previous adaptations regarding formulating $h$ by the classic relation for values of Prandtl greater than 0.5 and by the asymptotic relation for Prandtl numbers lower than 0.5, may not be so suitable since the latter relation is an asymptotic one and not an algebraic relation. Therefore, the new proposed relation may be considered as a replacement as it is derived regardless of any numerical solution to the similarity energy equation.

5. An especial case

An especial case which can be deduced from Eq.13 is with respect to the flow around a 90 degree corner where the magnetic parameter is equal to one ($\beta = -1$ and $M = 1$). Furthermore, remind that Eq. (13) for $\beta = -1$ and $M = 0$ has been previously solved and among some existing contributions in this case, Ref. [29] stands as a more detailed one. But the present case has not yet been investigated in the literature.

For the present case, subject to mass injection/suction and whether the surface is fixed or moving, we have:

$$f''(\eta) + f'(\eta)f''(\eta) + f''(\eta) - f'(\eta) = 0$$
$$\theta''(\eta) + Prf'(\eta)\theta'(\eta) = 0$$

$$f(\eta) = 0 = f'(0), f'(\eta) = 0 = f'(0), f'(\infty) = 1$$
$$\theta(0) = 1, \theta(\infty) = 0$$

Integrating momentum equation once, one obtains:

$$f''(\eta) + f(\eta)f'(\eta) - f(\eta) = c$$ (27)

Indefinitely integrating Eq. (27) results:

$$f'(\eta) + \frac{1}{2}f'(\eta) - \int f(\eta)d\eta = c\eta + d$$ (28)

Boundary conditions at the infinity can be considered as:

$$f(\eta) \to \infty = \eta + A$$
$$f'(\eta) \to \infty = 1$$

where $A$ is an unknown constant. Substituting (29) into Eq. (28) (for $\eta \to \infty$), yields:

$$1 + \frac{1}{2}(\eta + A)^{2} - \int (\eta + A)d\eta = c\eta + d$$ (30)

Eq. (30) deduces that $c = 0$ and $d = 1 + \frac{1}{2}A^{2}$. Therefore Eq. (27) can be rewritten as:

$$f''(\eta) + f(\eta)f'(\eta) - f(\eta) = 0$$ (31)

As it is crystal clear, this case results in an immediate identification of $f''(0)$ (the normalized factor of wall shear stress) as:

![Fig. 2. Comparison of the proposed formulation for $f'(\eta)$ with that of by numerical calculation.](image-url)
\begin{equation}
\tilde{f}''(0) = f(0)(1 - f'(0))
\end{equation}

Besides, based on Eq. (31), we have:

\begin{equation}
\tilde{f}'(\eta) + f(\eta)f'(\eta) - f(\eta) = 0 \Rightarrow f(\eta) = \frac{-\tilde{f}'(\eta)}{\tilde{f}(\eta) - 1}
\end{equation}

Now, reconstructing the integral solution for energy equation (Eqs. (14) and (15)), it is easy to obtain for \( Pr = 1 \):

\begin{equation}
\theta'(0) = 1 - \frac{f(\eta) - \eta - f(0)}{A - f(0)}
\end{equation}

And

\begin{equation}
\theta(\eta) = 1 - \frac{f(\eta) - \eta - f(0)}{A - f(0)}
\end{equation}

Eqs. (34) and (35) reveal a completely different structure for dimensionless temperature as compared to that of for Blasius case. Unfortunately, Eq. (31) is not within the so far known nonlinear system; yet it can be simply shown that this equation might be transformed to an Abel equation but even in this type, one may not be able to deduce the explicit closed form solution. Here, we have alternatively proposed a parametric implicit solution to Eq. (31) (which will be later transformed into Lambert W-function) and then the explicit solution is given using Homotopy Perturbation Method (HPM).

6. Implicit parametric solution

Consider Eq. (31) which by defining \( f'(\eta) = y \), it can be reduced to:

\begin{equation}
\frac{dy}{y-1} = -f(\eta)df(\eta)
\end{equation}

Please note that now \( y = y(f) \).

Solution to Eq. (36) is written as:

\begin{equation}
y + \ln|y - 1| = \frac{1}{2}f^2(\eta) + C_1.
\end{equation}

On using boundary conditions: \( C_1 = f'(0) + \ln f'(0) - 1 + \frac{1}{2}f'(0)^2 \).

Reconstructing Eq. (31) using Eq. (37), we have:

\begin{equation}
\tilde{f}'(\eta) + f(\eta)f'(\eta) - f(\eta) = 0 \Rightarrow f(\eta) = \frac{-\tilde{f}'(\eta)}{\tilde{f}(\eta) - 1}
\end{equation}

Therefore, we may reach the following equations which parametrically describe the solution to Eq. (31):

\begin{equation}
\int \frac{df'(\eta)}{\pm(f'(\eta) - 1)\sqrt{-2f'(\eta) + \ln f'(\eta) - 1} - C_1} = \eta + C_2
\end{equation}

\begin{equation}
f'(\eta) + \ln f'(\eta) - 1 = \frac{12}{f(\eta) + C_1}
\end{equation}

Please note that \( \pm \) signs may not represent the existence of dual solutions. Based on Eq. (38), either + or − should be selected in such a way to certify \( f'(0) = f(0)(1 - f'(0)) \).

Luckily enough, the parametric implicit solution given in (39) can be simplified further by using the inverse operator of Lambert W-function. For this we first note the main property of Lambert W-function (also known as product log function) as:

\begin{equation}
Y = Xe^X \iff X = W(Y)
\end{equation}

where \( W \) is the notation of Lambert W-function.

In this regard, Eq. (37) (or the corresponding equation denoted in (39)) can be written as:

\begin{equation}
(f'(\eta) - 1)e^{f'(\eta) - 1} = e^{\frac{1}{2}f(\eta) - 1} - C_1
\end{equation}

On Using (40), it can be deduced that:

\begin{equation}
f'(\eta) - 1 = W\left(e^{\frac{1}{2}f(\eta) - 1} - C_1\right)
\end{equation}

And finally we obtain:

\begin{equation}
\eta + C_2 = \int_{f(0)}^{1} \frac{1}{W\left(e^{\frac{1}{2}f(\eta) - 1} - C_1\right) + 1}df(\eta)
\end{equation}

In (43) \( C_2 \) is the second integrating constant.

Eq. (43) can be simply used to functionalize the thermal solution as well. For this, we have:

\begin{equation}
\theta'(0) = -\frac{1}{\Sigma_{\eta=0}^{\infty} e^{f(\eta) - \eta - f(0)} \frac{df(\eta)}{f(\eta) - 1}} + \frac{1}{\Sigma_{\eta=0}^{\infty} e^{-\eta - f(0)} \frac{df(\eta)}{f(\eta) - 1}}
\end{equation}

And:

\begin{equation}
\theta(\eta) = 1 - \frac{1}{\Sigma_{\eta=0}^{\infty} e^{f(\eta) - \eta - f(0)} \frac{df(\eta)}{f(\eta) - 1}} + \frac{1}{\Sigma_{\eta=0}^{\infty} e^{-\eta - f(0)} \frac{df(\eta)}{f(\eta) - 1}}
\end{equation}

7. Explicit solution by HPM: Introduction and application

In this section, we consider Eq. (31) along with the boundary conditions denoted in (26) to be solved using HPM in order to obtain the explicit solution. To indicate the basic idea of this method, the following form of nonlinear equation is considered:

\begin{equation}
A(U) - f(r) = 0, \ r \in \Omega
\end{equation}

Along with the following boundary condition:

\begin{equation}
B\left(u, \frac{\partial u}{\partial n}\right) = 0, \ r \in \Gamma
\end{equation}

where \( A \) is a general function operator, \( B \) is a boundary condition operator and \( \Gamma \) is a boundary of domain \( \Omega \) and \( f(r) \) is a known function.

The general operator in Eq. (46) may be decomposed into a linear and a nonlinear operator as:

\begin{equation}
L(U) + N(U) - f(r) = 0, \ r \in \Omega
\end{equation}

On using the homotopy \( U(r, p) : \Omega \times [0, 1] \rightarrow R \) satisfying:

\begin{equation}
H(U, p) = (1 - p)(L(U) - L(U_0)) + p(A(U) - f(r)) = 0, \ p \in [0, 1], \ r \in \Omega
\end{equation}

Or:

\begin{equation}
H(U, p) = (L(U) - L(U_0)) + p(L(U_0) + N(U) - f(r)) = 0
\end{equation}

where \( p \) is an embedding parameter and \( U_0 \) is an initial approximate solution to Eq. (46), the following properties would be deduced:

\begin{equation}
H(U, 0) = (L(U) - L(U_0)) = 0
\end{equation}

And

\begin{equation}
H(U, 1) = (A(U) - f(r)) = 0
\end{equation}
Obviously, upon changing the embedding parameter from zero to one, the solution moves from the initial guess to the actual one (the concept is called homotopy in topology). According to HPM, one may expand the solution on the basis of perturbation technique:

\[ U = U_0 + pU_1 + p^2U_2 + p^3U_3 + p^4U_4 + \cdots \]  

(53)

On choosing \( p = 1 \), the solution of Eq. (46) can be approximated as:

\[ U = \lim_{p \to 1} U = U_0 + U_1 + U_2 + U_3 + U_4 + \cdots \]  

(54)

Now, the method is applied to Eq. (31). Let \( \frac{\partial}{\partial p} \) be the linear operator. Therefore, we have:

\[ \frac{\partial^2 F}{\partial p^2} + F + \frac{\partial F}{\partial p} = 0 \]  

(55)

Please note that we could choose other linear operators such as: \( \frac{\partial}{\partial p} - 1 \); but the resulting linear equations would not be straightforward; meaning that for each step, a solution to a linear second order differential equation with constant coefficients respecting both \( F^* \) and \( F \) would be required which in particular would result in higher computational costs. Nevertheless, by the previous linear operator, one can construct the following homotopy:

\[(1 - p) \left( \frac{\partial^2 F}{\partial p^2} - \frac{\partial F}{\partial p} \right) + p \left( \frac{\partial^2 F}{\partial p^2} + \frac{\partial F}{\partial p} - F \right) = 0 \]  

(56)

Or:

\[ \left( \frac{\partial^2 F}{\partial p^2} + \frac{\partial F}{\partial p} \right) + p \left( \frac{\partial^2 F}{\partial p^2} + \frac{\partial F}{\partial p} - F \right) = 0 \]  

(57)

Assume that the solution can be expanded using perturbation series as:

\[ F = F_0 + pF_1 + p^2F_2 + p^3F_3 + p^4F_4 + p^5F_5 + p^6F_6 + p^7F_7 + p^8F_8 + p^9F_9 + p^{10}F_{10} + \cdots \]  

(58)

On substituting Eq. (58) into Eq. (57) or Eq. (56), and rearranging the algebraic terms with identical powers of \( p \), one reaches:

\[ p^1 \left( \frac{\partial^2 F_1}{\partial p^2} + \frac{\partial F_1}{\partial p} \right) + p \left( \frac{\partial^2 F_1}{\partial p^2} + \frac{\partial F_1}{\partial p} - F_1 \right) = 0 \]  

(59)

For the sake of simplicity and also to guarantee the boundary conditions at zero point, we choose the following zeroth function:

\[ F_0 = f_0 = f'(0)\eta + f(0) \]  

(60)

On applying Eq. (61) into Eq. (60) and only for a few first steps of perturbation analysis, we simply obtain:

\[ \frac{\partial^2 F_1}{\partial p^2} = \eta(f'(0) - f'(0)^2) + f(0)(1 - f'(0)) \]  

\[ \frac{\partial^2 F_2}{\partial p^2} = (\eta(f'(0) - 1)4f'(0)^2 + f(0)\eta) \]  

\[ \frac{\partial^2 F_3}{\partial p^2} = -\eta^2(f'(0) - 1)(34f'(0)^3) + 170f'(0)^2f(0)\eta^2 \]  

(61)

\[ -20f'(0)^2\eta + 220f'(0)^3f(0)^2 - 100f'(0)^2f(0)\eta^2 \]  

\[ + f'(0)\eta^2 + 60f'(0) - 100f'(0)^2 + 5f'(0)^2\eta^2 \]  

\[ \frac{\partial^2 F_4}{\partial p^2} = \cdots \]  

Therefore, it is easy to verify that our initial guess (zeroth deformation/function) is suitable since upon letting \( \eta = 0 \), all the second derivatives, except the first one \( \frac{\partial^2 F}{\partial p^2} \) become zero. In other words, the true value for the normalized skin friction coefficient

\[ \left( \frac{f'(0)}{0} \right) = \frac{\partial^2 F_1(0)}{\partial p^2} + \frac{\partial F_1(0)}{\partial p} + \frac{\partial^2 F_2(0)}{\partial p^2} + \frac{\partial F_2(0)}{\partial p} + \frac{\partial^2 F_3(0)}{\partial p^2} + \cdots = f(0)(1 - f'(0)) \]  

is recovered (see Eq. (32)). Furthermore, it should be noted that this type of zeroth function could be simply supported on using Adomian Decomposition Method (ADM).

Similarly, for the thermal problem, we have:

\[ (1 - p) \left( \frac{\partial^2 \theta}{\partial p^2} - \frac{\partial \theta}{\partial p} \right) + p \left( \frac{\partial^2 \theta}{\partial p^2} + \text{Pr} \frac{\partial \theta}{\partial p} \right) = 0 \]  

(62)

Or:

\[ \left( \frac{\partial^2 \theta}{\partial p^2} + \frac{\partial \theta}{\partial p} \right) + p \left( \frac{\partial^2 \theta}{\partial p^2} + \text{Pr} \frac{\partial \theta}{\partial p} \right) = 0 \]  

(63)

Which by assuming the following perturbation series as the solution:

\[ \theta = \theta_0 + p\theta_1 + p^2\theta_2 + p^3\theta_3 + p^4\theta_4 + p^5\theta_5 + p^6\theta_6 + p^7\theta_7 + p^8\theta_8 + p^9\theta_9 + p^{10}\theta_{10} + \cdots \]  

(64)

The following recursive equations are obtained:

\[ p^1 \left( \frac{\partial^2 \theta_1}{\partial p^2} + \frac{\partial \theta_1}{\partial p} \right) + p \left( \frac{\partial^2 \theta_1}{\partial p^2} + \text{Pr} \frac{\partial \theta_1}{\partial p} \right) = 0 \]  

(65)

In the same manner, the zeroth function can be considered as:

\[ \theta_0 = \theta_0 = \theta'(0)\eta + 1 \]  

(66)
Finally, the following series solution is obtained for the hydrodynamic problem:

$$F_0 = f(0) + f'(0)\eta$$
$$F_1 = (\eta^2)(3f(0) + f'(0)\eta)/6 - (f'(0)\eta^2)(3f(0) + f'(0)\eta)/6$$
$$F_2 = (\eta^2)(f'(0) - 1)(f'(0)^2\eta^2 + 5f'(0)f(0)\eta + 5f(0)^2))/30 - (\eta^2)(5f(0) + f'(0)\eta)(f'(0) - 1))/120$$
$$F_3 = (\eta^2)(f'(0) - 1)(f'(0)^2\eta^2 + 5f'(0)f(0)\eta + 5f(0)^2))/30 - (\eta^2)(5f(0) + f'(0)\eta)(f'(0) - 1))/120$$
$$+ (f'(0)^2\eta^3)(f'(0) - 1))/120 - \eta^2(\eta^2)((102f'(0)f(0) - 48f(0)f(0))(f'(0) - 1))/2160 - \eta^2(8f'(0)^2)$$
$$- 17f'(0)^2)(f'(0) - 1)/2520 + (f'(0) - 1)(165f'(0)f(0)^2 - 60f(0)^2))/1800 - \eta^2((f'(0) - 1)(6f(0))$$
$$- 24f'(0)f(0)))/4320 + (f'(0) - 1)(f'(0) - 4f'(0)^2))/5040 - (f(0)^3\eta(f'(0) - 1))/24$$

$$F_4 = \eta^4(\eta((f'(0) - 1)(1260f(0)f(0)^2 - 1015f(0)f(0)^2 + 112f(0)^2))/35280 + \eta^2((f'(0) - 1)(496f(0)f(0)^2)$$
$$- 440f(0)f(0)^2 + 64f(0)(f(0))/40320 + (\eta^2)(f'(0) - 1)(62f'(0)^4 - 55f(0)^3 + 8f(0)^2))/45360))$$
$$+ ((f'(0) - 1)(1092f'(0)f(0)^2 - 60f(0)^2))/2720 - \eta^2(\eta((f'(0) - 1)(272f(0)f(0)^2)$$
$$- 160f(0)f(0) + 8f(0))/322560 + (\eta^2)(f'(0) - 1)(34f'(0)^3 - 20f'(0)^2 + f(0))/362880)$$
$$+ ((f'(0) - 1)(616f'(0)f(0)^2 - 280f(0)^2))/282240 - (f(0)^3\eta^2(f'(0) - 1))/720 + (f(0)^4\eta^2(f'(0) - 1))/120$$

(67)

And so on. The finalized solution is then simply obtained by

$$f(\eta) = \sum_{n=0}^{N} a_n \eta^n$$

In a similar manner, for the thermal solution, we reach:

$$\theta_0 = \theta(0)\eta + 1$$
$$\theta_1 = -((\theta(0)(Pr\eta^2)(3f(0) + f'(0)\eta))/6$$
$$\theta_2 = ((\theta(0)^2)(\theta(0)^2)(Pr^2\eta^2))/6 - \eta^2((\theta(0)(\theta(0)(f(0)/2 - (f'(0)f(0)^2)))/2 - (f'(0)\eta^2)(f(0))/2))/12 + (\theta(0)(\theta(0)(\theta(0)(f(0))/6 - (f'(0)/2))/2))/20$$
$$\theta_3 = -((\theta(0)(\theta(0)^3)(15f'(0)^2(Pr^2\eta^2) + 15f'(0)^2(Pr^2)^2\eta^2 + 105f'(0)^2f(0)Pr^2\eta^2 + 15f'(0)^2f(0)Pr^2\eta^2 + 28f'(0)^2f(0)^2\eta^2)$$
$$- 15f'(0)^2Pr^2\eta^2 - 5f'(0)^2Pr^2\eta^2 + 252f'(0)f(0)^2Pr^2\eta^2 + 168f'(0)f(0)^2Pr^2\eta^2 + 42f'(0)f(0)^2Pr^2\eta^2 - 35f'(0)f(0)^2Pr^2\eta^2)$$
$$+ f'(0)^2\eta^3 + 210f'(0)^2Pr^2 - 168f'(0)^2Pr\eta - 42f'(0)^2Pr\eta + 7f'(0)^2\eta^2))/5040$$
$$\theta_4 = ((\theta(0)^3)(\theta(0)^3)(Pr^4\eta^4))/3456 + (\theta(0)^3)(\theta(0)^3)(Pr^4\eta^4))/17282 + (7\theta(0)^3)(\theta(0)^3)(Pr^2\eta^3))/17280 + (17\theta(0)^3)(\theta(0)^3)(Pr^2\eta^3))/181440$$
$$+ ((\theta(0)^3)(\theta(0)^3)(Pr^2\eta^3))/384 + (\theta(0)^3)(\theta(0)^3)(Pr^2\eta^3))/1920 + (7\theta(0)^3)(\theta(0)^3)(Pr^4\eta^3))/20160 - (\theta(0)^3)(\theta(0)^3)(Pr^4\eta^3))/20160$$

(68)

And so on. The finalized solution is then simply obtained by

$$\theta(\eta) = \sum_{n=0}^{N} a_n \eta^n$$

Here, to express the strength of the HPM solution, we have arbitrarily chosen a case where \(f(0) = 1\) and \(f'(0) = 0.5\). Using a parametric code developed by MATLAB, we obtained the first 50 perturbation terms both for the hydrodynamic and thermal func-
8. Asymptotic solution at the Infinity

At the infinity we can express the functions as:

\[ f \eta \to \infty = \eta + A + F(\eta) \eta \to \infty = 0 \]
\[ f'(\eta) \to \infty = 1 + F'(\eta) \eta \to \infty = 0 \]
\[ f''(\eta) \to \infty = F''(\eta) \eta \to \infty = 0 \]

Substituting Eq. (70) into Eq. (31), one gets:

\[ F'' + \eta F' = 0 \]  

Eq. (71) can be simply solved and then choosing appropriate values for integration constants, we may reach:

\[ f \eta \to \infty = \eta + A + \frac{\sqrt{\pi}}{2} \left( \text{erf} \left( \frac{\sqrt{2}}{2} \eta \right) - 1 \right) \]  

\[ f'(\eta) \to \infty = 1 + e^{-\frac{\eta^2}{2}} \]  

\[ f''(\eta) \to \infty = -\eta e^{-\frac{\eta^2}{2}} \]  

In the next section, following our previous discussion, the conservative equations will be constructed on the basis of pure base fluids properties and both in the context of classic Eulerian model and the new one to be studied further regarding Cu/Water nanofluids.
9. Nanofluids analysis

9.1. The classic Eulerian framework

In the context of the classic Eulerian framework, the governing conservation equations for the present problem can be obtained as:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho \mathbf{u})}{\partial x} = 0
\]

where the nanofluids properties are considered as:

\[
\rho_{nf} = \rho_p + (1 - \phi)\rho_{nf}, \quad C_{pf} = \phi C_{pf} + (1 - \phi)\rho_{nf}C_{pf}
\]

\[
\mu_{nf} = \frac{\mu_f}{(1 - \phi)^2}, \quad \sigma_{nf} = \sigma_f(1 + 3\frac{(\sigma - 1)\phi}{(\sigma - 2) - (\sigma - 1)\phi}), \quad \sigma = \frac{\sigma_{nf}}{\sigma_f}
\]  

9.2. The new Eulerian framework

Here, a realistic Eulerian framework is used in order to account the effect of nanofluids against the pure base fluids. For this, empirical based relations proposed in [1] are brought into account. Although the relations cover a wide domain of nanofluids but yet, the restrictions are noted. Basically, the correlations are on the basis of some certain base fluids with different nanoparticles within (for more information see [1]). Moreover, bands for temperature, nanoparticles diameter and volumetric fraction of nanoparticles should be also considered. While implementing this new model, one should combine the proposed bands both for the correlated thermal conductivity and viscosity. Combining these bands, we may reach a band of 25–150 nm for nanoparticles size, 294–323 K for temperature and 0.002–0.071 for the volumetric fraction of nanoparticles. Under the considerations of this model, thermo-physical properties of nanofluids can be written as:

\[
\rho_{nf2} = \rho_{nf} + (1 - \phi)\rho_{nf}, \quad C_{pf2} = \phi C_{pf} + (1 - \phi)\rho_{nf}C_{pf}
\]

\[
k_{nf}(T) = k_{nf}(1 + AT^{0.4})
\]

\[
A = 4.4 \left( \frac{2\rho_{nf}k_{nf}}{\pi \mu_{nf}^2 l_f} \right)^{0.4} \left( \frac{1}{T_f} \right)^{10} \left( \frac{k_{nf}}{k_{nf}} \right)^{0.03} \phi^{0.66}
\]

In the context of this model, we can consider thermal conductivity as:

\[
U_{nf} = f'(\eta), \quad C = C\eta, \quad B(x) = B\eta^{0.5}, \quad \eta = \sqrt{\frac{C(1 + 1.5\eta)}{2h_{nf}}}
\]

where the nanofluids properties are considered as:

\[
\rho_{nf} = \phi \rho_p + (1 - \phi)\rho_{nf}, \quad C_{pf} = \phi \rho_p C_{pf} + (1 - \phi)\rho_{nf}C_{pf}
\]

\[
\mu_{nf} = \frac{\mu_f}{(1 - \phi)^2}, \quad \sigma_{nf} = \sigma_f(1 + 3\frac{(\sigma - 1)\phi}{(\sigma - 2) - (\sigma - 1)\phi}), \quad \sigma = \frac{\sigma_{nf}}{\sigma_f}
\]  

10. Main results and discussion

As it is crystal clear, structure of energy similarity equation is different under the considerations of the so-called realistic Eulerian model. In other words, thermal conditions now play role in energy similarity equation. Therefore, one may expect different responses subject to cooling and/or heating applications. Moreover, different nanoparticles sizes affect the thermal behavior of nanofluids by changing the factor of \(A\) in Eq. (76). Generally, it is simple to conclude that a single type of nanofluids may express different thermal behaviors with respect to different thermal boundary conditions. It is equivalent as saying that the structure of heat transfer breaks down in this model. This means that although for the base fluids, convective heat transfer coefficient was only reliant on the fluid type, but now, different thermal conditions (here is to be cooling and/or heating) result in different convective heat transfer coefficients. This particularly reveals that choosing nanofluids for special applications requires more attention. For Cu/Water nanofluids, we illustrate different responses arise out of using Eq. (77). Cooling case is referred to \(T_w = 323\) and \(T_e = 294\) whilst Heating case reflects \(T_w = 294\) and \(T_e = 323\). In addition, two different nanoparticles sizes of 25 nm and 150 nm are selected. Furthermore, eight different volumetric fractions of 0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06 and 0.07 are studied. Please note that the selected bands are the largest ones we could choose respecting the limitations within the applied correlations given above.

Before representing the outcome numeric results, the accuracy of the numerical algorithm is checked for the pure base fluids in different wedge geometries of \(\beta = 0\) and \(\beta = 1\). For this, dimensionless heat transfer coefficient of \(\frac{h_{nf}}{h_n}\) is compared to those previously obtained by Evans [30,31] and expressed in Tables 1 and 2.
As it is obvious from Tables 1 and 2, the accuracy of the developed R-K code is certified. Fig. 5 indicates the normalized velocity function of \( f'(\eta) \) in different magnetic parameters and wedge geometries for the basic equation (Eq. (8) and/or base fluid case). According to Fig. 5, an increase in magnetic field (magnetic parameter) shrinks the boundary layer thickness \( (\eta_{0.99}) \) for each wedge geometry which was already expected. Physically, the body force due to the presence of magnetic effect is implemented respecting a relative velocity. This relative velocity is referred to the difference between the velocity in the boundary layer and the velocity at the far field where the flow is supposed to be inherently inviscid. Hence, shrinkage of the boundary layer thickness is justified.

In order to shorten the content of the present work, we have only visualized the factors of engineering interest which are the most important information required with respect to pertinent applications and hence skipped from representing velocity and temperature distributions for the different cases as nothing interesting in those skipped figures was observed (the trends of both velocity and temperature functions were obviously analogous to those for the pure base fluids where velocity function tends to unity at the infinity and temperature function goes from unity to zero far from the wall only with minor deviations. Therefore, in order to provide engineers with the major characteristics of Cu/Water nanofluids, the figures with the highest priority are expressed).

The ratio of skin friction coefficient of nanofluids to the pure base fluids can be written as:

\[
\frac{C_f(nf)}{C_f(bf)} = \frac{f'(0)|nf| H_{nf}}{f'(0)|bf| H_{bf}}
\]  

(77)

Besides, the ratio of convective heat transfer coefficient of nanofluids to the pure base fluids is expressed as:

\[
\frac{h_{nf}}{h_{bf}} = \frac{\vartheta(0)|nf| k_{nf}}{\vartheta(0)|bf| k_{bf}}
\]  

(78)

For Cu/Water nanofluids, the related properties are tabulated in Table 3.

Table 3
Properties of Copper and Water.

<table>
<thead>
<tr>
<th>Properties</th>
<th>Copper</th>
<th>Water</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>8960 (kg/m(^3))</td>
<td>998</td>
</tr>
<tr>
<td>( \mu )</td>
<td>3046.4 (N s/m(^2))</td>
<td>653.4</td>
</tr>
<tr>
<td>( C_p )</td>
<td>384.5 (J/kg( ^\circ )K)</td>
<td>4200</td>
</tr>
<tr>
<td>( k )</td>
<td>401 (W/m( ^\circ )K)</td>
<td>630</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>59,170,000 (( ^\circ )C m(^3/)N)</td>
<td>0.05</td>
</tr>
</tbody>
</table>

For the present Eulerian model, also note the important simplified factors as described in Table 4.

Table 4
Important factors regarding the new Eulerian model.

<table>
<thead>
<tr>
<th>Cu/Water</th>
<th>( d_p = 25 ) nm</th>
<th>( d_p = 150 ) nm</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>( 2.189 \times 10^{-2\phi^{0.66}} )</td>
<td>( 1.069 \times 10^{2\phi^{0.66}} )</td>
</tr>
<tr>
<td>( \mu )</td>
<td>( 9.93 \times 10^{-4}(1 - 4.998\phi^{0.66})^{-1} )</td>
<td>( 9.93 \times 10^{-4}(1 - 2.92\phi^{0.66})^{-1} )</td>
</tr>
</tbody>
</table>

![Fig. 5. \( f'(\eta) \) (Y-Axis), \( \eta \) (X-Axis) for different values of wedge angle and magnetic parameter.](image-url)
In the present study, the results for Cu/Water nanofluids are visualized in different wedge geometries of $\beta = 0$ and $\beta = 0.5$ as well as different magnetic parameters of $M = 0$ and $M = 0.5$.

Please also note that based on the many numerical results obtained through the present work, only the most distinguished ones are expressed.

Fig. 6 shows the ratio for skin friction coefficient in the case of $\beta = 0$ and $M = 0$. The difference between the results via the classic model and those by the present model is clear in this figure. According to Fig. 6, one can simply note that the classic model is relatively in accordance with nanofluids with larger nanoparticles size. This is whilst smaller nanoparticles show different behaviors. Fig. 7 betrays the ratio of convective heat transfer coefficient in the case of $\beta = 0$ and $M = 0$. Again, it is noted that larger nanoparticles follow relatively a linear trend which is similar to those for the classic model; whilst the smaller nanoparticles express a different trend.

Based on Fig. 7, it is clear that nanofluids may express different behaviors in convective heat transfer coefficient depending on both nanoparticles size (which manipulates the factor of $A$ in energy equation) and thermal conditions (which is treated as heating or cooling in the present work). Figs. 8 and 9 show analogous ratios but now for $\beta = 0.5$ and $M = 0$. It is clear that for skin friction coefficient in this case (Fig. 8), there is no difference compared to Fig. 6 ($\beta = 0$ and $M = 0$). The reason for this has been discussed earlier in this paper by referring to this fact that if $f(\eta)$ is a solution to Eq. (8) with $M = 0$, so $\sqrt{\frac{c_{0}(0)}{j_{0}(0)}} f(\eta)$ would be the solution to Eq. (9). This means that $\sqrt{\frac{c_{0}(0)}{j_{0}(0)}}$ in Eq. (78) is equal to a constant regardless of the wedge geometry in the case of $M = 0$. Therefore the analogy was already expected. More importantly, considering Figs. 7 and 9, one may note a very slight difference. This might be an important result which states that convective heat transfer coefficient may be...
separable in characteristics. More precisely, results depicted in Fig. 9 are only slightly shifted with the same trend compared to Fig. 7. In other words, if we are heading to formulate convective heat transfer coefficient in the case of wedge flow subject to normal boundary conditions ($\theta'(0) = g(\beta, Pr, A, T_\infty, T_{sat})$), it might be possible to deduce that the trend of convective heat transfer coefficient (in ratio form) is not affected by the wedge geometry; meaning that the corresponding term for Prandtl (or $A, T_1$ and $T_{inf}$) in $\theta'(0)$ might be separable from the wedge index.

For example, considering the general form denoted in Eq. (13) without magnetic parameter (in which convective heat transfer coefficient is on the basis of $Pr^{1/3}$ for Blasius case), one may deduce that for each geometry, it is reasonable to seek for $\theta'(0)$ in the following form: $\theta'(0) = g_1(\beta)Pr^{1/3} + g_2(\beta)$.

Fig. 10 shows the ratio of skin friction coefficient for the case of $\beta = 0$ and $M = 0.5$. Comparing the results to Fig. 6 which was for $\beta = 0$ and $M = 0$, it can be concluded that the ratio of skin friction coefficient decreases with the increase of magnetic parameter. Please note that the results indicate the ratio of factors while both with increasing wedge angle and magnetic parameter, normalized skin friction coefficient increases accordingly. Fig. 11 depicts the ratio of convective heat transfer coefficient in the case of $\beta = 0$ and $M = 0.5$. Comparing this figure to Fig. 7 reveals that the ratio of convective heat transfer coefficient also decreases with the increase of magnetic parameter. The last two figures (Figs. 12 and 13) show the same ratios for the case of $\beta = 0.5$ and $M = 0.5$.

It is clear based on these figures that the ratios of interest are enhanced compared to Figs. 10 and 11 because of the wedge effect.
11. Engineering discussion

In this seemingly new section, we are headed to briefly indicate the outcome of the present work, those which may specifically appear in engineering applications.

First is that the present simulation is mostly valid for nanoparticles concentrations below 4%; as by increasing the volumetric fraction of nanoparticles to reach around 4%, one may expect encountering the non-Newtonian regime (it is not a certain phenomenon but a possible one).

Second is that the classic Eulerian modeling of nanofluids may not reveal some facts behind the applicability or uselessness of nanofluids. Unlike many existing reports regarding the suitability of nanofluids for engineering purposes, it is now clear that Cu/Water nanofluids may not be really promising for heat transfer purposes at least for the specific problem studied here. If we take the results below 4% into account, it is simple to note that heat transfer is generally enhanced by about 10% whilst skin friction coefficient is also increased about 20% both compared to those for the pure base fluids.

Besides, both the type of application (saying cooling or heating) and nanoparticles size affect these ratios noticeably which have been almost neglected in many previous studies.

12. Conclusion

The Magnetohydrodynamic (MHD) wedge flow of Cu/Water nanofluids was studied analytically. An interesting insight to the similarity energy equation in the case of Blasius flow was revealed...
in the first place where we deduced the convective heat transfer coefficient regardless of any numerical solution to the energy equation. The finalized derived relation may be of use as a replacement for the classic one. Before proceeding with the nanofluids analysis, an especial case was found to be interesting, seemingly untapped which was treated analytically in both implicit and explicit manners. The case was referred to the MHD wedge flow around a 90 degree corner where the magnetic parameter was equal to one. We constructed the analytic implicit solution on the basis of Lambert W-function, whilst the explicit solution was given via Homotopy Perturbation Method (HPM). After this section, we moved to study the MHD Cu/Water nanofluids respecting the geometry of interest. Following a very recently published paper, one-phase modeling of nanofluids was found to be more realistic. In this regard, we constructed the nanofluids properties both in the context of the classic Eulerian framework and a new one; where it was noted that the classic Eulerian modeling fails to describe the two significant anomalies within the nanofluids system. The anomalies were referred to the effects imposed by different nanoparticles sizes as well as various thermal boundary conditions. The most practical outcome refers to this fact that the effectiveness of nanofluids is strongly conditional. Upon some conditions (different magnetic fields as well as different wedge geometries, cooling or heating applications and different nanoparticles sizes), Cu/Water nanofluids may not be really useful for heat transfer purposes. Based on the main plotted figures, it is easy to verify that there exist cases where skin friction coefficient rises about 20% whilst convective heat transfer coefficient is enhanced only about 6%. Finally, it is worth mentioning that the results via the classic Eulerian framework were somewhat analogous to those for the cooling case with larger nanoparticles size; whilst other cases expressed different behaviors.