Entanglement Constrained by Superselection Rules.

Author

Bartlett, SD, Wiseman, HM

Published

2003

Journal Title

Physical Review Letters

DOI

10.1103/PhysRevLett.91.097903

Entanglement constrained by superselection rules

Stephen D. Bartlett[1,*] and H. M. Wiseman[2,†]

[1]Department of Physics, Macquarie University, Sydney, New South Wales 2109, Australia
[2]Centre for Quantum Computer Technology, Centre for Quantum Dynamics,
School of Science, Griffith University, Brisbane, Queensland 4111, Australia

(Dated: 26 August 2003)

Bipartite entanglement may be reduced if there are restrictions on allowed local operations. We introduce the concept of a generalized superselection rule (SSR) to describe such restrictions, and quantify the entanglement constrained by it. We show that ensemble quantum information processing, where elements in the ensemble are not individually addressable, is subject to the SSR associated with the symmetric group (the group of permutations of elements). We prove that even for an ensemble comprising many pairs of qubits, each pair described by a pure Bell state, the entanglement per element constrained by this SSR goes to zero for a large number of elements.

PACS numbers: 03.67.Mn, 02.20.-a, 03.65.Fd, 03.65.Ta

Entanglement lies at the heart of quantum information processing (QIP) [1], and quantifying entanglement as a physical resource is a primary goal of this field [2]. Recently, it has been shown that the existence of superselection rules (SSRs) [3] requires us to reassess traditional entanglement measures [4] and the allowed bipartite operations [5]: the SSRs enforce additional restrictions on what Alice and Bob can accomplish using only local operations and classical communication (LOCC).

In this Letter, we quantify entanglement constrained by a generalized SSR and show that this entanglement is typically less than the amount given by any standard measure. To accomplish this task, we first introduce the concept of a generalized SSR as a rule associated with some group of physical transformations of a system. The rule is defined operationally: it restricts the allowed operations on the system to those that are covariant with respect to that group. Our definition encompasses traditional SSRs such as charge and particle number as well as effective SSRs for quantities such as angular momentum or photon number (which may arise due to practical restrictions on operations, the lack of an appropriate shared reference frame, or through interaction with an environment). Our measure of entanglement constrained by SSRs is also operational in that it describes the accessible entanglement that Alice and Bob can distill into standard quantum registers through allowed LOCC.

As an explicit example of entanglement constrained by a SSR, we describe ensemble QIP where access to individual elements of the ensemble is not possible. The relevant SSR here restricts the allowed operations to be "collective" in that they act identically on all elements of the ensemble. We find that our operational measure of entanglement constrained by this SSR can be hugely smaller than that found from standard entanglement measures. In particular, we prove the remarkable result that, even if each element of the ensemble consists of two qubits described by a *pure* Bell state, the entanglement per element constrained by this SSR is zero in the limit of a large number of elements. We discuss how this result places a powerful constraint on QIP in liquid-state NMR [6] and spin-squeezing experiments [7].

We begin by providing an operational definition of a SSR, and show that this definition is compatible with colloquial uses. *A SSR is a restriction on the allowed local operations on a system, and is associated with a group of physical transformations.* This restriction could be imposed by properties of the underlying theory (e.g., a SSR for charge required in a Lorentz-invariant quantum field theory [8]), but we also consider SSRs that arise due to practical restrictions. Consider a local quantum system with Hilbert space $\mathbb{H}$. The set of physical operations on this quantum system is given by the semigroup of completely positive (CP) trace-preserving maps $\{\mathcal{E}\}_{\rm CP}$. These CP maps describe not only unitary (closed) operations but also open processes such as state preparation, dissipation and measurement. Let G be a group of physical transformations acting on $\mathbb{H}$ through a unitary representation T. We define an operation $\mathcal{O} \in \{\mathcal{E}\}_{\rm CP}$ to be *G-covariant* if

$$\mathcal{O}[T(g)\rho T^\dagger(g)] = T(g)\mathcal{O}[\rho]T^\dagger(g)\,, \tag{1}$$

for all group elements $g \in G$ and all density operators ρ. We then define the *SSR associated with G*, or G-SSR, to be to be a restriction on the allowed operations on the system to those CP maps $\{\mathcal{O}\}_{G\text{-SSR}} \subset \{\mathcal{E}\}_{\rm CP}$ that are G-covariant. The following examples reveal how this definition is compatible with some traditional SSRs:

Example 1: Charge. Let G be a one-dimensional Lie group U(1) generated by a Hermitian operator Q; i.e., $T(\xi) = \exp(\mathrm{i}\xi Q)$. If Q is a local charge operator then this U(1)-SSR is usually referred to as a SSR for charge. Similar SSRs can be developed for particle number. When such a SSR applies, one cannot, for instance, *locally create* superpositions of charge eigenstates because the required operations are not G-covariant. Note that this SSR does not forbid the creation of superpositions where, for example, one charge can be found at two different locations, as in the twin-slit experiment for electrons.

Example 2: Angular momentum. Let $G = \mathrm{SO}(3)$ be the rotation group generated by the total angular momentum operators $\{L_{x,y,z}\}$. The associated SSR ensures that all allowed operations are rotationally invariant. This SSR may apply, for instance, when there is no reference frame for orientation and thus all observables commute with the total angular momentum. A reference frame would establish operators that specify a direction; such operators do not commute with total angular momentum and thus violate the SSR. In this example, the SSR is a practical rather than fundamental consideration: the lack of a reference frame leads to a SSR.

Example 3: Environmentally-induced SSR. Let $\hat{H}_{\rm int}$ be a coupling Hamiltonian between the system and an environment, and let $G = \mathrm{U}(1)$ be the group generated by this Hamiltonian. Einselection [9], which is often expressed as the condition that the only allowed states of the system are those that commute with this Hamiltonian, has the form of a U(1)-SSR.

It should be noted that the existence of a SSR is not equivalent to a conservation law; in fact, the only interesting SSRs are those that apply to *non-conserved* quantities [10]. Also note that a SSR does not restrict the allowed states of the system. However, the restrictions imposed on the allowed operations by the G-SSR mean that a state ρ is indistinguishable from the states $T(g)\rho T^\dagger(g)$ for all $g \in G$. Because of this indistinguishability, it is operationally appropriate to describe ρ by the state

$$\mathcal{G}[\rho] \equiv \begin{cases} (\dim G)^{-1} \sum_{g\in G} T(g)\rho T^\dagger(g)\,, & \text{finite groups} \\ \int_G \mathrm{d}v(g)\, T(g)\rho T^\dagger(g)\,, & \text{Lie groups} \end{cases} \tag{2}$$

where $\mathrm{d}v$ is the group-invariant (Haar) measure [11]. The state $\mathcal{G}[\rho]$ is invariant under the action of G,

$$T(g)\mathcal{G}[\rho]T^\dagger(g) = \mathcal{G}[\rho]\,, \quad \forall\, g \in G\,, \tag{3}$$

so we call this state the *G-invariant state*.

Consider a SSR for charge as an example. States that are superpositions of charge eigenstates are not *a priori* prohibited. For a state that is a superposition of charge eigenstates $|\psi\rangle = \alpha|q_1\rangle + \beta|q_2\rangle$, with $Q|q_i\rangle = q_i|q_i\rangle$, the effect of the superoperator $\mathcal{G}$ on this state is

$$\mathcal{G}\big[|\psi\rangle\langle\psi|\big] = |\alpha|^2|q_1\rangle\langle q_1| + |\beta|^2|q_2\rangle\langle q_2|\,. \tag{4}$$

Thus, in the presence of the SSR, a superposition of charge eigenstates is operationally equivalent to a *mixture* of charge eigenstates. The effect of $\mathcal{G}$ is to project onto eigenspaces of the group generator Q. SSRs associated with one-dimensional Lie groups are often defined this way (*cf.* [5]). However, for general SSRs (including the example associated with the rotation group), there is not necessarily an equivalent expression.

We now consider imposing a SSR in a bipartite setting; that is, both parties (Alice and Bob) are restricted to local operations obeying Eq. (1). Consider a bipartite state ρ^{ab} shared by Alice and Bob. This state may have been prepared by a third party under conditions where no SSR applies. The G-invariant state constrained by these local SSRs is $\mathcal{G}[\rho^{ab}] = \mathcal{G}^a \otimes \mathcal{G}^b[\rho^{ab}]$. To quantify the entanglement of this state we assume that, in addition to this bipartite system, Alice and Bob each possess a quantum register with Hilbert space dimension equal to or greater than that of their respective systems. These registers are initially in the pure product state ϱ_0^{ab} and are not subject to any SSR. (For example, these registers could be standard qubits over which Alice and Bob have complete control.) We quantify the entanglement $E_{G\text{-SSR}}(\rho^{ab})$ constrained by the G-SSR as the maximum amount of entanglement that Alice and Bob can produce between their registers by LOCC [4]. The latter can be quantified by an appropriate standard measure E, e.g., the entanglement of distillation [1]. The following theorem quantifies the entanglement constrained by an arbitrary SSR for pure or mixed states, generalizing the result of [4]:

Theorem: The entanglement $E_{G\text{-SSR}}(\rho^{ab})$ that Alice and Bob can produce between their registers from the state ρ^{ab} by LOCC constrained by a G-SSR is given by the entanglement $E(\mathcal{G}[\rho^{ab}])$ that they can produce from the state $\mathcal{G}[\rho^{ab}]$ by *unconstrained* LOCC, where E is a standard measure of entanglement.

Proof: First, note that any CP map can be composed with $\mathcal{G}$ to yield a G-invariant operation, i.e.,

$$\mathcal{G}\circ\mathcal{E}\circ\mathcal{G} \in \{\mathcal{O}\}_{G\text{-SSR}} \quad \forall\, \mathcal{E} \in \{\mathcal{E}\}_{\rm CP}\,, \tag{5}$$

which follows from the definitions (1) and (2). Let $\mathcal{O}$ be a G-invariant operation in LOCC acting on the initial state $\rho^{ab}\otimes\varrho_0^{ab}$. The final state of the registers is given by $\varrho_1^{ab} = \mathrm{Tr}_{\rm sys}\big(\mathcal{O}[\rho^{ab}\otimes\varrho_0^{ab}]\big)$, where the trace is over the shared system. The maximum entanglement produced between the registers is given by maximizing $E(\varrho_1^{ab})$ over all LOCC obeying the G-SSR. Thus, using (5),

$$\begin{aligned} E_{G\text{-SSR}}(\rho^{ab}) &= \max_{\mathcal{O}}\ E\big(\mathrm{Tr}_{\rm sys}\big(\mathcal{O}[\rho^{ab}\otimes\varrho_0^{ab}]\big)\big) \\ &= \max_{\mathcal{O}}\ E\big(\mathrm{Tr}_{\rm sys}\big((\mathcal{G}\circ\mathcal{O}\circ\mathcal{G})[\rho^{ab}\otimes\varrho_0^{ab}]\big)\big) \\ &= \max_{\mathcal{E}}\ E\big(\mathrm{Tr}_{\rm sys}\big((\mathcal{G}\circ\mathcal{E}\circ\mathcal{G})[\rho^{ab}\otimes\varrho_0^{ab}]\big)\big) \\ &= \max_{\mathcal{E}}\ E\big(\mathrm{Tr}_{\rm sys}\big(\mathcal{E}\big[\mathcal{G}[\rho^{ab}]\otimes\varrho_0^{ab}\big]\big)\big)\,, \end{aligned} \tag{6}$$

where the second line follows from the properties of trace and the definition (2), and the last line follows from the properties of trace. The latter maximization is over *all* LOCC (not just operations in $\{\mathcal{O}\}_{G\text{-SSR}}$), and gives the entanglement $E(\mathcal{G}[\rho^{ab}])$ that Alice and Bob can produce between their registers from the state $\mathcal{G}[\rho^{ab}]$ by unconstrained LOCC. □

We now turn to a specific application of the above result that yields a striking difference between the amount of entanglement with and without the SSR. Ensemble QIP describes N identical copies of a system of qubits, where N is usually taken to be very large. We consider a situation where access to individual elements of the ensemble is not possible, and thus only *collective* transformations and measurements (i.e., operations which affect each element identically) are allowed. In the following, we formulate this restriction as a SSR, and show that it severely limits the entanglement in the system.

Consider an ensemble consisting of N copies of a single qubit. (For convenience, we assume N is even.) The Hilbert space $\mathbb{H}_2^{\otimes N}$ carries a collective tensor representation R of SU(2), by which a rotation $\Omega \in \mathrm{SU}(2)$ acts identically on each of the N qubits. The Hilbert space also carries a representation P of the symmetric group S_N, which is the group of permutations of the N qubits. The action of these two groups commute, and Schur-Weyl duality [11] states that the Hilbert space $\mathbb{H}_2^{\otimes N}$ carries a multiplicity-free direct sum of SU(2)$\times S_N$ irreducible representations (irreps), each of which can be labelled by the SU(2) total angular momentum quantum number j. Each of these irreps can be factored into a tensor product $\mathbb{H}_{jR}\otimes\mathbb{H}_{jP}$, such that SU(2) acts irreducibly on $\mathbb{H}_{jR}$ and trivially on $\mathbb{H}_{jP}$, and S_N acts irreducibly on $\mathbb{H}_{jP}$ and trivially on $\mathbb{H}_{jR}$. Thus,

$$\mathbb{H}_2^{\otimes N} = \bigoplus_{j=0}^{N/2} \mathbb{H}_{jR}\otimes\mathbb{H}_{jP}\,. \tag{7}$$

The dimension of $\mathbb{H}_{jR}$ is $2j+1$, and that of $\mathbb{H}_{jP}$ is [12]

$$c_j^{(N)} = \binom{N}{N/2-j}\frac{2j+1}{N/2+j+1}. \tag{8}$$

Consider a basis $|j,m\rangle_R \otimes |j,r\rangle_P$ for $\mathbb{H}_{jR}\otimes\mathbb{H}_{jP}$, with $\{|j,m\rangle_R, m=-j,\dots,j\}$ the standard angular momentum basis for $\mathbb{H}_{jR}$ and $\{|j,r\rangle_P, r=1,\dots,c_j^{(N)}\}$ a basis for $\mathbb{H}_{jP}$. The group SU(2)$\times S_N$ acts on this basis as $R(\Omega)|j,m\rangle_R\otimes P(p)|j,r\rangle_P$ for $\Omega\in\mathrm{SU}(2)$ and $p\in S_N$.

In ensemble QIP without individual addressability, the only allowed operations $\mathcal{O}$ are those that are invariant under permutations of elements and thus must satisfy

$$\mathcal{O}[P(p)\rho P^\dagger(p)] = P(p)\mathcal{O}[\rho]P^\dagger(p)\,, \tag{9}$$

for all $p\in S_N$. (Note that these allowed operations include all transformations generated by Hamiltonians in the enveloping algebra of su(2), i.e., which are polynomials in J_x, J_y and J_z. In liquid-NMR QIP the operations are more restricted because there are no controllable inter-molecular interactions. We will not consider that additional restriction here.) Thus, ensemble QIP with this restriction must respect the SSR associated with the symmetric group S_N. Unlike previous examples involving Lie groups, this SSR is associated with a finite group. We define the superoperator

$$\mathcal{P}[\rho] = \frac{1}{N!}\sum_{p\in S_N} P(p)\rho P^\dagger(p)\,, \tag{10}$$

which can be extended to act on states of n qubits on N elements. The action of $\mathcal{P}$ is best seen in the decomposition of Eq. (7): it completely mixes states in $\mathbb{H}_{jP}$ while leaving states in $\mathbb{H}_{jR}$ invariant. The spaces $\mathbb{H}_{jR}$ are called noiseless subsystems (NSs) [13], and are free from the decohering effect of $\mathcal{P}$. These NSs are dual to the collective NSs $\mathbb{H}_{jP}$, which have been explored in the context of quantum computation [13, 14] and quantum communication without shared orientation reference frames [12] and which are free from collective SU(2) decoherence.

We now quantify bipartite entanglement in ensemble QIP constrained by the S_N-SSR; specifically, we show that the standard (unconstrained) measure of entanglement can grossly overestimate the amount of entanglement accessible to Alice and Bob. Consider the following example, where each of N elements possesses two qubits, a and b. These qubits are separated such that all the qubits of type a are given to Alice and b to Bob. If the S_N-SSR is enforced, Alice and Bob are both restricted to local S_N-covariant operations, and any state ρ^{ab} is indistinguishable from the state $\mathcal{P}[\rho^{ab}] = \mathcal{P}^a\otimes\mathcal{P}^b[\rho^{ab}]$.

If the two qubits on every element are described by the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(|uu\rangle^{ab}+|dd\rangle^{ab}\big)$, the state of the total ensemble is $|\Phi^{(N)}\rangle \equiv |\Phi^+\rangle^{\otimes N}$. A naive quantification of entanglement of this pure state gives N ebits. However the constrained entanglement is much less. Expressing this state in terms of the decomposition of Eq. (7) yields

$$\begin{aligned}|\Phi^{(N)}\rangle &= \frac{1}{\sqrt{2^N}}\sum_{j=0}^{N/2}\sum_{m=-j}^{j}\sum_r |j,m\rangle_R^a|j,r\rangle_P^a|j,m\rangle_R^b|j,r\rangle_P^b \\ &= \sum_{j=0}^{N/2}\sqrt{\frac{(2j+1)c_j^{(N)}}{2^N}}|\phi_j\rangle^{ab}|\chi_j\rangle^{ab}\,,\end{aligned} \tag{11}$$

where

$$|\phi_j\rangle^{ab} = \frac{1}{\sqrt{2j+1}}\sum_{m=-j}^{j}|j,m\rangle_R^a|j,m\rangle_R^b\,, \tag{12}$$

$$|\chi_j\rangle^{ab} = \frac{1}{\sqrt{c_j^{(N)}}}\sum_r|j,r\rangle_P^a|j,r\rangle_P^b\,, \tag{13}$$

are (normalised) maximally entangled states in $\mathbb{H}_{jR}^a\otimes\mathbb{H}_{jR}^b$ and $\mathbb{H}_{jP}^a\otimes\mathbb{H}_{jP}^b$, respectively.

The action of $\mathcal{P}$ on the state (11) is

$$\mathcal{P}\big[|\Phi^{(N)}\rangle\langle\Phi^{(N)}|\big] = \sum_{j=0}^{N/2}\frac{(2j+1)c_j^{(N)}}{2^N}|\phi_j\rangle^{ab}\langle\phi_j|\otimes\sigma_j^{ab}\,, \tag{14}$$

where σ_j^{ab} is the (normalised) completely mixed state on $\mathbb{H}_{jP}^a\otimes\mathbb{H}_{jP}^b$. The resulting state is an incoherent sum of maximally-entangled states on each irrep $\mathbb{H}_{jR}^a\otimes\mathbb{H}_{jR}^b$. The entanglement of this state can be easily calculated because both Alice and Bob can *locally* perform a measurement of total J^2, which determines j and yields a pure state in $\mathbb{H}^a_{jR}\otimes\mathbb{H}^b_{jR}$. Thus, the entanglement of this state constrained by the S_N-SSR is

$$E_{S_N\text{-SSR}}\big(|\Phi^{(N)}\rangle\langle\Phi^{(N)}|\big) = \sum_{j=0}^{N/2}\frac{(2j+1)c_j^{(N)}}{2^N}\log_2(2j+1)\,, \tag{15}$$

which, for large N, behaves as $\frac{1}{2}\log_2 N$. Thus, although the state $|\Phi^{(N)}\rangle$ possesses N ebits of unconstrained entanglement, its entanglement constrained by the SSR is only $\frac{1}{2}\log_2 N$ ebits (asymptotically). This result is remarkable: for a pure state of the ensemble, each element consisting of two qubits in a Bell state, the constrained entanglement per element rapidly approaches zero for large N.

Note that the state $|\Phi^{(N)}\rangle$, describing each element as a Bell state, is *not* a maximally entangled state under the S_N-SSR constraint; we now identify such a state. Observing the form of the decohering mechanism $\mathcal{P}$, this state is clearly a pure maximally entangled state in the NS $\mathbb{H}^a_{jR}\otimes\mathbb{H}^b_{jR}$ with the largest dimension, given by $j_0 = N/2$. The entanglement per element of this state is $N^{-1}\log_2(N+1)$, which approaches zero for large N. As this state is the maximally entangled state under the SSR constraint, we have proved that the maximum entanglement per element in the large N limit is zero.

A remarkable duality is evident between the SSR associated with the symmetric group and that associated with the rotation group when one considers the maximum entanglement these SSRs allow. The latter describes a situation where Alice and Bob do not share an orientation reference frame for their qubits [12]. In the Hilbert space decomposition of Eq. (7), the SSR for the rotation group is described by a decohering superoperator on the SU(2) irreps, as opposed to the S_N irreps described above. As proven in [12], the maximum entanglement between N pairs of qubits constrained by the SU(2)-SSR behaves asymptotically as $N-\log_2 N$; in this letter, we proved that the maximum entanglement between N pairs of qubits constrained by the S_N-SSR behaves asymptotically as $\log_2 N$. Note that these two values asymptotically sum to N, which is the maximum unconstrained entanglement for N qubit pairs.

In summary, we have defined generalized SSRs, and quantified the constrained entanglement of a bipartite state as the amount of entanglement that can be distilled into quantum registers using only LOCC that obey the appropriate SSR. Our example of ensemble QIP reveals that systems with apparently large amounts of entanglement can in fact possess very little under the appropriate SSR constraint. We note that this result for ensemble QIP applies to liquid-state NMR [6], where qubits are realized as nuclear spins on a molecule and a sample generally contains $N\sim 10^{20}$ molecules [16]. Our operational definition of entanglement constrained by a S_N-SSR is also applicable to spin-squeezed atomic gases [7]; because the NS corresponding to $j_0 = N/2$ is used in such experiments, our result shows that the present measures of entanglement used for this system [15] *are* appropriate.

Another question related to our operational definition of entanglement is: What constraints does a SSR impose on the entangled states that can be created (formed) from a set amount of entanglement in the quantum registers? As pointed out in [5], it is not even possible to create certain *separable* states in the presence of a SSR. It is clear that SSRs place severe contraints on QIP, and our operational definitions of SSRs and entanglement constrained by them provide a new understanding and a valuable tool to quantum information science.

We thank E. Knill, R. W. Spekkens, F. Verstraete and P. Zanardi for helpful comments on our manuscript, and thank M. A. Nielsen, T. Rudolph, B. C. Sanders, and R. W. Spekkens for valuable prior discussions.

* Electronic address: bartlett@physics.uq.edu.au
† Electronic address: H.Wiseman@griffith.edu.au

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information,* (Cambridge University Press, Cambridge, 2000).
[2] M. Horodecki *et al*, Phys. Rev. Lett. **84**, 2014 (2000).
[3] G. C. Wick *et al*, Phys. Rev. **88**, 101 (1952).
[4] H. M. Wiseman and J. A. Vaccaro, Phys. Rev. Lett. **91**, 097902 (2003).
[5] F. Verstraete and J. I. Cirac, Phys. Rev. Lett. **91**, 010404 (2003).
[6] D. G. Cory *et al*, Proc. Natl. Acad. Sci. USA **94**, 1634 (1997); N. Gershenfeld and I. L. Chuang, Science **275**, 350 (1997).
[7] B. Julsgaard *et al*, Nature (London) **413**, 400 (2001).
[8] F. Strocchi and A. S. Wightman, J. Math. Phys. **15**, 2198 (1974).
[9] W. H. Zurek, Rev. Mod. Phys. **75**, 715 (2003).
[10] H. M. Wiseman, Proceedings of SPIE Vol. 5111 Fluctuations and Noise in Photonics and Quantum Optics, Eds. D. Abbott, J. H. Shapiro, and Y. Yamamoto (SPIE, Bellingham, WA, 2003), pp 78-91. `quant-ph/0303116`.
[11] W. Fulton and J. Harris, *Representation theory: a first course*, (Springer-Verlag, Berlin, 1991).
[12] S. D. Bartlett *et al*, Phys. Rev. Lett. **91**, 027901 (2003).
[13] E. Knill *et al*, Phys. Rev. Lett. **84**, 2525 (2000).
[14] P. Zanardi and M. Rasetti, Phys. Rev. Lett. **79**, 3306 (1997); P. Zanardi, Phys. Rev. A **63**, 012301 (2000).
[15] J. K. Stockton *et al*, Phys. Rev. A **67**, 022112 (2003).
[16] However, NMR QIP possesses additional constraints due to impure initial conditions, restrictions on the allowed operations, and inefficient read-out.