

**Transformations between isospectral membranes yield  
conformal maps**

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In two dimensions, in converse to the known situation that a conformal transformation yields isospectral density functions for vibrating membranes, it is shown that a coordinate transformation leading to isospectral densities must be conformal.

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## 1. Introduction

It is well known (e.g. Kuttler & Sigillito, 1984; Schinzinger & Laura, 1991, chapter 9) that a conformal transformation of two-dimensional coordinates (with vibration displacement functions related by substitution) results in isospectral densities for vibrating membranes. Applications and examples have recently been given by Cureton & Kuttler (1999) and Laura *et al.* (2000).

In Gottlieb (2002), transformations which generate isospectral strings were found, and the question then naturally arises whether this approach for the one-dimensional case of strings may in general be extended to the two-dimensional case of isospectral membranes.

As a converse to the situation stated in the opening sentence above, we show here that a transformation of two-dimensional coordinates which results in isospectral densities must be conformal. The displacement functions are only related by simple substitution (even if multiplication by a coordinate-dependent function, an important ingredient of Gottlieb, 2002, is a priori allowed). The treatment is straightforward, but fills an important gap concerning isospectrality. The result has significant implications in restricting searches for isospectral membrane density functions.

## 2. Analysis

We start with the non-homogeneous wave equation in Cartesian coordinates

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} = \phi(\xi, \eta) \frac{\partial^2 v}{\partial t^2} \quad (2.1)$$

representing a vibrating (unit-tension) membrane with displacement  $v$  and density function  $\phi(\xi, \eta)$  in that coordinate system. Then the coordinate and displacement transformations are made (c.f. Gottlieb, 2002, for the one-dimensional string case)

$$x = x(\xi, \eta) \quad ; \quad y = y(\xi, \eta) \quad , \quad (2.2)$$

$$v(\xi, \eta; t) = \gamma(x, y) u(x, y; t) \quad , \quad (2.3)$$

in order to attempt to obtain the form again involving a Laplacian

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \frac{\partial^2 u}{\partial t^2} \quad (2.4)$$

representing a membrane in  $(x, y)$  coordinates with density function  $f$ . The transformation (2.3), which includes the possibility of a multiplicative coordinate-dependent function in the displacement relation, preserves the Dirichlet boundary condition: when  $u = 0$ ,  $v = 0$ .

Let

$$P = \left( \frac{\partial x}{\partial \xi} \right)^2 + \left( \frac{\partial x}{\partial \eta} \right)^2 , \quad (2.5a)$$

$$Q = \left( \frac{\partial y}{\partial \xi} \right)^2 + \left( \frac{\partial y}{\partial \eta} \right)^2 , \quad (2.5b)$$

$$R = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \xi} + \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \eta} , \quad (2.5c)$$

$$S = \frac{\partial^2 x}{\partial \xi^2} + \frac{\partial^2 x}{\partial \eta^2} , \quad (2.5d)$$

$$T = \frac{\partial^2 y}{\partial \xi^2} + \frac{\partial^2 y}{\partial \eta^2} . \quad (2.5e)$$

In terms of these, the left hand side of (2.1) becomes

$$\begin{aligned}
& \gamma P \frac{\partial^2 u}{\partial x^2} + \gamma Q \frac{\partial^2 u}{\partial y^2} + 2\gamma R \frac{\partial^2 u}{\partial x \partial y} \\
& + \left[ \gamma S + 2 \frac{\partial \gamma}{\partial x} P + 2 \frac{\partial \gamma}{\partial y} R \right] \frac{\partial u}{\partial x} + \left[ \gamma T + 2 \frac{\partial \gamma}{\partial y} Q + 2 \frac{\partial \gamma}{\partial x} R \right] \frac{\partial u}{\partial y} \\
& + \left[ \frac{\partial^2 \gamma}{\partial x^2} P + \frac{\partial^2 \gamma}{\partial y^2} Q + 2 \frac{\partial^2 \gamma}{\partial x \partial y} R + \frac{\partial \gamma}{\partial x} S + \frac{\partial \gamma}{\partial y} T \right] u .
\end{aligned} \tag{2.6}$$

To achieve (2.4), first of all the coefficient of  $\partial^2 u / \partial x \partial y$  must vanish, i.e.

$$R=0. \tag{2.7}$$

Secondly, the coefficients of  $\partial^2 u / \partial x^2$  and  $\partial^2 u / \partial y^2$  must be equal, i.e.

$$P=Q. \tag{2.8}$$

Suppose now that  $\partial x / \partial \eta = 0$ , so  $x = x(\xi)$ . Then, by (2.7) with (2.5c), either  $\partial x / \partial \xi = 0$  so  $x$  is a constant and not a genuine coordinate, or  $\partial y / \partial \xi = 0$ . In the latter case,  $x = x(\xi)$  and  $y = y(\eta)$ . But then, by (2.8) with (2.5a,b),  $(\partial x / \partial \xi)^2 = (\partial y / \partial \eta)^2 =$  constant, so each coordinate transformation is merely linear in one variable. This corresponds to just a constant scaling (possibly with shift of origin) and so is trivial (and is a trivial conformal mapping).

Suppose on the other hand that  $\partial x/\partial \xi = 0$ , so  $x = x(\eta)$ . Then, by (2.7) with (2.5c), either  $\partial x/\partial \eta = 0$ , so  $x$  is constant, or  $\partial y/\partial \eta = 0$  so  $y = y(\xi)$ . The above arguments are repeated with coordinates interchanged.

For non-trivial coordinate transformations, it may therefore be deduced that all partial derivatives appearing in P,Q,R ((2.5a-c)) are non-zero.

Now (2.7) and (2.8) are of the form  $ab + cd = 0$ ,  $a^2 + c^2 = b^2 + d^2$ , with all terms non-zero. These imply that either  $a = d$  and  $b = -c$ , or  $a = -d$  and  $b = c$ . The first possibility here reads

$$\frac{\partial x}{\partial \xi} = \frac{\partial y}{\partial \eta} \quad , \quad \frac{\partial y}{\partial \xi} = -\frac{\partial x}{\partial \eta} \quad (2.9)$$

(and the second possibility simply corresponds to a change of sign of one coordinate, which leaves the Laplacian unchanged). It then follows from (2.9) that

$$S = 0 = T \quad . \quad (2.10)$$

It is known (e.g. Saff & Snider, 1976, pp 45-46) that if the relations (2.9) hold (with suitable continuity of functions and partial derivatives which may be assumed for physical systems) then the complex mapping

$$\Psi(\zeta) = x(\xi,\eta) + i y(\xi,\eta) \quad , \quad (2.11)$$

with  $\zeta = \xi + i\eta$ , is in fact analytic, and equations (2.9) may be identified as the Cauchy-Riemann equations for this map.

When  $R=0$  and  $P=Q$  and  $S=0=T$ , as derived above, are now used, the left hand side of (2.1), re-expressed by (2.6), simplifies to

$$J \left[ \gamma \nabla^2 u + 2 \frac{\partial \gamma}{\partial x} \frac{\partial u}{\partial x} + 2 \frac{\partial \gamma}{\partial y} \frac{\partial u}{\partial y} + (\nabla^2 \gamma) u \right] \quad (2.12)$$

where  $\nabla^2$  is the two-dimensional Laplacian in  $x, y$ , coordinates. Here  $J$  is equal to the expressions  $P$  on either side of (2.8), viz (2.5a) or (2.5b). Because of (2.9), it may also be written as

$$J = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta} \quad (2.13a)$$

$$\equiv J_{\xi, \eta}[x, y] \quad (2.13b)$$

where the notation in (2.13b) identifies  $J$  as the Jacobian of the transformation of  $x$  and  $y$  in terms of  $\xi$  and  $\eta$ . In fact, for (2.11),  $J = |d\Psi/d\zeta|^2$ .

For (2.12) now to lead to (2.4),  $J$  must be non-zero, so its inverse exists. This means that the coordinate transformation (2.11) is not merely analytic but is conformal. By a well-known property of Jacobians,

$$J \equiv J_{\xi, \eta}[x, y] = \{J_{x, y}[\xi, \eta]\}^{-1} \equiv \left[ \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} - \frac{\partial \eta}{\partial x} \frac{\partial \xi}{\partial y} \right]^{-1} \quad (2.14)$$

for the inverse transform.

Further comparison of (2.12) with (2.4) requires that the coefficients of  $\partial u/\partial x$  and  $\partial u/\partial y$  in (2.12) must vanish. Because  $J \neq 0$ , this implies

$$\frac{\partial \gamma}{\partial x} = 0 = \frac{\partial \gamma}{\partial y} \quad (2.15)$$

so

$$\gamma(x,y) = \gamma = \text{constant}, \quad (2.16)$$

which may without loss of generality be set equal to unity for the displacement functions in (2.3) appearing in the linear p.d.e.s (2.1) and (2.4). Then the coefficient of  $u$  in (2.12) also vanishes.

Now (2.1) simply becomes

$$J \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \phi \frac{\partial^2 u}{\partial t^2} . \quad (2.17)$$

In view of (2.14), this becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = J_{x,y}[\xi, \eta] \phi(\xi, \eta) \frac{\partial^2 u}{\partial t^2} . \quad (2.18)$$

Comparison with (2.4) now shows that the membrane density corresponding to  $\phi(\xi, \eta)$  is given explicitly by



$$f(x,y) = J_{x,y}[\xi(x,y),\eta(x,y)] \phi(\xi(x,y),\eta(x,y)) \quad (2.19)$$

where  $J_{x,y}$  in (2.19) is the Jacobian of the inverse transformation to (2.2) (as in (2.14)), i.e. it equals  $|\psi'(z)|^2$  in the inverse conformal transformation to (2.11) namely

$$\psi(z) = \xi(x,y) + i \eta(x,y) \quad (2.20)$$

where  $z = x + iy$ .

The displacement functions are related via (2.3), with (2.16) with  $\gamma=1$ , by the simple substitution

$$v(\xi,\eta;t) = u(x,y;t) . \quad (2.21)$$

Isospectrality is now manifested by setting the time dependence of  $v$  explicitly to  $\exp(i\omega t)$  where  $\omega$  denotes the eigenfrequencies. Then by (2.21)  $u$  has the same time dependence. The isospectral systems are expressed by the non-homogeneous Helmholtz equations corresponding to (2.1) and (2.4) with common eigenvalues  $\omega^2$ :

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} + \omega^2 \phi(\xi,\eta) v = 0 \quad ; \quad (2.22)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \omega^2 f(x,y) u = 0 \quad ; \quad (2.23)$$

with displacements  $u, v$  related by (2.21), and isospectral densities  $f, \phi$  related by (2.19) via the conformal transformation (2.20).

### 3. Conclusion

It has thus been shown that for isospectral two-dimensional densities obtainable via (2.3) with nonsingular coordinate transformations, the displacement function relation cannot be generalized to include a non-trivial multiplicative function of coordinates, and the coordinate transformation itself corresponds to a conformal map, with the isospectral densities being related by (2.19). The result holds regardless of whether the initial and final shapes are different, or the same.

It may now be noted that, because of (2.21) (in place of the no longer operative more general relation (2.3)), isospectrality holds not only for Dirichlet ("fixed") boundary conditions ( $u = 0 = v$ ), but also for Neumann ("free") boundary conditions  $\partial u/\partial n = 0 = \partial v/\partial n$  (where  $n$  denotes the normal to the boundary) which are preserved by conformal maps (c.f. Churchill *et al.*, 1974, p. 206).

For completely isospectral circular membranes in particular, the above indicates that the results for instance of Gottlieb (1988) and Gottlieb (2004) cannot be extended using the technique of (2.3) with non-constant multiplicative function  $\gamma(x,y)$  relating the displacement functions.

Furthermore, one could not expect, for example, to extend the results of works such as Gottlieb (2004) for circular membranes or Gottlieb (1992) for a specific annular membrane by working directly in plane polar coordinates and attempting a general transformation analogous to (2.3) involving  $\gamma(r,\theta)$ , since one could initially convert to Cartesian coordinates and then proceed as above.

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