Learning Objectives:
Through reading and engaging with this chapter, the reader will:

1. Develop a broad understanding of the Australian Curriculum: Mathematics
2. Appreciate numeracy as the authentic application of mathematics
3. Understand how children learn mathematics and numeracy
4. Develop an appreciation of the powerful role of digital technologies in mathematics education
5. Learn some principles for assessing mathematical learning
Chapter 9: Teaching Primary Mathematics and Numeracy

Introduction

Learning mathematics is an important and integral part of the primary school curriculum. For many, mathematics achievement is typically associated with skills and formulae application. However, an unintended outcome of learning mathematics in the primary years of schooling is that many students learn to dislike mathematics and regard it as dull and useless (Attard, Ingram, Forgasz, Leder, & Grootenboer, 2016). Indeed, this remains one of the consistent problems in primary school education, as it seems to resist change despite intense focus and effort. Due to the persistence of negative attitudes towards mathematics (Grootenboer & Marshman, 2017; Jorgensen & Larkin, 2017), primary school teachers need to appreciate that, in addition to the need to teach mathematical concepts and ideas well, they also need to ensure that students find that the learning of mathematics is enjoyable and interesting, and that they regard it as useful. Therefore, in this chapter we will discuss the who of mathematics education (i.e., the students), the how of mathematics education (i.e., the teaching, or pedagogy) and the what of mathematics education (i.e., the curriculum).

We of course recognise that these dimensions of mathematics education are integrated and related; however, for the purposes of this text we consider them in a sequential manner. Therefore, we start with the learner and consider mathematics education as fundamental in developing positive mathematical identities. This perspective foregrounds mathematical learning as encompassing both knowledge and skills and also attitudes and values. We then move on to the curriculum – specifically the Australian Curriculum: Mathematics (ACM), and also numeracy as an integral part of the broader Australian Curriculum. We also address how primary children learn mathematics, by specifically connecting to the role of technology and briefly linking this to prominent learning theories as well as rich, practical, classroom activities. Throughout we present small vignettes of actual classroom practices and activities to illustrate the who, how and what of mathematics education “in action”.
Developing Mathematical Identities

When children are learning mathematics, they are learning more than just knowledge. As would be clear from earlier chapters, when children are learning in the classroom, they are learning knowledge and skills, but they are also developing attitudes, values and feelings both about the content they are learning and also about themselves as learners. This is no different in mathematics classrooms. Indeed, it seems that for many people, the most prominent thing they learned from their mathematics education is that it is boring, irrelevant, and to be feared and avoided (see Grootenboer & Marshman, 2017; Larkin & Jorgensen, 2016). Furthermore, many people define their experience of mathematics education as one that entails remembering countless rules and formulae. They admit to having little to no conceptual understanding of how or why methods are applied and used. And further, they often believe that success in mathematics is only attainable by the elite and gifted few (Schoenfeld, 2013). The sad thing is that they probably learned this in their mathematics classrooms – the very place that you would hope that they would learn to love and enjoy mathematics, and appreciate its beauty and usefulness.

With this in mind, it is important to think of mathematics teaching and learning in a more holistic manner – to conceptualise mathematics education as developing mathematical identities. This means simultaneously thinking about developing students’ mathematical knowledge, their mathematical skills, and their mathematical attitudes and beliefs. This can be thought of as a head, hand, and heart approach. Simply put, this means that when planning and undertaking mathematics teaching, attention needs to be given to the mathematical topic simultaneously with the feelings and dispositions that students might develop. This does not mean that we ignore the importance of learning essential mathematical concepts and ideas, but rather it demands careful attention to the broader things students learn and experience while they are engaging with the mathematics.

In mathematics curriculum documents across the world, including in Australia, it is clear that mathematics education aims to promote positive mathematical identities. For example, in the

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1 The idea of ‘head, heart, and hand’ initially came from Sergiovanni (1996) when discussing school leadership.
Aims and Rationale section of the Australian Curriculum: Mathematics (ACM), the following statements are included:

- Learning mathematics … develops the numeracy capabilities that all students need in their personal, work and civic life;
- Mathematics has its own value and beauty and the Australian Curriculum: Mathematics aims to instil in students an appreciation of the elegance and power of mathematical reasoning; and
- The mathematics curriculum … encourages teachers to help students become self-motivated, confident learners through inquiry and active participation in challenging and engaging experiences. (ACARA, 2017e, p. 4)

While the mandated ACM includes details of mathematical knowledge and skills to be learned, it is clear that the curriculum must develop much more. The ACM demands that attention is paid to the development of positive student values, beliefs, attitudes and dispositions towards mathematics.

A common understanding regarding primary education in general is that students’ learning builds on their pre-existing knowledge. It has been common practice for teachers to assess students’ prior knowledge before embarking on a new mathematics topic (often with a pre-test), but if mathematics learning encompasses more than just knowledge, then consideration needs to be given to this broadened understanding of mathematics. Students’ identities - their mathematical identities (i.e., their current knowledge, skills and attitudes) but also their general broader identities (e.g., their experiences, interests, family life, and the like) should be integral to determining the foundational knowledge upon which to build for further mathematical learning. Therefore, when planning learning experiences for primary students, teachers need to think carefully about both the students’ mathematical backgrounds as well as their wider experiential background.

**Provocation**: As a primary school teacher you are a teacher of mathematics.

- What were your experiences of mathematics at school?
- How do you feel about mathematics? What is your attitude and feelings about mathematics? What is your mathematical identity?
- As a mathematics teacher, what do you need to do to enhance and develop your own mathematical identity?
How will you ensure that students develop a healthy mathematical identity?

Vignette One: Learning about Location

In the ACM, one of the main concepts underpinning the Measurement and Geometry strand is location. The formal development of the concept of location commences in Year 1. It is then further developed and refined as students progress through the primary years. By way of example, Year 5 students are expected to be able to demonstrate that they can:

- Use a grid reference system to describe locations. Describe routes using landmarks and directional language (ACMMG113)

However, while learning about “grid reference systems” and how to “describe routes” in Year 5 it would also be hoped that students would also develop an appreciation that:

- being able to do this is valuable and that this is a useful and practical mathematical knowledge and skill;
- it is fun and engaging; and
- they are able to learn and be confident in their mathematics.

It seems unlikely that they would learn these things by solely sitting at their desks and only working from textbooks and worksheets. Indeed, if this is the only sort of pedagogy employed, students may develop some knowledge and skills, but such experiences may simultaneously reinforce perceptions that mathematics is useless and boring. Conversely, if the learning experience included activities such as orienteering or geocaching as part of the program, then students would be more likely to see mathematics as useful, fun and engaging (Sollervall, Gil de la Iglesia & Zbick, 2018 in press). It is also important to remember that any sort of mathematics experience that can be taken outdoors, where school or local surroundings are utilised for learning purposes, could also foster positive mathematical identities. Such experiences can assist students in making real-world connections with their mathematics learning.

Australian curriculum

Launched in 2013, the ACM is relatively new curriculum. In this section, we provide a brief overview of the structure of the ACM and further discuss how a focus on key proficiencies and capabilities can assist students to develop positive dispositions towards mathematics.
Mathematics is fundamental to, and enhances, the lives of all Australians and is an essential tool in becoming and being an active member of society. Effective learning and teaching of mathematics is cultivated by sound planning that is collaborative in nature, requiring a collective notion of how children learn and understand mathematics, and this planning and teaching in primary schools must be seen in terms of the “big picture”, and governed by an interrelated framework of mathematics. In Australia, this big picture is provided by the ACM (Downton, 2015).

The Australian Curriculum, Assessment and Reporting Authority (ACARA) is an independent statutory authority responsible for the national curriculum for school age students from the first year of schooling (Foundation) to Year 10. The development of a national curriculum was a milestone in mathematics education in Australia, as it provided a standardised curriculum where previously the mathematics curriculum in each State and Territory differed. The ACM is published as a fully online document and is regularly updated. At time of publication (2018) the ACM is in its 8th iteration. It is important to note that as the ACM is aligned to National Educational goals, the curriculum is in a constant state of flux (Atweh, Miller & Thornton, 2012). Teachers therefore need to remain conversant with the regular updates to the online version of the ACM.

The ACM comprises six content strands that are presented as three conceptual pairs:

- Number and Algebra;
- Measurement and Geometry; and
- Probability and Statistics (ACARA, 2017e).

These conceptual pairs are regarded as the three strands of the ACM. As students engage with topics in each of these strands, they are expected to develop mathematics knowledge that is more than just content. As such, the ACM includes the four proficiencies of understanding, fluency, problem-solving and reasoning (ACARA, 2017c). According to Askew (2012), understanding in the ACM involves an internal process of connecting ideas to develop an evolving comprehension and awareness of specific knowledge. Askew further stipulates that understanding is the “relationship between the why and the how of mathematics” (p. 20). Fluency, a second proficiency, is the process of strategically following a set of procedures with factual recall (Askew, 2012). Furthermore, fluency entails more than just efficient computation and the memorisation of facts and includes the capacity to choose
efficient strategies that work interchangeably between the concept and known knowledge (Hurrell, 2014). Problem-solving is a complex proficiency and, according to ACARA (2017c), involves the utilisation of mathematics to decipher contextual quandaries. Problem-solving involves making sense of problems, having persistence to solve them, and can entail the use of heuristics (e.g., guess and checking, looking for patterns, etc.) (Clarke, Goos, & Morony, 2007; Schoenfeld, 2013). Lastly, reasoning mathematically involves the ability to explain and justify thinking by evaluating, deducing and summarising to draw inferences (Clarke, Clarke, & Sullivan, 2012). The intent of the ACM is to cultivate an intricate and in-depth mathematical knowledge via the proficiencies, which means that teachers must design mathematics learning experiences so that students develop rich conceptual understanding of the topic; that they become fluent with the skills and procedures associated with that topic; and that they develop competency in problem solving and reasoning associated with that topic. A focus on the four proficiencies is fundamental in the development of positive attitudes towards mathematics (Lowrie, 2005).

In addition to the ACM comprising three content strands and four proficiencies, the teaching of mathematics, as is the case for all learning areas, is to develop students’ general capabilities. In the Australian Curriculum, there are seven named general capabilities that are to be developed through all learning areas. These are: Literacy; Numeracy; Information and Communication Technology; Critical and Creative Thinking; Personal and Social Capabilities; Ethical Understanding; and Intercultural Understanding (ACARA, 2017b). It is important to note here that numeracy is not only developed in Mathematics as a general capability, it is expected that numeracy be incorporated into all curriculum learning areas. Numeracy is the capacity to apply mathematics in a context for a specific purpose and all learning areas have inherent numeracy demands (Human Capital Working Group, 2008). For example, in the Australian Curriculum: Science, Year 2 students are required to use informal means for measuring, such as using cups (cooking), hand spans (length), walking paces (distance) to make, compare and record observations utilising digital technologies as appropriate (ACARA, 2017a). Using informal units for measuring is a feature of the mathematics curriculum; in the science curriculum the students are applying their mathematics in another context, and thus they are engaging in a numeracy activity. Although each of the general capabilities is vital to a comprehensive education, we believe that numeracy is of particular importance because it involves a great deal more than a generous mathematics understanding. Numeracy connects the mandated mathematics taught at school
to everyday situations that require “problem solving and critical judgement” (Geiger, Forgasz & Goos, 2015, p. 330). We will discuss Numeracy in detail later in this chapter.

While the ACM provides organisation and structure, with an extensive quantity of content endorsed, some researchers and teachers have noted potential deficiencies and limitations in its design and functionality. Sullivan, et al. (2013) suggest that teachers can experience complications in the planning process when attempting to identify the significant ideas within the curriculum due to the 152 content descriptors and 251 elaborations that comprise the three strands in the primary years of schooling. Furthermore, while the ACM incorporates a conventional viewpoint with considerable focus upon content, some critics suggest that it is lacking in its integration of complex problem-solving opportunities that relate to real life contexts via cross-disciplinary approaches (Atweh & Goos, 2011). Also, the structure of the curriculum may lead to the teaching of small and discrete mathematical ideas and concepts, and this can militate against the well documented benefits of rich tasks (Goos, Geiger & Dole, 2013) and use of inquiry-based teaching approaches (Beswick, Muir, & Callingham, 2014).

While the ACM is considered to incorporate some aspects that require further consideration; overall the curriculum is a useful and necessary resource and a significant step forward in a coordinated approach to teaching mathematics Australia wide. The ACM affords students the opportunity of a comprehensive study of critical mathematical skills and concepts. Importantly, for the development of an appreciation of mathematics, it is comforting to note that the ACM stipulates that mathematics has a unique worth and elegance and therefore requires teachers to attend to the affective dimension noted earlier. Additionally, it requires that mathematics be developed as a useful and applied field, and this latter requirement has implications for developing students’ numeracy.

**Numeracy**

Numeracy is not equivalent to mathematics. This observation causes confusion for many people, as often the terms numeracy and mathematics are often used interchangeably. Indeed, in some primary school classrooms, you may find that students do not normally study mathematics as they are deemed to be completing numeracy lessons instead. As noted earlier,
the Australian Curriculum consists of a number of learning areas (subjects), such as Mathematics, English, the Arts, and so on. Through the teaching of each subject, the general capabilities are to be developed. Thus, in the Australian education context, Mathematics is a learning area (subject) and numeracy is a general capability, which means that a primary school that only teaches numeracy is overlooking mathematics as a subject.

A potential source of confusion regarding mathematics as a subject, and numeracy as a general capability, may stem from the National Assessment Program – Literacy and Numeracy (NAPLAN) that focuses on assessments of literacy and numeracy in Years 3, 5, 7 and 9. Despite the label, the numeracy assessment is not technically a test of numeracy as it actually tests students’ mathematics knowledge derived from the ACM – i.e. the questions are often stripped of any direct relevance to the lived experiences of many children. To illustrate the difference between mathematics and numeracy, consider the following quote:

Mathematics climbs the ladder of abstraction to see, from sufficient height, common patterns in seemingly different things. Abstraction is what gives mathematics its power; it is what enables methods derived from one context to be applied in others. But abstraction is not the focus of numeracy. Instead, numeracy clings to specifics, marshalling all relevant aspects of setting and context to reach conclusions. (Steen, 2001, p. 17-18)

This quote is suggestive of mathematics as a science of patterns, of a powerful, connected body of knowledge that makes up the discipline of the subject. In contrast, numeracy is associated with context; it is about using mathematical knowledge and applying it. The following description taken from the Australian Curriculum elaborates this notion:

…students become numerate as they develop the knowledge and skills to use mathematics confidently across all learning areas at school and in their lives more broadly. Numeracy involves students in recognising and understanding the role of mathematics in the world and having the dispositions and capacities to use mathematical knowledge and skills purposefully. (ACARA, 2017d, para 1)
Numeracy, therefore, is much richer than the attainment of mathematics skills and knowledge. It is the capacity to apply that knowledge in authentic contexts. In terms of the NAPLAN numeracy test, the capacity for a test situation to be regarded as a real-world context for applying mathematics is questioned. Numeracy is about being able to apply mathematics in real situations. It occurs when you are adjusting ingredients in baking a cake and your cake is perfect; when you are making estimations of the amount of fuel you have left in your car and you manage to get to the service station before your car runs out of fuel; when you count out the money to pay cash for a purchase; or when checking the bus timetable and you make it on that bus. Whilst we can prepare students for numeracy in the real world through engaging in rich investigations grounded in authentic contexts, the NAPLAN numeracy test, with its series of questions that link to topics within the Australian Curriculum: Mathematics appears to be more a test of mathematics rather than a test of numeracy.

Mathematics knowledge and skill is therefore only one, albeit an important one, aspect of numeracy. A 1997 Australian definition of numeracy read as follows:

To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life. In school education, numeracy is a fundamental component of learning, performance, discourse and critique across all areas of the curriculum. It involves the disposition to use, in context, a combination of:

- Underpinning mathematical concepts and skills from across the discipline (numerical, spatial, graphical, statistical, and algebraic);
- Mathematical thinking and strategies;
- General thinking skills; and
- Grounded appreciation of context (DETYA, 1997, p. 15).

From this description, the emphasis on context is clear. More recently, a substantial body of work has been undertaken by researchers working extensively in schools to support teachers in embedding numeracy across the curriculum and thus building children’s numeracy as a general capability (see for example, Cooper, Dole, Geiger, & Goos, 2012; Geiger, Fergasz, & Goos, 2015; Geiger, Goos, & Dole, 2014; Goos, Geiger, & Dole, 2014; Geiger, Goos, & Dole, 2013). One of the fundamental bases for the work undertaken by these researchers with
teachers in their schools was the presentation of a distinct model of numeracy. The model, developed by Goos (2007) (See Fig 9.1), was found to assist teachers in building their own rich understanding of what numeracy is and what it means to be numerate. This conceptualisation of numeracy supports teachers in making the distinction between numeracy and mathematics and provides a common language to describe numeracy practices and the numeracy of their students.

Fig 9.1. Model for numeracy (Goos, 2007)

Goos’ (2007) model for numeracy highlights the key elements of numeracy: real-life contexts, mathematical knowledge, representational, physical and digital tools, positive dispositions and a critical orientation. The model consists of four connected, equilateral triangles that actually presents as a net for the construction of a tetrahedron. In the 2-dimensional model, Context is located in the centre and serves to emphasise the centrality of context to numeracy. The other triangles depict other elements of numeracy. Mathematical knowledge includes concepts, skills, problem solving strategies and the capability to make valid estimations. Tools can be subcategorised as representational, physical and digital. Representational tools assist mathematical understanding and are often a way of communicating mathematical information. These include symbol systems, graphs, maps, diagrams, drawings, tables, ready reckoners and so on. Physical tools are objects that are used in everyday situations such as measuring instruments (rulers, measuring jugs, scales) and
models. Digital tools include computers, graphing software, calculators, tablet Applications (Apps), and the Internet.

Although dispositions are critical to our understanding of both mathematics and numeracy, it is sometimes an overlooked aspect of numeracy. For children to be numerate, that is, to use and apply mathematics in real situations in their everyday life, they must have the disposition to do so. That is, the confidence to apply the mathematics they have learned in school. As outlined previously in this chapter, many children leave primary school with negative attitudes to mathematics and this also limits the development of their numeracy capability. As teachers, supporting the development of a positive disposition towards mathematics must be paramount. Disposition clearly relates to confidence in learning and using mathematics but also relates to attitudinal factors such as perseverance in problem solving; in being prepared to reason clearly to determine solutions; the capacity to apply strategies flexibly in unknown situations; and being prepared to take risks in applying mathematics, rather than relying on others to provide a strategy or solution.

A core element of numeracy is a critical orientation. This means having the capacity to analyse information presented mathematically and to make well-founded judgments based on the outcomes of the analysis. This includes checking solutions to see if they make sense, but also interpreting information and taking action as required. A simple example of a numerate person taking a critical orientation is considering all information regarding an advertised product to ensure that it is truly value for money. A critical orientation includes the capacity to challenge an argument or position. In the 2D numeracy model, a critical orientation is depicted as a type of cloud overlaying all other elements to highlight its importance to each element. The model also includes three very important labels: work, citizenship, and personal and social. These three labels are included in the model to ensure that the purpose of numeracy is always in focus. As numeracy is fundamental to the realisation of full, active, citizenship, it becomes the responsibility of all teachers in all learning areas. As stated by the Australian government, “while mathematics can be taught in the context of mathematics lessons, the development of numeracy requires experience in the use of mathematics beyond the mathematics classroom, and hence requires an across the curriculum commitment” (Human Capital Working Group, Council of Australian Governments, 2008, p. 7). One strategy for ensuring a broad, cross disciplinary understanding of numeracy is the use of rich tasks.
Rich Tasks

Students’ numeracy development can be enhanced through engaging them in rich tasks that incorporate the utilisation of “good” mathematical questions (Sullivan & Lilburn, 2002). Good mathematics questions are those that have a range of answers and that generally allow students to take a range of possible pathways to generate solutions. They immerse all learners at a range of levels and enable students to access and use some mathematics along the way. In a similar way, rich tasks are those that engage students at all levels of mathematics in a way that challenges them to think mathematically; often encouraging them to work collaboratively with peers to develop solutions (Goos, Geiger, & Dole, 2013). Rich tasks often start with a question. One example of a rich task, that is also a rich numeracy task, is based upon the following question: Why do penguins huddle? This rich task connects to the Science curriculum and exposes students to the concept of homeostasis; i.e. maintaining a constant body temperature. It explores the ratio of surface area to volume in a manner that is accessible to students of many ages. Even children in junior primary years can start to think of this big concept through exploring why penguins huddle by constructing some penguins out of blocks and then determining surface area, volume, and ratio as outlined in the activity below.

Why do penguins huddle?

1. Take three wooden blocks and create a penguin in an upright position (i.e. one cube for the head, one cube for the upper body, one cube for the lower body).

2. Determine the volume of the penguin (count the cubes). Determine the surface area of the penguin (count the number of square faces that are on the outside of the penguin).

3. Construct a data table, and enter this data in the data table. Calculate the surface area to volume ratio (See Table 1).

4. Construct more penguins and completely surround the first penguin with the new penguins. You should have a total of nine penguins in the huddle this time.

5. Calculate the surface area, volume and determine the ratio.

6. Create more penguins and determine the surface area, volume and ratio.

7. Examine the data and discuss what is happening as the number of penguins in the huddle increases. What can you say about the surface area and volume ratio as the huddle increases? Write a statement explaining why penguins huddle, making mention of the ratio between surface area and volume.
**Table 9.1. Is there a pattern? How many penguins before the ratio is 1:1?**

<table>
<thead>
<tr>
<th>Number of penguins</th>
<th>Surface area</th>
<th>Volume</th>
<th>SA:V ratio</th>
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</tbody>
</table>

Rich tasks can be readily designed using everyday materials and considering everyday situations. For example, numeracy can be developed through exploring the labels of food, e.g. determining the relative salt and sugar content of a range of foods. Children can investigate the “health” of the foods in their lunchbox and can design healthy breakfast or lunch menus. A significant resource that primary school teachers now have for working with rich tasks is the use of manipulatives, initially concrete and then more broadly, digital in nature.

**Provocation:** The children in the Year 5 classroom were undertaking an investigation to answer the question: Do Year 5 children have healthy lunches? One group decided to undertake an ‘audit’ of every child’s lunchbox by examining the contents of each lunchbox that their fellow classmates brought to school on that day. As they opened the first lunchbox, they engaged in considerable discussion about how to record the information that was presented to them. They created a list of the lunchbox contents and drew some pictures. When they came to the second lunchbox, their list began to grow. Their recording system started to take up a lot of paper. They realised that they did not have an efficient approach for collecting and collating data. They engaged in extensive discussion to determine a suitable approach.

In your group, brainstorm a system for recording the data that these Year 5 children would find. Imagine that you are a Year 5 child. What would be the process for determining an efficient system? Imagine that you are the Year 5 teacher of these children. What would you do to assist or intervene, or not? Is there merit in providing the children with space to develop, trial and modify their own system for collecting data? Thinking more holistically on this task, discuss whether this is a rich task and justify your response.

Consider this task from a numeracy perspective. Using the elements of the numeracy model (context, mathematical knowledge, tools, dispositions, critical orientation), analyse the extent to which this task could be considered a rich numeracy task. As a teacher, what actions would
you need to take as the children undertook this investigation, to ensure that this was a valuable numeracy learning experience that connected to the elements of the numeracy model? Consider specifically how this investigation might develop in students a critical orientation and consider the extent to which this investigation connects to other learning areas in the curriculum.

**Digital Technology and mathematics**

Technologies (in mathematics often termed manipulatives) of various forms have always played a critical role in mathematics education - Cuisenaire rods, Multi-base Arithmetic Blocks (MAB), ten frames, place value charts, to name but a few – and activities using such manipulatives have long formed the basis of enhancing student learning in mathematics. A wealth of literature exists on the benefits of their use to support student learning in mathematics (e.g. Moyer-Packenham et al., 2015; Sarama & Clements, 2009). Such literature likely reinforces the experience of most primary school teachers and it would be a rare classroom where at least the use of basic manipulatives such as MAB, counters and the like are not used. However, of more interest to us here is an examination of the last few decades that have seen an explosion in the availability and use of digital technology in mathematics classrooms to support student learning.

Increasingly it is being overtly recognised in numerous international curricula that mathematics manipulatives do not necessarily have to be of a physical nature but can be virtual as well (Ladel & Kortenkamp, 2016). For example, the ACM explicitly states that “Digital technologies allow new approaches to explaining and presenting mathematics, as well as assisting in connecting representations and thus deepening understanding” (ACARA, 2009, p. 12, para 7). Due to the push to incorporate digital technology, it is perhaps unsurprising that a range of digital technologies – Interactive Whiteboards, Laptops, Tablets, Visualisers etc. – are now likely to be found in many primary mathematics classrooms (Moyer-Packenham et al., 2015) and used to provide opportunity for students to use digital (i.e. virtual) manipulatives. According to Moyer-Packenham and Bolyard (2016), a virtual manipulative is “an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated that presents opportunities for constructing mathematical knowledge” (p. 1). Although we acknowledge that external representations can never exactly represent a
student’s internalisation of knowledge, we follow Zbiek, Heid, Blume and Dick (2007), who suggest that manipulatives “are useful as visible phenomena that can be shared and discussed with others (e.g. other learners or the teacher and thus help teachers understand conceptual difficulties that students face)” (p. 1173).

At their core, virtual manipulatives provide students with opportunities to combine pictorial and symbolic representations, with the actions that they perform on them, to better understand mathematical concepts. A recent meta-analysis conducted by Moyer-Packenham and Westenskow (2013) found that good quality virtual manipulatives: allow exploration in a different manner to concrete manipulatives; support the development of individual representations as the learner is in control; and have a moderate, positive effect on student achievement. Özel (2012) reported on affective effects of the use of virtual manipulatives and noted that immediate feedback enhanced student self-efficacy. In addition, Özel reported that the use of virtual manipulatives was identified with reducing the time it takes to learn the affordances of a related concrete manipulative and also with enhancing student enjoyment, attitudes, and interest in mathematics. Apps for tablet devices are the latest iteration of virtual manipulatives (see Ladel & Kortenkamp, 2016; Larkin, 2015). Whilst it is apparent that not all iPad apps meet the definition to qualify as a virtual manipulative, i.e. they may be static and merely convey knowledge via text-based definitions, this merely reinforces the historical legacy regarding the availability of both high quality and low quality concrete and digital manipulatives in general (Larkin, 2016). On balance, it appears clear that virtual manipulatives, if used in a pedagogically appropriate manner, play an important role in mathematics education.

Given that the use of manipulatives to teach mathematics is supported by the literature, consideration of the practical classroom component of the equation i.e. the which, when, and how of manipulatives use, comes to the fore. Bruner (1966) proposed that students learn through three experiential stages: Enactive (direct sensory) experience where students take an active part in their learning through the manipulation of their learning environment; Iconic representation of experience where enacted experiences are represented via diagrams, tables, charts etc. and; Symbolic representation including written language symbols such as words and mathematical symbols. More recently Cooper (2012) has used the terms “concrete”, “representational” and “abstract” to describe these three stages and indicating that primary school students will predominantly be working in the concrete and representational stages.
Larkin (2016) has proposed a framework (See Fig 9.2) that can assist primary school teachers in making decisions regarding the which, when and how of virtual and concrete manipulatives use to best support student learning.

**Figure 9.2.** Mapping mathematics materials to Bruner’s experiential stages of learning (Larkin, 2016).

A number of important considerations arise in using the framework. Firstly, although the framework suggests the use of manipulatives across the span of conceptual development, incorrect use of manipulatives may in fact be harmful to student learning. For example, it would be inappropriate to only use square tiles when developing area concepts as this may detract from the development of an understanding of area as a “covering” and may also promote the misconception that only regular figures have area. Secondly, although it is normally the case that, to promote robust conceptual development of mathematics content, students should spend a significant amount of time at the Enactive stage of learning, some learners may more quickly develop Iconic or even Symbolic understanding of concepts and thus flexibility in the use of the framework is required. For example, when learning about area, the Enactive stage would incorporate a range of activities where students physically cover a range of shapes to determine their area – e.g. the rectangle is six stamps in area. The Iconic stage would include diagrammatic representations of their area – e.g. a drawn rectangle indicating three stamps by two stamps. The Symbolic stage would include the representation of the area using a formula. Finally, the framework depicts a separation of concrete (familiar and substituted) and digital manipulatives as the latter generally add an
additional level of abstractness and thus digital resources should normally be used after concrete materials in the development of conceptual understanding.

Although this brief discussion has argued that digital manipulatives can be gainfully utilised by teachers to promote student learning in mathematics education classrooms, a number of concerns still remain. Firstly, care must be taken to match the manipulative with the type of learning being undertaken. Secondly, attention must continually be paid to the equity issues that emerge when teaching and learning with digital manipulatives, particularly in rural and remote communities (Lowrie & Jorgensen, 2012) as these still remain a point of division between the advantaged and the disadvantaged. Thirdly, challenges still remain for teachers when introducing or extending their use of digital manipulatives as, despite technology marketing being quick to indicate the positive aspects of manipulatives, Geiger et al. (2016) note that teachers often adopt new technologies in a manner that reinforces their current practice rather than transforming it. Given these challenges, further research is required in relation to the use of digital manipulatives in mathematics classrooms.

**Provocation:** It is becoming more common in many jurisdictions for educational authorities to prescribe teaching and learning sequences for mathematics or to mandate the use of prescribed mathematics textbooks (paper based or eBooks). In addition, teachers are increasingly under pressure to continually assess student learning. Given these curriculum and time pressures:

- How do you, as a classroom teacher, incorporate manipulatives in your teaching, given that there use may appear to be trivial to school leadership?
- What mechanisms do you have in place as a classroom teacher to remain current with new developments in the digital technology space that can support student learning?
- How do you best utilise technology in your mathematics classroom given that, at one end of the spectrum it may be very limited through to the different issues of managing teaching in Bring Your Own Device (BYOD) contexts where each student may have access using their own device?
- How do you communicate your use of manipulatives, especially virtual manipulatives, to parents who may be concerned with issues of overuse of digital devices in place of “real” learning?
Learning Theories

Underpinning the use of manipulatives in mathematics education in particular, and the teaching of mathematics more generally, is the understanding by teachers of relevant theories of how students learn mathematics. Such theoretical knowledge assists teachers in the development of rich mathematical tasks to support student learning. In earlier chapters of this book you would have been introduced to some learning theories including [XXX – editors to add]. Here our intention is not to reiterate these theories, but rather to highlight how two learning theories can facilitate a clearer understanding of student learning in mathematics and numeracy. Although there are many learning theories and theorists, the focus in this section will be on two that we recognise are fundamental to mathematics education in primary schools:

- Concrete to abstract (from Piaget, 1972)
- Sociocultural theory (from Vygotsky, 1978)

Concrete to Abstract

A prominent theorist in the development of mathematics instruction is Piaget (1972), who developed a range of ideas that continue to exert a large influence on mathematics education in the primary years of schooling. In his early work, after undertaking some empirical work with a small group of children, he theorised that children pass through certain “stages of learning” at particular ages. This can be seen in the way the curriculum reinforces the use of “hands-on” materials and manipulatives leading to the treatment of mathematics concepts in an abstract manner by the end of primary school and certainly by the early secondary school years. Interestingly, later in his writing it is clear that Piaget moved away from his rigid linking of “ages with stages”; however, he still held to the notion that conceptual learning moves from the concrete to the abstract. As a theoretical principle, this has great practical relevance in planning and teaching mathematics.

As noted above, in the early primary years, there is a strong emphasis on manipulatives and concrete materials as these provide a good foundation for conceptual development in mathematics. Indeed, children using counters in learning to count is useful; however, this can cause later problems when we do not help children to move towards abstraction. In other words, although children start counting with their fingers or other readily available resources, as learning progresses we need to help them understand the underlying structure of numbers
without having to rely of these readily available manipulatives (e.g. gaining number sense to understand how large the number 123 is without the use of concrete representations).

A further example is using bundling sticks or Mathematics Attribute Blocks (MABs) to develop initial conceptual understanding of two-digit numbers. After beginning activities, where students can learn that “ten ones is one ten”, through various activities where they bundle groups of ten, the goal is then to help them move towards more abstract and powerful representations of two-digit numbers. To do this a teacher can use a simple laminated T-chart where the sticks are placed on the chart, and the digit for the corresponding number of bundles or MAB ‘ten sticks’ (in the tens column) and sticks or MAB ‘unit cubes’ (in the ones column) can be recorded just below (see Figure 9.3 below). In this way we work from the concrete (the sticks) towards the abstract (the digits), but the transition is bridged (by the materials) for the learners.

<Fig 9.3 insert photo of T-chart>

Then, to further enhance this move towards the numerical (symbolic / abstract) representation of two-digit numbers, a smaller T-chart could be drawn on the side where the students can write down the numbers (thus facilitating a further move away from the concrete materials), and then finally a space to write the number without a T-chart (see Figure 9.4).

<Fig 9.4 insert next T-chart photo>

Of course, there is a lot more that can be done with these materials and frames to help children understand two-digit numbers in an abstract way, but the details above suffice to illustrate our general point. Piaget’s theoretical idea about moving from concrete to abstract is an important tool for planning and facilitating mathematical learning in the primary years (and beyond), but as we have shown here, it is the teacher’s responsibility to help students make the conceptual move from the concrete to the abstract. Abstract representations of mathematical ideas are far more powerful and useful than concrete ones because they are transferable and can be examined and used more broadly. In short, facilitating development by bridging between representations for students is important and necessary.
**Sociocultural Theory**

As with the case of Piaget, we cannot do justice in this chapter to the extensive theoretical work of Vygotsky (1978); here the focus will be on only one dimension of his learning theory that relates to learning mathematics. Vygotsky’s theoretical work is a significant foundation to many of the sociocultural theories of learning. Three major themes can be identified, the first concerns *social interaction*. Vygotsky’s theory stresses that social interaction is central in the process of cognitive development. He explains that the various functions of a child’s cultural development appear first on the social level and then later on an individual level. He also argues that social learning generally precedes cognitive development. The second theme is labelled *the more knowledgeable other* which is somewhat self-explanatory. It refers to any individual who has a better understanding or more sophisticated understanding than the learner, usually thought of as teachers or adults, but also peers or even computers. The third theme is *the zone of proximal development*, referring to the ‘distance’ between a child’s ability to solve problems independently and their ability to perform tasks under adult supervision and/or peer collaboration. Vygotsky believes that learning occurs in this zone. Vygotsky’s theoretical ideas promote a shift from the traditional model of teaching where a teacher “transmits” information to a student, to a learning context where teacher and student, or student and student, work together to construct meaningful learning experiences. This would be done through the use of tools such as language, and this is why we have the children explain and justify their mathematical thinking. Learning then becomes a reciprocal experience for all involved.

**Assessing mathematics learning**

Regardless of the various theoretical perspectives that might underpin their teaching, a primary responsibility for all teachers is determining whether learning has occurred. We therefore conclude this chapter with a focus on appropriate assessment in primary mathematics education. As with any learning area, assessment is integral to teaching mathematics. When planning mathematics learning activities and lesson sequences, it is important that assessment experiences are thoughtfully planned and designed prior to teaching rather than added ad hoc as the teaching unfolds. Even more important is to determine the impact of assessment on students as assessment can, and often does, contribute to the development of negative dispositions towards mathematics.
Research suggests that, with many students exhibiting a dislike or fear of mathematics, inauthentic or solely summative assessment experiences can serve to compound or augment negative attitudes and feelings towards mathematics with subsequent negative impacts on the development of students’ numeracy capabilities (Grootenboer & Marshman, 2017). Whilst assessment of students’ mathematics learning, and providing feedback to students on their mathematics progress is vital, teachers can thoughtfully undertake mathematics assessments in ways that can minimise the potential negative impact of messages that assessment often gives children. For example, when students are engaged in rich tasks, they are exhibiting mathematical behaviours that can be observed by their teacher without disrupting the learning. Observation opportunities enable teachers to undertake informal assessment without interrupting the learning experience and such experiences are unlikely to be captured by an end of unit, written assessment task. A teacher cannot undertake full observations of all students during one lesson, but frequent use of rich tasks will provide opportunities for this to occur.

Such an approach to assessment is not new. Almost 30 years ago, the value of informal assessment was enunciated as follows:

Assessment must be more than testing; it must be a continuous, dynamic, and often informal process…Assessment is more than the establishment of definite conclusions. …Assessment should provide a biography of student’s learning, a basis for improving the quality of instruction. (National Council for Teachers of Mathematics, 1989, p. 203)

The primary school classroom provides great opportunity for inclusion of informal assessment of this nature through incorporation of rich tasks. Assessment approaches must also align with the intent of the ACM, particularly the four proficiencies of understanding, fluency, problem solving, and reasoning and the seven general capabilities. An authentic assessment plan should ensure that teachers can assess and provide feedback to students on their progress in demonstrating these proficiencies and general capabilities. This is unlikely to occur in learning environments characterised by a strict regime of Friday morning mathematics tests. Teaching mathematics in the primary school means creating classrooms where students are engaged in problem solving and reasoning so that these processes are valued. However, it must be acknowledged that assessing proficiencies and general capabilities is not an easy task. By way of example, one way to assess problem solving is to
develop rubrics that identify key aspects of the task (criteria) that you expect students will demonstrate. Such things may include: use of strategies; working in groups; communicating mathematically; using diagrams and other representations; developing plans of attack; justifying results; drawing conclusions; applying mathematical procedures and techniques. Standards and levels can then be developed to give students feedback on the extent to which they have demonstrated criteria according to particular levels. The resulting assessment rubric provides an overview of what is valued by the teacher in relation to the task. There are websites available that assist teachers in constructing rubrics (See, for example, the Queensland Curriculum and Assessment Authority website = https://www.qcaa.qld.edu.au/p-10/aciq/p-10-mathematics). Rubrics provide more detailed information for teachers to make judgments about students’ mathematics knowledge than using test results alone. We agree that data can provide teachers with information to better inform and shape subsequent teaching and learning. However, we remain concerned that sometimes this data is a) narrow in scope and focussed on skill development; and b) can be harmful to developing positive mathematics dispositions if used uncritically.

Concluding Comments

This chapter has argued that, although mathematics has always been seen as an important and integral part of the primary school curriculum, it has been dogged by a persistent and pervasive “wicked problem”; namely, the long history of students learning to dislike mathematics and seeing it as dull and useless (Attard et al., 2016). What we have therefore attempted in this chapter is to foster a developing appreciation by primary school teachers that, as an integral life skill, children need to develop mathematical concepts and ideas. However, this is to be achieved in an environment where students learn that mathematics is enjoyable, interesting and useful. Thus, in this chapter we have focussed on the who of mathematics education (i.e., the students), then on the how of mathematics education (i.e., the pedagogy), with less emphasis on the what of mathematics education (i.e., the curriculum). We are confident that, via the small vignettes of actual classroom practices and activities, we have been able to model an approach to teaching mathematics education that demonstrates the who, how and what philosophy ‘in action’. We invite primary school teachers to join with us in our new approach to primary mathematics education.
Reference List


Australian Primary Mathematics Classroom, 21(1), 12-17.


