We derive a quantum Cramér-Rao bound (QCRB) on the error of estimating a time-changing signal. The QCRB provides a fundamental limit to the performance of general quantum sensors, such as gravitational-wave detectors, force sensors, and atomic magnetometers. We apply the QCRB to the problem of force estimation via continuous monitoring of the position of a harmonic oscillator, in which case the QCRB takes the form of a spectral uncertainty principle. The bound on the force-estimation error can be achieved by implementing quantum noise cancellation in the experimental setup and applying smoothing to the observations.

We discretize time as $t = t_0 + j\delta t$, $j = 0, 1, \ldots, J$, and assume that $\delta t$ is small enough that we can treat $x(t)$ as piecewise-constant, i.e., $x(t) = x_j$ for $t_j \leq t < t_{j+1}$. The prior probability density $P[x]$ for the vector $x \equiv (x_{j-1}, \ldots, x_0)^T$ characterizes what is known or assumed about the waveform prior to the measurements. For a vector of observations $y \equiv (y_{N-1}, \ldots, y_1, y_0)^T$ made any time during the interval $t_0 < t \leq t_f$, we define a conditional probability density $P[y|x]$. The joint probability density is $P[y, x] = P[y|x]P[x]$. Finally, we define the estimate of $x_j$ as $\hat{x}_j[y]$ and the estimate bias, given signal $x$, as $\int D y (\hat{x}_j - x_j)P[y|x] = b_j[x]$, where $D y \equiv \prod_{n=0}^{N-1} dy_n$.
with respect to \( x_k \), and then integrating over all \( x \) using
\[
Dx = \prod_{j=0}^{J-1} dx_j,
\]
we obtain
\[
-\delta_{jk} + \int Dx Dy (\tilde{x}_j - x_j) \frac{\partial P[y,x]}{\partial x_k} \]
\[
= \int Dx \frac{\partial}{\partial x_k} (b_j[x] P[x]) = 0,
\]
where the final equality assumes \( b_j[x] P[x]|_{x_k = \pm \infty} = 0 \). This assumption, also used in the proof of the classical CRB \([1]\), is satisfied as long as the prior density approaches zero at the infinite endpoints (as it must for any probability density) and the bias there is not infinite.

Quantum mechanics enters this description, which till now is classical, by determining the conditional probability of the observations. Given a quantum system, we can describe any measurement protocol, including sequential measurements and excess decoherence, during the interval \( t_0 \leq t < t_f \) by introducing appropriate ancillae, in accord with the Kraus representation theorem \([2] [3] [4]\). This also accounts for any feedback during the interval, based on the measurement outcomes, because the principle of deferred measurement \([1\) allows one to put off the measurements on the ancillae till time \( t_f \); measurement-based feedback is replaced by controlled unitaries prior to the measurements, as schematically shown in Fig. 1.

**FIG. 1:** Any quantum dynamics and sequential measurements described by completely positive (CP) maps, including feedback based on measurement outcomes, as illustrated in (a), can be reproduced by unitary evolution of an enlarged system that includes appropriate ancillae, coherent controlled unitaries, and deferred measurements of the ancillae, as shown in (b).

In this approach, the overall system dynamics is described by unitary evolution of the enlarged system; the conditional probability of observations is given by
\[
P[y|x] = \text{tr}(E[y]|L_k)\rho_x,
\]
where \( \rho_x \) is the density operator of the enlarged system at time \( t_f \), conditioned upon \( x \), and \( E[y] \) is the positive-operator-valued measure (POVM) that describes the (deferred) measurements up to time \( t_f \). We denote expectation values with respect to \( \rho_x \) by angle brackets subscripted by \( x \), so that \( \langle E[y] \rangle_x = \text{tr}(E[y]|\rho_x) \). Continuous measurements can be modeled as the limit of a sequence of infinitesimally weak measurements \([2]\).

We now follow a procedure similar to the one used by Helstrom \([2]\) to derive the QCRB. We introduce an operator \( Q_k \) that satisfies \( \partial \rho_x/\partial x_k = (Q_k \rho_x + \rho_x Q_k^\dagger)/2 \). Unlike Helstrom, we do not require \( Q_k \) to be Hermitian. Note that the vanishing trace of \( \partial \rho_x/\partial x_k \) in the definition of \( Q_k \) implies that \( \text{Re}(Q_k)_x = 0 \).

It is convenient to incorporate the prior information by working in terms of a density operator \( \rho[x] \equiv \rho_x P[x] \) in a hybrid quantum-classical space and introducing an operator \( L_k = Q_k + \partial \ln P[x]/\partial x_k \), which satisfies \( \partial \rho[x]/\partial x_k = (L_k[x] \rho[x] + \rho[x] L_k^\dagger[x])/2 \). In terms of \( L_k \), Eq. 1 takes the form that we use to derive the QCRB:
\[
\delta_{jk} = \text{Re} \int Dx Dy (\tilde{x}_j - x_j) \text{tr}(E[y] L_k[x] \rho[x]).
\]

Multiplying Eq. 2 by \( u_j v_k \), where \( u_j \) and \( v_k \) are the components of arbitrary real column vectors \( u \) and \( v \), and then summing over all \( j \) and \( k \), we obtain
\[
u^T u = \sum_j u_j v_j = \text{Re} \int Dx Dy \text{tr}(A^\dagger B)
\]
where \( A^\dagger \equiv \sum_k v_k \sqrt{E[y]} L_k \sqrt{\rho[x]}, \ B \equiv \sum_j u_j (\tilde{x}_j - x_j) \sqrt{\rho[x]} \sqrt{E[y]} \), and \( T \) denotes transposition. It follows from Eq. 3 that
\[
(v^T u)^2 \leq \int Dx Dy \text{tr}(A^\dagger A) \int Dx Dy \text{tr}(B^\dagger B),
\]
where the second inequality is the Schwarz inequality. The second integral in Eq. 4 is \( \int Dx Dy \text{tr}(B^\dagger B) = u^T \Sigma u \), where
\[
\Sigma_{jk} \equiv \int Dx Dy P[x,y](\tilde{x}_j - x_j)(\tilde{x}_k - x_k)
\]
is the estimation-error covariance matrix. The first integral in Eq. 4 is, using the completeness of the POVM, \( \int Dx Dy \text{tr}(A^\dagger A) = v^T F v \), where \( F \) is a (real, symmetric) Fisher-information matrix,
\[
F_{jk} \equiv \frac{1}{2} \int Dx P[x] \text{tr}((L_j^{\dagger} L_k + L_k^{\dagger} L_j)[\rho_x]).
\]
Since \( \text{Re}(Q_k)_x = 0 \), \( F \) separates neatly into a quantum and a classical, prior-information component, i.e., \( F = F(Q) + F(C) \), where
\[
F^{(Q)}_{jk} = \frac{1}{2} \int Dx P[x] \text{tr}((Q_k^{\dagger} Q_k + Q_k^{\dagger} Q_j)[\rho_x]) \quad (7)
\]
\[
F^{(C)}_{jk} = \int Dx P[x] \frac{\partial \ln P[x]}{\partial x_j} \frac{\partial \ln P[x]}{\partial x_k}.
\]
When these results are substituted into Eq. 4, we find that \( (v^T F v)(u^T \Sigma u) \geq (v^T u)(u^T v) \). Setting \( v = F^{-1} u \) implies that \( u^T (\Sigma - F^{-1}) u \geq 0 \) for arbitrary real vectors \( u \). Since \( \Sigma - F^{-1} \) is real and symmetric, this implies that \( \Sigma - F^{-1} \) is positive-semidefinite; the matrix inequality
\[
\Sigma \geq F^{-1}
\]
(9)
is the QCRB in its most general form. To use a CRB in practice, it is customary to define a non-negative, quadratic cost function $C \equiv \text{tr}(A^T \Sigma)$ using a positive-semidefinite (Hermitian) cost matrix $A$ suited to the application. The matrix QCRB is equivalent to a lower bound, $C \geq \text{tr}(A^T F^{-1})$, on all such cost functions.

To calculate the QCRB, we must be more specific about the evolution of the enlarged quantum system. The Hamiltonian that governs overall system dynamics over the interval $t_j \leq t \leq t_{j+1}$, of duration $\Delta t$, is $H_j(x_j)$, with corresponding evolution operator $U_j = \exp[-iH_j(x_j) \Delta t/\hbar]$. We have $\partial U_j/\partial x_j = U_j(-i\hbar \partial H_j/\partial x_j)$, where $\hbar \partial H_j/\partial x_j$. Let $U_{kj} \equiv U_{k-1} \cdots U_{j}$ denote the evolution operator over the interval $t_j \leq t \leq t_k$. The density operator $\rho_x$ is related to the initial density operator $\rho_0$ by $\rho_x = U_{j0} \rho_0 U_{j0}^\dagger$, which gives $\partial \rho_x/\partial x_k = -i[M_k, \rho_x]$, where

$$M_k \equiv \frac{\partial U_{j0}^\dagger}{\partial x_k} U_{j0} = \frac{\partial}{\partial t} U_{jk} \hbar \hat{h}_k U_{j0}^\dagger \hbar \hat{h}_j,$$  

with $\hat{h}_k \equiv \int \hat{h}_k \hbar \hat{U}_{k0} = \hbar(t_k)$ being the Heisenberg-picture version of $h_k$. An obvious choice for $Q_k$ is the anti-Hermitian $Q_k = -i\hbar \partial M_k/\partial \hat{x}_k$, where $\Delta M_k \equiv M_k - \langle \hat{M}_k \rangle$. The quantum part of the Fisher matrix then becomes

$$F_{jk}(t,t') = \frac{4(\delta t)^2}{\hbar^2} \int dx P[x] \frac{1}{2} \text{tr} \left( (\Delta \hat{h}_j \Delta \hat{h}_k + \Delta \hat{h}_k \Delta \hat{h}_j) \rho_0 \right),$$  

(11)

where $\Delta \hat{h}_k \equiv \hat{h}_k - \langle \hat{h}_k \rangle_0$. Angle brackets with subscript 0 denote an expectation value with respect to $\rho_0$. The quantum Fisher information is thus a two-time covariance function, averaged over $P[x]$.

To take the continuous-time limit, we let $\Delta t \rightarrow 0$, $\Sigma_{jk} \rightarrow \Sigma(t_j, t_k)$, $F_{jk}(\delta t)^2 \rightarrow F(t_j, t_k)$, and $\Lambda_{jk}(\delta t)^2 \rightarrow \Lambda(t_j, t_k)$. The estimation-error covariance matrix then becomes the two-time covariance function of estimation error, $\Sigma(t, t')$, and the Fisher matrix becomes $F(t, t') = F^{(Q)}(t, t') + F^{(C)}(t, t')$, with

$$F^{(Q)}(t, t') = \frac{4}{\hbar^2} \int dx P[x] \left[ \frac{1}{2} \left\langle \Delta \hat{h}(t) \Delta \hat{h}(t') + \Delta \hat{h}(t') \Delta \hat{h}(t) \right\rangle_0 \right.$$

$$\left. \times \frac{\delta \ln P[x]}{\delta x(t)} \right] \frac{\delta \ln P[x]}{\delta x(t')},$$  

(12)

$$F^{(C)}(t, t') = \int dx P[x] \frac{\delta \ln P[x]}{\delta x(t)} \frac{\delta \ln P[x]}{\delta x(t')} \delta \hat{x}(t) \text{ being the functional derivative.}$$

In the continuous-time limit, the matrix QCRB retains the same form as Eq. (11), where the continuous-time inverse is defined by $\int_{t_0}^{t_j} dt'' F(t, t'') F^{-1}(t', t') = \delta(t - t')$. The bound on a cost function becomes

$$C \equiv \int dt dt' \Lambda(t, t') \Sigma(t, t') \geq \int dt dt' \Lambda(t, t') F^{-1}(t, t'),$$  

(14)
\(\xi(t)\), and becomes stationary. In the limit of negligible damping, the spectrum of \(\Delta q\) becomes \(S_{\Delta q}(\omega) = |G(\omega)|^2 \xi(\omega)\), where \(G(\omega) \equiv 1/m(\omega^2_n - \omega^2)\) is the oscillator transfer function. The spectral uncertainty principle \(17\) now takes the form

\[
C(\omega)\left(|G(\omega)|^2 \xi(\omega) + \frac{\hbar^2}{4S_{\Delta x}(\omega)}\right) \geq \frac{\hbar^2}{4} . \tag{18}
\]

The corresponding bound on point estimation error is

\[
\Pi \geq \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\frac{4}{\hbar^2} |G(\omega)|^2 \xi(\omega) + \frac{1}{S_{\Delta x}(\omega)}\right)^{-1} . \tag{19}
\]

Notice that a bandwidth constraint on \(x(t)\) is incorporated in the prior information: \(S_{\Delta x}(\omega)\) goes to zero outside the relevant bandwidth, thus allowing \(C(\omega)\) to be zero there and making the integral \(17\) finite.

We can elucidate the meaning of the QCRB \(18\) by considering how to estimate the force from the observations in this scenario. In the frequency domain, the observation process \(y(t)\) reads \(y(\omega) = G(\omega)[x(\omega) + z(\omega)]\), \(z\) being a noise term that depends on \(\xi\) and \(\eta\). Using smoothing \(17, 18\) to estimate \(x\) from \(y\) yields an error

\[
\Pi = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\frac{1}{S_z(\omega)} + \frac{1}{S_{\Delta x}(\omega)}\right)^{-1} . \tag{20}
\]

This is the minimum achievable error for a given noise spectrum \(S_z(\omega)\). It cannot be reached by the more well-known technique of filtering \(18\), as filtering does not make use of the entire observation record. If \(\xi\) and \(\eta\) are uncorrelated and quantum limited, we have

\[
S_z(\omega) = \frac{S_\eta(\omega)}{|G(\omega)|^2} + S_\xi(\omega) \geq \frac{\hbar}{|G(\omega)|} \equiv S_{SQL}(\omega), \tag{21}
\]

where the power spectrum \(S_{SQL}(\omega)\) is known as the standard quantum limit (SQL) for force detection \(18\).

It is now evident that to attain the QCRB \(18\), it is necessary to beat the SQL. This requires evading or tempering the effects of the backaction \(\xi\). One way to do this is to correlate \(\xi\) and \(\eta\), as was proposed for interferometric gravitational-wave detectors by Unruh \(11\). An alternative is to use quantum noise cancelation (QNC) \(8\), which has the advantage of making the QCRB \(18\) achievable, as we now show. One QNC approach, discussed in \(8\), adds an auxiliary oscillator with position \(q'\) and momentum \(p'\). One monitors continuously the collective position \(Q = q + q'\), giving a process observable \(y = Q + \eta\); the backaction force \(\xi\) acts on \(P = (p + p')/2\) and thus equally, with strength \(\xi\), on each of the two oscillators. Suppose the auxiliary oscillator has the same resonant frequency and equal, but opposite mass (the negative mass can be simulated by an optical mode at the red sideband of the optical probe). The dynamics of the collective position is then determined by \(dQ/dt = \delta p/m\) and \(d\delta p/dt = -m\omega_n^2 Q + x\), where \(\delta p = p - p'\). There being no backaction noise in \(z(t)\), one easily finds that

\[
S_z(\omega) = \frac{S_\eta(\omega)}{|G(\omega)|^2} \geq \frac{\hbar^2}{4S_\xi(\omega)} \frac{1}{|G(\omega)|^2} , \tag{22}
\]

with equality for quantum-limited noise. This quantum-noise-cancellation scheme beats the SQL and if the noise is quantum limited, does so optimally: the smoothing error given by Eq. \(20\) achieves the QCRB \(18\), which implies that the spectral uncertainty principle \(17\) is saturated. Our force-sensing QCRB, rigorously proven and demonstrably achievable, thus serves as a fundamental quantum limit, against which the optimality of future force sensing schemes should be tested. More generally, our QCRB for arbitrary cost functions \(14\) will find application whenever quantum-limited estimation of temporally varying waveforms is attempted.

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