On the Calculation of Fields in Three Dimensions using the Cauchy Integral

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***Abstract*—This paper describes a method for calculating the three dimensional monochromatic electromagnetic fields scattered by conducting and dielectric objects using the Cauchy integral cast in a multi-dimensional form based on Clifford algebra. Formal relationships to methods based on quaternions and vector calculus are presented. The characteristics of solutions based on the Cauchy method are described and its advantages over comparable methods involving Greens functions are discussed.**

I. Introduction

For many problems which involve physical phenomena and their representation in mathematical structures there is no particularly strong reason for choosing one differential geometric algebra over another. For problems involving electromagnetism in the form of Maxwell's equations that has been but is no longer the case. There *is* a very definite advantage in one particular of these geometric algebras over the others. Here we review that particular geometric algebra in the context of Maxwell's equations, relate it to approaches using quaternions and vector calculus, and describe its advantages and the characteristics of the solutions it offers.

II. Background

At the time Maxwell introduced the displacement current and formulated the set of equations named after him, he had three differential geometric algebras from which to choose: (1) Cartesian coordinates, (2) complex numbers and (3) quaternions. Gibb's vector calculus, Cartan's differential geometry and tensors had not yet been invented.

Maxwell himself used quaternions [1] but with his students, for most of whom quaternions were too much of a challenge, wrote everything in Cartesian coordinates. Complex numbers were not sufficiently general because they were at that time restricted to problems in two dimensions only.

In developing quaternions Hamilton [2] had been intending to find a generalisation of complex numbers which carried them into three dimensions. However, the algebra he did create did not do so. That had to wait (not very long) for Clifford to take two independent and commutative sets of quaternions written in terms of Grassmann's linear algebra, and from them construct what Clifford called his even 5 way algebra [3] and what we today call a four dimensional Clifford algebra.

For electromagnetism the key here is (1) the generalisation from *analytic* functions of a complex variable in two dimensions to *monogenic* functions of a Clifford variable in any number of dimensions, and (2) the generalisation from the Cauchy integral of complex variables in two dimensions to the Cauchy integral of Clifford variables in any number of dimensions. These generalisations were properly formulated and documented only within the last few years [4], [5].

III. Formulation

A. Time Domain

Equation 1 shows a four dimensional Clifford number $\mathbb{X}$ (which has 16 independent complex components) in terms of an equivalent matrix of quaternions (centre) and in terms of an equivalent matrix written in the notation of vector calculus (right). Table I shows, with Clifford numbers $\mathbb{X}=\mathbb{D},\mathbb{F},\mathbb{S},\mathbb{P}$ on the left and quaternions $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ on the right, the necessary substitutions to encode Maxwell's equations into the Clifford–quaternion–vector formulation of equation 1.

The differential operator $\mathbb{D}$ is fixed by the need to accommodate Einstein's theory of special relativity. The field $\mathbb{F}$ is then fixed under the influence of the differential operator by the need to faithfully reproduce all the various parts of Maxwell's equations in regions void of sources. Finally the source $\mathbb{S}$ is fixed to reproduce Maxwell's equations in the presence of sources. It then follows by simple algebraic verification (using the rules $e_j^2=-1$ and $e_j e_{k\neq j}=-e_k e_j$ for the Clifford units) that Maxwell's equations are written:

$$\mathbb{D}\mathbb{F} = \mathbb{S} \quad \text{(field to source)} \tag{2}$$

$$\mathbb{D}\mathbb{P} = \mathbb{F} \quad \text{(potential to field)} \tag{3}$$

$$\mathbb{D}^2\mathbb{P} = \mathbb{S} \quad \text{(potential to source)} \tag{4}$$

The square of the gradient: $\mathbb{D}^2=-(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2})$ is recognised the negative of the d'Alembertian (wave) operator.

B. Frequency Domain

For monochromatic electromagnetic fields $\mathbb{F}(\mathbb{R},t) = \mathbb{F}_k(\mathbb{R})e^{i\omega t}$ with angular frequency ω and wavenumber $k=\omega/c$ Fourier transformation gives equations similar in form to those in the time domain. In particular the gradient: $\mathbb{D}_k=\nabla+ke_0$ and its square: $\mathbb{D}_k^2=-(\nabla^2+k^2)$, the latter of which is recognised as the negative of the Helmholtz operator. The Cauchy integral in multiple dimensions is then written [5]:

$$\mathbb{F}_k(\mathbb{R}) = \int_\Sigma \mathbb{E}_k(\mathbb{R}'-\mathbb{R})\mathbb{N}(\mathbb{R}')\mathbb{F}_k(\mathbb{R}')\,d\sigma(\mathbb{R}') \tag{5}$$

Here $\mathbb{F}_k$ is a Clifford-valued function which is monogenic at all points $\mathbb{R}=R_x e_1+R_y e_2+R_z e_3$ in a region Ω of three dimensional space enclosed by a surface Σ, $\mathbb{N}$ is a Clifford number representing the outward normal vector to the surface Σ, $d\sigma$ is the scalar elemental measure of surface and:

$$\mathbb{E}_k(\mathbb{R}) = -\left\{\frac{\mathbb{R}}{|\mathbb{R}|^2}+ik\left(\frac{\mathbb{R}}{|\mathbb{R}|}-ie_0\right)\right\}\frac{e^{-ik|\mathbb{R}|}}{4\pi|\mathbb{R}|} \tag{6}$$

is the multi-dimensional Cauchy kernel.

$$\mathbb{X} \;=\; \left(\begin{array}{cc|cc} \mathcal{A} & -\mathcal{D} & -\mathcal{C} & -\mathcal{B} \\ \mathcal{D} & \mathcal{A} & \mathcal{B} & -\mathcal{C} \\ \hline \mathcal{C} & -\mathcal{B} & \mathcal{A} & \mathcal{D} \\ \mathcal{B} & \mathcal{C} & -\mathcal{D} & \mathcal{A} \end{array}\right) \;=\; \left(\begin{array}{cc|cc|cc|cc} a & -\mathbf{A}\cdot & -d & \mathbf{D}\cdot & -c & \mathbf{C}\cdot & -b & \mathbf{B}\cdot \\ \mathbf{A} & (aI+\mathbf{A}\times) & -\mathbf{D} & -(dI+\mathbf{D}\times) & -\mathbf{C} & -(cI+\mathbf{C}\times) & -\mathbf{B} & -(bI+\mathbf{B}\times) \\ \hline d & -\mathbf{D}\cdot & a & -\mathbf{A}\cdot & b & -\mathbf{B}\cdot & -c & \mathbf{C}\cdot \\ \mathbf{D} & (dI+\mathbf{D}\times) & \mathbf{A} & (aI+\mathbf{A}\times) & \mathbf{B} & (bI+\mathbf{B}\times) & -\mathbf{C} & -(cI+\mathbf{C}\times) \\ \hline c & -\mathbf{C}\cdot & -b & \mathbf{B}\cdot & a & -\mathbf{A}\cdot & d & -\mathbf{D}\cdot \\ \mathbf{C} & (cI+\mathbf{C}\times) & -\mathbf{B} & -(bI+\mathbf{B}\times) & \mathbf{A} & (aI+\mathbf{A}\times) & \mathbf{D} & (dI+\mathbf{D}\times) \\ \hline b & -\mathbf{B}\cdot & c & -\mathbf{C}\cdot & -d & \mathbf{D}\cdot & a & -\mathbf{A}\cdot \\ \mathbf{B} & (bI+\mathbf{B}\times) & \mathbf{C} & (cI+\mathbf{C}\times) & -\mathbf{D} & -(dI+\mathbf{D}\times) & \mathbf{A} & (aI+\mathbf{A}\times) \end{array}\right) \tag{1}$$

Equation 5 plays the role of a direct inverse in the frequency domain to the Maxwell's homogeneous (source free) equations: $\mathbb{D}_k\mathbb{F}_k = 0$ when cast in the Clifford formalism. Static solutions are recovered by setting $k = 0$.

IV. DISCUSSION

The relationship between the Clifford, the quaternion and the vector representations in equation 1 makes explicit the additional structure of the Clifford over the quaternion and of the quaternion over the vector. To match with the Clifford, structure is imposed on the four quaternions by embedding them in a matrix and adopting the usual rules of matrix addition and multiplication. Similarly, to match with the quaternion additional structure is imposed on the scalars and vectors as used in vector calculus by embedding them in a matrix:

$$\mathcal{A} \;=\; \begin{pmatrix} a & -\mathbf{A}\cdot \\ \mathbf{A} & (aI+\mathbf{A}\times) \end{pmatrix} \;=\; \begin{pmatrix} a & -A_x & -A_y & -A_z \\ A_x & a & -A_z & A_y \\ A_y & A_z & a & -A_x \\ A_z & -A_y & A_x & a \end{pmatrix} \tag{7}$$

where A_x, A_y, A_z are the Cartesian components of vector $\mathbf{A}$. The additional structure afforded by the Clifford or the matrix-quaternion or the matrix-vector approaches properly represents all aspects of three dimensional geometry, *i.e.* scalars, vectors, bivectors (oriented areas) and trivectors (oriented volumes). Of the various choices in geometric algebras the Clifford choice has the definite advantage of providing access to the Cauchy integral in a manageable form. That by itself is sufficient as a path to a new method for calculating the electromagnetic field scattered from conducting and non-conducting objects.

For conducting objects (simpler than dielectrics) the new method reduces (somewhat surprisingly) to the intersection of two straight lines in a particular functional space (Banach space). That result holds regardless of the shape of the scatterer. Examples of applying the technique in solving for the fields scattered from a conducting cubic scatterer are found in [6].

The new method follows a different approach and therefore behaves differently from other methods. In comparison to existing boundary integral formulations note that the Cauchy kernel and integral are used instead of the Green's function and its integral form. The formulation involves only first order differential and integral operators, not second order ones. The singular integrals involved are tamer, and easier to work with. Potentials and surface currents play no part in the solution. The latter point means the method is not restricted to conductors and applies equally well to dielectrics. Furthermore, solutions are formulated only in terms of fields. That means the solution is constructed only in a single functional space. There is no need to have multiple distinct functional spaces for potential, field and current, unless one particularly wishes to calculate the potential or current.

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TABLE I

ELECTROMAGNETIC PARAMETERS IN THE FORM OF CLIFFORD VARIABLES $\mathbb{D}, \mathbb{F}, \mathbb{S}, \mathbb{P}$ AND QUATERNIONS $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$.

	Clifford	Quaternion							
		$\mathcal{A}$		$\mathcal{B}$		$\mathcal{C}$		$\mathcal{D}$	
		a	A	b	B	c	C	d	D
gradient	$\mathbb{D} = \nabla - \frac{i}{c} e_0 \frac{\partial}{\partial t}$	0	0	$-\frac{i}{c}\frac{\partial}{\partial t}$	0	0	$i\nabla$	0	0
field	$\mathbb{F} = \sqrt{\mu} H \sigma - i\sqrt{\epsilon} E e_0$	0	$\sqrt{\mu} H$	0	0	0	0	0	$-\sqrt{\epsilon} E$
source	$\mathbb{S} = \sqrt{\mu} J + \frac{i}{\sqrt{\epsilon}} \rho e_0$	0	0	$\frac{i}{\sqrt{\epsilon}} \rho$	0	0	$i\sqrt{\mu} J$	0	0
potential	$\mathbb{P} = \frac{1}{\sqrt{\mu}} A + i\sqrt{\epsilon}\phi e_0$	0	0	$i\sqrt{\epsilon}\phi$	0	0	$\frac{i}{\sqrt{\mu}} A$	0	0