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Teaching Mathematics: The Long Journey for Teachers

Jenni Way, Judy Anderson, Janette Bobis

For the vast majority of teachers, the journey to being a really effective mathematics teacher is a long and often challenging one. Continuous learning about mathematics and mathematics education is required if a teacher is to arrive at the desired destination (though some would say that if we stop striving toward the goal of being the best teacher we can be, it is time to retire!). The articles in this issue of MTED tell some stories about teacher journeying.

Continuing the metaphor ….

The journey begins with our own schooling, where we are equipped with the basic kit of knowledge and understanding that we will need to take with us out onto the open road. As explained by Norton in the first article, the more mathematics we do in high school, the more successful we are likely to be in the next phase of the journey – as the road passes through ‘teacher-preparation land’. Here the road can become quite slippery as we discover that we don’t know the mathematics as well as we thought we did, particularly when put into the context of teaching it to someone else. This ‘slipperiness’ is illustrated through Chesler’s article about the deficiencies in teacher education students’ ability to work with mathematical definitions.

The next part of the journey takes us back into the realm of schools, but with vastly different roles and responsibilities to our student days. Schools are of course situated within a world of imposed policies, curricula and standardised testing. When faced with pressures to implement particular teaching approaches, teachers often need to develop new knowledge and skills. The Harvey and Averill article explores the complexity of planning and implementing context-based mathematics teaching in response to an initiative for increasing the connection of mathematics to real-life problems.

Responding to external pressures can locate teachers at a fork in the road. One path may lead onwards and upwards to higher-quality teaching, while another might lead to a dead-end or divert them onto a longer route to the journey’s goal. Sometimes we need to pause and ask for some directions. Anderson and White’s article demonstrates how – with a little guidance from professional conversations – the information from national tests can lead to informed decisions about teaching directions, rather than leading to ‘teaching to the test’.

Clearly the key to a successful journey is ongoing learning by teachers. Like the students they teach, teachers learn more effectively when supported by colleagues, or, as described by Polly in the final article, scaffolded by a professional coach. And sometimes, gaining advice by reading a good professional journal like MTED can help too!
Prior Study of Mathematics as a Predictor of Pre-service Teachers’ Success on Tests of Mathematics and Pedagogical Content Knowledge

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There remains a lack of empirical evidence about the relationship between the level of mathematics studied at high school and within tertiary degrees and primary school pre-service teachers’ success in curriculum subjects. Further, there is little evidence to inform the structure and delivery of mathematics teacher preparation. In this study, the content and pedagogical knowledge of pre-service primary teachers were examined, as was their view of the effectiveness of a unit of study based on mathematics content and pedagogy. The cohort comprised 122 graduate diploma primary teacher preparation students; the unit’s assessment required them to know the mathematics they were expected to teach as well as describe how to teach it. It was found that the level of high school mathematics undertaken was highly correlated with success in the teacher education unit designed to prepare prospective teachers to teach primary (elementary) mathematics. The findings have implications for enrolment in pre-service primary teacher preparation courses as well as for the structure of mathematics curriculum units.

Introduction

The study reported here examined the level of mathematics content knowledge that pre-service teachers brought to primary (elementary) teacher preparation. The importance of pre-service teachers’ knowledge of subject matter has been recognised as central to their teaching (e.g., Ball, Hill, & Bass, 2005; Goulding, Rowland, & Barber, 2002; Silverman & Thompson, 2008). Internationally, a number of authors have expressed concern that many pre-service teachers have learnt limited mathematics at school (e.g., Adler, Ball, Krainer, Lin, & Novotna, 2005; Henderson & Rodrigues, 2008). Yet it has also been reported that teacher preparation sometimes does not focus on remediating deficiencies in teacher knowledge of mathematics because there are so many competing agendas (Kane, 2005). A number of authorities have identified as a research priority an investigation of what pre-service teachers know and how best to equip them to teach primary mathematics (Ball, 1988; Goulding et al., 2002; United States [U.S.] Department of Education, 2008). With this background in mind, this study examines one mathematics curriculum unit of study in an Australian university to examine what knowledge the pre-service teachers arrived with, how it was related to their previous study, and how they improved through completion of the unit.
Importance of Subject Area Content Knowledge in Primary Teacher Preparation

In 2002, Goulding et al. made the following comment about the mathematical subject knowledge that pre-service teachers bring to teacher preparation: “For pre-service teachers ... what they bring to training courses would seem to be critical” (p. 690). The authors believed that, in the main, tertiary teacher education courses did little to modify pre-service teachers’ content or pedagogical knowledge in relation to mathematics teaching. The authors held that mostly, pre-service teachers would teach as they were taught. The reason for this was that pre-service teacher education units were a relatively weak intervention, in part because of the time demands in university education due to competing priorities (Kane, 2005). The ineffectiveness of initial training upon subsequent pedagogy was also reported by Askew, Rhodes, Brown, Wiliam, and Johnson (1997). Goulding et al. (2002) believed that effective teacher preparation ought to be based upon empirical evidence, including knowledge of the mathematical understandings with which pre-service teachers entered teacher preparation programs and how various programs impacted on their competency and confidence.

There is considerable debate about what constitutes critical knowledge for the preparation of pre-service teachers. For example, the recently released Professional Standards for Teachers (National Standards Expert Working Group, 2010) in Australia lists seven key standards, only one of which relates to a knowledge of content and how to teach it. Within this one standard there are nine sub-standards that relate to knowledge of: skills and pedagogy; stages of development; current research related to remediation; different communication strategies; sequencing and links to broader curriculum; assessment; reporting; ICT usage; and knowledge of Australia’s Indigenous peoples. Addressing the list of priorities above illustrates the diversity of competing demands that Kane (2005) reported as leaving little time for transforming students’ understanding of mathematics and how to teach it. Among all the standards and sub-standards of skills and pedagogy, content seems to be de-emphasised. This might be because there is an assumption that prospective teachers entering teacher education programs understand primary mathematics concepts, an argument noted by Henderson and Rodrigues (2008).

Mathematics Curriculum Knowledge

In regard to “skills and pedagogy”, the importance of content knowledge in the teaching of mathematics has long been recognised as central to successful teaching at all levels (e.g., Ball et al., 2005; Ma, 1999; Osana, Lacroix, Tucker, & Desrosiers, 2006; Shulman, 1987, 1999; Warren, 2009). This relationship was articulated by the U.S. Department of Education (2008, p. 37): “Teachers must know in detail the mathematical content they are responsible for teaching and its connections to other important mathematics, both prior and beyond the level they are assigned to teach.”
How knowledge to teach mathematics is best developed in primary teacher preparation courses is a matter for debate. Ball et al. (2005) list some of the most common recommendations:

- that teachers study more mathematics, either by requiring additional mathematics coursework or a subject matter major;
- that there be a focus on mathematics methods coursework, particularly related to the mathematics expected of the classroom teacher and curriculum materials; and
- that prospective teachers be chosen from selected colleges, anticipating that they are more likely to succeed in mathematics teaching “betting that overall intelligence and mathematics competence will prove effective in producing student learning” (p. 16).

Ball et al. (2005) question whether teachers need knowledge of advanced calculus or linear algebra in order to teach secondary, middle, or elementary school students. The assumption is that the study of more advanced mathematics ought to become increasingly less relevant to mathematics teaching towards the lower grade levels. Knowledge of calculus seems less relevant to the teaching of counting than to middle school algebra. There is some research to support this assumption. Ma (1999) noted that it was possible to pass advanced courses in mathematics without understanding how they might inform the teaching of primary mathematics but that, none the less, a deep conceptual knowledge of mathematics plays a vital role in mathematics teaching and learning. At a macro level, most researchers agree with the U.S. Department of Education (2008, p. xxi) statement: “It is self-evident that teachers can not teach what they do not know.”

Knowledge of mathematics content and how to teach it are intertwined in complex ways (Shulman, 1999). Shulman (1987) used the term pedagogical content knowledge (PCK) and described it as an intersection of subject knowledge and pedagogical knowledge. Askew et al. (1997) reported that highly effective teachers had knowledge and awareness of inter-relations between the areas of the primary mathematics curriculum they taught. However, “being highly effective was not associated with having an A level or degree in mathematics” (p. 5). Ma (1999) also noted that high levels of teacher content knowledge do not necessarily imply that individuals understand the material in a way that enables them to impart or teach it to students. Ma describes what is needed to teach as profound understanding of fundamental mathematics (PUFM). That is, teachers need to understand the material and ways of representing it to students. This has recently been described as mathematical knowledge of teaching (MKT) (Silverman & Thompson, 2008). Essentially, PCK and MKT are dependent upon a fundamental understanding of underlying mathematical structures (Silverman & Thompson, 2008). Goulding et al. (2002) suggested that there is a direct correlation between subject matter knowledge (SMK) and teaching mathematics, with teachers with strong SMK being more likely to be assessed as strong numeracy teachers and teachers with low SMK being more likely to be assessed as weak numeracy teachers. Goulding et al. reported that higher levels of pedagogical subject knowledge were linked to the
systematic presentation of new ideas and making explicit links between different representations (verbal, concrete, numerical, and pictorial). Ball and McDiarmid (1988) argued that teachers’ subject knowledge influenced the nature of questions they asked their classes, the types of tasks they allocated students, and teachers’ ability to respond to questions.

Hill, Rowan, and Ball (2005) found that teachers’ mathematical knowledge was significantly related to student achievement gains in first and third grades. In particular, teachers with higher content knowledge produced the students who demonstrated the greatest improvement. Hill et al. (2005) also noted that the total number of mathematics methods and mathematics content courses taken as part of teachers’ pre-service and post-service graduate higher education were highly correlated. They were surprised to find that teachers’ mathematical content knowledge predicted student gains in mathematics even in first grade. Hill et al. recommended content-focused professional preparation and pre-service programs as valid ways to improve student achievement.

A number of authors have noted that the level of pre-service teachers’ mathematics and PCK is very important since there is little development of this on school placement (e.g., Brown, McNamara, Hanley, & Jones, 1999). The explanation for this is that mathematics PCK becomes subsumed in the pragmatics of general pedagogic concerns and that supervising teacher mentoring focused on classroom management, especially when their mentees are in survival mode. Once a teacher commences classroom practice there is likely to be limited opportunity to develop deeper mathematical PCK, reportedly in part because collaboration between teachers is limited (e.g., Bakkenes, De Brabander, & Imants, 2011; Weissglass, 1994) and there is a tendency for teachers to be resistant to change (e.g., Cuban, 1984; Gregg, 1995).

Diverse Approaches to Mathematics in Primary Teaching Preparation

It is to be expected that different pre-service preparation programs have different emphases upon mathematics curriculum and different ways to meet the various certification standards. Even within the domain of mathematics curriculum education, the focus upon content and pedagogical content knowledge compared to other content domains differs between institutions and even within an institution. These differences include considerable differences in the contact time allocated to mathematics curriculum across institutions.

In some jurisdictions there are multiple pathways to primary teacher certification. For example, New York State has five pathways (Boyd, Grossman, Lankford, Loeb, & Wyckoff, 2009) with a range of mathematics prerequisite requirements prior to teacher preparation entry. Most primary school teachers in Australia complete an undergraduate degree, usually full time over 4 years. This pathway is common across many countries, including China (Li, Zhao, Huang, & Ma, 2008). The alternative pathway in Australia, the United Kingdom, USA and elsewhere is a graduate diploma, usually completed in 1 year subsequent to
the completion of an undergraduate degree. Throughout Australia students have increasingly favoured the 1-year, 18-month, or 2-year graduate pathways, with enrolments increasing proportionally at the expense of 4-year undergraduate degrees in a number of universities including the author’s institution. An analysis of the time allocated to learning to teach mathematics across five teacher preparation institutions (University of Melbourne; University of Sydney; Griffith University, Gold Coast and Mount Gravatt Campuses; Queensland University of Technology) ranged from 90 hours down to only 24 hours of contact. About 40 hours of contact time was found in three institutions. Another provider, the Wesley Institute, offers online courses with no face-to-face contact time. None of the above have a pre-requisite level of mathematics content knowledge for enrolment.

It is difficult to readily determine how the allocated hours of contact for mathematics-related teaching are used, in part because course outlines tend to be generic in nature and do not list what is actually taught. In some institutions there is greater emphasis on theories of learning and social issues; in others the focus is on specific pedagogical approaches to teaching the content for the primary years. Henderson and Rodrigues (2008) suggest that the relative lack of focus on content and specific pedagogy for mathematics is because there is “an assumption that skills possessed need simply to be added to pedagogical content knowledge and other curriculum knowledge to produce effective teachers” (p. 104). Thus, in some Australian states, for example Queensland, there may be no systematic accounting of what is taught about teaching mathematics or what standards content or PCK is attained upon graduation.

Further, Australian primary teachers are not at present required to undertake registration examinations. Instead, state-based accrediting bodies review university course structures, and students are accredited on the basis of assessments of their university. The added criterion is that the student demonstrates “reasonable classroom practice,” a judgment made by the primary school in which the pre-service teachers gain classroom experience.

Testing prior to registration exists in New York State, where prospective teachers must pass specific tests (e.g., New York State Liberal Arts and Science Test –LAST, and Assessment of Teaching Skills-Written-ATS-W, and possibly an appropriate Content Specific Test – CST). However, these tests do not focus on PCK, not even the content essential for teaching primary mathematics. Henderson and Rodrigues (2008) report that in England and Wales, teachers must achieve a minimum standard in numeracy, literacy, and information handling before qualifying. Similar standards are required in some Australian states, for example New South Wales.
Rationale for this Study

Particular shortcomings in the research literature confirm the need for this study. First, there is limited empirical research to guide primary preparation providers as to what level of mathematics ought to be considered essential for entry into primary teacher preparation courses (e.g., Goulding et al., 2002). Second, there is limited research into how mathematical understanding is best developed in primary teacher preparation programs and what relationships exist between the pre-service teachers’ content knowledge and PCK. These problems have been recognised for some time. For example, Ball (1988) reported:

This lack of attention to what teachers bring with them to learning to teach mathematics may help to account for why teacher education is often such a weak intervention – why teachers, in spite of courses and workshops, are most likely to teach math just as they were taught. (p. 40)

More recently, Ball et al. (2005) reported the problem described above remains, and that part of the reason for the limited empirical data informing these questions is that “testing teachers, studying teaching or teacher learning, using standardised student achievement measures – each of these draws sharp criticism from some quarters” (p. 45). The U.S. Department of Education (2008) noted:

Most studies have relied on proxies for teacher’s mathematical knowledge (such as teacher certification or course taken) [and that] existing research does not reveal the specific mathematical knowledge and instructional skill needed for effective teaching … Direct assessments of teachers’ actual mathematical knowledge provide the strongest indication of a relation between teachers’ content knowledge and their students’ achievement. (p. xxi)

In short, empirical data on the depth or extent of pre-service teachers’ content knowledge are relatively scarce; this is also the case in Australia. However, it is generally accepted internationally that many primary school teachers have less than ideal mathematical knowledge upon which to base their pedagogy (e.g., Ball et al., 2005; Brown & Benken, 2009; Ma, 1999). Such deficiency has also been reported in Australia (e.g., Masters, 2009). Further, although a lack of confidence in mathematics and teaching mathematics has been documented (e.g., Bursal & Paznokas, 2006; Henderson & Rodrigues, 2008), ways to remediate this situation in teacher preparation units have received scant attention.

Aims of this Study

The study had two guiding questions:

1. What relationships exist between high school and prior tertiary subject selection of mathematics and pre-service teacher success on primary mathematics content and pedagogical content knowledge?
2. What relationships exist between demonstrated content knowledge and demonstrated pedagogical content knowledge upon completion of a particular pre-service teacher mathematics preparation unit of study?
Methodology

The method chosen for this study was mixed-mode. Data were used inferentially and qualitatively, that is, the raw data were examined to determine the relationships between the variables. The following data were collected from the pre-service teachers:

1. The level of mathematics studied at high school. (Survey)
2. The form of mathematics studied during their undergraduate degrees or prior tertiary study. (Survey)
3. The level of mathematics upon entry to the course as measured by a standard Year 9 test of numeracy (MCEETYA, 2009). (Pre-test)
4. The level of mathematics upon exit from the course as measured by a standard Year 9 test of numeracy (MCEETYA, 2009). (Post-test)
5. A measure of pre-service teachers’ ability to describe how they would teach specific mathematics to primary students. This was in effect an estimate of students’ PCK at exit. (Post-test)

Test Procedures and Analyses

The pre-tests were administered in the first week of the mathematics curriculum unit and the post-tests in the last week of tutorials. The pre- and post-test NAPLAN data and the students’ PCK were mapped to the pre-service teachers’ prior mathematics learning. The relationships between prior study and student content and PCK tested in the mathematics curriculum unit were analysed using an analysis of variance. PCK was assessed upon completion of the unit; there was no pre-test of PCK since the specific pedagogy for teaching the number and algebra components of primary mathematics had not been taught to students.

Subjects

Almost the entire cohort of 129 students from the Graduate Diploma in Primary Education 2010 participated in the study (n=129 for the pre-test and n=122 for the post-test). The percentage of females at the start of the study was 85%. The majority of students had completed high school since 2000 and with few exceptions had undertaken a degree before commencing teacher pre-service education. The cohort was chosen on the basis of convenience: the researcher had the opportunity to collect data from its members. The subjects of this cohort were similar in entry numeracy and exit results to cohorts in the past two years. These pre-service teachers may well be similar to student intakes for similar courses at other teacher preparation institutions at least in the state of Queensland, potentially across Australia, and internationally such as in the United Kingdom. In Australia and the United Kingdom at least, teacher preparation courses do not stipulate pre-requisite knowledge of mathematics.
Curriculum Unit Description

The curriculum course structure included the teaching of numeration, whole number computation, fraction computation, and introductory algebra, and there was an emphasis on teaching proportional reasoning across the strands of number, space, and measurement. Teaching sequences emphasised the use of specific language to make links between various models, material and diagrammatic and symbolic representations. This approach to teaching and learning mathematics is supported widely (e.g., Goulding et al., 2002; Reys, Lindquist, Lambdin, & Smith, 2009; Van de Walle, 2007). The explicit approach has the support of a number of education bodies (e.g., U.S. Department of Education, 2008) and mathematics education researchers (e.g., Kirschner, Sweller, & Clark, 2006).

The underlying goal of the curriculum unit was to teach the underpinning mathematical concepts to the pre-service teachers while teaching them how to teach the concepts. For example, by modelling how to teach division with the use of specific language, materials, and linking these representations to symbolic recording, it was anticipated that the pre-service teachers would understand division as well as know how to teach it. The curriculum unit in this study had been approved by the teacher registration body in the state (Queensland College of Teachers, 2006) as meeting the requirements for teacher preparation such that the graduating students are eligible to be registered as teachers in the state of Queensland.

Instruments

Categorising the level of mathematics studied at high school

Assessing and categorising the level of high school mathematics was relatively unambiguous since each level was described to the students. The categorisation mirrors the form of mathematics studied at high school. Students who cease study of mathematics at Year 10 or 11 generally have had limited exposure to abstract mathematics associated with algebra, proportional reasoning in number and geometry contexts, or logic associated with proof. These students who had not completed any senior mathematics were classified as Level 1.

Students who study senior Mathematics A similarly have limited exposure to abstract mathematics; rather, they study units that focus on the application of mathematics in financial contexts, applied geometry such as navigation or building construction and plans, and relatively simple presentation and analysis of data. Mathematics A does not assume knowledge of calculus and the applications of algebra and geometry are relatively simple. Students who had studied Mathematics A or its equivalent were classified as Level 2.

Pre-service teachers who completed Mathematics B or its equivalent were classified as Level 3. Mathematics B (or its equivalent) is generally the minimum level of school mathematics needed to enter science-based courses at tertiary institutions and is undertaken by about 20% of senior school students in
Australia (Barrington, 2006). Mathematics B or its equivalent typically has core units such as introduction to functions, rates of change, periodic functions and applications, exponential and logarithmic functions, optimisations, integration, and statistics. The subject matter is mostly calculus and there is some statistics including hypothesis testing.

At a higher level, students who study senior Mathematics C study core topics including groups, real and complex number systems, matrices and applications, vectors and applications, the application of calculus, and a range of optional topics including linear programming, conics, dynamics, and advanced periodic functions and exponential functions. Generally only students who intend to enter tertiary study associated with the hard sciences such as engineering, actuarial studies, or pure mathematics study Mathematics C. Barrington (2006) reported that across Australia about 10% of graduating high school students complete Mathematics C type courses. Students who had studied Mathematics C were classified as Level 4.

The completion of the various levels above provides a reasonable guide to the level of mathematics undertaken, and presumably understood, by the students. For example, a student can gain a pass result in Level 1 or 2 with very limited understanding of abstract mathematics, proof, algebraic processes, or even good number sense. This is not the case with Levels 3 and 4. It is for this reason that most tertiary institutions assume the equivalent of Levels 3 or 4 knowledge for entry to most science-based tertiary courses and frequently offer bridging courses for those lacking in this level of mathematical competence.

Categorising the form of mathematics studied at university

Assessing and categorising the level of tertiary mathematics studied was problematic. It was difficult to estimate accurately the level of tertiary mathematics embedded in courses that varied from “mathematics associated with nursing,” “mathematics associated with health sciences,” “health science statistics,” or “business mathematics.” The categories of tertiary mathematics levels that emerged from the tertiary data were “no mathematics,” “health science mathematics,” “business mathematics,” mathematics associated with business, accounting, or economics, and “advanced mathematics” associated with the study of subjects including physics, engineering, and computer sciences. “No mathematics” indicates that the tertiary experience did not add to the mathematics the students learned in high school. “Health science mathematics” tends to be dominated by specific mathematics associated with measurement and is not very dissimilar from aspects of Mathematics A in terms of the level of abstraction required. It is to be expected that “business mathematics” might extend upon what students had studied in high school mathematics to Levels 2 and 3, since a typical business degree contains up to three 10-credit point subjects in research methods and statistics as well as two or three subjects in which mathematics plays an important role, for example accounting-based subjects or economic modelling.
Assessing entry and exit content knowledge (numeracy)

In order to gain a measure of students’ content knowledge of mathematics at the beginning of the course, students completed the 2009 Year 9 NAPLAN non-calculator test (MCEETYA, 2009) under examination conditions. At the end of the course the students completed the second of the two Year 9 NAPLAN tests. In both instances the pre-service teachers were not allowed to use a calculating device. A test analysis of the NAPLAN items shows that, due to the structure of test items developed by MCEETYA (2009), students with a reasonable knowledge of primary computation ought not to have been disadvantaged by not having access to a calculating device (Norton, 2009). NAPLAN test papers are designed to assign students to particular band levels, and thus test a range of difficulty levels with questions that are of a standard lower than what is expected of a year level as well as some more challenging questions. Teachers of upper primary years would be expected to teach most of the concepts tested in these tests and few educators would argue that teachers do not need to know at least middle years mathematics.

Assessing Exit Pedagogical Content Knowledge

In order to assess students’ grasp of PCK, students completed 10 questions under examination conditions. The structure of the written exam is presented in Table 1.

Table 1
Structure of the Post-Test Exam including Extended Answer Questions

<table>
<thead>
<tr>
<th>Question</th>
<th>Concept</th>
<th>Marks</th>
</tr>
</thead>
<tbody>
<tr>
<td>NAPLAN Year 9 Numeracy test</td>
<td>Short answer test of PCK</td>
<td>/31</td>
</tr>
<tr>
<td>1</td>
<td>Teaching naming numbers</td>
<td>/5</td>
</tr>
<tr>
<td>2</td>
<td>Teaching the addition concept</td>
<td>/5</td>
</tr>
<tr>
<td>3</td>
<td>Teaching subtraction with renaming</td>
<td>/7</td>
</tr>
<tr>
<td>4</td>
<td>Teaching the multiplication algorithm</td>
<td>/7</td>
</tr>
<tr>
<td>5</td>
<td>Teaching the area model of multiplication</td>
<td>/7</td>
</tr>
<tr>
<td>6</td>
<td>Teaching the division algorithm</td>
<td>/7</td>
</tr>
<tr>
<td>7</td>
<td>Teaching fraction and decimal representations</td>
<td>/7</td>
</tr>
<tr>
<td>8</td>
<td>Teaching mixed number subtraction</td>
<td>/7</td>
</tr>
<tr>
<td>9</td>
<td>Teaching problem solving in the context of fractions and decimals</td>
<td>/7</td>
</tr>
<tr>
<td>10</td>
<td>Teaching algebra problem solving</td>
<td>/10</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>/100</td>
</tr>
</tbody>
</table>
Two sample items testing PCK for a lower primary and a middle primary concept are presented in Figure 1 and Figure 2.

A Year 3 student carried out the following addition.

\[
\begin{array}{c}
4578 \\
\hline
12413
\end{array}
\]

a) What was his conceptual error and what teaching might have led to that error?
b) Set out a teaching sequence clearly linking materials and formal symbols with clear connecting language.

\[\text{Figure 1. Question 2}\]

Examine the student working below showing the computation 45 - 18.

\[3.3\]

a) What teaching and strategies might have led to this method? What are the limitations of the method?
b) Set out a teaching sequence clearly linking materials and formal symbols with clear connecting language.

\[\text{Figure 2. Question 3}\]

Appendix A contains a sample of a good script where full marks were awarded for Question 3. The solution presented in Appendix A illustrates that the pre-service teacher is able to recognise error patterns in student scripts and design a teaching sequence to assist in remediation of this error. The pre-service teacher’s solution in Appendix B illustrates that the student can link the equivalent representations of 75%, .75 and \(\frac{3}{4}\). The marking criteria are documented in Appendix C.

SPSS 18 was used to undertake all analyses. Significance was assessed with type 1 error, \(\alpha = 0.05\) for 2-sided tests, and significance set at significant * < 0.05, highly significant ** < 0.01, very highly significant *** < 0.001.
Comments on the Instruments and Potential Limitations

There were additional hours of study related to mathematics curriculum after this unit, but they were limited and the focus was upon general pedagogical principles, planning, and designing assessment rather than providing specific strategies for the diagnosis and remediation of key aspects associated with the number strand, which was the focus of this course.

The use of the NAPLAN tests as a measure of numeracy has the support of the Department of Education and Training (2010a, 2010b). The authors cite consistent matching of scores in sample and test populations of up to a million students in any year. NAPLAN reports student achievement according to bands, that is, raw scores are scaled to 1 with a mean of 500 and standard deviation of 100. In this study raw scores are used, but this does not detract from the validity of the results or the comparisons made.

It could be considered that the test of PCK is problematic in that it essentially asked pre-service teachers to replicate the pedagogy for teaching numeration, algorithms, and problem-solving models that they had studied in lectures and workshops. However, the use of the instruments such as those described above is supported by Council of Australian Governments [COAG] (2008) who reported valid teacher assessment should not be remote from what teachers do in the classroom. In terms of the teaching and assessment approach, most teacher educators would concur that systematic linking of various representations of mathematical concepts is central to teacher planning (e.g., Goulding et al., 2002; Reys et al., 2009; U.S. Department of Education, 2008; Van de Walle, 2007). Thus, it is reasonable to expect teachers to be able to describe what they would get students to do, what language they would use, what materials they would use, and how they would assist students to connect various representations of mathematical concepts. From this point of view the test of PCK has content validity.

Results

Level of Mathematics Studied at High School and at University

The first level of data reporting and analysis seeks to answer the first research question:

What relationships exist between high school and prior tertiary subject selection of mathematics and pre-service teacher success on primary mathematics content and pedagogical content knowledge?

Initially the data on high school and tertiary mathematics are presented, then pre-service teachers’ results in tests of upper primary content and mathematics PCK are documented. The levels of senior high school mathematics completed by commencement of pre-service teaching, and the level of tertiary mathematics undertaken, are documented in Table 2.
Table 2
High School and Tertiary Mathematics Completed (N=119)

<table>
<thead>
<tr>
<th>University Mathematics</th>
<th>Categories of high school mathematics completed by pre-service teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Level 1</td>
</tr>
<tr>
<td>None (54.4%)</td>
<td>7.5%</td>
</tr>
<tr>
<td>Health statistics (5.9%)</td>
<td>3%</td>
</tr>
<tr>
<td>Business mathematics (31%)</td>
<td>1.5%</td>
</tr>
<tr>
<td>Advanced mathematics (8.4%)</td>
<td>0%</td>
</tr>
<tr>
<td>Total</td>
<td>12.6%</td>
</tr>
<tr>
<td>(N=15)</td>
<td>(N=55)</td>
</tr>
</tbody>
</table>

The survey data indicate that most students had studied relatively low levels of high school mathematics (about 59% Level 1 or 2), about a third had studied intermediate mathematics (Level 3), and about 11% had studied advanced mathematics, that is, both Mathematics B and C (Level 4). Most students had not studied any mathematics as part of their tertiary courses, about 37% had completed mathematics as part of health sciences or basic business statistics, and few (8.4%) had studied advanced mathematics at a tertiary institution.

The following results are reported in terms of previous high school mathematics, without taking into account any tertiary mathematics studied by students. The possible effects on tertiary mathematics results are discussed at the end of this section.

Results on Tests of Content and Pedagogical Content Knowledge

In the sections below students’ results on the test items are reported and major findings described. The pre-service teachers in this study were found to have a level of mathematical understanding not significantly different from the average Year 9 student in the state of Queensland.

In regard to the first research question, the relationship between the level of high school mathematics studied and success on a test of primary mathematics content, the data indicate that higher levels of high school mathematics are associated with higher scores on both the pre- and post-test NAPLAN tests and the written test of PCK (see Table 3). In terms of the pre-test of content knowledge, the mean differences between Level 1 and 2 students were 1.91 marks ($p=0.940$); between Level 2 and Level 3 students it was 3.15 ($p=0.008$); and between Levels 3 and 4 the mean difference was 2.85 ($p=0.401$). There was an
increase in NAPLAN scores for each high school mathematics category, which was statistically significant (df, 21, 117; F = 4.734; \( p = 0.000 \)). Analysis of scripts indicated that upper primary concepts such as division of two-digit numbers, operations with fractions, and questions related to proportional reasoning were the most challenging to the pre-service teachers, especially for pre-service teachers with high school mathematics at Levels 1 and 2.

It is worth noting that the variation of scores was much more extensive among students who completed lower levels of high school mathematics. This was the case for each assessment instrument. The data indicate that more mathematics studied in high school was not only associated with higher marks on these tests, but that this was consistently the case.

The data in Table 4 sum up relationships between the pre and post-tests of content knowledge and the post-test of PCK and levels of high school mathematics completed.

Table 3

<table>
<thead>
<tr>
<th>High school mathematics</th>
<th>Pre CK/31 mean (sd)</th>
<th>Post CK/31 mean (sd)</th>
<th>Post PCK/69 Mean (sd)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1 No senior</td>
<td>15.63 (5.42)</td>
<td>17.90 (6.82)</td>
<td>43.53 (14.13)</td>
</tr>
<tr>
<td>Level 2 (Maths A)</td>
<td>17.54 (4.11)</td>
<td>19.45 (4.59)</td>
<td>47.63 (11.25)</td>
</tr>
<tr>
<td>Level 3 (Maths B)</td>
<td>20.69 (4.56)</td>
<td>22.94 (3.61)</td>
<td>52.69 (9.53)</td>
</tr>
<tr>
<td>Level 4 (Maths C)</td>
<td>23.54 (4.70)</td>
<td>25.27 (3.66)</td>
<td>60.42 (4.75)</td>
</tr>
</tbody>
</table>

Prior Study of Mathematics as a Predictor of Pre-service Teachers’ Success on Tests

Table 4

Summary of ANOVAs on the Pre-Test for Content Knowledge (CK), Post-Test for Content Knowledge and Post-Test for Pedagogical Content Knowledge According to High School Mathematics Studied

<table>
<thead>
<tr>
<th>Test</th>
<th>Df</th>
<th>F</th>
<th>Sig</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-CK</td>
<td>2, 114</td>
<td>12.497</td>
<td>&lt;.000</td>
<td>There was no significant difference between scores of Level 1 and Level 2 groups. Students who studied more advanced mathematics (Level 3 and Level 4) achieved much higher scores.</td>
</tr>
<tr>
<td>Post-CK</td>
<td>2, 117</td>
<td>17.474</td>
<td>&lt;.000</td>
<td>There was little to distinguish between Levels 1 and 2 and between Levels 3 and 4, but the latter groups had much higher scores than the students who studied lower levels of high school mathematics.</td>
</tr>
</tbody>
</table>
Those pre-service teachers who had studied low levels of high school mathematics equivalent to Mathematics A (Level 2) were similar to those studying no mathematics in senior years of high school (Level 1). Further, the effect of studying high school mathematics at Level 3 – which contains a strong emphasis on calculus – was indistinguishable from that of studying at Level 4. That is, doing the extra abstract mathematics at high school did not seem to confer any advantage; passing the equivalent of Mathematics B was sufficient.

The study of the equivalent of Mathematics B at high school seems to be a defining feature of success on tests of primary content and PCK, which is explaining how to teach it. This finding is supported by data contained in Appendices D and E showing the ranking of the top and bottom quartiles. Almost without exception, students who had studied to Level 3 (Mathematics B) at high school occupied the top quartile of results. The data in Appendix D illustrate that, when final content and ability to explain how primary mathematics is taught is tallied and students ranked according to this total, almost universally the top 30 students had studied calculus and most of the top 20% of students had also studied advanced or business mathematics at university. The top ranked pre-service teacher who had studied high school at Level 1 (Year 10) was ranked 14th overall. However, it should be noted that this student had studied computing mathematics at university and was graded with a distinction. The highest ranking achieved by a pre-service teacher who had studied Level 2 mathematics was 12th overall and s/he had achieved a credit in university statistics. Of the top ranked pre-service teachers, a colleague and experienced mathematics educator who moderated the course results commented, “Wow, I agree this student really knows a lot about how to teach the various concepts.”

The data in Appendix E illustrate that students who had studied Level 1 and Level 2 high school mathematics dominate the bottom quartile.

The second aspect of the first research question focuses on students’ selection and completion of tertiary mathematics courses and probes any relationship that might exist between this and their subsequent success on the pre-service tests of content and PCK. It is very difficult to make much of this because so many variables are unknown. It is not known exactly what content was taught in the tertiary courses or how well it was learnt. Importantly, it is not known how tertiary study might be associated with an increased primary mathematics content of PCK mark or what interaction might exist between the

<table>
<thead>
<tr>
<th>Test</th>
<th>Df</th>
<th>F</th>
<th>Sig</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Post-PCK</td>
<td>2, 117</td>
<td>11.032</td>
<td>&lt;.000</td>
<td>There was no difference between those who studied Level 1 to Level 2. Level 3 students’ scores were significantly better than Level 1 but not Level 2 and while Level 4 were significantly better than Levels 1 and 2 students’ scores, they were not significantly better than Level 3 students.</td>
</tr>
</tbody>
</table>
tertiary mathematics studied and the level of mathematics studied at school. Most of the top 20 students had studied statistics of some form and a few had studied engineering or computing. Still, the data in Appendix E indicate that most of the bottom quartile had not studied mathematics at university. However 13 out of 30 had studied some form of statistics, sometimes associated with finance or health science. It was clear that this study of basic statistics did not compensate for not having done at least Level 3 high school mathematics.

The second research question sought to examine the relationship between demonstrated content knowledge and PCK upon completion of the course. Most students who studied the higher levels of mathematics at high school achieved relatively well on all tests. High scores on mathematics content were associated with high scores on pedagogy. It could be said that these pre-service teachers were sufficiently literate to explain what they understood. Similarly, pre-service teachers who did not know the mathematics could not explain it no matter how many non-mathematics-based subjects they had undertaken at a tertiary level.

The 10 students who achieved less than 50% on the examination were granted a supplementary examination after several weeks of further study. All students who attempted the supplementary examination attained at least 50%. The student who scored 23% on her first attempt at describing pedagogy subsequently attained 88% on similar tasks.

Discussion and Conclusions

The review of pre-service teacher program requirements and outlines indicates that within Australia, and internationally, there is considerable diversity in terms of what is taught and what time is taken to teach it. Face to face learning time varies form zero for study options offered online to close to 100 hours. Without access to their examination scripts it is difficult to determine what is taught in the various courses and what emphasis there is upon content and PCK. Readers are asked to decide for themselves if the findings have any relevance to their own situation.

In regard to the first research question, this study begins to document what content and PCK pre-service teachers from one postgraduate unit on one campus at one institution have demonstrated. As a cohort, the students entered the unit with content knowledge similar to the average Year 9 student (age 13 to 14 years). Relatively low levels of mathematics prior to entry to primary teaching preparation are not unique to this sample: Adler et al. (2005, p. 361), for example, reported that in many countries “prospective elementary teachers have learned limited mathematics in school.” This finding supports the concerns expressed by Henderson and Rodrigues (2008) who reported teachers’ understanding of mathematics was shaped by school and informal experiences and that teacher education programs tend to assume that prospective teachers bring with them sufficient mathematical understanding to enable them to promote effective classroom practice.

The data presented here show that most pre-service teachers who have completed limited mathematics study in high school, know less when they commence tertiary teacher preparation study and exit with lower levels of content and PCK than other pre-service teachers. That is, having completed no
senior high school mathematics, or having studied mathematics without calculus, is strongly associated with lower marks on tests for primary mathematics content and PCK.

A few pre-service teachers who undertook Level 1 and Level 2 high school mathematics did achieve high scores on the pre-service teacher preparation tests. This may be because many Queensland students who are quite good at mathematics have received advice not to take the equivalent of Mathematics B and C unless they intend to enter tertiary courses that specifically require these, such as engineering and the hard sciences. This is especially the case with Mathematics C. A further factor that discourages mathematically capable students from undertaking the more exacting mathematics subjects is that low-level mathematics subjects (Mathematics A) have the same weighting reward as high-level mathematics subjects (Mathematics B and C) for tertiary entrance. In terms of final tertiary entrance ranking scores, a student with high achievement in Mathematics A might well gain similar credit to a person with a high achievement on the much more demanding Mathematics B or C subjects. These channelling factors may help to explain the wide range of mathematical achievement among Level 2 pre-service teachers.

It is difficult to determine what effect undergraduate tertiary study of mathematics has upon the level of relevant mathematics a pre-service teacher brings to teacher preparation. This is in part due to the observation that most pre-service teachers who studied relatively rigorous tertiary mathematics associated with science, finance, or computing had previously studied high school mathematics at least to Level 3. However, the data indicate that the study of tertiary mathematics associated with health sciences or statistics did not seem to compensate for the lack of study of Level 3 mathematics in high school. In short, if a pre-service teacher did not study mathematics to Level 3 and did not study advanced mathematics at university, but rather did no tertiary mathematics or only mathematics units associated with health sciences such as nursing or basic statistics, it was highly likely they would fail or nearly fail tests of primary content and PCK, even after 40 hours of focused tertiary learning. There is a substantial body of research indicating that teachers’ confidence in teaching is strongly correlated to their confidence with the subject matter of mathematics (Ball, 1988; Bursal & Paznokas, 2006) and teacher confidence affects their practice (e.g., Stipek, Givven, Salmon, & MacGyvers, 2001).

Almost half the pre-service teachers exited this unit with relatively strong knowledge of content and how to teach it (refer to Table 4). Some possible contributing factors include the structure of the unit, its content, how it was taught, the time it was implemented, and the nature of the intake.

In regard to the second research question, the results indicate that pre-service teachers who were proficient at mathematics were effective at explaining how to teach it. This finding provides empirical support for the arguments of those who consider there is a strong link between content knowledge and teaching knowledge (e.g., Ball et al., 2005; Goulding et al., 2002; Ma, 1999; Silverman & Thompson, 2008; U.S. Department of Education, 2008). It is
interesting that a test designed to assess primary and lower middle school students’ knowledge of mathematics (NAPLAN) should be such a strong predictor of success on a pre-service test of PCK and overall success on the curriculum unit. This finding supports the claims from the Department of Education and Training (2010a, 2010b) that the NAPLAN tests are a reliable assessment of primary and middle school mathematics across a range of student ability.

The study raises another interesting question. Why would the study of calculus, particularly advanced calculus, be such a robust predictor of high marks on both content and pedagogical knowledge tests designed for primary students? There is no evidence in the data to answer this question and the finding contradicts earlier research (e.g., Askew et al., 1997). It may be that those pre-service teachers who had selected to study high levels of high school mathematics were in the main generally more competent or more intelligent. Alternatively, the study of advanced mathematics may have assisted these pre-service teachers in becoming analytical beyond the domains of calculus or more advanced statistics, such as in concise writing of explanations about how to teach mathematics. A third and related possibility is that knowing calculus helped these pre-service teachers to quickly develop a profound understanding of primary and early middle year mathematics, particularly in regard to the content of the NAPLAN tests.

The data indicate there is merit in exploring the use of the level of high school mathematics completed as a partial filter for teacher preparation programs. At least knowing the level of high school mathematics completed by the applicant would alert the tertiary preparation provider to the need for additional testing in order to signal the need for early intervention. Widely available tests such as NAPLAN could be used to provide additional data.

The major finding of this study suggests the following recommendations for further study. First, a more in-depth study of the relationship between content and pedagogical knowledge is needed. Second, ongoing research into the effectiveness of various mathematics pre-service teacher programs is warranted, as are instruments to study progress. The data indicate that further research is needed on the content, duration, and delivery methods of units preparing pre-service teachers to teach mathematics. It is clear that in this and potentially many other instances, too little is done too quickly for the many students who enter teacher preparation with limited mathematical background.

References


**Author**

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Appendix A
Sample of a good response to a PCK question.

**Question 3**
Examine the student working below showing the computation of 45 - 18.

\[ \text{Question 3. What is } 45 \text{ takeaway } 18? = 27 \]

(a) What teaching and strategies might have led to this method? What are the limitations of the method? (1 mark)

This student may have been taught using rainbow facts and also taking the smallest from the biggest. His method does not teach the concept of renaming and students do not understand the algorithm.

(b) Set out a teaching sequence clearly linking materials and formal symbols with clear connecting language. (4 marks)

<table>
<thead>
<tr>
<th>Language</th>
<th>Materials</th>
<th>Symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>Takeaway the ones.</td>
<td>[ ]</td>
<td>[ \text{3}\text{15} ]</td>
</tr>
<tr>
<td>I have 5 ones can I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>takeaway 8 ones? NO!</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rename one of the tens as 10 ones. Record the 3 tens and 15 ones.</td>
<td>[ ]</td>
<td>[-18]</td>
</tr>
<tr>
<td>I have 15 ones can I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>takeaway 8 ones? Yes. Do it.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15 takeaway 8 is 7. Record.</td>
<td>[ ]</td>
<td>[ ]</td>
</tr>
<tr>
<td>Takeaway the tens.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>I have 3 tens can I</td>
<td></td>
<td></td>
</tr>
<tr>
<td>takeaway 1 ten? Yes. Do it 3 tens takeaway 1 ten is 2 tens. Record</td>
<td>[ ]</td>
<td>[ \text{3}\text{15} ]</td>
</tr>
<tr>
<td>45 takeaway 18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>is 27</td>
<td>[ ]</td>
<td>[ ]</td>
</tr>
</tbody>
</table>
Appendix B

Question 7 (7 Marks)

Explain how you would develop and understand that \( \frac{75}{100} \) was equal to 0.75 and 75%.

\[
\frac{75}{100} \rightarrow 0.75 \quad \text{and} \quad \frac{3}{4} \text{ and } 75%.
\]

![Mathematical calculations and explanations]

\[
\frac{3}{4} \text{ converted into percent:} \quad \frac{25}{4} = \frac{75}{100} \quad \text{per cent per 100}
\]

\[
75 \text{ per 100} \quad 75\%
\]

Apply unit rule to rename fraction to hundredths.
Appendix C

Marking Criteria for Written Examination Tasks assessing PCK.

<table>
<thead>
<tr>
<th>Grade</th>
<th>Criteria description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>The problem is fully solved. The response shows evidence of interpretation, analysis, identification of assumptions, use of appropriate strategies and procedures for teaching while showing initiative. All choices and explanations are justified and all steps well explained. Teaching has been explicit with appropriate use of various representations. Full marks.</td>
</tr>
<tr>
<td>B</td>
<td>The problem is fully solved. The response shows evidence of interpretation, analysis, identification of assumptions, use of appropriate strategies and procedures for teaching while showing initiative. There may be minor errors in choices and explanations or justification of steps contains minor omissions. Teaching has been explicit with only minor omissions in the use of appropriate use of various representations. High marks e.g., 8/10 or 6/7 etc.</td>
</tr>
<tr>
<td>C</td>
<td>The problem has been solved. However, while there is evidence of use of appropriate strategies for teaching; justification, explanations or use of representations, they have not been appropriate in significant ways or choices and explanations have not been well explained. A peer would likely have difficulty following the teaching steps. Approximately half marks.</td>
</tr>
<tr>
<td>D</td>
<td>The problem has not been solved. There are significant flaws in methodology for working out the solution and explaining its teaching with poor communication or lack of use of appropriate use of representations. Few or zero marks.</td>
</tr>
</tbody>
</table>
Appendix D

Raw Scores on Tests, High School Mathematics and Tertiary Mathematics Studied for Top Quartile, Ranked According to Final Test Score. (Ranked according to final test score).

<table>
<thead>
<tr>
<th>N1/31</th>
<th>N2/31</th>
<th>Written /69</th>
<th>Final %</th>
<th>High School Mathematics</th>
<th>Math Level</th>
<th>Tertiary Mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>28</td>
<td>66.5</td>
<td>94.5</td>
<td>Maths B (HA)</td>
<td>3</td>
<td>None</td>
</tr>
<tr>
<td>25</td>
<td>30.5</td>
<td>63.5</td>
<td>94</td>
<td>Maths B, C</td>
<td>4</td>
<td>Commerce statistics</td>
</tr>
<tr>
<td>24</td>
<td>26</td>
<td>67.5</td>
<td>93.5</td>
<td>Maths B</td>
<td>3</td>
<td>Statistics</td>
</tr>
<tr>
<td>27</td>
<td>30</td>
<td>63.5</td>
<td>93.5</td>
<td>Maths B, C</td>
<td>4</td>
<td>BSC Hons physics</td>
</tr>
<tr>
<td>25</td>
<td>26.5</td>
<td>66.5</td>
<td>93</td>
<td>Maths A (VHA)</td>
<td>2</td>
<td>Business finance (D)</td>
</tr>
<tr>
<td>28</td>
<td>27</td>
<td>65.5</td>
<td>92.5</td>
<td>Maths B, C HA</td>
<td>4</td>
<td>None</td>
</tr>
<tr>
<td>26</td>
<td>28.5</td>
<td>63.5</td>
<td>92</td>
<td>Maths B (Vic)</td>
<td>3</td>
<td>None</td>
</tr>
<tr>
<td>23</td>
<td>26</td>
<td>65.5</td>
<td>91.5</td>
<td>Maths B</td>
<td>3</td>
<td>Statistics</td>
</tr>
<tr>
<td>14</td>
<td>25</td>
<td>66</td>
<td>91</td>
<td>Maths B, C</td>
<td>4</td>
<td>Finance statistics</td>
</tr>
<tr>
<td>29</td>
<td>29.5</td>
<td>61</td>
<td>90.5</td>
<td>Maths B, C Dist</td>
<td>4</td>
<td>Engineering</td>
</tr>
<tr>
<td>28</td>
<td>27</td>
<td>63</td>
<td>90</td>
<td>Maths B C VHA</td>
<td>4</td>
<td>Economics</td>
</tr>
<tr>
<td>17</td>
<td>26.5</td>
<td>62.5</td>
<td>89</td>
<td>Maths A (HA)</td>
<td>2</td>
<td>Statistics Credit</td>
</tr>
<tr>
<td>26</td>
<td>24.5</td>
<td>62.5</td>
<td>87</td>
<td>Maths B</td>
<td>3</td>
<td>None</td>
</tr>
<tr>
<td>21</td>
<td>23</td>
<td>63.5</td>
<td>86.5</td>
<td>Year 10</td>
<td>1</td>
<td>Computer maths (D)</td>
</tr>
<tr>
<td>18</td>
<td>25</td>
<td>61.5</td>
<td>86.5</td>
<td>Maths B</td>
<td>3</td>
<td>None</td>
</tr>
<tr>
<td>25</td>
<td>26</td>
<td>60.5</td>
<td>86.5</td>
<td>Maths B</td>
<td>3</td>
<td>Probability credit</td>
</tr>
<tr>
<td>24</td>
<td>29</td>
<td>57.5</td>
<td>86.5</td>
<td>Maths B, C (D)</td>
<td>4</td>
<td>MSc.</td>
</tr>
<tr>
<td>27</td>
<td>25</td>
<td>61</td>
<td>86</td>
<td>Maths B</td>
<td>3</td>
<td>Statistics (D)</td>
</tr>
<tr>
<td>20</td>
<td>23.5</td>
<td>62.5</td>
<td>86</td>
<td>Maths B</td>
<td>3</td>
<td>Physics (D)</td>
</tr>
<tr>
<td>22</td>
<td>23</td>
<td>62.5</td>
<td>85.5</td>
<td>Maths B (HA)</td>
<td>3</td>
<td>None</td>
</tr>
<tr>
<td>22</td>
<td>26</td>
<td>59</td>
<td>85</td>
<td>Yr 12 Canada adv</td>
<td>2</td>
<td>None</td>
</tr>
<tr>
<td>23</td>
<td>23</td>
<td>61.5</td>
<td>84.5</td>
<td>Maths B (HA)</td>
<td>3</td>
<td>Business risk assess</td>
</tr>
<tr>
<td>19</td>
<td>22</td>
<td>62</td>
<td>84</td>
<td>Maths A</td>
<td>2</td>
<td>None</td>
</tr>
<tr>
<td>26</td>
<td>26</td>
<td>57.5</td>
<td>83.5</td>
<td>Finite math Canada</td>
<td>3</td>
<td>Statistics maths</td>
</tr>
<tr>
<td>16</td>
<td>20</td>
<td>63</td>
<td>83</td>
<td>Year 11</td>
<td>2</td>
<td>None</td>
</tr>
<tr>
<td>25</td>
<td>27.5</td>
<td>55.5</td>
<td>83</td>
<td>Maths A (HA)</td>
<td>2</td>
<td>None</td>
</tr>
<tr>
<td>15</td>
<td>19.5</td>
<td>63</td>
<td>82.5</td>
<td>Maths A</td>
<td>2</td>
<td>None</td>
</tr>
<tr>
<td>18</td>
<td>20</td>
<td>60.5</td>
<td>80.5</td>
<td>Maths B</td>
<td>4</td>
<td>None</td>
</tr>
<tr>
<td>28</td>
<td>26</td>
<td>53</td>
<td>79</td>
<td>Maths B</td>
<td>3</td>
<td>Statistics (Pass)</td>
</tr>
<tr>
<td>16</td>
<td>23</td>
<td>54.5</td>
<td>77.5</td>
<td>Maths B</td>
<td>3</td>
<td>Accounting</td>
</tr>
</tbody>
</table>

Key:
N1 – Score on pre-test for content knowledge with Year 9 NAPLAN test.
N2 – Score on post-test for content knowledge with Year 9 NAPLAN test.
Written – Score on post-test for content pedagogical knowledge
VHA – very high achievement
HA – high achievement
D – distinction
Appendix E

Raw Scores on Tests, High School Mathematics and Tertiary Mathematics Studied for Bottom Quartile (ranked according to final test score)

<table>
<thead>
<tr>
<th>N1</th>
<th>N2</th>
<th>Wri</th>
<th>%</th>
<th>High School Math Lev</th>
<th>Tertiary Mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>24</td>
<td>44</td>
<td>68</td>
<td>Maths A (VHA)</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>19</td>
<td>46.5</td>
<td>65.5</td>
<td>Year 10</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>22</td>
<td>42.5</td>
<td>64.5</td>
<td>Maths B Canada</td>
<td>3</td>
</tr>
<tr>
<td>24</td>
<td>27.5</td>
<td>36.5</td>
<td>64</td>
<td>Maths B 1968</td>
<td>3</td>
</tr>
<tr>
<td>24</td>
<td>20</td>
<td>43</td>
<td>63</td>
<td>Maths B (LA)</td>
<td>3</td>
</tr>
<tr>
<td>15</td>
<td>23</td>
<td>38.5</td>
<td>61.5</td>
<td>none</td>
<td>1</td>
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Key:
N1-Score on pre-test for content knowledge with Year 9 NAPLAN test.
N2- Score on post-test for content knowledge with Year 9 NAPLAN test.
Written-Score on post-test for content pedagogical knowledge
VHA- very high achievement
HA- High achievement
D- Distinction
SA- Sound achievement
LA- Low achievement
Pre-service Secondary Mathematics Teachers
Making Sense of Definitions of Functions

Joshua Chesler
California State University, Long Beach

Definitions play an essential role in mathematics. As such, mathematics teachers and students need to flexibly and productively interact with mathematical definitions in the classroom. However, there has been little research about mathematics teachers' understanding of definitions. At an even more basic level, there is little clarity about what teachers must know about mathematical definitions in order to support the development of mathematically proficient students. This paper reports on a qualitative study of pre-service secondary mathematics teachers choosing, using, evaluating, and interpreting definitions. In an undergraduate capstone course for mathematics majors, these future teachers were assigned three tasks which required them to (1) choose and apply definitions of functions, (2) evaluate the equivalence of definitions of functions, and (3) interpret and critique a secondary school textbook's definition of a specific type of function. Their performances indicated that many of these pre-service mathematics teachers had deficiencies reasoning with and about mathematical definitions. The implications of these deficiencies are discussed and suggestions for teacher educators are proposed.

Introduction

Definitions matter in mathematics. They introduce ideas, they describe objects and concepts, they identify fundamental and essential properties of mathematical objects, they support problem solving and proof, and they facilitate communication of mathematics (Zaslavsky & Shir, 2005). Accordingly, the United States' Common Core State Standards for Mathematics (CCSSM) acknowledges the importance of definitions in the mathematics education of all K-12 students (Common Core State Standards Initiative, 2010). Within their eight “Standards for Mathematical Proficiency”, the CCSSM note that mathematically proficient students understand and use definitions in constructing arguments, in their reasoning, and in communication about mathematics. However, there has been relatively little research either on student learning or on teacher knowledge of the roles and uses of mathematical definitions (de Villiers, 1998; Moore-Russo, 2008; Vinner, 1991; Zaslavsky & Shir, 2005). Moreover, the research that does exist on in- and pre-service mathematics teachers interacting with mathematical definitions indicates that many have deficiencies in this area (e.g., Leikin & Winicki-Landman, 2001; Linchevsky, Vinner, & Karsenty, 1992; Moore-Russo, 2008; Vinner & Dreyfus, 1989; Zazkis & Liekin, 2008).

The study reported herein is an examination of pre-service secondary mathematics teachers (PSMTs) choosing, analysing, evaluating, and using definitions in a mathematics capstone course taken in the final semester of their undergraduate mathematics program. The broad goal is to help illuminate the task and challenges of training PSMTs to prepare mathematically proficient
students who reason from, with, and about definitions, as envisioned in the CCSSM and elsewhere. The present focus is on pre-service secondary teachers, however, it should be noted that elementary teacher preparation with mathematical definitions is likewise important. For example, the mod4 project at University of Michigan recently published professional development materials entitled *Using Definitions in Learning and Teaching Mathematics* for elementary teachers (mod4, 2009). The activities focus on identifying the roles and emphasizing the importance of definitions in teaching elementary school and, more generally, in mathematical reasoning and in the discipline of mathematics. Furthermore, they address the question, “What makes a good mathematical definition?” (p. 1). This question is one which the PSMT in the present study had to confront and, fortunately, it is one about which there is some clarity.

A mathematical definition must not be self-contradicting or ambiguous; it must be invariant under choice of representation and it must be hierarchical (i.e., based on prior concepts) (Zaslavsky & Shir, 2005). Furthermore, mathematical definitions are arbitrary; for any particular object or concept, there are many equivalent ways to define it. On this point, Winicki-Landman and Leikin (2000) note that, “teachers’ professional development should include activities focusing on the issue of equivalent and non-equivalent definitions” (p. 21). Zaslavsky and Shir (2005) characterize definitions as either (1) procedural, describing how an object is constructed, or (2) structural, identifying essential properties of an object. In addition to these mathematical considerations, there are didactic considerations when evaluating a definition for use in a classroom setting. This interplay between mathematical and classroom considerations is at the heart of the present study which investigates 23 PSMTs working with definitions about functions and identifies some of the challenges they encountered.

**Literature Review**

Teachers must draw upon various types of knowledge to effectively interact with mathematical definitions in the classroom. The Mathematical Knowledge for Teaching (MKT) model proposed by Hill, Ball, and Schilling (2008) provides a useful framework for unpacking this sort of teacher knowledge and distinguishes between subject matter knowledge and pedagogical content knowledge (PCK). For instance, teachers must have sufficient PCK to choose age-appropriate definitions, or to respond to student questions about or work with definitions. The present focus, however, is more on PSMTs’ subject matter knowledge; particularly, there is a specialized content knowledge (SCK) which teachers must draw upon to interpret, evaluate, choose, or use definitions. This includes, but is not limited to, an awareness of the features of mathematical definitions discussed above. This type of knowledge overlaps with what Ernest (1999) described as the meta-mathematical knowledge about definitions; i.e, the largely tacit norms and standards of definition use within the mathematical community.

Educational research and perspectives on mathematical definitions complement this outlook on teacher knowledge. Notably, Tall and Vinner (1981) drew a contrast between the concept image as the “total cognitive structure” (p.
152) that an individual associates with a concept and the concept definition as the words used to describe a concept. They further distinguished between a personal concept definition constructed by the individual and formal concept definitions accepted by the mathematics community. They noted that concept images need not be coherent and that concept images can be, and often are, in conflict with concept definitions. Indeed, Vinner and Dreyfus (1989) found this to be the case in their survey of college students’ and junior high teachers’ conceptions of mathematical functions. The 307 respondents were asked to define functions; of them, 82 supplied a Dirichlet-Bourbaki definition of functions as a correspondence between two nonempty sets that assigns exactly one element of the second set (the co-domain) to every element of the first set (the domain). However, when working on other function tasks, these 82 respondents displayed inconsistent behaviour; 56% of them did not use this conception of functions when answering other questions about functions. This was described as the respondents having potentially conflicting cognitive schemes for which concept images and definitions were not mutually supportive; a phenomenon also described as compartmentalization (Vinner, Hershkowitz, & Bruckheimer, 1981). Vinner (1991) advised that, in negotiating these conflicts, the roles of definitions in a mathematics class should be determined by the educational goals.

Other researchers have demonstrated this difficulty interacting with definitions amongst pre- or in-service mathematics teachers. Linchevsky et al. (1992) reported that out of a group of 82 pre-service teachers, all of whom expressed an interest in potentially teaching junior high school mathematics, only 21 were “aware of the arbitrariness aspect of definition” (p. 53). Moore-Russo (2008) found that, among the 14 pre- and in-service secondary mathematics teachers in her study none had any prior experience with definition construction. She reported that definition construction activities helped the study participants develop a deeper understanding of slope. Leikin and Winicki-Landman (2001) reported on professional development activities for secondary mathematics teachers which focused on “what is definition” and “how to define” in order to deepen the participants’ subject matter and meta-mathematical knowledge (p. 63). The researchers noted that many teachers were unaware of the arbitrariness aspect and of the consequences of particular definition choices. Elsewhere, they described the teachers’ strategies for evaluating the equivalence of definitions as either based on logical relationships between the definitions (“the properties strategy”) or by comparing the sets of objects determined by each definition (“the sets strategy”). A third, but rarely-used, strategy was based on referencing a representation of the object (“the representation strategy”) (Leikin & Winicki-Landman, 2000, p. 25). Shir and Zaslavsky (2001) noted inconsistencies amongst mathematics teachers evaluating the equivalence of definitions of squares; the teachers were particularly unsuccessful in evaluating procedural definitions. The 24 teachers in their study considered both mathematical and pedagogical concerns in determining equivalence. Zazkis and Leikin (2008) reported on pre-service secondary mathematics teachers creating definitions of squares; of the 140 definitions provided, about 40% were found to
be inappropriate. Other studies have documented limited understanding of definitions among pre-service elementary teachers interacting with geometrical objects (e.g., Chesler & McGraw, 2007; Fujita & Jones, 2007; Pickreign, 2007).

The studies referenced in the preceding paragraph indicate that many pre- and in-service teachers have deficiencies in their understandings of definitions. However, in general, there has been relatively little attention given to definitions in mathematics education research (deVilliers, 1998; Moore-Russo, 2008; Vinner, 1991; Zaslavsky & Shir, 2005). Notable amongst the few studies which examine student understanding of definitions at the K-12 level is that of Zaslavsky and Shir (2005) who studied conceptions of mathematical definitions among four advanced 12th-grade mathematics students. They noted that students evaluated definitions according to mathematical, communicative, or figurative considerations. That is, in determining if a proposed definition is acceptable, they focused, respectively, on logical concerns, on clarity, or, in the case of geometrical definitions, on some prototypical mental picture(s) of the object being defined. Certainly, there is some overlap between these three types of considerations and the strategies described by Leikin and Winiki-Landman (2000); the “properties strategy” perhaps aligns with mathematical considerations and the “representations strategy” with figurative considerations. Zaslavsky and Shir also reported that, in evaluating definitions, the students justified their responses either by referencing examples or by referencing features or roles of the definitions.

Despite the few studies that directly address conceptions and understanding of mathematical definitions, many have acknowledged K-12 students’ meaningful interactions with definitions as important or essential. De Villiers (1998) wrote of definition construction as “a mathematical activity of no less importance than other processes such as solving problems, making conjectures, generalizing, specializing, proving, etc., and it is therefore strange that it has been neglected in most mathematics teaching” (p. 249). Ouvrier-Buffet (2006) and Harel, Selden and Selden (2006) likewise noted that constructing definitions can foster both students’ and teachers’ productive reflection on mathematics and can deepen teachers’ insight into student understanding. Zaslavsky and Shir (2005) similarly noted that considering alternate definitions can help refine students’ conceptual understanding.

Thus, the limited research on in- and pre-service mathematics teachers’ indicate that many struggle with constructing definitions, evaluating alternative definitions, and using definitions to reason and justify. However, the importance and value of definitions throughout mathematics education has been widely acknowledged both by researchers (e.g., Harel et al., 2006; Vinner, 1991; Winicki-Landman & Leikin, 2000; Zaslavsky & Shir, 2005) and in standards documents (Common Core State Standards Initiative, 2010). Interactions with definitions are built into the secondary mathematics teacher’s role as she/he must evaluate, interpret, and model the use of definitions. Further complicating this, the definitions which teachers encounter in curricular materials often do not foster conceptual understanding or help build a logical foundation for future mathematics studies.
(Harel & Wilson, 2011). Vinner (1991) advises teachers and textbook writers to be cognizant of the “cognitive power that [a] definition has on the student’s mathematical thinking”; something, which he warns, is often neglected (p. 80).

Methodology
Data are comprised of student work on three problems assigned in a capstone course in the mathematics department at a large masters-granting university in the western United States of America. The author of this paper was also the instructor. The course is required for all undergraduate mathematics majors who intend to be secondary mathematics teachers. Of the 23 students enrolled in the course, 19 were in their last semester of undergraduate study (working toward a BSc in Mathematics with an option in secondary education) and 4 had already completed an undergraduate mathematics degree (3 were enrolled in the teaching credential program, 1 was enrolled as a graduate mathematics student). Throughout this paper, the students are referred to as pre-service secondary mathematics teachers (PSMTs) as each intended to (or at least was keeping the option open to) follow that career path. Each of the three problems presented to the students required them to answer questions about definitions of functions. Two of the questions were assigned as homework problems and one was assigned on a take-home exam. Not all students answered each of the three problems. Students were encouraged to work together on homework problems (though there is no data about the extent to which this occurred) and were forbidden to collaborate on the take-home examination.

The three problems were not specifically designed as research instruments. However, in the prior semester of this course, similar problems had yielded interesting results that inspired data collection and a refinement of the problems in an attempt to explore the themes described above. The analysis process was an iterative search for patterns through coding of student responses (Coffey & Atkinson, 1996). The first round of coding recorded the degree to which students successfully completed the required tasks which explored PSMTs’ knowledge of the roles of definitions, the arbitrariness of definitions, and the pedagogic dimension of definition choice. Subsequent rounds were driven by emergent themes. Further details about coding strategies are embedded in the Results section of this paper.

Results
This section explores and is organized around the PSMTs’ responses to the three problems that required them to choose, use, evaluate, and analyse definitions.

Problem 1
On a homework assignment from the second week of the semester, students were required to select a definition for function and then to use that definition to justify why sequences are functions (see Figure 1). At the time of this assignment, some class time had been devoted to discussions of functions and their
representations; however, sequences had not been discussed nor had there been explicit discussion about the roles of mathematical definitions.

This question has 2 parts:

a) Write down a definition of functions. You may use a definition you’re familiar with or you may find one somewhere but, in either case, note the source of your definition.

b) Use this definition to justify why sequences are functions.

Figure 1. Problem 1

The definitions that students chose came from various sources such as websites, textbooks, and dictionaries. Nineteen of the 21 students for whom data were collected described functions as mappings, rules, relations, correspondences, or relationships between sets, variables, or inputs/outputs. In general, students chose correct definitions, which aligned with the Dirichlet-Bourbaki definition, yet their choices, often not in their own words, gave little insight into their understandings of functions or of mathematical definitions. However, the choices that students made are not of primary focus herein; see Vinner and Dreyfus (1989) for a categorization of students’ definitions of functions. Though, it is noteworthy that two of the 21 students’ chosen definitions defined functions as types of equations; these definitions were separated from the others because, even at the high school level, not all functions can be defined by equations. For example, PSMT #5 adapted a definition which was labelled as a “working definition” on a website titled “The Definition of a Function” (Dawkins, 2011). Both students described functions as equations in which an \( x \) is “plugged in”, and a unique \( y \) is the result. Neither of these two students answered part (b) correctly and were unable to identify the domain or what equation would determine the terms of the sequence (which both identified as “the \( y \)’s”).

Indeed, more insight was gained from part (b) as students attempted to use their chosen definitions. In order to determine how or if students used their definitions, focus was given to the type of object which each student defined a function to be. Each definition was in the form, or could simply be reformulated as the form, “A function is a [direct object]”. If the student either explicitly or implicitly referenced that object in part (b) then their response was coded as having referenced the object of the definition. For example, PSMT #13, who defined a function as an “association”, noted that each “element” of a sequence has “a specific number associated with it” and gave a clarifying example. Since this student used a verb-form of the appropriate object, his response was coded as having referenced the object. An implicit use of the object occurred if, for example, functions were defined as relations in part (a) and reference was made to ordered pairs in part (b); two students did this.
Responses were also coded as either correct or incorrect; a correct answer would need to, in some way, identify the domain and range. Table 1 summarizes the results. Only five of 21 students referenced the object of their definition in part (b) and got the answer correct; the majority of correct answers did not make implicit or explicit reference to the definition supplied in part (a). Six students referenced their definition yet gave an incorrect answer. PSMT #1, for example, described functions as types of relations and, though he referenced that definition, his answer for part (b) was incorrect: “Since in (a) we get a set of ordered pairs, a sequence \( \{a_n\} \) is also a set of numbers written in a definite order.”

Another student, PSMT #12, who described functions as relations, also made reference to ordered pairs yet incorrectly and vaguely wrote, “Sequences are functions because each term, which lives in the domain, is paired with exactly one element in the range.”

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Summary of Responses to Problem 1</th>
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<tr>
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</tr>
<tr>
<td>Incorrect</td>
<td>6</td>
</tr>
<tr>
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Seven students had correct answers for part (b) but did not reference the object in their definitions or, as was the case with PSMT #9, chose a definition that was too restrictive and was at odds with an otherwise correct answer in part (b). She supplied a definition, attributed to Dirichlet and historically significant, which, in the context of sequences, is too restrictive, as it required the domain to be defined on an open interval:

\[
y \text{ is a function of a variable } x, \text{ defined on an interval } a < x < b, \text{ if to every value of the variable } x \text{ in this interval there corresponds a definite value of the variable } y. \text{ Also, it is irrelevant in what way this correspondence is established.}
\]

(Luzin (1998) provides an extended discussion of this definition.)

If a PSMT response was coded as “No Reference to Object”, it did not mean that there was no reference to the definition at all. PSMT #19, for example, described functions as “rules” which “link” elements of sets and, though she made no explicit or implicit references to rules in part (b), she correctly identified what was being linked: “any element of [the set of Natural Numbers] can be linked to one and only one element of the sequence”. In general, in the analysis of student responses, it was difficult to separate their knowledge of functions from their habits using definitions. This challenge will be revisited in the Discussion section.
Problem 2

During class, students had, in groups, compared and discussed several definitions of functions. Though not formalized, there was a discussion about what constitutes equivalent definitions. This was intended to be an activity that would both deepen their understanding of functions and help future teachers develop a more critical and analytical view of textbook definitions. A follow-up activity, Problem 2 (see Figure 2), was assigned as homework in the third week of the semester in which two definitions were to be compared. The primary intention was for students to notice that Definition I requires the domain and range to be sets of numbers whereas Definition II is less restrictive.

Of the 22 students who answered this question, 12 said that the definitions were equivalent, one said "yes and no", and nine said that they were not equivalent. Of these nine, only five attributed the lack of equivalence to the sets of numbers required in Definition I. The other four students said the definitions were not equivalent because, as PSMT #3 put it, Definition I “fails to clearly state that each input is assigned to a unique output”. That is, the “definite output” in Definition I was not interpreted as a requirement for a “unique” output. PSMT #20 said that the definitions are “equivalent in one sense and not equivalent in another sense” and was perhaps distracted by the fact that Definition I came from a calculus textbook; he stated that this definition allowed for multivariate functions whereas Definition II did not. The success rate for part (b) was better, though five out of 22 students provided incorrect answers. These five students all erroneously made a connection between the word “special” and the condition that each input is paired with a unique output.

Here are some definitions of functions:

i) “A function is a rule that takes certain numbers as inputs and assigns to each a definite output number.” From *Calculus* by Hughes-Hallet et al. (2006)

ii) “A function is a special type of relation in which each element of the domain is paired with exactly one element of the range.” Relation had been previously defined as “a set of ordered pairs ... The domain of a relation is the set of all first coordinates from the ordered pairs, and the range is the set of all second coordinates of the ordered pairs.” From *Algebra 2* by Holliday et al. (2005)

Answer these questions:

a) Are these definitions equivalent? Explain.

b) What is the word “special” referring to in the second definition?

(Hint: Think about what is meant when we say that a square is a special type of rectangle.)

Figure 2. Problem 2

Of the 22 students who answered this question, 12 said that the definitions were equivalent, one said “yes and no”, and nine said that they were not equivalent. Of these nine, only five attributed the lack of equivalence to the sets of numbers required in Definition I. The other four students said the definitions were not equivalent because, as PSMT #3 put it, Definition I “fails to clearly state that each input is assigned to a unique output”. That is, the “definite output” in Definition I was not interpreted as a requirement for a “unique” output. PSMT #20 said that the definitions are “equivalent in one sense and not equivalent in another sense” and was perhaps distracted by the fact that Definition I came from a calculus textbook; he stated that this definition allowed for multivariate functions whereas Definition II did not. The success rate for part (b) was better, though five out of 22 students provided incorrect answers. These five students all erroneously made a connection between the word “special” and the condition that each input is paired with a unique output.
Problem 3

Problem 3 (see Figure 3) was assigned on a take-home examination distributed in the fourth week of the course. The intention was for students to engage in an authentic activity for secondary mathematics teachers, analysing a definition from a mathematics textbook. The definition was chosen because it was from an actual high school textbook and because it has some interesting issues; namely, the definition is dependent upon choice of representation and the condition “q(x) ≠ 0” may not be presented with sufficient clarity.

Here’s a quote from the *Glencoe Mathematics Algebra II* textbook (Holliday et al., 2005, p. 485):

A **rational function** is an equation of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and “$q(x) ≠ 0$”. (p. 485)

On page 60 of (Cooney, Brown, Dossey, Schrage, & Wittmann, 1996), Mr Washington gives the following (in the chart) as an example of a rational function: $3x^{-3} + 2x^{-1} - 5x^2$.

The equation $y = 3x^{-3} + 2x^{-1} - 5x^2$ is not of the form $f(x) = \frac{p(x)}{q(x)}$.

a) Using the Glencoe definition and an equivalent statement Mr Washington’s function, show that $y = 3x^{-3} + 2x^{-1} - 5x^2$ is a rational function.

b) Answer just one of these two related questions:

i) Change the Glencoe definition so that it is more clear that functions such as $y = 3x^{-3} + 2x^{-1} - 5x^2$ or $y = \frac{5x + 2}{x^2 - 3} + x^3$ are rational functions.

(The more minor the change the better.)

ii) What does a student need to understand to be able to realize that $y = 3x^{-3} + 2x^{-1} - 5x^2$ is a rational function, even though it is not written as the ratio of two polynomials?

a) Consider the Glencoe definition and do both of the following related questions/tasks:

i) Explain why this definition includes the condition that “$q(x) ≠ 0$”.

ii) What is meant by “$q(x) ≠ 0$” in the definition? Keep in mind, as you construct your answer, that $f(x) = \frac{1}{x}$ is a rational function (because the numerator and denominator can both be thought of as polynomial functions) but the denominator of $\frac{1}{x}$ is zero sometimes! Also keep in mind that the author(s) of that definition thought that the condition “$q(x) ≠ 0$” was necessary – so your answer should probably not imply that the condition was unnecessary.

Figure 3. Problem 3
All 22 students correctly answered part (a), which was intended to scaffold the other parts. Eight students chose to answer b) part i), six of these students changed the definition to something like the following:

A rational function is an equation which can be written in the form \( f(x) = \frac{p(x)}{q(x)} \), where \( p(x) \) and \( q(x) \) are polynomial functions and \( q(x) \neq 0 \).

The other two students wrote something that was inaccurate; PSMT #20 wrote something incorrect (“Every polynomial function is a rational function written in the form …”), and PSMT #8 wrote something which was also dependent on representation. Of the 14 students who answered b) part ii) nine responded in a way similar to PSMT #3, writing that a student would need to “realize that \( 3x^{-3} + 2x^{-1} - 5x^2 \) can be rewritten as the quotient of polynomial expressions”. Four listed the skills that a student would need to manipulate the expression and one student noted that the field of rational expressions is closed.

The verbosity of part c) was an attempt to give the PSMTs enough clues to reason through why “\( q(x) \neq 0 \)” is in the definition. Only six out of 22 students noted that this condition meant that \( q(x) \) could not be the zero polynomial. This may provide more insight about the communicative power of the textbook’s definition than about the PSMTs’ knowledge. For example, a definition of rational functions is communicated with greater clarity in a college-level algebra textbook:

A rational function is a function that can be put in the form \( f(x) = \frac{a(x)}{b(x)} \), where \( a(x) \) and \( b(x) \) are polynomials, and \( b(x) \) is not the zero polynomial (McCallum et al., 2010, p. 407)

Furthermore, this definition has the advantage of paralleling a common definition of rational numbers (i.e., A rational number is a number that can be put in the form \( \frac{a}{b} \), where \( a \) and \( b \) are integers, and \( b \) is not zero. The other 14 students answered that the “\( q(x) \neq 0 \)” condition in the Glencoe definition was included as a domain restriction.

Discussion

It was often difficult to determine the causes of student errors on the three problems. Were their incorrect answers the result of deficiencies in subject matter knowledge about functions, in general, more meta-level, knowledge about mathematical definitions, or in some combination of both? There are, however, some instances of relatively greater clarity. For example, 12 PSMTs correctly answered part b) of Problem 1 (Why is a sequence a function?) but seven of them did not reference the object which they defined function as even though they were explicitly asked to use their definition. Though this was perhaps an imperfect way of determining if a PSMT had “used” the definition, the results align with what Vinner and Dreyfus (1989) reported; they found that more than half of the students who gave a Dirichlet-Bourbaki definition of function did not
use that definition when answering other questions about functions. Their description of this as a gap between concept image and concept definition is certainly relevant to the PSMTs who made these errors.

However, it is also likely that many of these PSMTs lacked the appropriate meta-mathematical knowledge about the roles of definitions in mathematics. Student work on Problem 1, in particular, may indicate that the relationship between this meta-mathematical knowledge and subject matter knowledge may be both complicated and context-dependent. For example, unlike the students who correctly justified why a sequence is a function, the majority of students (6 out of 9) who incorrectly answered that part of Problem 1 actually did reference the object of their definition. That is, they knew and conformed to that convention of “using a definition” but fell short on their content knowledge. Another notable example may be PSMT #5’s choice of a definition which was clearly marked as “a working definition”; she may not have understood the limitations of such a definition. It would be worthwhile to further examine and explicate the relationship between subject matter knowledge (e.g., functions) and meta-mathematical knowledge (e.g., the role of definitions).

A similar question can be formulated about the relationship between pedagogical content knowledge (PCK) as conceptualized in (Hill, Ball, & Schilling, 2008) and knowledge of the role of definitions. By the nature of their craft, mathematics teachers interpret, model the use of, and build upon definitions in their instruction. They also may need to reconcile equivalent (or, at times, non-equivalent) definitions of the same object that appear in different curricular materials or in student work. The three-capstone problems explored PSMTs using, choosing, comparing, and evaluating definitions; for mathematics teachers, these are didactic actions, which could help or hinder student learning. For example, on Problem 3, only six out of 22 PSMTs correctly interpreted the “$q(x) \neq 0$” condition in a definition of rational functions from a high school textbook. On part b) of Problem 2, despite the hint, a quarter of PSMTs did not know what the word “special” meant in the context of “a function is a special type of relationship”; this is similar to what Chesler and McGraw (2007) noted about pre-service elementary teachers’ difficulty interpreting the phrase “a special kind of”. Furthermore, some PSMTs in this study had difficulty choosing a definition of function that could support required tasks (as with PSMT #9 on part b) of Problem 2).

Indeed, choice of definition mattered. Of the five students who described functions as types of “rules” in Problem 1, only one of them made any attempt to describe how a sequence can be thought of as a type of “rule”. Perhaps defining functions as a different, more clearly defined type of object would have better supported the follow-up task. Of the 11 PSMTs who referenced the object in their definition in part b) of Problem 1, five of them used a verb-form of the object to explain why sequences are functions (e.g., if a function is an association then a sequence “associates”). It is possible, and worthy of study, that definitions which accommodate this action-object connection help narrow the gap between concept image and concept definition or even between action and object levels of understanding (Dubinsky & McDonald, 2001).
Unfortunately, the definitions that appear in textbooks do not always support harmony between concept image and definition. Harel and Wilson (2011), in reviewing a high school textbook, lamented that, “it is difficult to learn from this text what a mathematical definition is or to distinguish between a necessary condition and a sufficient condition. Students are also expected to discover definitions given pictures as hints” (p. 826). This was offered as one of many examples of “the sorry state of high school textbooks”. Indeed, there is a difference in clarity between the two textbook definitions for rational functions which were presented above; the definition from the high school textbook (Holliday et al., 2005) had issues with two essential properties of a good definition: invariance under choice of representation and non-ambiguity. Choice of definition can support or undermine both teaching and learning.

Indeed, the PSMTs’ performances on the three tasks highlight the notion that the definitions which these future teachers encounter in their classrooms, and how the PSMTs interpret and use these definitions, will be impactful. As Vinner (1991) wrote,

Definition creates a serious problem in mathematics learning. It represents, perhaps, more than anything else the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition. (p. 65)

Moreover, as exemplified by the textbook definition which was examined in Problem 3, definitions in curricular materials often may not help teachers and/or students resolve this conflict. Even the United States’ Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010) section on functions echoes the definition equivalence issues encountered in Problem 2. The CCSSM begin this section by declaring that functions “describe situations where one quantity determines another” yet, on the same page, they provide an example of a function in which a state name determines its capital city (emphasis added, p. 67).

Many of the PSMTs’ difficulties on the three problems may be broadly, and perhaps vaguely, described as a lack of recognition of details or nuance. For example, on Problem 2, only five of the 22 respondents correctly noted that the two definitions of functions were not equivalent because one of them defined the domain and range more restrictively. The results reported herein indicate that many PSMTs may have difficulty with choosing, interpreting, comparing and evaluating definitions, which appear in secondary mathematics curricular materials. It seems likely that these are, at least in part, symptoms of a lack of flexibility and expertise in interpreting and using mathematical definitions. The PSMTs who did not acknowledge that “definite output”, in Problem 2, was communicating the same thing as “unique output” (1) did not have the flexibility to make sense of this alternative word-choice, and (2) may not have had the knowledge about definitions to properly assess the equivalence of the two definitions. In sum, pre-service secondary mathematics teachers may benefit from thoughtful modelling of and explicit attention to definition use by teacher educators. This task would be aided by a deeper understanding of how
knowledge about mathematical definitions interacts with or is subsumed by subject matter knowledge and pedagogical content knowledge.

References


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A Lesson Based on the Use of Contexts:
An Example of Effective Practice in Secondary School Mathematics

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The importance of using real-life contexts in teaching mathematics is emphasised in many policy and curriculum statements. The literature indicates using contexts to teach mathematics can be difficult and few detailed exemplars exist. This article describes the use of real-life contexts in one New Zealand Year 11 algebra lesson. Data included a video recording of one lesson and the teacher’s reflections on the lesson. Analysis of the lesson revealed the importance for its success of the ways in which the learning tasks and their contexts were introduced, ongoing referral to the contexts, consolidation of prior mathematics learning, and teacher questioning. The lesson described illustrates how meaningful links to real-life contexts can be developed to promote mathematical understanding, how a balance between focusing on the mathematics and the context can be achieved, and that these require careful planning. The lesson example and its analysis indicate that awareness of the complexity of implementing context-based mathematics learning is important for those who promote or want to implement context-based mathematics teaching, including policy makers, teacher educators, and teachers.

Background

Teaching mathematics through context-based examples is endorsed by professional mathematics teaching bodies (e.g., National Council of Teachers of Mathematics, 2000) in official curriculum documents (e.g., Ministry of Education, 2007), and through curriculum exemplars (e.g., Australian Curriculum, Assessment and Reporting Authority, n.d.). Scrutiny of literature about the use of contexts in teaching mathematics reveals that teaching in real-world contexts can be problematic (e.g., Inoue, 2009; Verschaffel, Greer, & De Corte, 2000) and the productive use of contexts requires pedagogical skill (Doorman et al., 2007). Few examples of context-rich mathematics lessons have been documented to date. However, detailed descriptions of effective context-based lessons together with teacher commentary about the lesson have the potential to contribute to understanding the complexities of using real-life contexts within mathematical instruction.

A range of meanings for the term context exists in the mathematics education literature; in this paper we use the term to refer to real-life situations. The literature examining the use of word problems will be used to highlight issues that are pertinent to the discussion of the use of contexts.

Shulman (1986) argued for the exemplification of principles of good practice through dissemination of accounts of successful mathematics teaching practice. In order to illustrate effective use of contexts, this article reports on the analysis
of a lesson in which students were required to develop and make sense of algebraic relationships that were found by exploring real-world contexts. We begin by describing literature most closely related to using contexts to help the students understand mathematical ideas. This is followed by information describing the New Zealand setting of the lesson and then its analysis. We conclude with a discussion of the implications for those involved in the provision of mathematics education of using context-based teaching.

**Contexts**

There are a variety of ways in which contexts can be used in mathematics instruction. One approach is to use “pure mathematical tasks that have been ‘dressed up’ in a real-world context that for their solution merely require that the students ‘undress’ these tasks and solve them” (Palm, 2009, p. 3). Many textbook problems exemplify this approach. Other tasks can require more extensive investigation by students such as those that more faithfully represent the mathematical problems people solve in situations outside school (Organisation for Economic Co-operation and Development, 1999).

Typically, teachers use links to contexts to motivate students and support the learning of mathematics content, rather than to develop the ability to explore real-world contexts through the use of mathematics (Gainsburg, 2008). Most commonly, problems involving contexts are presented as direct applications of mathematical techniques. In these cases the students merely need to follow the procedures developed in recent lessons (Llinares & Roig, 2008) rather than have to grapple with the realities of the context.

A study of teaching eighth grade in seven industrially developed countries found that the proportion of problems with real-world connections posed in mathematics classrooms varied between the countries from 9% to 42% (Hiebert et al., 2003). Mathematics teachers in the Netherlands made greater use of contexts than in other countries in this study (Hiebert et al., 2003). The Dutch advocates of the Realistic Mathematics Education [RME] approach to mathematics (e.g., Gravemeijer & Doorman, 1999) argue for extensive use of “experientially real” (p. 111) contexts as vehicles for the development of mathematics. The RME approach includes requiring students to grapple with contextual problems and in the process of doing so, creating mathematical tools for the solving of problems. Contextual problems are chosen carefully to match the learning needs of the students and to potentially enable students to create mathematical models that can then be used as objects to assist the development of mathematical thinking (Gravemeijer & Doorman, 1999; van den Heuvel-Panhuizen, 2003).

While mathematics educators continue to advocate context-based mathematics instruction, examples in textbooks and classrooms often employ a surface level approach in that the use of contexts may not be associated with a strong focus on the development of mathematical thinking (Doorman et al., 2007). Because context-based problems are most often framed using words, the literature identifying particular difficulties that students have in solving word problems is now discussed.
Greer (1993) tested 13 and 14-year-old students with context-based word problems and found that frequently no consideration of context was used when answering the questions. For example, over 90% of students attempted to solve the following problem by direct proportion.

A girl is writing down names of animals that begin with the letter C. In one minute she writes down 9 names. About how many will she write in the next 3 minutes? (p. 245).

Similarly, Belgian 10 and 11-year-old students typically ignored the real-world considerations when solving word problems and used the numbers to calculate answers that were unrealistic (e.g., Verschaffel, De Corte, & Lasure, 1994). Such results have been replicated in many similar studies with pre-service teachers (e.g., Verschaffel, De Corte, & Borghart, 1997) and primary school students (e.g., Palm, 2008; Reusser & Stebler, 1997; Yoshida, Verschaffel, & De Corte, 1997).

Verschaffel et al. (2000) argued that the students ignoring the details of the context was a consequence of their past histories in the mathematics classroom.

[Students’ responses to word problems that apparently disregard considerations of reality should be interpreted as showing that they are adhering to conventions learned and reinforced over a considerable period of time (p. 66).

In related work with university students, Inoue (2005, 2009) found that fewer than half of the student responses took real-life considerations into account when solving problems such as “John’s best time to run 100 metres is 17 seconds. How long will it take him to run 1 kilometre?” (Inoue, 2005, p. 70). Questioning students who had calculated answers without reference to contextual factors revealed that some spontaneously indicated that they would answer differently in a real-world setting; however, a greater proportion required further prodding to recognise that their response may not be correct. Inoue (2009) concluded that actions that could assist students to incorporate the practicalities of the context when solving problems include discussing problems where the context must be taken into account to create realistic solutions, and discussing the assumptions that need to be made in specific situations before attempting to generate solutions.

In assessment items students may encounter word problems that require them to take account of particular realistic considerations, but penalise those students who take more general realistic considerations (Boaler, 1994; Cooper & Harries, 2002). English 11-year-old students were asked “There is a lift in the office block. The lift can carry up to 14 people. In the morning rush, 269 people want to go up in this lift. How many times must the lift go up? (Cooper & Harries, 2002, p. 7). In conventional testing the answer 20 would be the only answer seen as correct as the student has recognised that after division by 14, the fractional answer needs to be rounded up. Later in the same questionnaire the students were asked to comment on the answers: 19.21, 25 and 15 times. The question asked was to think about how each answer may have been calculated and to consider whether or not each answer was a feasible solution to the context
problem. The answer of 25 could be valid if the lift was not completely full each time, and the answer of 15 recognises that some people used the stairs instead of the lift. When presented with the question in this manner, some students were prepared to consider a broader range of real-world considerations than would normally be rewarded in mathematics classes, and the researchers argue that word problems used in teaching and assessment should include emphasis on realistic considerations in general and not just on very specific mathematical considerations (Cooper & Harries, 2002). Palm (2008) conducted research with Swedish 11-year-old students and found that requiring solutions to word problems to be acted out increased the likelihood that students would use their real-world knowledge when solving the problems.

The literature regarding the difficulties associated with the use of word problems has largely been informed by research that has been conducted through purpose-designed tests rather than having been centred in classrooms. Next we consider the literature reporting students’ and teachers’ views about the classroom implementation of context-based teaching. This literature indicates that many students support the use of carefully chosen contexts in mathematics teaching, and that secondary school mathematics teachers find it difficult to develop suitable contexts.

The majority of the British secondary school students reported in Boaler’s (2000) study stated they found mathematics classes boring and the content meaningless. In contrast to these experiences, when describing subjects they enjoyed, they commented on the meaningfulness and links of subject matter to their world. For students in the initial years in an Australian secondary school, Attard (2010) found that lessons that integrated the mathematics content with material from other subjects increased student engagement; however, some students wanted lessons focusing directly on mathematical content to be taught alongside the context-based lessons. A British study found that 13 and 14-year-olds listed the use of interesting contexts when asked to identify features of ‘fun’ lessons; however, they regarded some contexts used in mathematics lessons as unappealing and “did not see through fence designing or table manufacturing an opportunity for practising certain algebraic skills that are transferable to contexts that are personally relevant to them” (Nardi & Steward, 2003, p. 352).

Gainsburg (2008) found that 80% of American middle school and secondary teachers reported that they typically sourced “real world connections” (p. 201) in their mathematics teaching from their own ideas or experiences, and the teachers reported that many of the examples presented in textbooks were inadequate. Although the majority of the teachers in Gainsburg’s study indicated that they used real-world connections at least weekly, many of the connections were brief and extended context-based activities were seldom used. When asked to explain why they didn’t make greater use of real-world connections, the most common responses were that this approach was too time consuming and that resources and training were needed to assist them to make such connections. It is possible that the teachers’ stated need for training may be related to feelings of lack of success with teaching mathematics using context-based problems.
In a study that attempted to provide such training, Canadian secondary school mathematics student teachers visited workplace sites to observe and interview staff, and were required to develop classroom activities based on this experience. The prospective teachers found it difficult to identify mathematics in the workplace and, when identified, to incorporate such mathematical ideas into teaching sequences at an appropriate level for their classes (Nicol, 2002) which may indicate that successful development of real-life contexts in mathematics lessons requires deep knowledge of curriculum content, practice and experience.

The Study

The New Zealand Ministry of Education’s (2007) rationale for studying mathematics and statistics includes the statement “Mathematics and statistics have a broad range of practical applications in everyday life, in other learning areas, and in workplaces” (p. 26) which indicates an expectation that students will be able to apply the mathematics that they learn in context-based situations. Additionally, there is an expectation that mathematics teaching at all levels will be set in a “range of meaningful contexts” (Ministry of Education, 2007, fold-out pages). Year 11 is the first year in which New Zealand students are assessed towards national qualifications. At this level the assessment of mathematics is done through problems set in real-life or mathematics contexts (e.g., New Zealand Qualifications Authority, 2011). This is exemplified in a sample examination paper (New Zealand Qualifications Authority, 2011) that set graphs and relationships problems in the context of using a sausage sizzle to raise funds.

At the beginning of the 21st century the Ministry of Education undertook an initiative to enhance the teaching of mathematics in New Zealand schools, specifically focused on building teachers’ knowledge of students’ development of mathematical ideas, and using assessment information to further understanding of their students’ mathematical progress (Ministry of Education, 2001). In secondary schools the initiative focused on the teaching of Years 9 and 10 mathematics (e.g., Harvey & Higgins, 2007). In each school one teacher was given a time allowance to lead the professional development of the mathematics-teaching colleagues in their school and these leaders were supported with training and external mentoring. The leadership role included running workshops, appraisal of teaching, and mentoring peers.

Teachers involved in the initiative reported a moderate increase in the use of real-life contexts in Year 9-11 classrooms and attributed this change in practice to the professional development (Harvey & Averill, 2009). The lesson described in this paper was drawn from a study aimed at investigating and reporting examples of effective mathematics teaching in senior secondary schools that took part in the initiative. Full ethical approval was granted for the study. This article focuses on describing the elements that appeared to lead to the successful use of context-based mathematics teaching in one Year 11 mathematics lesson. The lesson was video-recorded and teacher reflections were audio-recorded and transcribed.
Participants and method

The lesson was based in one all girls’ secondary school with approximately 1300 students and serving a mid to low-income urban community. Craig (pseudonym), the assistant to the head of the mathematics department at the secondary school, understood the purpose of the wider investigation for which the data were collected. As an experienced teacher with responsibility for leading the Secondary Numeracy Project (Ministry of Education, n.d.) in his school, Craig volunteered to teach a lesson and was given several weeks’ notice of the timing of the recording. He was videotaped teaching a class of 22 Year 11 students of average ability in mathematics. Effective teacher-student relationships appeared to be in place and the students seemed confident they could learn well in the teacher’s class, happy to seek assistance and to offer answers.

The lesson was videotaped from the back of the room so that the evidence of student involvement and actions could be gathered. Immediately after the lesson Craig was audio taped as he reflected on the lesson with prompts from the interviewer. The lesson was viewed many times by the lead author to build understanding of the actions and purpose of the teacher. The audiotape of teacher reflections was transcribed and analysed. Themes that emerged from the lesson and teacher reflections are reported.

The videotaped lesson

Two contexts were explored in the lesson. The first context, the focus of the first part of the lesson analysis, led to finding relationships between the length of a Bailey bridge (one quickly assembled from prefabricated steel girders) and the number of triangles used to construct its sides. Craig used photographs he had taken in a nearby region, and compared that region’s terrain and climate with those of the school, and used photographs of Bailey bridges to introduce the mathematical tasks (see Figure 1).
Local links were exploited in the second context by exploring a situation from within the school, that of carpeting a senior staff-member’s office. The carpet consisted of plain carpet squares surrounding a square central feature of a rose motif. The motif could have been purchased in a range of sizes, and the task for the students was to calculate the number of plain tiles required for different dimensions of the motif. This context gave rise to a quadratic relationship.

Examples of effective practice during the lesson (introduction of the task and context, ongoing referral to the context, review of previous knowledge, questioning, consolidating and extending, and interactions with individuals) are discussed in turn below. Transcripts of parts of the lesson and interview are used to illustrate the features. Craig is denoted by C, and S indicates a contribution by a student. Sections in italics provide the authors’ commentary on the lesson.

Introduction of the context and the task

Each context was introduced in a way that preserved the links to the real-life situation with a focus on a broad range of ideas rather than a cursory treatment of the context merely in order to introduce the mathematical ideas (see Figure 2).

The Bailey bridges context was introduced using photographs Craig had taken in a nearby region.

C: As you drive down the West coast, you cross a lot of bridges. Does anyone know why you cross a lot of bridges on the West coast?

S: A lot of rivers.

C: Great, there are a lot of rivers. Why are there a lot of rivers?

S: Because of the mountains.

C: Yes it’s got to do with mountains. What is it to do with mountains?

S: Because the water comes from them.

C: Actually you’re right. You know when we have nor-westers here and they are hot and dry, does anyone know what happens on the West Coast?

Craig discussed orographic rainfall, the frequency of floods and the need to use Bailey bridges to provide temporary access, after access is lost because of flooding. The focus moved to discussion of the construction of the sides of Bailey bridges which he simplified to being made up of congruent equilateral triangles with each side being 10 metres long. The mathematical task given was to create three different representations of the number of triangles required to construct the sides of the bridges for different lengths of span. The representations required were: a table; a graph; and an algebraic rule.

Figure 2. Craig’s introduction of the context-based problems
Craig supported the students in their exploration of the tasks by circulating around the desks and privately questioning individuals to help them continue and extend their exploration to help solve the problems. Craig’s introduction and development of the context (see Figure 2) illustrates the way that he encouraged, and built on, student contributions to set up the context-based investigations. Student work produced after the bridges context had been introduced and prior to class discussion showed that many students were able to explore the context algebraically (see Figure 3). The progress of students through this task varied and many students completed the table and graph but did not find the algebraic relationship.

<table>
<thead>
<tr>
<th>Length of bridge (L) (metres)</th>
<th>Number of triangles (T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2</td>
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<tr>
<td>20</td>
<td>6</td>
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<tr>
<td>30</td>
<td>10</td>
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<td>50</td>
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<tr>
<td>60</td>
<td>14</td>
</tr>
<tr>
<td>18</td>
<td>22</td>
</tr>
</tbody>
</table>

Relationship between length of bridge (L) (in metres) and number of triangles (T) \( T = \frac{4}{10}L - 2 \)

Figure 3. Replication of initial student answers to the Bailey bridge task developed from video footage
Ongoing referral to the context

Greer (1993) and Inoue (2009) are among others who have highlighted problems associated with students ignoring the context when dealing with mathematical tasks. Once Craig had introduced the contexts and mathematical tasks, a factor that appeared to contribute to the success of the lesson was his emphasis on referring back to the context when students were involved in solving the tasks, rather than focusing solely on the mathematical aspect of the tasks. The following extract (see Figure 4) shows how Craig introduced, and worked with the class to resolve, the mathematical complexities arising from the Bailey bridge context. Also apparent is the frequency with which he emphasised how the bridge context affected possible answers.

After 15 minutes of student independent work Craig asked two volunteers to come to the board: one recorded her table of results on the board, while the second provided Craig with points to plot to create the graph. He used this student work as a basis for discussing progress on the problem. Initially he dealt with bridges that spanned less than 10 metres, before asking the students to consider a bridge spanning a 25 metre gap.

C: We have to be careful with what we just did. I’m going to ask you something now. How many triangles are required for a bridge that is 25 metres long?

S: 8

C: Good Teri. How did you come up with 8?

The student explained how she got the point off the graph.

C: Here’s what Teri did and I want you to tell me whether practically if this is actually OK.

C: If we have a bridge that is 25 metres long. It means – and you went like this didn’t you Teri – you need 8 triangles (Craig shows how reading off the graph gives a value of 8.)

Replication of Craig’s graph showing how a 25 metre long bridge would appear to need 8 triangles to construct the sides:

Relationship between length of bridge and number of triangles needed

Continued next page.
During discussion of the bridge example, Craig structured questions and required students to consider the answer to reveal that graphs had to be treated with care since not all values that could be read from the graph would be feasible in the context (e.g., non-whole number values). This enabled him to discuss the modification of the table to model the situation more accurately (Table 1). The ongoing reference back to the context and the preparedness to expose the complexities of the situation appeared to help give this teaching episode an authentic flavour.

Craig wove the context and the mathematics together throughout the lesson (see Figure 4). Through careful questioning he built on student contributions to identify aspects of the initial student answers that did not accurately portray the context. Investigating the number of triangles required when the bridge length was 25 metres illustrated that the algebraic relationships that had been developed were only valid when the bridge length was a multiple of the length of the sides of the triangle. Similarly, examining the values generated by the

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C: Show me on the bridge. If I have a bridge that is 25 metres long, we need 8 triangles. Show me on the bridge how I would arrange my 8 triangles. Show me on the bridge how Teri’s 8 triangles can be organised.

S: Has to be an odd number.

C: Oh has to be an odd number. Good call. Why does it have to be an odd number?

Pause

S: Because otherwise it will be ending like this (gesture with arm indicating that the bridge would be incomplete at one end.)

C: Uh oh. Watch this please, 10 metre bridge, 20 metre bridge. This is how long it needed to be. But we forgot to put this triangle on the end. What is going to happen? Can you see that?

Replication of Craig’s diagram showing an attempt to make a bridge using 4 triangles on each side:

Actually, although these points are lined up in a straight line, the in-between things can’t be read off it, can they? In which case, actually, the table makes a lot of sense, doesn’t it?

Building on this, Craig went on to show how to provide intervals for the domain, so that the number of triangles required could be read off directly (see Table 1).

Craig returned to a discussion of the equation of the line. Through questioning, the slope of the line was linked to the extra triangles required as the length of the bridge increases. Questioning was used again to find the y-intercept of the graph and to establish the fact that a y-intercept of -2 has no relevance to the context.

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Figure 4. Development of the mathematical relationships with links to context
relationship when the bridge length was zero illustrated that the relationship was not valid when considering a bridge of length zero metres.

Table 1.

<table>
<thead>
<tr>
<th>Length of bridge (L) (metres)</th>
<th>Number of triangles (T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>≤ 10</td>
<td>2</td>
</tr>
<tr>
<td>≤ 20</td>
<td>6</td>
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<tr>
<td>≤ 30</td>
<td>10</td>
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<tr>
<td>≤ 40</td>
<td>14</td>
</tr>
<tr>
<td>≤ 50</td>
<td>18</td>
</tr>
<tr>
<td>≤ 60</td>
<td>22</td>
</tr>
</tbody>
</table>

Reflecting on the interview after the lesson, Craig explained:

I am now trying to do as much as possible to make it real so that the students have something to hook onto and to support the move from number to algebra and to generalisation ... I think we’ve learnt here not to be too quick about jumping away from the context and jumping in to doing tables and rules but allowing them to maintain the context. The context actually gives all the clues as to how it fits.

Reviewing of previous mathematics knowledge

Through the lesson Craig took opportunities to review concepts from topics the class had met previously. These included: the concepts and calculation of the gradient and y-intercept of the relationship (see Figure 5); substitution of coordinates that satisfied the relationship to check that the equation was correct; and showing how to substitute values into an expression that included a fraction. During whole class teaching, Craig took voluntary contributions from many students and directly questioned others to check for student understanding. Engaging two students to help him in constructing the table, graph, and algebraic relationship on the board also served as a review of prior knowledge. Reviewing this knowledge provided support for those students who had found the work challenging and enabled them to make sense of the lesson.
C: With straight lines we have been talking about some other things, haven’t we? What other things have we talked about?
S: We need a y-intercept.
C: Yes we need a y-intercept. Brianna, where is the y-intercept?
S: Um . . . Um.
C: With this line here, where does it cut the y-axis? At the moment I haven’t, have I? How am I going to get the y-intercept on this graph?
S: You need to extend the line.
C: You want me to take my ruler, and extend it out a bit further. There it goes, down there. And it’s going to cut. Um. Weird? -2
C: By the way Stevie, you answered the question and so did several other people but I am going to pick on you for a moment. Stevie, -2. How did you come up with exactly -2?
S: On my graph each of my little squares is 2.
C: So you have read it straight off the graph.
S: Yep.
C: You’ve said there it is (gesture to graph). Has anyone come up with another way of coming up with -2?
S: Could it be from the first number that you started with?
C: Have a look at this. (points to the pair 10, 2 on the table) Think about this for a moment, You’ve got it Liana I can see your eyes light up. Where do we get -2?
S: Because it goes up in fours.
C: Because it goes up in fours so how do I get -2 on the y-axis?
S: Because you went back 4.
C: Because I went backwards 4. So it’s actually where we start.
   If I have a zero bridge to make. Zero length; I need -2 triangles. Doesn’t quite do it, does it?
C: So sometimes where we start on the y-axis actually has no relevance to the situation whatsoever. Can you see that? It’s got absolutely no relevance to the y-intercept but it’s going to help us. Let’s have a think about how it helps us.

Figure 5. Lesson extract showing consolidation of the concept of y-intercept through questioning
Questioning

When working with students as individuals and when conducting discussion, Craig probed student understanding through questioning, and used the answers to develop the discussion. This use of questioning is illustrated in each of the lesson extracts (see Figure 5 in particular). Craig’s purposeful use of questioning to invite student contributions and to inform him of how the students were thinking was made explicit in his reflection:

In order for me to know what is going on I need their cues and I need to ask lots of questions ... The lesson might go in lots of different directions based on how the students think about it rather than the way I think about it.

Consolidating and extending

In order to consolidate and extend the learning, Craig introduced a second context-based task (see Figure 6). This task again required the students to find and represent an algebraic relationship; however, in this case the context gave rise to a quadratic relationship. The process of creating a table through systematically calculating specific values was followed by drawing a graph, and attempting to find its equation. Students were required to consider the constraints of the situation in deciding the appropriate domain for the relationship.
Craig informed the students that the motif could be obtained as any sized square. The task was to represent an algebraic relationship between the edge length of the rose motif and the number of plain tiles in three ways. Students who asked Craig to supply dimensions for the graph were asked to work those out for themselves by considering the information in the context.

The interactions that Craig had with the students in this time included:

- re-explaining the task to get students started,
- assisting students to fill in their table of values
- challenging a student to find the equation of the curve, and
- reminding a student of the restrictions on the domain of the graph caused by the dimensions of the room.

This is exemplified by the following interaction which started with Craig working with one student, but soon all four students in the group were participating:

C I’ve got a question for you. Does the graph carry on and on, and where should the graph stop? I think your graph has gone too far. Why has it gone too far?

S Because of the room.

C Yes, the room is only 7 squares wide so you can’t have a rose that is 8 squares by 8 squares.

The first student showed understanding, another student appeared to show partial understanding, so Craig directed his next question to her:

C Did you understand what I just said?

S The room is only 84 squares so the rose can’t go bigger than that.

C The room is only 7 squares wide so we can’t possibly have a graph that is 8 squares wide (hand gestures to support this).

At this point all four students showed understanding and Craig continued to the next part of the lesson.

Figure 6. Lesson extract showing consolidation and extending of task, and interactions with individuals
The carpet task consolidated the work done in the bridges task as it required the students to again show the algebraic relationship in three different ways. The students appeared to progress through this task at a greater rate than they did through the bridge task. The carpet context gave rise to a more demanding algebraic relationship than the linear relationship which arose in the bridges context, and hence use of this context extended student skills. Craig elaborated how both contexts consolidated material that had been taught in previous lessons:

In previous lessons we have worked from number to algebra and to graphs to create patterns, and we have used graphs to see what happens. In the carpet example, the taking away squares had a meaning. It wasn’t just that if you have $x$ squared, it is a parabola, but the concept of the square related to real-life.

Interaction with individuals

During the time that students worked individually on the carpet task, Craig circulated and assisted. In addition to the silent observations that he made of student work, Craig had 13 different interactions with students in the 11 minutes that the class worked on the problem. Each interaction was tailored to the progress and needs of the individual student. In each interaction he was unhurried, gave his full attention to the student, and posed questions and prompts to engage the students in thinking. Craig’s reflection indicates that Craig deliberately used the time when the class was working as individuals to give individually focused feedback and support to the range of students.

… knowing conceptually where they’re at and their ability to move, it’s not a class anymore, it’s a group of individual students and they are all at different places ... although today’s lesson wasn’t necessarily differentiated delivery, I can’t just do one size fits all ... the individual conversations that I had with individuals around the room gave me opportunities to do different things with the examples ...

Craig reported the value of the professional development in relation to pedagogy that enabled identifying specific learning needs:

I think the professional development has given us much greater understanding of the different needs of the individual girls. The graph work has enabled me to ask lots of different questions and problems in different ways around the room.

Discussion

The bridge and carpet contexts provide excellent examples of Inoue’s (2009) conditions in that solving the problems requires use of the practical aspects of the problems’ contexts. In the observed lesson, both of the contexts were introduced in an unhurried way. Craig’s sharing of non-mathematical information about the contexts and use of photographs enabled an holistic treatment of the contexts.
The Bailey bridge context was introduced through questions and discussion of terrain and climate that could also be studied in other curriculum areas, an illustration of successful integration of content from other subjects to enhance the lesson as described by Attard (2010). Whereas this setting up of the lesson may have been described as time consuming by the participants in Gainsburg’s (2008) study, in this case it may have contributed to the effectiveness of the lesson.

The lesson exhibited elements of teaching called for by Inoue (2005) in that links to the context were maintained throughout the teaching sequence and the relationships developed were tested against the context, and discussed as mathematical solutions that were only true under certain conditions. Craig’s deliberate focus on the potential to read erroneous values from the graph may have assisted the students to be vigilant in taking real-world considerations into account when solving problems, thus avoiding difficulties noted by researchers (e.g., Greer, 1993; Verschaffel et al., 1994; Verschaffel et al., 1997) regarding the context merely being used to introduce the content and the final answer not necessarily relating to the richness of the context. While keeping the context under consideration, the teacher ensured that the key focus of teaching and learning mathematics (i.e., consolidating students’ algebraic skills) was maintained throughout the lesson.

This weaving together of the context and the mathematics associated with it showed the potential to both support the learning of mathematics and to give insight into how mathematics can be used. Simplifications to the context were made to make the context more mathematically manageable and although the context appeared to be from the real world, it is likely that the approach to finding the number of triangles required differed from what would actually occur in a bridge-building situation in that it is unlikely that Bailey bridges are supplied in the form of prefabricated triangular sections. Rather than striving to use contexts where the mathematics done in the classroom is the same as that used in the real world, it may only be necessary for teachers to use contexts in such a way as to be mainly faithful to the context.

The lesson illustrates how Craig was able to use the context-based lesson to build on mathematical ideas that the class had met earlier. The increased mathematical complexity of the second context enabled students to consolidate and extend skills developed during work on the first context. Craig’s careful questioning during class discussion and his work with individuals enabled him to build understanding of the progress of the students, which he used to inform his teaching.

The presentation and discussion of this lesson serves as one example of effective practice. The rich episodes may serve as models of context-based mathematical problems for other practitioners. It is possible that the classroom learning environment, established teacher-student relationships, passion for the subject, and depth of knowledge which enabled Craig to develop this lesson based on these contexts, may be key elements in the confident and successful implementation of this lesson.
Conclusions and recommendations

Given that teaching mathematics through context-based examples is endorsed by many professional mathematics teaching and policy bodies as well as by students, and the challenges teachers report regarding teaching mathematics using contexts, it is essential that effective context-based mathematics teaching is explored and described. Difficulties for teachers and students have been associated both with context-based teaching approaches and use of word problems. The videotaped lesson illustrates that weaving together of the mathematics and the context can lead to purposeful mathematics teaching. It is unlikely that students in studies conducted by Nardi and Stewart (2003) would have nominated the context of Bailey bridges as engaging; however, the lesson by Craig illustrates that a context which may seem to be very dry in the eyes of teenagers can be used as an interesting and engaging lesson when executed by a skilled and passionate teacher.

This article adds to the literature by providing analysis of an example of the use of contexts to support effective teaching that highlights key features that appeared to contribute to its success through:

- introduction of the context and the task;
- ongoing referral to the real-life context;
- timely reminders of previous mathematical knowledge necessary for the task;
- adept questioning;
- consolidation and extension of mathematical ideas; and
- effective teacher-student interactions.

To assist with curriculum implementation and to develop students’ perceptions of the relevance of mathematics to everyday life, it is recommended that further lessons illustrating the productive use of contexts be documented. Furthermore, the lesson and teacher reflection indicate that teachers can be encouraged and assisted to develop context-based mathematics teaching through professional development. Implications of this research for pre and in-service education and resource development include ensuring teachers possess a bank of tasks linked to contexts known to be realistic, purposeful, of high interest and effective in supporting students’ mathematical learning and understand ways of maximising the effectiveness of such tasks when implementing them. Further research is required to ascertain whether this, or similar lessons, can be replicated by other teachers with the same level of success. However, documenting this lesson provides a model teachers and teacher educators can use towards developing expertise in context-based mathematics teaching.

Acknowledgements

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References


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Pressure to Perform: Reviewing the Use of Data through Professional Learning Conversations

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Australian Catholic University  The University of Sydney

With increased accountability attached to students’ results on national testing in Australia, teachers feel under pressure to prepare students for the tests. However, this can lead to shallow teaching of a narrowed curriculum. An alternative approach involves using data to identify common errors and misconceptions, discussing strategies aimed at building understanding of important mathematical ideas as well as students’ confidence in answering context-based mathematics questions. This study explored the use of a learning model based on professional conversations about national testing results as well as school-based assessment data with junior secondary mathematics teachers in one school. The teachers identified the learning needs of students and chose to implement mental computation and estimation approaches as well as a strategy to address the literacy demands of numeracy test items to support student learning before and after the NAPLAN test. An analysis of the professional learning model identified approaches to enhance both student learning and teaching practice.

Background

In Australia, the debate surrounding mathematics and numeracy achievement has been similar to that experienced elsewhere. There is a growing recognition of the need for greater numeracy proficiency and that early intervention provides the best chance of success for children at risk of failure. The concern about numeracy by Australian governments was first highlighted in the National Literacy and Numeracy Plan (DETYA, 2000), which provided a framework for improving the literacy and numeracy outcomes of all students. This plan embraced the development of the national benchmarks for students in Years 3, 5 and 7, as well as the need for assessment and reporting against these benchmarks. Until recently, each state and territory in Australia collected student achievement data for the Federal Government. Concern about the proportion of students not meeting the minimum national benchmark standards (Curriculum Corporation, 2000) has continued with large investments by governments to address the needs of students at risk.

To better standardise the monitoring of student achievement the National Assessment Program in Literacy and Numeracy (NAPLAN) was introduced in 2008 (DETYA, 2000). The same tests in literacy and numeracy are now administered nationally to all students in Years 3, 5, 7 and 9. Testing early in the school year potentially provides diagnostic information to teachers about their students’ performance in mathematics topics common to all states and territories (Curriculum Corporation, 2006).

Whether we approve of a national testing regime or not, this level of accountability is in place for the foreseeable future with pressure on school
principals and teachers to improve results. While the information may be useful after the results are released, teachers of Years 3, 5, 7 and 9 are experiencing increased pressure early in the school year to prepare students for the test. Principals, school systems personnel and parents are scrutinising the results to determine whether schools and their teachers are ‘measuring up’. Public comparisons between ‘statistically similar’ schools are now possible with the Federal Government sponsored My School website which presents statistical and contextual information about schools.

The results from the NAPLAN assessments are reported in individual student reports to parents, as well as school and aggregate reports with substantial information including results for each item and for each student. The school reports enable teachers to analyse the results for each year group to determine which items appear to be understood and which are problematic. In addition, school data can be compared to the Australian student data. The information is useful to address common errors and misconceptions as well as to aid planning and programming of future learning (Perso, 2009). Rather than abandon good pedagogical practices and have students individually practise test items, NAPLAN items can be used to address key issues in students’ understanding and develop appropriate quality-teaching approaches (Anderson, 2009).

The purpose of the project reported here was to engage teachers in professional learning conversations about using evidence from their own NAPLAN results to identify their students’ needs and collaboratively develop pedagogical practices which research has shown to be beneficial in building understanding. In particular, this paper describes and analyses the outcomes of a professional learning program conducted in one school by addressing the following research questions.

1. What strategies did teachers choose to use to support student preparation for NAPLAN and how was this different to previous practice?
2. Did the professional learning support have an impact on student learning and on teaching practice (including attitudes)?

Literature Review

Teaching to the Test!

High-stakes testing has been criticised for encouraging teachers to limit the curriculum to what is assessed (Abrams, Pedulla & Madaus, 2003) and resulting in the “corruption of indicators and educators” (Nichols & Berliner, 2005, p. 1). While the types of testing being conducted in some states in the United States of America in recent years could be considered higher stakes than the NAPLAN testing in Australia, systems, principals and teachers feel under pressure to prepare students for the tests and achieve good results, particularly given the publishing of data on the My School website. The pressure to raise scores has the potential to distort teaching and learning but there are ways teachers can support students’ preparation for high-stakes tests without detracting from real learning
Miyasaka (2000) identified five types of test preparation practices that support student learning and improve achievement – teaching the mathematics content, using a variety of assessment approaches, teaching time management skills with practise in test-taking, reviewing and assessing content throughout the year, as well as fostering student motivation and reducing test anxiety. In addition, Marzano, Kendall and Gaddy (1999) found knowledge of test vocabulary and terminology improves student performance.

Compulsory testing of students in Years 3, 5, 7 and 9 in Australia has the potential to focus teachers’ efforts on preparing students for the test by using past papers for practise and limiting learning to technical support such as how to fill in answers (Nisbet, 2004). However, balancing this narrow approach is the potential benefit of identifying students’ strengths and weaknesses with data informing planning and teaching. In a survey of 56 primary schools, Nisbet (2004) reported two thirds of the schools in his study used data to identify topics causing difficulties but only 40% of teachers used the results to identify individual students who were having difficulty. Further, only 22% used the results to plan their teaching. The low proportion of primary school teachers using the data to inform teaching and learning represents a missed opportunity and there is little evidence that secondary mathematics teachers are analysing NAPLAN data in meaningful ways.

An Alternative Approach – Engaging Teachers in Professional Conversations about Data

There is an alternative approach to ‘teaching to the test’ but the evidence above suggests teachers require support to analyse and interpret the data and consider alternative practices, to address common student misconceptions and difficulties (Anderson, 2009). Gulek (2003, p. 42) refers to the need for “school practitioners to become assessment literate in order to make the maximum use of test results” and Thomson and Buckley (2009) describe the potential of test item analysis to inform pedagogy. It should be noted the test preparation practices that we are advocating are aimed at improving students’ knowledge, skills and understanding of numeracy and mathematics and not at artificially increasing students’ test scores. Unlike Dimarco (2009) who criticises giving any attention to such tests, we believe teachers’ professional standing does not need to be compromised by considering how NAPLAN items can be used to improve student learning.

Planning professional learning opportunities for teachers in relation to new assessment regimes, or new approaches to teaching and learning, requires consideration of several factors which impact on teachers’ practice in classrooms such as teachers’ knowledge, beliefs and attitudes (Wilson & Cooney, 2002). Rather than change in beliefs and attitudes preceding change in practice, Guskey’s (2002) model of teacher change proposes professional learning precedes the implementation of new ideas in classrooms, which when implemented could lead to a positive change in student learning outcomes, and
subsequently, a change in teachers’ beliefs and attitudes. This model suggests that teachers need to try new ideas and witness positive student outcomes before they fully embrace such approaches.

Following Earl and Timperley’s (2009) research into the use of evidence to inform practice and building on Guskey’s (2002) model of teacher change, the professional learning model developed for this project aimed to engage secondary mathematics teachers’ in rich conversations about data including NAPLAN and whether NAPLAN items provide opportunities for learning and teaching. As noted by Earl and Timperley these conversations required more than just looking at their students’ results.

... conversations that are grounded in evidence and focused on learning from that evidence have considerable potential to influence what happens in schools and ultimately enhance the quality and the efficiency of student learning. We have also come to the conclusion that having conversations based on data in educational contexts is very hard to do. It is hard because productive use of evidence requires more than just adding data to the conversation; it involves a way of thinking and challenging ideas towards new knowledge. (p. 2)

The research design was based on a model of “productive evidence-based conversations” (Earl & Timperley, 2009, p. 3), which has particular qualities (see Figure 1). The conversations involve having an “inquiry habit of mind”, with discussions about a range of relevant evidence where relationships are respectful but allowing for challenge. The approach taken in this study involved a group of teachers from the same school discussing the evidence from the previous NAPLAN Numeracy test for their students, asking questions about the data informed by classroom-based knowledge of their students, identifying topic areas requiring further investigation, and developing strategies to address the particular learning needs of their students.

Figure 1. Processes for evidence-informed conversations (Earl & Timperley, 2009, p. 3)
**Pedagogical Approaches to Improve Students’ Engagement with Context-based Mathematics Questions**

National testing agendas can provide an opportunity if we use test items to assist students who have difficulty reading and interpreting mathematical text, to further develop students’ thinking skills, and to analyse common errors and misconceptions, frequently presented as alternative solutions in multiple-choice items. One practical approach to ‘teaching to the test’ while maintaining sound pedagogical practices is to use NAPLAN items as discussion starters so that students develop number sense, adopt new problem-solving strategies, and build confidence and resilience. Hence, teachers’ professionalism need not be compromised by national testing agendas as long as they adopt teaching strategies, which use the data in meaningful ways to inform their planning and teaching.

Research has advocated several teaching practices that have the potential to target particular aspects of students’ difficulties in mathematics and numeracy. While many strategies could be considered, in this project, the following strategies were chosen based on sources of students’ errors; mental computation, estimation and number sense; and the literacy demands of context-based mathematics questions.

Common student misconceptions have been identified as a major source of errors. For example, Ryan and Williams (2007, p. 23) use the term “intelligent overgeneralization” to refer to students’ predisposition to create inappropriate rules based on experiences. Some common generalisations include: multiplication makes bigger; division makes smaller; division is necessarily of a bigger number by a smaller number; and longer numbers are always greater in value. Figure 2 presents a NAPLAN Numeracy item where this type of over-generalisation occurs with few students selecting the correct answer of 22.

![Figure 2](image)

What is the answer to 6.6 + 0.3?

A) 0.022  B) 0.22  C) 2.2  D) 22

**Figure 2.** An item from the 2008 Year 7 non-calculator allowed numeracy NAPLAN test

A common fraction misconception occurs when area is not the feature students identify in regional models of fractions (Gould, Outhred & Mitchelmore, 2006). The “number of pieces” interpretation is a common response. This research explains the responses to the 2008 Year 7 NAPLAN item shown in Figure 3 where only 28% correctly selected the last option. The fact that three parts (though unequal) were shaded obviously prompted most students to see it representing three quarters. The most popular response was option c.
1. Mental Computation, Estimation and Number Sense

In dealing with misconceptions like these, Anderson (2009) points out that encouraging students to apply reasoning about numbers to evaluate answers can be a challenge. She argues that one way to support the development of students’ thinking strategies is to use test items that focus on mental computation, estimation and number sense (McIntosh, Reys & Reys, 1997). While students are frequently reluctant to estimate, this is an important first step. Options in multiple-choice items may often be eliminated after considering whether the solutions are reasonable. Anderson proposes that after students have estimated the answer teachers can pose questions such as the following.

1. What strategies could you use to check the solution?
2. What would the question need to be to obtain each of the alternative answers?
3. What happens when you multiply a whole number by a number less than one?

An estimation focus allows test items to provide a source of meaningful mathematical discussion.

2. Literacy Demands of Context-based Mathematics Questions

The contextual nature of many NAPLAN items and the associated language implications often lead to claims that these tests are more comprehension than mathematics. However, interpreting mathematical situations in context is what numeracy is all about. Hence, we claim the contextual nature of the items is at the heart of numeracy and deserving of special attention. Further, it seems pointless to pursue repetitive symbolic manipulation exercises to address poor responses to contextual items.

Newman (1983) developed an error analysis protocol to analyse student responses to contextual items. She identified five levels of difficulty (Table 1). Most errors occurred in the second and third levels of ‘comprehending’ and ‘transforming’ the text into an appropriate mathematical strategy, not applying the symbolic procedure. By translating each of the levels from Table 1 into a question for students, teachers are able to determine their first level of difficulty (White, 2005).
Table 1

<table>
<thead>
<tr>
<th>Newman’s Error Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reading the question</td>
</tr>
<tr>
<td>Comprehending what is read</td>
</tr>
<tr>
<td>Transforming the words into an appropriate mathematical strategy</td>
</tr>
<tr>
<td>Applying the mathematical process skills</td>
</tr>
<tr>
<td>Encoding the answer into an acceptable form</td>
</tr>
</tbody>
</table>

Methodology

One school with a high NESB enrolment and a history of NAPLAN results below state and national average volunteered to participate in the project. Ten teachers of Years 7 and 9 (12 classes in total) were involved. The Professional Learning Model had five stages.

Stage 1 involved teachers collecting data about their own students’ ability in NAPLAN style items. In May each year, Years 7 and 9 students complete two 32-item test papers for Numeracy, one with and one without the use of a calculator. With the teachers, the authors used the 2008 NAPLAN numeracy test results for the school to identify specific areas of the curriculum requiring review and consolidation. Items from the 2008 NAPLAN papers in these areas were used to compile a short pre-test for diagnostic purposes for each of Years 7 and 9 consisting of 5 non-calculator and 5 calculator items. Though the results from 2008 were those of the current Year 8 and 10, not the cohorts involved in the project, they were still considered reflective of teaching approaches in the school because the teachers were the same. Furthermore, the value of the selected items would be gauged by how the students responded to them.

Teachers administered the tests in early March, slightly more than two months before the NAPLAN tests in May 2009. Each teacher corrected their class responses to reveal the number of students selecting each option in multiple-choice items or the common solutions to the free-response items. In the six Year 7 classes, only one class had more than 50% of total responses correct in the calculator and non-calculator pre-tests (same class). In the six Year 9 classes, two had more than 50% of total responses correct in the non-calculator pre-test and no class had more than 50% of total responses correct in the calculator pre-test. These data support the items chosen as being areas of difficulty for the students.

Stage 2 involved a one-day meeting (two months before the NAPLAN tests) between the teachers and the authors. The day consisted of professional learning conversations to review the students’ pre-test responses, consider the key mathematical ideas and misconceptions in the tasks, compare this to data collected using school-based assessment procedures, and explore a range of possible research-based teaching approaches identified by the authors. Teachers
were encouraged to pose questions about the data. They also contributed suggestions about the mathematical issues they saw as relevant and strategies they believed could be used to address the identified student difficulties. From these conversations, a list of possible strategies was jointly constructed. Each teacher then nominated one or more to implement in their general teaching as well as specifically with targeted NAPLAN items.

Stage 3 where teachers implemented their chosen strategies occurred over the next two months prior to the NAPLAN tests and continued beyond the tests. In this stage, lesson observations by a trained research assistant who is a qualified mathematics teacher were conducted.

Stages 4 and 5 involved further professional learning conversations about the effectiveness and learning from the project in October 2009 and September 2010. Data collected in Stage 4 involved teacher questionnaires and interviews from the original 10 teachers and in Stage 5, eight teachers provided data about their use of their nominated strategies and reactions to the professional learning. An interview with the principal also occurred in Stage 5. In addition, student learning was analysed by comparing NAPLAN results for the Year 7 and 9 students in 2009 with their Year 5 and Year 7 results respectively in 2007 aligned with the corresponding New South Wales data.

Results and Discussion

The results are reported in two sections. The first looks at the preferred teaching strategies identified and used by the teachers. These data confirm pedagogical practices and identify opportunities for teacher change supported by the professional learning model. The second section reports on student learning. Given there was only two months of teacher implementation between the professional conversations and the NAPLAN test, these data are seen as some indicator of the professional learning model’s success, but not in any way conclusive on its own.

Teaching Strategies

Teaching strategies data were collected in Stages 2, 3, 4 and 5. Stage 4 questionnaire and interview data are reported before Stage 3 lesson observation data. This order allows for a better comparison of the observations against the teachers’ reporting.

Stage 2

During the professional learning discussions, the teachers reported giving their students practice on NAPLAN type items before the tests. However, there was no use of actual school data to inform their planning and practice to support student learning, or approaches to build desired understanding in their general teaching. When each pre-test item was discussed, teachers were asked to estimate the proportion of the school cohort correctly answering each item. They tended to overestimate and were frequently surprised by the low proportion of correct responses.
From looking at the mathematics involved in the identified areas and the incorrect answers chosen by students, the teachers and authors chose eight strategies as potentially useful for improving students’ mathematics and numeracy proficiency. These strategies contained a mix of general teaching strategies and some which are appropriate when conducting class discussions based around NAPLAN style items. The teachers indicated during the professional learning discussions in Stage 2 that they intended to focus on the areas of concern and use strategies to address these from the professional learning day not only in their general teaching, but also to use NAPLAN items as stimuli for constructive class discussion.

**Stage 4**

After implementation, teachers completed a short questionnaire where they ranked the strategies in their preferred order of usefulness. Table 2 shows the results from the eight teachers who responded to the questionnaire. Scores were calculated by assigning 1 to the first choice, 2 to the second choice and so on, hence the lowest score indicates the most preferred strategy and the highest score indicates the least preferred (scores could range from 8 to 64).

Their ranking must be interpreted realising they may not have tried some at all and only chose from the specific strategies they did implement. None the less, the attractiveness of the ones they did choose to try is a factor in determining effective strategies, which promote good pedagogy and are seen as comfortable for use by teachers.

Table 2

*Preferred strategies as selected by the teachers to address students’ numeracy learning needs*

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Promoting interpretation of context-based mathematics questions using Newman’s error analysis questions</td>
<td>20</td>
</tr>
<tr>
<td>2. Developing efficient mental computation strategies</td>
<td>29</td>
</tr>
<tr>
<td>3. Using estimation strategies with all questions</td>
<td>36</td>
</tr>
<tr>
<td>4. Eliminating possibilities in multiple choice questions</td>
<td>41</td>
</tr>
<tr>
<td>5. Checking reasonableness of answers</td>
<td>43</td>
</tr>
<tr>
<td>6. Developing visualisation strategies in geometry (2D to 3D and 3D to 2D representations)</td>
<td>47</td>
</tr>
<tr>
<td>7. Identifying irrelevant information in mathematics questions and problems</td>
<td>52</td>
</tr>
<tr>
<td>8. Developing strategies for answering open-ended questions</td>
<td>58</td>
</tr>
</tbody>
</table>
Newman’s questions and mental computation emerged as the most popular choices with 7 teachers ranking Newman’s in the top 3. The final teacher ranked it last – so among 7 of the teachers, the preference was strong. Teachers’ comments revealed some believed they were already using such strategies. For example:

The majority of the strategies I already used prior to the PD except for the Newman’s Method.

Others found the opportunity to consider new approaches was beneficial to both their teaching and student learning as shown by the comments below from three different teachers.

Identified their need for mental computation and to read all the question.

I found the Newman’s questions are very useful. I went through that with all my classes.

Newman’s strategies – worked – ensuring read all of the question.

Three teachers’ comments suggest their knowledge and understanding of the potential of NAPLAN items and data have improved:

It gives me an idea of which kind of questions students found hard so I would focus more on those areas.

Next year I intend to show students a variety of strategies for approaching the numeracy tests. I will also target some specific areas of knowledge that students in the past have had difficulties with.

The pre-test identified common areas of weakness in my class. Common misconceptions were easily identified by the alternate choices students made when choosing the answer.

Professional dialogue between teachers and the researchers enabled the identification of a range of strategies for implementation in classrooms, an approach acknowledged as successful by the following three teachers’ comments:

It was good to gather with colleagues and to discuss alternate teaching strategies.

It was especially good to get the chance to do practical maths questions and be the “student” ourselves.

Focusing on mental computation, visualisation, Newman’s as part of each unit, from beginning of the year – encouraging this as a normal part of doing Maths.

Even though teachers indicated they already used some of the teaching strategies in regular lessons, their awareness of the strategies and ability to identify when they were using them increased. Further, they had not used them as a focus for supporting NAPLAN preparation nor in taking items and through these strategies making them a source of constructive class discussion rather than
right/wrong drill and practice. The data here show they were still using some of the learning from the professional conversations three months after the NAPLAN tests.

**Stage 3**

Table 3 shows the strategies identified in the professional learning conversation, which were planned for and actually used by the teachers in the eight observed lesson. Some teachers used more than one strategy.

The data set here is not large but still allows for some inference about the classroom practices of the participating teachers. The top two strategies (Newman’s analysis and mental computation) figured prominently but a specific focus on estimation did not. All three teachers who used Newman’s analysis actually went through the steps with the class. Visualisation, though not an original popular choice, was used as the basis for three of the lessons. The specific test strategy of eliminating possibilities in multiple choice questions was planned but not widely used indicating lessons became more involved with the mathematics and appropriate procedures rather than test based strategies. As one teacher said to her class:

Does the answer actually fit the question? Have confidence in your ability.

**Table 3**

*Teachers’ planned and observed aimed at addressing students’ numeracy learning needs*

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Planned</th>
<th>Observed</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Promoting interpretation of context-based mathematics questions using Newman’s error analysis questions</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>2. Developing efficient mental computation strategies</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3. Using estimation strategies with all questions</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4. Eliminating possibilities in multiple choice questions</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>5. Checking reasonableness of answers</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>6. Developing visualisation strategies in geometry (2D to 3D and 3D to 2D representations)</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>7. Identifying irrelevant information in mathematics questions and problems</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>8. Developing strategies for answering open-ended questions</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Four of the lessons involved NAPLAN items as a source of class discussion and group work. In all these lessons, teaching went beyond right/wrong answers and looked at procedures. Three involved group work while one was more teacher centred. The visualisation lessons were three of the four, which did not
use NAPLAN items. The teachers chose other activities to involve groups of students building objects given specific properties (for example, can you build the shape which looks like this from the front and has the most or least number of cubes). In one visualisation lesson, one group was noted as definitely not being engaged. The level of student engagement was commented on positively in six of the other seven lessons.

In summary, the teachers in all eight lessons planned for one of the identified strategies and in seven of the lessons implemented one or more. In all lessons, the focus was on procedures and strategies, not just right answers. Newman’s analysis appears to have provided a new lens for dealing with mathematics in a context. The focus on mental computation supported student thinking rather than memory based approaches.

**Stage 5**

The data from stage 5 gave some mixed messages. Table 4 shows the eight teachers responses about effects of involvement in the project in September 2010.

These results suggest the project had some effect on the teachers over a year after participation but the effect does not seem emphatic. Further, when asked what they did differently now as a result of participation, five of the eight said ‘nothing really’. The three who nominated some change identified ‘targeted review of questions students found difficult in previous years’; ‘problem solving including Newman’; ‘start with 5 questions (Naplan style)’. Except for Newman, these changes do not reflect the intended focus on strategies and the mathematics involved.

**Table 4**

*Teachers’ responses about effects of involvement in September 2010*

<table>
<thead>
<tr>
<th>Statement</th>
<th>Agree</th>
<th>Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>My involvement in the NAPLAN project has impacted on the strategies I use in my general mathematics teaching</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>Because of the project, I felt more confident in preparing my classes for NAPLAN this year</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>Since the NAPLAN project, the classroom environment in mathematics lessons promotes students’ willingness to engage more with word problems</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Because of the project, I am more aware of the types of errors and misconceptions students have in learning mathematics</td>
<td>5</td>
<td>3</td>
</tr>
</tbody>
</table>

However, when asked about the strategies, which had been identified in Stage 4, the eight teachers indicated sustained substantial use as shown in Table 5.
Table 5

Frequency of use of identified strategies aimed at addressing students’ numeracy learning needs

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Regularly (weekly)</th>
<th>Sometimes (monthly)</th>
<th>Never</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Promoting interpretation of context-based mathematics questions using Newman’s error analysis questions</td>
<td>3</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>2. Developing efficient mental computation strategies</td>
<td>7</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3. Using estimation strategies with all questions</td>
<td>7</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4. Eliminating possibilities in multiple choice questions</td>
<td>2</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>5. Checking reasonableness of answers</td>
<td>6</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>6. Developing visualisation strategies in geometry (2D to 3D and 3D to 2D representations)</td>
<td>2</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>7. Identifying irrelevant information in mathematics questions and problems</td>
<td>1</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>8. Developing strategies for answering open-ended questions</td>
<td>5</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

The frequency of use is consistent with both the nominated preference of strategy in Stage 4 and observed strategies in Stage 3. Estimation here matches Stage 4 nominations and suggests the Stage 3 non-observation of estimation was just a chance occurrence. Newman figured less regularly which can be explained by the strategy being very specific to contextualised questions which may not be a focus in class much of the time. Checking reasonableness, providing irrelevant information and open-ended questions are more prominent than in the earlier stages. The high level reported for mental computation and estimation shows a positive shift to engaging with working mathematically rather than with rote routine procedures.

The identification of specific strategies suggests more had been taken from involvement in the professional learning conversations than was indicated by the questionnaire. Interviews with the eight teachers confirm the stronger influence of the professional learning model. One teacher commented that ‘PD offered was an intense time’ – this included visits for lesson observation as well. Reflecting on changes to their own practices brought comments like:

Personally use Newman’s method as it works well

Too much and too little information encourages students to think
Hands on resources used including centicubes

Newman’s all the time in all subject areas (including RE)

Logic questions such as are all squares parallelograms? Are all parallelograms squares?

Other comments also indicated that the teachers were making more use of data and trends in the data. For example, all the teachers identified for themselves that tables and graphs as an area needing attention.

The interview with the principal indicated she was very pleased with the whole professional model saying there was evidence of a positive change in classroom practice. She cited an intensive intervention to work on school programs as one example, but more importantly their own awareness of strategies to use and a heightened consciousness and control of their use of these strategies. In addition, she acknowledged teachers increased understanding of ways to use data to inform decision-making. She stated:

The project has energized them and got the discussion going. There is more energy and discussion around teaching and learning. The teachers now seem to have the language to talk about these things. Staff members were ignoring NAPLAN but now they are starting to engage with it. We need to have strategies and practices based on, and informed by, data.

While mixed, the Stage 5 data suggest teachers are more aware of ways to implement different pedagogical practices in their classroom even though they may be of the opinion that they were doing so all along. In particular, the language they use to describe their practice would seem to indicate that they have in fact moved to a higher level of awareness about their own practice and the potential of evidence to inform their planning and programming.

**Student learning**

Student data from 2007 and 2009 at the sample school for each student were compared to the total New South Wales data. The New South Wales data set was readily available with the schools data and was seen as an appropriate standard to use for comparison. The group of students used in the comparisons was exactly the same group in both 2007 and 2009. The mean gain for each group was calculated by averaging the entire individual gains. The data need to be interpreted realising that an expected mean improvement from Years 5 to 7 is 50 points and Years 7 to 9 is 40 points and that the professional learning model intervention only occurred for two months prior to NAPLAN in 2009. The impacts of other unknown factors cannot be ignored.

To address the impact of other factors, Cohen’s coefficient for effect size has been calculated for each group. Effect size for statistically significant findings attempts to “quantify the importance or substantive influence of the mean differences observed” (Kline, 2004, p. 132). Tables 6 and 7 show the results of comparing the mean gains using a one tailed t-test along with the associated Cohen’s coefficient.
Table 6
Year 7 mean gains for the school compared to the NSW means from 2007 to 2009

<table>
<thead>
<tr>
<th></th>
<th>Yr 5-7</th>
<th>p value</th>
<th>t value</th>
<th>1 tail</th>
<th>Cohen’s d</th>
</tr>
</thead>
<tbody>
<tr>
<td>2007 – 2009 N=123</td>
<td>Sample School Mean Gain 66.08</td>
<td>2.508</td>
<td>0.008</td>
<td>0.23</td>
<td></td>
</tr>
<tr>
<td></td>
<td>NSW Mean Gain 55.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The results in Table 6 show that the gains by the sample school compared to the state are significant at the 1% level for Year 7. The results in Table 7 show that the gains by the sample school compared to the state are significant at the 2% level for Year 9.

Table 7
Year 9 mean gains for the school compared to the NSW means from 2007 to 2009

<table>
<thead>
<tr>
<th></th>
<th>Yr 5-7</th>
<th>p value</th>
<th>t value</th>
<th>1 tail</th>
<th>Cohen’s d</th>
</tr>
</thead>
<tbody>
<tr>
<td>2007 – 2009 N=126</td>
<td>Sample School Mean Gain 45.5</td>
<td>2.138</td>
<td>0.017</td>
<td>0.19</td>
<td></td>
</tr>
<tr>
<td></td>
<td>NSW Mean Gain 38.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Cohen’s coefficient for effect size (0.23 for Year 7 and 0.19 for year 9) supports that the intervention alone is not likely to be responsible for the statistically significant differences in mean achievements. The sample values of 0.19 and 0.23 for Cohen’s d Effect Size suggest that the mean differences are of a small rather than a large substantive difference accounting for a small proportion of the variation. A Cohen’s d of “0.2 or greater corresponds to a small-sized mean difference” (Kline, 2004, p. 132). A coefficient of 0.5 represents a medium effect size. This proportion of the variation between the means, however, is statistically very unlikely to have occurred by chance alone (Kline, 2004).

The analyses of the mean differences reported in the NAPLAN comparative data do support a positive impact of the professional learning model on student learning but the professional learning can only be viewed as one factor impacting on the gains. Analysis of the model needs to be much wider than just statistical measures of student improvement.

Another source related to student learning is the teacher comments about the students’ approach to solving problems and their overall attitude to engaging with mathematics. The eight teachers who responded to questions about what strategies worked and how these impacted on student attitudes indicated the chosen teaching approaches encouraged students to be more confident. Three different teacher comments from the eight are:
Using a variety of strategies empowers students with ways to better respond to set questions in class tests and exams.

Students gain confidence when they feel that they have been well prepared for tests and they perceive that it is important they try their best. They need to get used to the language used and the style of questions as well as improving their numeracy knowledge.

Students seem a bit more confident and are more inclined to have a go now.

Both the quantitative and qualitative data support an improvement in learning though this improvement cannot be directly linked to involvement in the project alone. Importantly, students’ experiences were positive and they gained confidence in tackling NAPLAN-style questions.

**Conclusions**

There is evidence that engagement in the professional learning model by teachers coincided with some positive student learning outcomes. The school thus saw the project as successful. The mix of using clearly identified strategies in general class teaching with NAPLAN items as a stimulus for discussion, appears to be an effective pedagogical combination. The results here are consistent with Martin’s (2003) observation that showing students test items and discussing strategies for thinking about questions and responses promotes student confidence and resilience, and enables a greater sense of student control over their learning. In addition, the professional learning model aimed to improve the assessment literacy (Gulek, 2003) of teachers, and develop their attitudes and beliefs about the potential of using NAPLAN data for planning and teaching. Using data to inform teaching certainly became apparent as part of teaching practice where no indication of doing so previously was evident. However, there is no conclusive evidence about the way the data were used and the degree to which such use impacted on teaching practice and student learning.

Overall, teachers’ comments (especially ones immediately after the implementation) supported the efficacy of the professional learning model. Interestingly, teachers’ general comments in interviews one year on indicated they felt involvement had no real impact on their teaching practice and confidence but principal comments and their identification of their teaching strategies suggest there were long term changes to their practice. We conclude that a professional learning model like this does have a positive impact on mathematics learning and teaching but that unless conversations are revisited regularly, can mean awareness of the impact is lost in the day-to-day hustle and bustle of school life and teacher involvement in a range of initiatives.

The problem which can arise with high stakes testing where comparative results are in the public domain is that the tests become the curriculum and teaching strategies become restricted to improving test performance regardless of whether any actual learning takes place. Such a position being taken by
teachers is understandable, but, as shown in this paper, not the only approach available to improving test performance. National testing programs provide challenges and opportunities for mathematics teachers. One challenge is to focus on the diverse learning needs of students while preparing them for national testing early in the school year.

In the project reported here, a professional learning model was implemented with a fair degree of success with two teams of teachers (those teaching Year 7 and those teaching Year 9), which aimed to turn NAPLAN into a teaching resource and a means of taking control of the testing agenda.

Ideally, the opportunity to form collaborative professional learning teams is desirable where some expertise about research into teaching mathematics can be accessed. However, the professional learning model described above need not be dependent on such external input and can be engaged with individually or in small groups within a school. As noted by Perso (2009, p. 11), “teacher reflection on student results becomes a powerful tool to guide the teaching of mathematics for numeracy by students”. In fact, the results indicate more teacher input rather than less is desirable.

The model presented here is not advocating ‘teaching to the test’, rather it supports the notion that there is much to learn from using data available from a school’s NAPLAN results and items to develop discipline knowledge as well as pedagogical content knowledge about important mathematics concepts. Nor does the approach presented here advocate national testing as the most desirable approach to assessing students’ knowledge, skills and understanding. Teachers best carry out assessment as they talk to and observe their students (AAMT, 2008). However, given the reality we face and the fact that many teachers do feel pressure to actively prepare their students for the tests, the model presented here offers some ideas for a constructive way to do so. Future development of the model is therefore indicated and, in particular, the results suggest looking for ways to increase long-term positive beliefs, awareness of the impact of the model and ownership by teachers.

References


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Supporting Mathematics Instruction with an Expert Coaching Model

Drew Polly

University of North Carolina, Charlotte

This article presents findings from a study in which the author served as an expert coach and provided ongoing support to four elementary school teachers related to employing standards-based pedagogies in their mathematics classrooms. In addition to assisting teachers, the author examined which supports they sought and the impact of them on mathematics instruction. Data were collected through participant interviews, classroom observations, and anecdotal notes. Inductive qualitative analysis indicated that teachers who sought more in-class support and co-teaching opportunities showed more enactments of standards-based pedagogies than teachers who asked for resources and support outside of their mathematics classroom. Implications for models of teacher support related to mathematics instruction are provided.

Introduction

Most professionals agree that teachers require worthwhile professional learning experiences in order to effectively implement reform-based pedagogies that embody current reforms in mathematics education (Bobis, 2010; Darling-Hammond, Wei, Andree, Richardson, & Orphanos, 2009; Higgins & Parsons, 2010). Numerous empirical and theoretical recommendations have been made about effective teacher learning (cf. Desimone, Porter, Garet, Yoon, & Birman, 2002; Heck, Banilower, Weiss, & Rosenberg, 2008; Loucks-Horsley, Love, Stiles, Mundry, & Hewson, 2009). Effective professional development designers focus on issues related to student learning (Yoon, Duncan, Lee, Scarloss, & Shapley, 2007), giving teachers ownership of their learning (Loucks-Horsley et al., 2009), addressing specific content and pedagogies (Desimone et al., 2002); providing opportunities for teachers to reflect and learn from their own practice (Loucks-Horsley et al., 2009), allowing teachers to collaborate with their colleagues and others (Putnam & Borko, 2000; DuFour, Eaker, & DuFour, 2005), and embedding activities in a comprehensive, ongoing project (Heck et al., 2008). Best practice approaches call for learner-centered approaches to professional development (Polly & Hannafin, 2010; National Partnership for Excellence and Accountability in Teaching [NPEAT], 2000).

While these theoretical and empirically based recommendations for professional development have promise, professional development research includes mixed results, especially in the area of mathematics. In a large-scale professional development study with middle grades mathematics teachers, researchers found that the professional development positively influenced teachers’ use of learner-centered practices in some cases, but with little evidence
of influence on student learning outcomes (Garet et al., 2010). In the seminal Cognitively Guided Instruction project, teachers spent the first year demonstrating no change in their instruction or beliefs, but in the second year of professional development started to drastically shift their teaching (Carpenter, Fennema, & Franke, 1996). Researchers in the Rational Number Project (Cramer, Post, & del Mas, 2002) found that professional development only changed teachers’ practice when it was paired with classroom-based support during and immediately after lessons. In summary, professional development that is content specific and develops teachers’ content knowledge in conjunction with teachers’ skills related to teaching with standards-based pedagogies can positively influence teachers’ instruction (Carpenter et al., 2006; Cohen, 2004).

Supporting Mathematics Teachers through Coaching

One type of professional development that has been empirically associated with gains in teacher performance and student achievement is site-based (or job-embedded) professional learning experiences (Joyce & Showers, 2002; Killion & Harrison, 2006). This approach focuses work between teachers and content experts, which could include instructional coaches, specialists, facilitators, administrators, or lead teachers who provide support with planning, teaching, assessment, and other duties related to instructional activities (Campbell & Malkus, 2010). In literacy, instructional coaches have had a positive influence on teachers’ use of reform-based pedagogies and student achievement (Mraz, Algozzine, & Kissell, 2009; Sailors & Shanklin, 2010). In mathematics, little research has been conducted to examine the influence of coaches on student achievement (Campbell & Malkus, 2010; Campbell & Malkus, in press). With the growing demand for the use of coaching models in mathematics classroom, the need for research evidence to support the efficacy of this approach is necessary.

Part of the need for research relates to the interaction between coaches and teachers in schools. Halai (1998) found that teachers were more likely to adapt instructional practices recommended by coaches when the relationship was built on mutual trust, rather than the coach taking on an evaluative or supervisory role. Males, Otten & Herbel-Eisenmann (2010) found that mathematics teachers in a critical lesson study group benefited from the experience when conversations focused on student learning and data, and the experience resulted in conflict when the conversations focused on anecdotal or personal experiences. This work extends the work of others (Doyle & Ponder, 1978; Guskey, 1985, Fullan, 1992, Fennema et al., 1996) who found that teachers’ beliefs change when they see how interventions benefit their students’ learning.

This study focuses on examining teachers’ use of two reform-based mathematics pedagogies: cognitively-demanding mathematical tasks and questions about students’ mathematical understanding. Cognitively-demanding mathematical tasks provide opportunities for students to engage in and explore complex mathematical situations that involve doing mathematics, or allowing students to make mathematical connections between mathematical concepts and
procedures (Henningsen & Stein, 1997; Smith & Stein, 1998; Stein, Grover, & Henningsen, 1996). As Henningsen and Stein (1997, p. 525) note

The nature of tasks can potentially influence and structure the way students think and can serve to limit or to broaden their views of the subject matter with which they are engaged.

In their study, Smith and Stein’s framework of cognitively demanding mathematical tasks was used to analyse the tasks posed during classroom observations. The researchers distinguished between four types of mathematical tasks. Table 1 provides descriptions and examples of the four different types of tasks.

Table 1

<table>
<thead>
<tr>
<th>Types of tasks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Types of Tasks and Description</td>
</tr>
<tr>
<td>Memorization – Students recall a simple calculation or definition</td>
</tr>
<tr>
<td>Procedures without Connections – Use of algorithm with no representation</td>
</tr>
<tr>
<td>Procedures with Connections – Use of algorithm with connection to multiple representations or other mathematical concepts</td>
</tr>
<tr>
<td>Doing Mathematics – Non-routine tasks that require the learner to devise a strategy and justify their approach</td>
</tr>
</tbody>
</table>

Teachers’ questions posed during mathematics instruction have also been found to be critical in understanding students’ mathematical thinking and supporting students’ understanding of mathematical concepts (Hufferd-Ackles, Fuson, & Sherin, 2004). Table 2 describes the various levels of questions that were used to analyze data during this study. This framework was developed after synthesizing frameworks from previous research (Hufferd-Ackles et al., 2004) and refined after a prior study (Polly & Hannafin, 2011).

Table 2

<table>
<thead>
<tr>
<th>Levels of questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Levels of Questions</td>
</tr>
<tr>
<td>0 – does not ask questions when the opportunity arises</td>
</tr>
<tr>
<td>1 – asks questions that elicit only a mathematical answer or definition</td>
</tr>
<tr>
<td>2 – asks questions and follow-up questions about students’ processes or steps towards finding a solution</td>
</tr>
<tr>
<td>3 – asks questions about students’ rationale for choosing certain steps or students’ mathematical thinking</td>
</tr>
</tbody>
</table>
Theoretical Framework: Zone of Proximal Development

Vygotsky’s (1978) construct of a zone of proximal development [ZPD] provides an empirically based framework for examining teacher support. Tharp and Gallimore (1989) explicited ZPD in the context of teaching and referred to the idea of teaching as assisted performance, where more knowledgeable others (i.e., coaches, specialists, or facilitators) support teachers in learning about the craft of designing, implementing and reflecting on their instruction. Tharp and Gallimore described four stages of ZPD for learners. During Stage I, within the ZPD assistance is provided by more capable others through modelling, coaching and other methods of scaffolding performance, while during Stage II learners become increasingly self-supported and able to carry out the task without assistance. Stage III focuses on internalization where assistance from more capable others can paradoxically hinder performance. Stage IV involves the recursive process back through the ZPD, during which learners have to frequently modify their actions based on the environmental surroundings and context (Tharp & Gallimore, 1989).

Research indicates that specific activities, such as co-teaching or providing in class support have a greater impact than less intensive activities, such as attending planning meetings or providing resources (Killion & Harrison, 2006). In recent years in the United States of America [USA], mathematics coaches have been referred to as coaches, facilitators, or specialists (U.S. Department of Education, 2008). Recent research shows that these school personnel can positively impact teachers’ practices (Campbell & Malkus, 2010; Haver, 2008) and student learning outcomes (Campbell, 2008; Campbell & Malkus, in press). In this article, the terms coach, facilitator and specialist are used synonymously to refer to a professional who supports classroom teachers with their mathematics teaching.

Methods

The purpose of this study was to examine the types of support that elementary school teachers seek from more knowledgeable others and the influence of various types of support on their teaching while attempting to implement standards-based pedagogies. Two research questions guided this naturalistic, qualitative study (Bogden & Biklen, 2003):

1. What types of support did teachers seek out while attempting to implement standards-based mathematics instruction in their classroom?
2. What was the influence of mathematics support on mathematics instruction?

Participants and Setting

All participants had a bachelor’s degree and were licensed to teach Kindergarten to 6th grade. Pam and Lynda taught in inclusion mathematics classrooms with a combination of general education and special needs children, and occasionally had support from a special education teacher. Ruth and Sarah taught general
education students. All four teachers worked at an urban elementary school in southern USA. The school was located a mile away from a major university. Sixty-five percent of the students were minority (51% African American and 14% Latino) and 76% qualified for free or reduced lunch.

Teachers were participants in grade-level learning communities, which met weekly for 90 minutes. During meetings teachers shared instructional plans and discussed logistical issues, such as field trips and special events. At the time of the study, the school used a basal mathematics curriculum, but had sample copies of a standards-based mathematics curriculum that they were interested in teaching.

**Procedure**

While numerous models of professional development are found in the literature such as collaborative or content-specific coaching, the model used and examined in this study was grounded on principles of learner-centered professional development (Polly & Hannafin, 2010; NPEAT, 2000) as well as Tharp and Gallimore’s (1989) explication of the ZPD. All of the support provided was influenced by teachers’ requests for assistance related to their mathematics instruction. By giving teachers ownership of their learning, there was an increased likelihood that teachers would feel empowered to have ownership of this support, and be more receptive to ideas related to modifying their instructional practices. The goal of this effort was to examine how to best support teachers’ instructional practice, and better understand how those supports influence teachers’ practice; a teacher-requested model of coaching supported this goal.

**Recruitment**

At the beginning of the study the author recruited participants who were teaching 3rd and 5th grade. These grades were chosen since state testing is emphasized in these grade levels, and teachers in these grade levels had previously requested support in mathematics from their administration. The author recruited participants by telling them about the characteristics of standards-based instruction (e.g., allowing students to explore worthwhile mathematical tasks, asking rich questions to gauge students’ mathematical understanding, etc.), and then gauged their interest in teaching mathematics in this manner. The author offered support for their mathematics instruction however they desired, including providing curricular resources, co-planning lessons, providing in-class support and feedback, co-teaching a lesson, or teaching a demonstration lesson. All four teachers who reported interest in using standards-based pedagogies were selected.

**Data collection**

Field notes from classroom observations were the primary data source in this study. Secondary data sources included conversations with participants and researcher memos, which were recorded after any interaction with participants.
The number of classroom observations ranged from 21 to 30, based on the requests of participants. During lessons, the author sat with a group of students and took field notes. In other instances, the author was invited to teach a model lesson or co-teach with the classroom teacher. In these cases, field notes were taken during breaks in the lesson or immediately afterwards. Field notes were recorded about the types of tasks posed, and the types of questions asked. The end-of-year interview lasted approximately 20 minutes and was transcribed verbatim.

Data Analysis

Data from field notes were entered into a spreadsheet and analysed using inductive analysis (Bogden & Biklen, 2003). Once the author had identified the types of support that teachers sought, data were revisited to confirm these types of support, in addition to examining what factors in the data might have led to teachers' specific requests (Question 1).

Using Vygotsky’s ZPD framework, data from classroom observations were examined (Question 2) with an explicit focus on the types of mathematical tasks and questions posed (see Table 3). Tasks were analysed using Smith and Stein’s (1998) framework for mathematical tasks. Teachers’ questions were analysed using a scale derived from prior studies (Polly & Hannafin, 2011; Hufferd-Ackles, et al., 2004). The author analysed instructional practices (i.e., tasks and questions) three times; each time tasks and questions were categorized into the various levels, and data from field notes were analysed to ensure that tasks and questions were correctly categorized.

In order to examine teachers’ instruction across the school year, data were analysed and presented for six observations: the first two observations, the middle two observations, and the final two observations. Data regarding instructional practices are presented in terms of percentages to illustrate potential shift during the study. Further qualitative descriptions are also provided to describe teachers’ instructional practices.

Table 3

<table>
<thead>
<tr>
<th>Tasks</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Memorization [M]</td>
<td>What is the product of 8 and 6?</td>
</tr>
<tr>
<td>Students recall a simple calculation or definition</td>
<td></td>
</tr>
</tbody>
</table>

Use of algorithm with no representation
Tasks

<table>
<thead>
<tr>
<th>Type and Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Procedures with Connections [PWC]</td>
<td>Find the product of 22 and 13. Find your answer in more than one way.</td>
</tr>
<tr>
<td>Use of algorithm with connection to multiple representations or other mathematical concepts</td>
<td></td>
</tr>
<tr>
<td>Doing Mathematics [DM] Non-routine tasks that require the learner to devise a strategy and justify their approach</td>
<td>There are 22 students in the class. During a canned food drive, each student brings in 10 cans on Monday and then 3 more cans on Friday. If the class’ goal is to donate 250 cans of food do they have enough?</td>
</tr>
</tbody>
</table>

Questions

| 0 – does not ask questions when the opportunity arises | Questions are not asked |
| 1 – asks questions that elicit only a mathematical answer or definition | “What did you get for an answer to 22 times 13?” |
| 2 – asks questions and follow-up questions about students’ processes or steps towards finding a solution | “Tell us how you found the answer.” |
| 3 – asks questions about students’ rationale for choosing certain steps or students’ mathematical thinking | “Why did you decide to multiply 22 by 10 and then 22 by 3?” |

Results

Types of Support Sought

During the study, participants sought various types of support from the author (see Table 4). These types of support included feedback on lessons, support during instruction, co-planning assistance, and providing curricular resources.

Feedback on lessons. After the first observation, each participant asked what I was focusing my attention on during observations. I showed both frameworks for analysing tasks and questions and then provided examples of high-level tasks and questions. All four participants sought feedback for every lesson for the rest of the year.
Table 4
Types of Support Requested

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Grade</th>
<th>Years in current teaching</th>
<th>Number of observations</th>
<th>Support Requested</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pam</td>
<td>3rd</td>
<td>1/1</td>
<td>25</td>
<td>Planning, resources, ideas for classroom management in her inclusion classroom</td>
</tr>
<tr>
<td>Ruth</td>
<td>3rd</td>
<td>1/1</td>
<td>30</td>
<td>Planning, resources, in class support posing word problems, co-teaching</td>
</tr>
<tr>
<td>Sarah</td>
<td>5th</td>
<td>1/5</td>
<td>21</td>
<td>Resources, clarification of content and what the standards mean</td>
</tr>
<tr>
<td>Lynda</td>
<td>5th</td>
<td>6/6</td>
<td>28</td>
<td>Ideas for hands-on activities, higher-level thinking skills, co-teaching</td>
</tr>
</tbody>
</table>

Participants requested different types of feedback. Pam, in her first year of teaching, always asked for feedback about how she should deal with classroom management problems. Rarely did she want feedback about her mathematics teaching. Sarah, who was new to 5th grade, also asked for a lot of feedback about management rather than her teaching. Primarily, teachers requested more feedback about tasks. When asked about receiving feedback, Ruth explained, “We have a choice about the curriculum and the activities. I want to make sure that I am challenging my students appropriately.”

In some cases, teachers were reflective about the tasks that they posed. During a lesson on ordering fractions, Lynda had posed the task:

You have \(1 \frac{3}{8}\) pieces of pie, your mom has \(1 \frac{1}{2}\) pieces of pie, and your sister has \(1 \frac{1}{8}\) pieces. Who has the most pie? Who has the least amount of pie?

How do you know?

Lynda’s entire class successfully completed the task. After the lesson Lynda said,

I think the last problem was too easy. They had just been successful with 2 tasks with three different denominators, and after I gave this one I realized that it was easier than the two that they had just gotten right.

Support during instruction. During classroom observations participants asked for assistance in a variety of ways. Each time I visited Ruth’s class, she asked me to pose a few mathematical tasks and questions to her students based on the concepts they were learning. As the year continued, Ruth became more
independent and asked for my assistance less frequently. Occasionally Lynda asked me to look at specific students’ work and discuss students’ error patterns. Lynda and Ruth asked me to situate myself near specific students to lend a hand during a lesson if they had problems. Sarah did not request in-class support and preferred receiving resources rather than having me in her classroom during her mathematics teaching.

The content also influenced the amount of in-class support that participants requested. All four participants requested extensive support while they were teaching fractions. Even Sarah, who did not typically ask for in-class support, requested me to teach a model lesson about fractions. She commented, “I’ve tried to teach this concept for three days and most of my class still doesn’t understand. I figured that you could handle this one.” However, during the model lesson, Sarah worked on other activities.

When I asked about their reason for the in-class support, Ruth said,

Fractions are difficult for me and I want feedback from you to make sure that I’m teaching it correctly. Also, I am unsure if I’m teaching in a way that makes the most sense to them.

Co-planning. Ruth and her grade level mentor planned together during the entire school year. Pam, who taught third grade with Ruth, did not receive much mentorship and independently planned lessons primarily from her basal curricula. In November, Pam sought my guidance about planning, and after talking with Ruth; a planning group was formed among some of the third grade teachers at the school. Each week between 4 and 9 teachers shared resources and ideas. In order to provide teachers with ownership, I attended meetings and contributed ideas when asked. For each meeting, I had chosen some lessons from the standards-based curricula that the school had copies of, and gave them to teachers as an option to use. By February, every teacher that attended the planning group was using either units or lessons from the standards-based curricula.

The fifth grade teachers, Sarah and Lynda sought assistance pacing out the lengths of units and long-term planning, rather than specific lessons. The district provided teachers with a broad pacing guide for topics to cover every quarter, but both teachers were unsure how long to spend on specific concepts. Further, Lynda had asked me to examine student data and make decisions based on her students’ progress each month. Lynda said,

The confusion was trying to determine whether students were ready to move on or not. By looking at some of their work, I feel more comfortable making the decision to move on if I have data that my students understand the concept.

Providing curricular resources. The teachers had access to sample units of a standards-based curriculum that the district was considering to adopt. Typically a week before starting a new concept, all four participants asked if I knew specific lessons from the curriculum that would be easy to implement. In the 3rd grade planning group with Pam and Ruth, both teachers taught several lessons later in the year. Sarah and Lynda were more reluctant to use the curricula; Sarah
tried a few lessons after I had spent time reviewing the activities and had taught a model lesson. Lynda saw little alignment between the curricula and the fifth grade state test, and was not interested in using it.

**Influence on Teachers’ Mathematics Instruction**

Observations illustrated two features related to how supporting teacher-participants influenced their instruction. The types of mathematical tasks and the questions posed during teachers’ mathematics instruction are described below.

*Mathematical tasks.* Overall, the quality of tasks that teachers posed improved throughout the year (Table 5). Specifically, teachers enacted more tasks that allowed students to generate multiple representations and explore mathematical connections within a task. For example, Pam and Ruth both enacted the following task from the standards-based curricula,

> You have 5 brownies and you want to share them equally among 4 people. How many brownies does each person receive?

Ruth kept the task integrity high by allowing students to explore with manipulatives. Meanwhile, Pam enacted this task as a procedure with connections task, as she walked her students through the process of splitting the leftover brownie into four equal pieces.

### Table 5

**Types of Tasks Posed**

<table>
<thead>
<tr>
<th></th>
<th>First Two Observations</th>
<th>Middle Two Observations</th>
<th>Last Two Observations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>M</td>
<td>PWoC</td>
<td>PWC</td>
</tr>
<tr>
<td>Pam</td>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Ruth</td>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sarah</td>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Lynda</td>
<td>25</td>
<td>75</td>
<td>0</td>
</tr>
</tbody>
</table>

Key: M=Memorization; PWoC=Procedures without connections; PWC=Procedures with connections; DM=Doing mathematics

Teachers’ enactments of more rich tasks were influenced by several factors. Pam, a 3rd grade teacher, started using more high-level tasks when she started co-planning with other 3rd grade teachers. Ruth, the other 3rd grade teacher, participated in the planning group, and sought several co-teaching opportunities with the author. However, she relied on more Memorization and Procedures without Connections tasks despite extensive co-planning and co-teaching support. She reported, “These are the types of tasks included on the end of grade tests so that has to be my focus. I don’t have a choice.”
Lynda, meanwhile, implemented a variety of Procedures without Connections and Procedures with Connections tasks during the school year. She shared her lesson plans with Sarah. However, Lynda’s tasks that were Procedures with Connections were enacted as Procedures without Connections tasks in Sarah’s classroom. Sarah frequently gave students algorithms and explicit steps for her students to follow. While co-planning with teachers improved the quality of the planned tasks, at times during implementation teachers provided too much structure, thus reducing the quality of enacted tasks.

Questions that teachers posed. All four teachers asked more higher-level questions as the year progressed (see Table 6). Specifically, teachers posed more and higher-level questions towards the end of a lesson as students were sharing their work on mathematical tasks.

Table 6  
Types of Questions Posed

<table>
<thead>
<tr>
<th></th>
<th>First Two Observations (%)</th>
<th>Middle Two Observations (%)</th>
<th>Final Two Observations (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Pam</td>
<td>42.0</td>
<td>58.0</td>
<td>0</td>
</tr>
<tr>
<td>Ruth</td>
<td>11.0</td>
<td>83.5</td>
<td>5.5</td>
</tr>
<tr>
<td>Sarah</td>
<td>10.0</td>
<td>90.0</td>
<td>0</td>
</tr>
<tr>
<td>Lynda</td>
<td>10.0</td>
<td>45.0</td>
<td>45.0</td>
</tr>
</tbody>
</table>

During the first month of observations, only Lynda (5th grade) asked students to share their mathematical thinking and strategies; the rest simply questioned for answers. During the year, Lynda asked me to pose questions during class discussions. As I posed questions in her class about her students’ mathematical thinking, she mimicked me and posed questions about students’ strategies during class-wide discussions and independent work time. For example, while teaching about the connection between fractions and decimals, Lynda’s students were shading representations of fractions on a decimal grid and naming the fraction. She asked students what they noticed about the decimal grids for $\frac{1}{4}$ and 0.8. When they commented that the same area was shaded, Lynda asked her class, “Why do you think that is the case?” Without a response from students, rather than giving an answer, Lynda then asked, “How can you represent each of those?” Over the course of the year, Lynda asked more how and why questions during her lessons. After an observation, Lynda commented,

I love my students’ responses when I pose these ‘why’ kinds of questions. Unfortunately, we have so much content to get through that we don’t have a lot of time to discuss the mathematics as much as I want.
Ruth began to ask higher-level questions, as well, after I had modelled how to facilitate a discussion by posing questions. In a lesson that we co-taught, I asked students to sort a set of 3-dimensional shapes anyway they wanted to, and then have their neighbour guess the rule for the sort. One student sorted the shapes into a pile of a prisms and non-prisms. During the class discussion, Ruth asked about this students’ rule:

Ruth: Let’s look at Angelica’s sort. What do you notice?

Ben: All of the shapes in this pile are stackable. You can put other shapes on top of them or underneath them.

Ruth: Okay. Other thoughts?

Austin: The stackable pile includes only prisms.

Ruth: How do you know they are all prisms?

Austin: It has two opposite faces that are identical and every face is flat.

Ruth: (as she picks up a hexagonal prism) Is this a prism? Why or why not?

Ben: Yes. Every face is flat and it has two congruent faces.

Through questioning, Ruth helped her students explore characteristics of prisms. Pam and Sarah rarely questioned students for information other than answers to tasks or descriptions of how students found answers to tasks that they had posed. When asked at the end of the year, Pam reported, “For me this year was about managing the classroom and teaching the standards. I hope that I can ask better questions next year.”

Discussion

Several findings from this study warrant further discussion. As education leaders continue to seek ways to support teachers’ use of standards-based pedagogies, expert coaching has promise to support teachers. Consistent with prior work (Banilower, Boyd, Pasley, & Weiss, 2006; Campbell & Markus, 2010), teachers desired support with curricular resources and areas explicitly connected to their daily practice. Three of the teachers sought feedback that they were enacting rich tasks. These desires were consistent with prior studies about teachers trying to use standards-based pedagogies (Polly, 2006; Polly & Hannafin, 2011; Tarr, Reys, Reys, Chávez, Shih, & Osterlind, 2008). Also, consistent with prior research (Peterson, 1990; Prawat, 1992), teachers who were more resistant to change (Pam and Lynda) sought less in-class support and preferred to limit their interactions with me to planning and receiving curricular resources.

Ruth and Sarah both sought more intensive supports during their mathematics teaching. As expected from prior work (Polly & Hannafin, 2011; Heck et al., 2008), both teachers demonstrated significant gains in the levels of
tasks and questions that they posed during the study. Ruth and Sarah’s primary request for support was to get reaffirmation and feedback about their instruction during lessons. Similar to earlier studies, the dialogue that occurred between the author and teachers about their instruction and their students’ learning was beneficial (Glazer & Hannafin, 2006).

Pam’s use of standards-based pedagogies improved when she collaboratively planned with myself, and her colleagues. Similar to other projects where collaboration led to increased enactment of standards-based pedagogies (Desimone et al., 2002; Heck et al., 2008; Polly, 2011) Pam grew as a result of her time co-planning with others.

This study was framed around Vygotsky’s (1978) concept of zone of proximal development, and the neo-Vygotskian view of teaching as assisted performance (Tharp & Gallimore, 1989). Vygotsky posited that learners need scaffolding and support until they are able to accomplish tasks independently. This holds true for teachers. As seen in this study, teachers spent most of the year in Stage I, requiring modelling and extensive coaching to support their mathematics instruction. Towards the end of the year, observations from Sarah’s classroom showed a shift to Stage II; she independently planned and enacted standards-based pedagogies without support before or during a lesson. Ruth also had slight shifts towards Stage II, as she became more independent during instruction; however, Ruth still requested extensive support during planning.

Implications for Research and Practice

In only one year of support, teachers started to pose higher-level tasks and questions. Future studies should collect and analyse data over multiple years, in order to provide a more comprehensive picture of teacher change through the various stages of ZPD. Further, future research should examine the best ways to efficiently move teachers through the various stages of ZPD. If studies continue to indicate that intensive supports, such as co-planning and co-teaching lead to higher enactments of standards-based pedagogies, subsequent studies should examine the issues with scaling up the model or having one coach work intensively with more teachers. One limitation of the study was teachers’ willingness to participate, and their interest in using these reform-based pedagogies in their classroom. Future studies should include a more diverse range of participants, including those teachers who are not interested or willing to immediately begin adopting these reform-based pedagogies.

This study indicates that expert coaching has promise to support mathematics instruction through activities such as co-planning, providing feedback on lessons, and co-teaching. The largest adoption of instructional practices occurred with teachers who requested and received extensive classroom-based support. Instructional coaches should be put in roles where they are able to support teachers during lessons through co-teaching and providing feedback after observations.
References


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