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THE APPLICATION OF NORMALITY RULE AND ENERGY BALANCE EQUATIONS FOR NORMALLY CONSOLIDATED CLAYS

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ABSTRACT: In this paper, it is reiterated that the Roscoe and Poorooshasb (1963) formulation of the stress strain behaviour of normally consolidated clays is indeed in a more generalized form which is easily amenable to incorporate deformations under various degrees of drainage and can be extended to include cyclic loading and time effects beyond the primary phase of deformation. Also, the formulation can be used for stress states below the state boundary surface to include lightly overconsolidated and heavily overconsolidated clays. Particularly, it is shown here that Cam Clay model of Roscoe *et al.* (1963) and Modified Cam Clay model of Roscoe and Burland (1968) as based on energy balance equations and the normality concept can be considered as the special cases of the original formulation of Roscoe and Poorooshasb (1963). In order to achieve this, all theories are presented in similar mathematical forms, adopting the same formulation of Roscoe and Poorooshasb (1963). Modified Cam Clay Model of Roscoe and Burland, and the Roscoe and Poorooshasb theory made identical predictions of the shape of the state boundary surface, the pore pressure development during undrained behaviour, and the volumetric strain in the drained tests for all types of applied stress paths. Also, Modified Cam Clay model was only successful in predicting the shear strains along radial stress paths. For non-radial stress paths, Modified Cam Clay model needed an additional set of constant deviator stress yield loci, and when such a set was incorporated, the prediction from Modified Cam Clay model was the same as the original prediction of Roscoe and Poorooshasb (1963).

Keywords: Energy balance equation, normality concept, incremental stress-strain theory.

INTRODUCTION

Soon after the original contribution of Roscoe and Poorooshasb (1963) on the correlation of the undrained and drained behaviour of normally consolidated clays in the axi-symmetric triaxial apparatus, extensive developments have taken place in formulating elasto-plastic stress strain models for cohesive and cohesionless soils, however, the relative importance of the more generalized form of Roscoe and Poorooshasb (1963) work is only presented in a very brief or cursory form in many of the subsequent textbooks by well known authors on "Critical State Soil Mechanics". Indeed nowadays, the term "Critical State Soil Mechanics" predominantly implies a particular form of the "plastic potential surface" which in conjunction with a coinciding "yield surface" leads to a rather special type of the constitutive equation describing the flow of granular media (Poorooshasb 2000). Prior to the classical work of Roscoe and Poorooshasb (1963), all the research work was primarily concerned with the stress-volumetric strain relationship for soils as measured in the various types of tests. The position in 1960 was well summarised by Henkel (1960), "….. so far it has not been possible to relate shear strains in the various types of tests. For a complete understanding of the stress-strain behaviour of clays, it is necessary that the shear stresses and the shear strains be related. Until this problem is solved it will not be possible to examine in any fundamental way, the deformation behaviour of clays". This clearly illustrates the milestone contribution of the Roscoe and Poorooshasb (1963) formulation.

This paper expresses the three classical theories of Roscoe and Poorooshasb (1963), Cam Clay model (Roscoe *et al.* 1963), and Modified Cam Clay model (Roscoe and Burland, 1968), as developed for normally consolidated clays, in similar forms and then examine the implication of the energy balance equation used in the Roscoe and Burland theory and the normality concept in achieving a set of volumetric yield loci from which contributions are derived for the volumetric and shear strains. Later work by Dafalias *et al.* (1987), Atkinson *et al.* (1987) and others, also formulated the energy balance equation in a form similar to Roscoe *et al.* (1963) and Roscoe and Burland (1968), so that the plastic strain increment ratio was still dependent on the stress ratio. Notable difference was the work of Pender (1978), where a realistic plastic strain increment ratio was formulated for clays to be dependent on both the mean normal stress and the stress ratio, for overconsolidated states. It appears that, Modified Cam Clay model could predict the strains in radial type of stress paths while for non radial types, the shear strains can only be predicted correctly by the incorporation of a second set of constant q yield loci. The use of the two sets of yield loci makes the predictions from Modified Cam Clay model the same as the original Roscoe and Poorooshasb (1963) theory. The concept of the bounding surface model is more a sophisticated version of the double yield loci approach used in the revised version of Modified Cam Clay model. With

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the use of only the volumetric yield loci, the strains in radial type of stress paths only can be predicted successfully using Modified Cam Clay model. These conclusions are well supported by the extensive laboratory tests performed by Balasubramaniam (1969) at Cambridge University on re-sedimented specimens of Kaolin tested under a wide range of applied stress paths with stress states below and on the state boundary surface. Also the subsequent work at the Asian Institute of Technology on undisturbed samples of Bangkok clay, has further confirmed these findings (Balasubramaniam, 1975; Balasubramaniam and Chaudhry, 1978; Balasubramaniam *et al.*, 1992; Balasubramaniam *et al.*, 1999). The Roscoe and Poorooshasb (1963) formulation for undrained behaviour under quasi-static repeated loadings were presented in Balasubramaniam and Chui (1976).

Thus, worldwide use of these theories and models in numerical analysis based on finite element analysis and other similar techniques must realize their inherent restrictions and limitations.

INCREMENTAL STRESS STRAIN THEORY OF ROSCOE AND POOROOSHASB

An incremental stress strain theory was developed for normally consolidated clays by Roscoe and Poorooshasb (1963) for stress states on the state boundary surface. In this formulation the incremental axial strain associated with a given infinitesimally small stress increment in a drained test can be considered as the sum of two components that occur in (i) a constant volume or undrained deformation and (ii) a consolidation process in which the stress ratio remains constant. The relevant strain characteristics of the clay are therefore determined from the undrained and anisotropic consolidation tests alone and from these, the strains that occur in any type of partially or fully drained tests can be predicted. The validity of this formulation was thoroughly verified by Balasubramaniam (1969) and his subsequent student researchers at the Asian institute of Technology.

Thus the classical contribution of Roscoe and Poorooshasb (1963) can be expressed as

$$(d\varepsilon_1)_{drained} = \left[\left(\frac{d\varepsilon_1}{d\eta}\right)_v d\eta\right] + \left[\left(\frac{d\varepsilon_1}{d\varepsilon_v}\right)_\eta d\varepsilon_v\right] \qquad (1)$$

where $(d\varepsilon_1)_{drained}$ is the incremental axial strain in a drained test. Also, the first term on the right hand side represents the variation of axial strain increment $d\varepsilon_1$ with the increment in stress ratio $d\eta$ in a constant volume shear (undrained shear) while the second term corresponds to the variation of $d\varepsilon_1$ with the volumetric strain increment $d\varepsilon_v$ in a constant η (anisotropic consolidation) stress path. Also, $d\varepsilon_v$ is the incremental volumetric strain along the drained path and is the same as component from the anisotropic consolidation. This will be explained in the latter sections. In terms of shear strain Equation (1) can be expressed as

$$(d\varepsilon_s)_{drained} = \left[\left(\frac{d\varepsilon_s}{d\eta}\right)_v d\eta\right] + \left[\left(\frac{d\varepsilon_s}{d\varepsilon_v}\right)_\eta d\varepsilon_v\right] \qquad (2)$$

In Equation (2), the first term on the right hand side refers to the undrained component of the shear strain and the second term refers to the shear strain during the anisotropic component of the drained stress increment. Appendix 1 elaborates these aspects in detail.

Equation (2) allows a variety of versatile tools which can be applied either in terms of total strains or in terms of plastic strains. The effect of prolonged time and creep in undrained or drained behaviour can be realised by studying the effect in each of the components $(d\varepsilon_s / d\eta)_\eta$, $(d\varepsilon_s / d\varepsilon_v)$ and $d\varepsilon_v$. Similarly, the effect of cyclic loading and stress reversals.

The original contribution of Roscoe and Poorooshasb (1963) restricted the validity of equation (2) to stress states on the state boundary surface. That is for normally consolidated clays. Also, the stress ratio must increase monotonically from any stress state to the critical state. However, latter work at the Asian Institute of technology by Kim (1991) and others (Balasubramaniam and Uddin 1977; Kim *et al.* 1994) show that the Roscoe and Poorooshasb (1963) formulation can be extended to overconsolidated clays as well, when the stress states lie below the state boundary surface.

Equation (2), when considered in terms of only plastic strains, with non-zero independent terms for $(d\varepsilon_s / d\eta)_\eta$ and $(d\varepsilon_s / d\eta) d\varepsilon_v$, indicates a strain hardening plastic behaviour with contributions of shear strains from two sets of yield loci. On the other hand when $(d\varepsilon_s / d\eta)_\eta$ equals zero, then the plastic shear strains derived is obeys an associated flow rule of the type encountered in Cam Clay Model (Roscoe *et al.* 1963) and Modified Cam Clay model (Roscoe and Burland 1968).

In this paper, it will be shown that the shear strain derived from Cam Clay Model and Modified Cam Clay model can be expressed in mathematical forms similar to Equation (2). Also, Modified Cam Clay model of Roscoe and Burland (1968) needs a set of constant q yield loci to contribute the term $(d\varepsilon_s / d\eta)_\eta$ which is needed for the satisfactory prediction of the shear strains for non radial type of stress paths. For the radial stress paths, the contribution from the volumetric yield loci as obtained using the energy balance concept, and the normality criterion are sufficient to predict the volumetric and shear strains.

ALL THEORIES EXPRESSED IN SIMILAR MATHEMATICAL FORM FOR EASE OF COMPARISON

Equation (2) of the Roscoe and Poorooshasb (1963) formulation can be expressed as

$$\left(d\varepsilon_s\right)_{drained} = \left(d\varepsilon_s\right)_{undrained} + \left(\frac{d\varepsilon_s}{d\varepsilon_v}\right)_{anisotropic} d\varepsilon_v \qquad (3)$$

Volumetric strain contours (see Fig. 1) plotted by Balasubramaniam (1969) in the (q, p) plot for stress paths with monotonically increasing stress ratio revealed that the volumetric strain, ε_v , can be expressed as a function of the mean normal stress, p and the stress ratio η = q/p, thus

$$\varepsilon_v = F(\eta, p) \qquad (4)$$

Equation (4) is the same as the state boundary surface expressed in (p,q ,e) plot, three dimensionally and then expressed as $\left(\frac{q}{p_e}, \frac{p}{p_e}\right)$ in a two dimensional plot.

Differentiating Equation (4),

$$d\varepsilon_v = dF(\eta, p) = \frac{\partial F}{\partial \eta} d\eta + \frac{\partial F}{\partial p} dp \qquad (5)$$

Also, the experimental observation on undrained tests in normally consolidated clays (Roscoe and Poorooshasb, 1963; Roscoe *et al.*, 1963; Roscoe and Burland, 1968; Balasubramaniam and Chaudhry, 1978) reveal that the undrained shear strain can be expressed as a continuous and differentiable function of η . Thus

$$\left(\varepsilon_s\right)_{undrained} = \int_0^{\eta} \phi(\eta) d\eta \qquad (6)$$

ie $\phi(\eta) = \frac{d}{d\eta}(f_1(\eta))$

Figures 2 and 3 express such behaviour in relation to the contribution from the constant q yield loci as proposed by Roscoe and Burland (1968). Similar observations were noted on soft Bangkok clay by many researchers (Balasubramaniam and Uddin, 1977; Kim *et al.*, 1994), and notably Kim (1991) who has investigated the stress strain behaviour both from the isotropic and anisotropic stress states with a variety of applied stress paths.

The slope $\left(d\varepsilon_v / d\varepsilon_s\right)_\eta$ during anisotropic and isotropic consolidation is dependent on the stress ratio, η (see Fig. 4) and can be expressed as

$$\left(\frac{d\varepsilon_v}{d\varepsilon_s}\right)_\eta = f_2(\eta) \qquad (7)$$

Experimental evidence in support of Equation (7) for the formulation of $\left(d\varepsilon_v / d\varepsilon_s\right)_\eta$ as a function of the stress ratio η is presented in Roscoe and Poorooshasb (1963), Roscoe *et al.* (1963), Roscoe and Burland (1968) and Balasubramaniam (1969) on Kaolin and the subsequent work at the Asian Institute of technology on undisturbed samples of soft Bangkok clay. The energy balance equations developed at Cambridge and elsewhere, revealed that the plastic dilatancy ratio, that is, $\left(d\varepsilon_v^p / d\varepsilon_s^p\right)$ along the volumetric yield locus at any stress ratio is a function of the stress ratio, η .

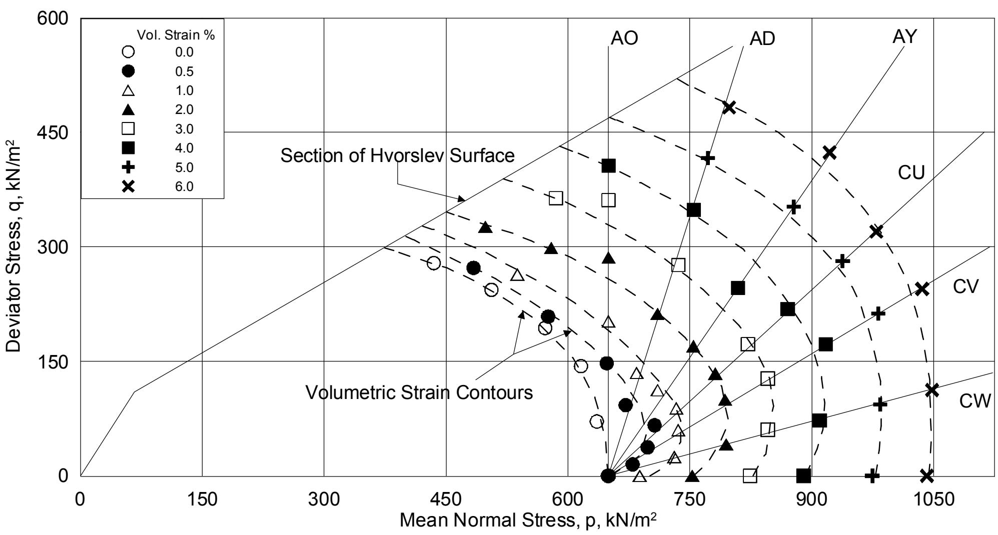

Fig. 1 Constant volumetric strain contours in (p, q) plot (Balasubramaniam, 1969)

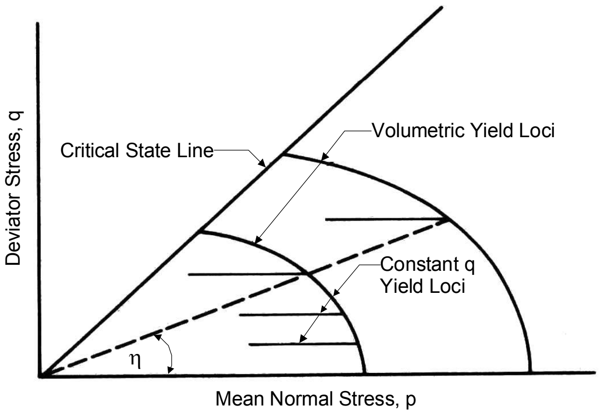

Fig. 2 Constant q yield loci

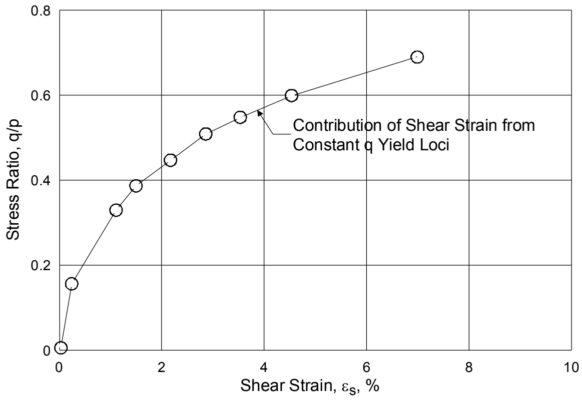

Fig. 3 Shear strain from constant q yield loci (virtually same as undrained shear strain) (Balasubramaniam, 1969)

In order to obtain the plastic dilatancy ratio, $d\varepsilon_v^p / d\varepsilon_s^p$, Cam Clay model used the energy balance equation:

$$pd\varepsilon_v^p + qd\varepsilon_s^p = Mpd\varepsilon_s^p \tag{8}$$

M is the slope of the critical state line in (q, p) plot.

From this equation, $d\varepsilon_v^p / d\varepsilon_s^p$ can be derived as:

$$\frac{d\varepsilon_v^p}{d\varepsilon_s^p} = M - \eta \tag{9}$$

This plastic dilatancy ratio, when expressed in terms of total strains for isotropic and anisotropic consolidation paths give, $\left(\frac{d\varepsilon_v}{d\varepsilon_s}\right)_\eta$ in Cam Clay model as

$$\left(\frac{d\varepsilon_v}{d\varepsilon_s}\right)_\eta = \frac{1}{\left(1 - \frac{k}{\lambda}\right)}(M - \eta) \tag{10}$$

k is the slope of the isotropic swelling line in (e, ln p) plot. Similarly, λ is the slope of the isotropic consolidation line in the (e, ln p) plot.

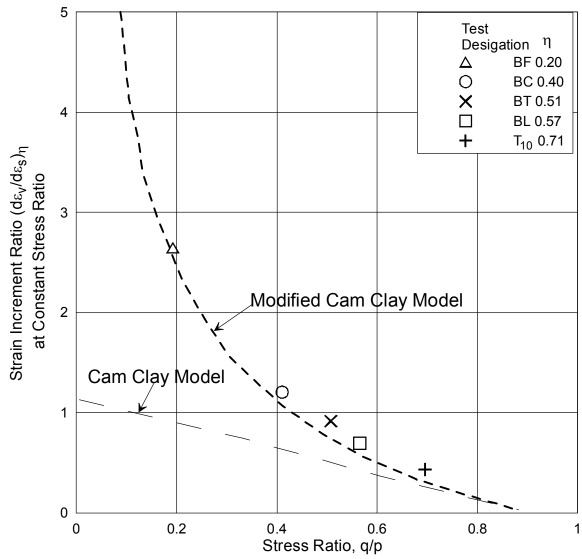

Fig. 4 Observed and predicted dilatancy ratio (Balasubramaniam, 1969)

For Modified Cam Clay Model of Roscoe and Burland, the energy balance equation is

$$pd\varepsilon_v^p + qd\varepsilon_s^p = p\sqrt{\left[\left(d\varepsilon_v^p\right)^2 + \left(Md\varepsilon_s^p\right)^2\right]} \tag{11}$$

This gives the plastic dilatancy ratio, $d\varepsilon_v^p / d\varepsilon_s^p$ as

$$\frac{d\varepsilon_v^p}{d\varepsilon_s^p} = \frac{M^2 - \eta^2}{2\eta} \tag{12}$$

The dilatancy ratio in terms of total strains for radial stress path in Modified Cam Clay Model becomes

$$\left(\frac{d\varepsilon_v}{d\varepsilon_s}\right)_\eta = \frac{1}{\left(1 - \frac{k}{\lambda}\right)}\left(\frac{M^2 - \eta^2}{2\eta}\right) \tag{13}$$

Equations (10) and (13) are strictly valid only for radial stress paths. For non-radial stress paths, $d\varepsilon_v / d\varepsilon_s$ depends both on the mean normal stress, p and the stress ratio, η.

ELASTIC WALL CONCEPT AND VOLUMETRIC YIELD LOCUS

Drucker *et al.* (1957) tried to associate the plastic strain rate vector to the Mohr Coulomb failure envelope, while the Cambridge researchers and Calladine (1963) used the normality concept of Drucker (1959) for a stable material and obtained the relation.

$$\frac{dq}{dp} = -\left[\frac{d\varepsilon_v^p}{d\varepsilon_s^p}\right] \tag{14}$$

Thus the following differential equation emerged for the volumetric yield locus:

For Cam Clay Model

$$\left(\frac{dq}{dp}\right)_y = -(M-\eta) \tag{15}$$

$\left(\frac{dq}{dp}\right)_y$ is the slope of the volumetric yield locus at any stress ratio, η in the (q, p) plot.

For Modified Cam Clay Model

$$\left(\frac{dq}{dp}\right)_y = -\left[\frac{M^2-\eta^2}{2\eta}\right] \tag{16}$$

These equations were then integrated to obtain the volumetric yield locus and then the state boundary surface. For Cam Clay Model the state boundary surface is given by

$$\frac{q}{p_e} = \left[\frac{M}{\left(1-\frac{k}{\lambda}\right)}\right]\ln\left(\frac{p_e}{p}\right) \tag{17}$$

p_e is the mean equivalent pressure and is defined as

$$p_e = p_0\left[\exp\left(\frac{e_0-e}{\lambda}\right)\right] \tag{18}$$

(e_0, p_0) corresponds to the voids ratio and the mean normal stress at the pre-shear consolidation pressure and e, refers to the voids ratio at the current state after drained shear with mean equivalent pressure, p_e.

For Modified Cam Clay model, the state boundary surface is given by

$$\frac{p}{p_e} = \left(\frac{M^2}{M^2+\eta^2}\right)^{\left(1-\frac{k}{\lambda}\right)} \tag{19}$$

It can be seen that Equations (17) and (19) for the state boundary surface are functions of η and p and are therefore in agreement with the formulation of Roscoe and Poorooshasb (1963) as given by Equation (3). In Equation (3), the volumetric strain ε_v is used instead of the voids ratio, e or the mean equivalent pressure p_e.

VOLUMETRIC AND SHEAR STRAINS IN DRAINED TESTS

The incremental expressions for the volumetric and shear strains in Cam Clay model and Modified Cam Clay model are given below.

For Cam Clay model

$$d\varepsilon_v = \left(\frac{\lambda}{1`+e}\right)\left(\frac{dp}{p}\right) + \left(\frac{\lambda-\kappa}{1+e}\right)\frac{1}{M}d\eta \tag{20}$$

$$d\varepsilon_s = \left(\frac{\lambda-\kappa}{1+e}\right)\left(\frac{1}{M-\eta}\right)\left[\frac{dp}{p}+\left(\frac{1}{M}\right)d\eta\right] \tag{21}$$

For Modified Cam Clay model

$$d\varepsilon_v = \left(\frac{\lambda}{1`+e}\right)\left(\frac{dp}{p}\right) + \left(\frac{\lambda-\kappa}{1+e}\right)\left(\frac{2\eta}{M^2+\eta^2}\right)d\eta \tag{22}$$

$$d\varepsilon_s = \left(\frac{\lambda-\kappa}{1+e}\right)\left(\frac{2\eta}{M^2-\eta^2}\right)\left[\frac{dp}{p}+\frac{2\eta d\eta}{M^2+\eta^2}\right] \tag{23}$$

The incremental stress–strain relationships given by Equations (20) to (23) are only dependent on the fundamental soil constants *M*, λ, and κ. *M* is the slope of the critical state line in the (*q*, *p*) plot. λ is the slope of the isotropic and anisotropic consolidation lines in the (*e*, *ln p*) plots. κ is the slope of the isotropic swelling line in the (*e*, *ln p*) plot. The values of *M*, λ, and κ are taken to be 0.9, 0.26 and 0.06 respectively.

Equation (3) of the Roscoe and Poorooshasb (1963) model can be expressed as

$$d\varepsilon_s = \phi(\eta)d\eta + f_2(\eta)dF(\eta, p) \tag{24}$$

In a similar manner the expression for the shear strain in Cam Clay model as given by Equation (21) can be expressed as

$$d\varepsilon_s = f_1^*(\eta)d\eta + f_2^*(\eta)dF^*(\eta, p) \tag{25}$$

Also,

$$dF^*(\eta, p) = \frac{\partial F^*}{\partial p}dp + \frac{\partial F^*}{\partial \eta}d\eta \tag{26}$$

For Modified Cam Clay model, the shear strain in Equation (23) can also be expressed as

$$d\varepsilon_s = f_1^{**}(\eta)d\eta + f_2^{**}(\eta)dF^{**}(\eta, p) \quad (27)$$

Also,

$$dF^{**}(\eta, p) = \frac{\partial F^{**}}{\partial p}dp + \frac{\partial F^{**}}{\partial \eta}d\eta \quad (28)$$

For Cam Clay model and Modified Cam Clay model the expressions are written with superfix * and ** respectively and function $f_1(\eta)$ for undrained shear strain. Then for Cam Clay model from Equations (21) and (25)

$$f_1^{*}(\eta) = \left(\frac{\lambda - \kappa}{1+e}\right)\left(\frac{k}{\lambda}\right)\left(\frac{1}{M}\right)\left(\frac{1}{M-\eta}\right) \quad (29)$$

For Modified Cam Clay model from Equations (23) and (27)

$$f_1^{**}(\eta) = \left(\frac{\lambda - \kappa}{1+e}\right)\left(\frac{\kappa}{\lambda}\right)\left(\frac{2\eta}{M^2 - \eta^2}\right)\left(\frac{2\eta}{M^2 + \eta^2}\right) \quad (30)$$

Also for Cam Clay model from Equations (10) and (25)

$$f_2^{*}(\eta) = \left(\frac{\lambda - \kappa}{\lambda}\right)\left(\frac{1}{M-\eta}\right) \quad (31)$$

For Modified Cam Clay model from Equations (13) and (27)

$$f_2^{**}(\eta) = \left(\frac{\lambda - \kappa}{\lambda}\right)\left(\frac{2\eta}{M^2 - \eta^2}\right) \quad (32)$$

Further from Equations (20), (22), (26) and (28)

$$\frac{\partial F^{*}}{\partial p} = \frac{\partial F^{**}}{\partial p} = \left(\frac{\lambda}{1+e}\right)\frac{1}{p} \quad (33)$$

All the theories seem to be able to predict the volumetric strains correctly in the constant stress ratio paths wherein the voids ratio-log mean normal stress relation is linear (see Fig. 5).

Further for Cam Clay model, from Equations (20) and (26)

$$\frac{\partial F^{*}}{\partial \eta} = \frac{1}{M}\left(\frac{\lambda - \kappa}{1+e}\right) \quad (34)$$

and for Modified Cam Clay model, from Equations (22) and (28)

$$\frac{\partial F^{**}}{\partial \eta} = \left(\frac{\lambda - \kappa}{1+e}\right)\left(\frac{2\eta}{M^2 + \eta^2}\right) \quad (35)$$

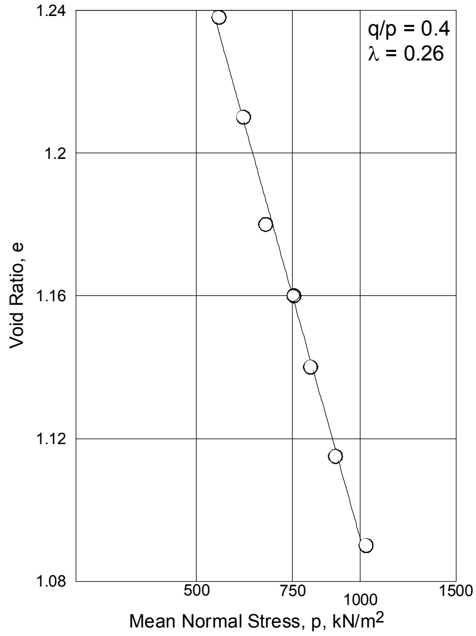

Fig. 5 Voids ratio-mean normal stress relation during isotropic and anisotropic consolidation (typical case for η =0.4) (Balasubramaniam 1969)

Experimental observations provided in Fig. 6, confirm that Modified Cam Clay model makes better prediction of the volumetric strain in constant p tests and as such

$$\frac{\partial F^{**}}{\partial \eta} = \frac{\partial F}{\partial \eta} \quad (36)$$

For Cam Clay model and Modified Cam Clay model the following similar expressions can be arrived as

$$f_1^{*}(\eta) = \left(\frac{k}{\lambda - \kappa}\right) f_2^{*}(\eta)\frac{\partial F^{*}}{\partial \eta} \quad (37)$$

and

$$f_1^{**}(\eta) = \left(\frac{k}{\lambda - \kappa}\right) f_2^{**}(\eta)\frac{\partial F^{**}}{\partial \eta} \quad (38)$$

$$\frac{\partial F^{**}}{\partial \eta} = \frac{\left(1 - \frac{\kappa}{\lambda}\right)\left(\frac{\lambda}{1+e}\right)\frac{1}{M}}{\left[1 + \left(\frac{1 - \frac{\kappa}{\lambda}}{M}\right)^2 \left(\frac{1}{f_2^{**}(\eta)}\right)^2\right]^{\frac{1}{2}}} \quad (39)$$

Equation (16) can be rearranged to give

$$f_2^{**}(\eta) = \frac{\dfrac{\partial F^{**}}{\partial \eta}}{\left[\left(\dfrac{\lambda}{1+e}\right)^2 - \left(\dfrac{M}{1-\dfrac{k}{\lambda}}\right)^2 \left(\dfrac{\partial F^{**}}{\partial \eta}\right)^2\right]^{\frac{1}{2}}} \qquad (40)$$

It thus appears, for Cam Clay model and Modified Cam Clay model the energy dissipating function when combined with the normality rule becomes the single most factor that controls the shape of the predicted state boundary surface, the volumetric strain in drained behaviour, the pore pressure development in undrained behaviour for all types of stress paths and in the case of shear strain, the prediction is only limited to radial stress paths.

Experimental evidence is provided (see Figures 4 and 6) to illustrate that Modified Cam Clay model can successfully predict the volumetric strains and the dilatancy ratio $\left((d\varepsilon_v / d\varepsilon_s)_\eta = f_2(\eta)\right)$. Cam Clay model was found to over predict the volumetric strain in drained test and hence the shear strain as well.

However, when it comes to the prediction of the shear strain, Modified Cam Clay model with a single set of volumetric yield loci satisfying the normality concept of Drucker (1959), is deficient in the prediction and an additional set of constant q yield loci based on the deviator stress are needed (see Fig.s 2 and 3).

Also it appears from experimental observations (see Figures 7 and 8) that

$$f_2(\eta) dF(\eta, p) = f_1^{**}(\eta) d\eta + f_2^{**} dF^{**}(\eta, p) \qquad (41)$$

Thus, Modified Cam Clay model lacks the undrained component of shear strain ($\phi(\eta) d\eta$) as used in Equation (6). This is why the second set of constant q yield loci was used by Roscoe and Burland so that in the revised version of Modified Cam Clay model the shear strain needs to be

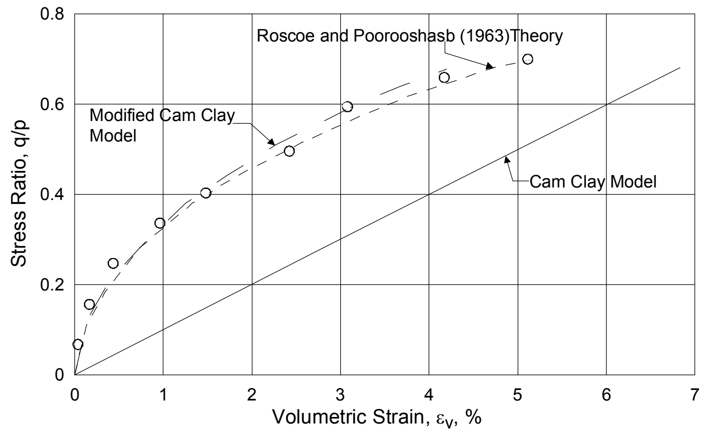

Fig. 6 Volumetric strains in constant p tests (Balasubramaniam, 1969)

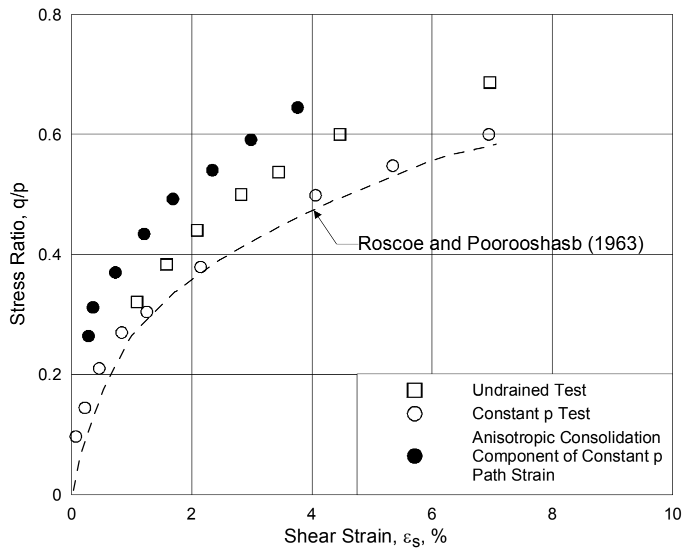

Fig. 7 Various components of shear strain (undrained, anisotropic and drained) (Balasubramaniam, 1969)

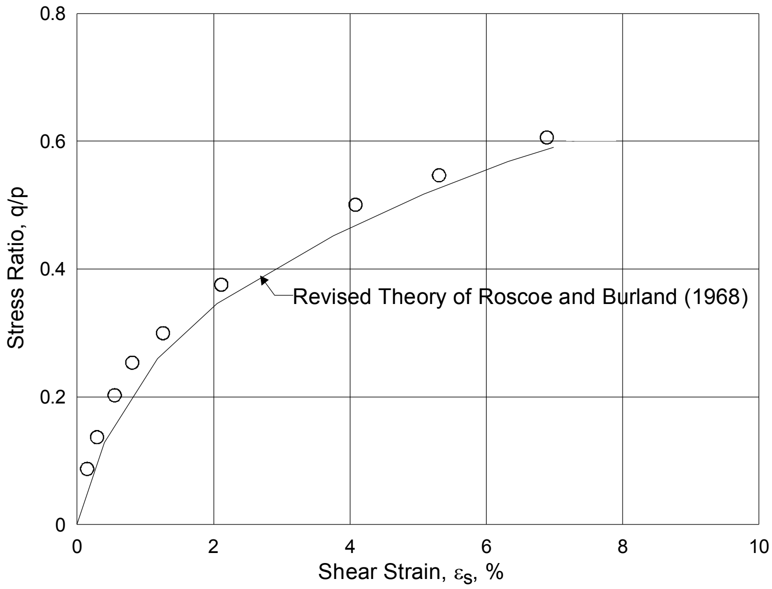

Fig. 8 Predicted shear strain by Roscoe and Burland (1968) after using contribution from constant q yield loci (Balasubramaniam, 1969)

obtained in two parts one from the constant q yield loci and the other from the volumetric yield loci as

$$d\varepsilon_s = [\phi(\eta)d\eta] + \left[f_1^{**}(\eta)d\eta + f_2^{**}(\eta)dF^{**}(\eta,p)\right] \quad (42)$$

The energy dissipation in the use of the constant q yield loci don't seem to be accounted for in the energy balance equation used to obtain the plastic dilatancy ratio which governs the shape of the volumetric yield loci through the normality rule and which can only make contributions for radial type of stress paths.

Experimental evidence is provided (see Figures 4 and 6) to illustrate that Modified Cam Clay model can successfully predict the volumetric strains and the dilatancy ratio $\left((d\varepsilon_v / d\varepsilon_s)_\eta = f_2(\eta)\right)$. Cam Clay model was found to over predict the volumetric strain in drained test and hence the shear strain as well.

However, when it comes to the prediction of the shear strain, Modified Cam Clay model with a single set of volumetric yield loci satisfying the normality concept of Drucker (1959), is deficient in the prediction and an additional set of constant q yield loci based on the deviator stress are needed (see Figures 2 and 3).

Also it appears from experimental observations (see Figures 7 and 8) that

$$f_2(\eta)dF(\eta,p) = f_1^{**}(\eta)d\eta + f_2^{**}dF^{**}(\eta,p) \quad (41)$$

Thus, Modified Cam Clay model lacks the undrained component of shear strain ($\phi(\eta)d\eta$) as used in Equation (6). This is why the second set of constant q yield loci was used by Roscoe and Burland so that in the revised version of Modified Cam Clay model the shear strain needs to be obtained in two parts one from the constant q yield loci and the other from the volumetric yield loci as

$$d\varepsilon_s = [\phi(\eta)d\eta] + \left[f_1^{**}(\eta)d\eta + f_2^{**}(\eta)dF^{**}(\eta,p)\right] \quad (42)$$

The energy dissipation in the use of the constant q yield loci don't seem to be accounted for in the energy balance equation used to obtain the plastic dilatancy ratio which governs the shape of the volumetric yield loci through the normality rule and which can only make contributions for radial type of stress paths.

CONCLUDING REMARKS

The classical theories of Roscoe and Poorooshasb (1963), and Cam Clay model (Roscoe *et al.*, 1963), and Modified Cam Clay model (Roscoe and Burland, 1968) are expressed in similar mathematical forms to understand the implications of the use of energy balance equations and the normality rule. All the theories seem to be able to predict the volumetric strains correct in the constant stress ratio paths wherein the voids ratio log mean normal stress relation is linear. It appears for Cam Clay model of Roscoe *et al.* (1963) and Modified Cam Clay model of Roscoe and Burland (1968), the energy dissipating function when combined with the normality rule becomes the single most factor that controls the shape of the predicted state boundary surface, the volumetric strain in drained behaviour, the pore pressure development in undrained behaviour for all types of stress paths while in the case of shear strain, the prediction is only limited to radial stress paths.

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APPENDIX A. SYMBOLS AND DEFINITIONS

e_0, e	initial and current voids ratio
F, f_1, f_2	functions defined with respect to Roscoe and Poorooshasb (1963) theory
F^*, f_1^*, f_2^*	functions defined with respect to Roscoe *et al.* (1963) theory
$F^{**}, f_1^{**}, f_2^{**}$	functions defined with respect to Roscoe and Burland (1968) theory
k	slope of isotropic swelling line in (*e*, *ln p*) plot
L_0, L	initial and current height of sample in triaxial test
M	slope of critical state line
p	mean normal stress
p_0	isotropic consolidation stress
p_e	mean equivalent pressure = $p_0 \exp\left(\frac{e_0 - e}{\lambda}\right)$
q	deviator stress
ε_v	volumetric strain
V_0, V	initial and current volume of sample
$\sigma_1', \sigma_2', \sigma_3'$	principal effective compressive stresses
$\varepsilon_1, \varepsilon_2, \varepsilon_3$	principal compressive strains
ε_v	volumetric strain
ε_s	shear strain
λ	slope of isotropic consolidation line in (*e*, *ln p*) plot
η	stress ratio = q/p
$\phi(\eta)$	function associated with the

shear strain from constant q yield loci

$\frac{d}{d\eta} f_1(\eta)$

$p = \frac{(\sigma_1' + 2\sigma_3')}{3}$ since $\sigma_2' = \sigma_3'$

$q = (\sigma_1' - \sigma_3')$

$\varepsilon_v = \varepsilon_1 + 2\varepsilon_3$

$\varepsilon_s = \frac{2}{3}(\varepsilon_1 - \varepsilon_3)$ since $\varepsilon_2 = \varepsilon_3$

Axial strain, $\varepsilon_1 = \int_L^{L_0} \frac{dL}{L}$

Volumetric strain, $\varepsilon_v = \ln\left(\frac{V_0}{V}\right) = \ln\left(\frac{1+e_0}{1+e}\right)$

$(d\varepsilon_1)_{drained}$ is the incremental axial strain in the drained stress path

$(d\varepsilon_1)_{undrained}$ is the incremental axial strain in the undrained stress path

$\left(\frac{d\varepsilon_1}{d\eta}\right)_v$ is the slope of the axial strain-stress ratio, η relationship during undrained shear with constant volume, v.

$\left(\frac{d\varepsilon_1}{d\varepsilon_v}\right)_\eta$ is the slope of the axial strain – volumetric strain relation during anisotropic and isotropic consolidation with constant stress ratio, η.

$\left(\frac{d\varepsilon_s}{d\eta}\right)_v$ is the slope of the shear strain- stress ratio, η relationship during undrained shear with constant volume, v.

$\left(\frac{d\varepsilon_s}{d\varepsilon_v}\right)_\eta = \left(\frac{d\varepsilon_s}{d\varepsilon_v}\right)_{anisotropic}$ is the slope of the shear strain – volumetric strain relation during anisotropic and isotropic consolidation with constant stress ratio, η.

$\left(\frac{dq}{dp}\right)_y$ is the gradient of the volumetric yield locus at any point in the (q, p) plot with stress ratio, η

APPENDIX B. DETERMINATION OF VOLUMETRIC AND SHEAR STRAIN

The procedure adopted for calculating the volumetric strain along any applied stress path is as follows. Let AB in Figure B1 be the applied stress path for which the volumetric strain stress ratio relationship is being determined. Then the undrained stress path AC through the point A is drawn. The path AB is divided into a large number of small steps AB_1, B_1B_2, B_2B_3, …B_nB. Then each step AB_1 is considered in two parts; AC_1, along an undrained stress path and C_1B_1 along an anisotropic consolidation path. The volumetric strain experienced by the specimen at states, B_1, B_2, ...B_n and B are denoted by $(\varepsilon_v)_{B_1}, (\varepsilon_v)_{B_2}, ...(\varepsilon_v)_{B_n}$ and $(\varepsilon_v)_B$ respectively. Similar notation is used for the shear strain as well as $(\varepsilon_s)_{B_1}, (\varepsilon_s)_{B_2}, ...(\varepsilon_s)_{B_n}$ and $(\varepsilon_s)_B$. The incremental strains for the steps AB_1, B_1B_2, ...B_nB will then be in the case of the volumetric strains as $(d\varepsilon_v)_{AB_1}, (d\varepsilon_v)_{B_1B_2}, ...(d\varepsilon_v)_{B_nB}$. Also the incremental shear strains are $(d\varepsilon_s)_{A_1B_1}, (d\varepsilon_s)_{B_1B_2}, ...(d\varepsilon_s)_{B_nB}$. The notations for the shear strains along the undrained stress path AC for the states C_1, C_2, ...C_n are $(\varepsilon_s)_{C_1}, (\varepsilon_s)_{C_2}, ...(\varepsilon_s)_{C_n}$, and for the incremental shear strains for the steps AC_1, C_1C_2, ... C_nC are $(d\varepsilon_s)_{AC_1}, (d\varepsilon_s)_{C_1C_2}, ...(d\varepsilon_s)_{C_nC}$.

As stated before, since an undrained stress path is a zero volumetric strain contour, it can be shown that:

$$(d\varepsilon_v)_{AB_1} = (d\varepsilon_v)_{C_1B_1} \quad \text{(B1)}$$

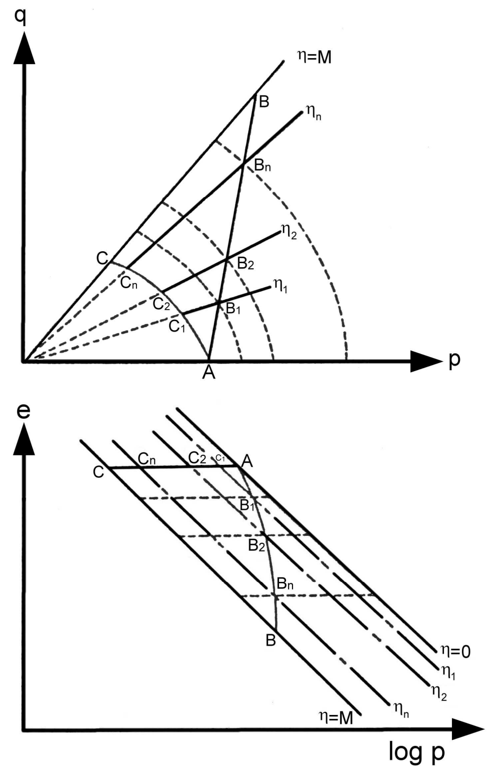

Fig. B1 Incremental steps in the calculation of volumetric and shear strains in drained tests

Let (e_0, p_0) be the voids ratio and the mean normal stress at point A on the isotropic consolidation line in Fig. B1. Then the isotropic consolidation can be expressed as

$$e_0 - e = -\lambda \ln\left(\frac{p_0}{p}\right) \tag{B2}$$

If $(p)_{C_1}, (p)_{C_2}$ denote the mean normal stress for states C_1 and B_1 and if $(e)_{B_1}$ is the void ratio corresponding to the state B_1, then:

$$e_0 - (e)_{B_1} = -\lambda \ln\left(\frac{(p)_{C_1}}{(p)_{B_1}}\right) \tag{B3}$$

Also,

$$(\varepsilon_v)_{B_1} = (d\varepsilon_v)_{C_1 B_1} = (d\varepsilon_v)_{AB_1} = \ln\left(\frac{1+e_0}{1+(e)_{B_1}}\right) \tag{B4}$$

Hence, knowing $e_0, \lambda, (p)_{C_1}$ and $(p)_{B_1}, (e)_{B_1}$ can be calculated, and therefore, $(\varepsilon\ )_{B_1}$, $(d\varepsilon_v)_{C_1 B_1}$ and $(d\varepsilon_v)_{AB_1}$ can be determined. Similarly considering the state points C_2 and B_2:

$$e_0 - (e)_{B_2} = -\lambda \ln\left(\frac{(p)_{C_2}}{(p)_{B_2}}\right) \tag{B5}$$

and

$$(\varepsilon_v)_{B_2} = \ln\left(\frac{1+e_0}{1+e_{B_2}}\right) \tag{B6}$$

From Equations (B4) and (B6), the following can be derived.

$$(d\varepsilon_v)_{B_1 B_2} = (\varepsilon_v)_{B_2} - (\varepsilon_v)_{B_1} = \ln\left(\frac{1+(e)_{B_1}}{1+(e)_{B_2}}\right) \tag{B}$$

The same procedure can be repeated to determine $(\varepsilon_v)_{B_3}$, $(\varepsilon_v)_{B_4} \ldots (\varepsilon_v)_{B_n}$ and $(d\varepsilon_v)_{B_2 B_3}, (d\varepsilon_v)_{B_3 B_4} \ldots (d\varepsilon_v)_{B_n B}$. Hence, the volumetric strain-stress ratio relationship can be established.

The undrained shear strain can be obtained from the (ε_s, η) relationship, which can also be expressed as

$$(\varepsilon_s)_{undrained} = \int_0^{\eta} f_1(\eta) d\eta \tag{B9}$$

Thus, the shear strain $(\varepsilon_s)_{C_1}, (\varepsilon_s)_{C_2} \ldots\ldots\ldots (\varepsilon_s)_{C_n}$ for stress ratio, $\eta_1, \eta_2, \ldots. \eta_n$ can be obtained for points $C_1, C_2, \ldots\ldots\ldots C_n$.

In the previous sections, the dilatancy ratio is given and can be expressed as

$$\left(\frac{d\varepsilon_v}{d\varepsilon_s}\right)_{\eta} = f_2(\eta) \tag{B10}$$

Hence, the shear strain in the drained path can be computed for various values of η using the relation

$$(\varepsilon_s)_{B_1} = (\varepsilon_s)_{C_1} + \left(\frac{1}{f_2(\eta)}\right)(d\varepsilon_v)_{AB_1} \tag{B11}$$

and

$$(\varepsilon_s)_{B_2} = (\varepsilon_s)_{C_2} + \left(\frac{1}{f_2(\eta)}\right)(d\varepsilon_v)_{AB_1} + \left(\frac{1}{f_2(\eta)}\right)(d\varepsilon_v)_{B_1 B_2} \tag{B12}$$

Thus,

$$(\varepsilon_s)_{B_j} = (\varepsilon_s)_{C_j} + \sum_{i=1}^{i=j} \left(\frac{1}{f_2(\eta)}\right)(d\varepsilon_v)_{B_{i-1} B_i} \tag{B13}$$

In Equation (B13) when i = 1, B_0 coincides with A.