An analytical solution for tidal fluctuations in unconfined aquifers with a vertical beach

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The perturbation technique has been commonly used to develop analytical solutions for simulating the dynamic response of tidal fluctuations in unconfined aquifers. However, the solutions obtained from the perturbation method might result in poor accuracy for the case of the perturbation parameter being not small enough. In this paper, we develop a new analytical model for describing the water table fluctuations in unconfined aquifers, based on Laplace and Fourier transforms. In the new approach, the mean sea level is used as the initial condition and a free surface equation, neglecting the second-order slope terms, as the upper boundary condition. Numerical results show that the present solution agrees well with the finite different model with the second-order surface terms. Unlike Teo et al.’s (2003) approximation which restricts on the case of shallow aquifers, the present model can be applied to most of the tidal aquifers except for the very shallow one. In addition, a large-time solution in terms of sine function is provided and examined graphically with four different tidal periods.


1. Introduction

The water table of the coastal unconfined aquifer usually fluctuates with periodical tides, which are in fact composed of several dozens of tidal components. The anisotropy of an aquifer in the horizontal and vertical directions is a consequence of long-time sediment migration and deposition. The unconfined aquifer has a free surface on the top of the aquifer system and this may introduce a vertical flow component. Most previous analytical approximations for unconfined or leaky coastal unconfined aquifers used a linear groundwater flow equation to approximate the unconfined flow [e.g., van der Kamp, 1972; Li and Chen, 1991; Jiao and Tang, 1999; Li et al., 2001; Li and Jiao, 2001; Jeng et al., 2002; Li et al., 2007; Xia et al., 2007; Chuang and Yeh, 2007; Sun et al., 2008]. Song et al. [2007] adopted the Boussinesq equation to derive the solution for groundwater fluctuations due to tidal waves in coastal unconfined aquifers. However, they neglected the vertical flow in their mathematical model. Li et al. [2000] developed the solution of one-dimensional (1-D) linearized Boussinesq equation for unconfined aquifers with moving shorelines in a sloping beach. Teo et al. [2003] presented a solution of two-dimensional (2-D) Laplace equation for unconfined aquifers with sloping beaches under the free surface boundary condition using the perturbation technique. They found that the second-order perturbation term and beach slope are important in the development of the solution for water table fluctuations. On the basis of the perturbation technique, some researches [e.g., Dentz et al., 2006; Bolster et al., 2007; Chang et al., 2010] decoupled the flow and transport equations and developed analytical models to describe the behavior of groundwater flow in coastal aquifers.

In this paper, a mathematical model, which is composed of 2-D groundwater flow equation with the mean sea level (MSL) as the initial condition and a first-order free surface equation as the upper boundary condition, is developed for describing the water level fluctuation in tidal unconfined aquifers. The time domain solution of the model is developed by applying Laplace and Fourier sine transforms. This new analytical solution is compared with the solution presented by Teo et al. [2003] and finite difference solution with the second-order slope terms in the free surface equation. The influences of the anisotropy and specific storage on the dynamic groundwater fluctuations are investigated. In addition, the advantages and possible applications of the new solution are also addressed.

2. Theoretical Formulations

2.1. Boundary Value Problem

The conceptual model for a 2-D homogeneous, anisotropic tidal unconfined aquifer is depicted in Figure 1, which is infinitely extended landward in horizontal direction and of a vertical beach. The origin of the coordinates is located at the intersection of the vertical beach and MSL which is considered as the reference datum. The thickness of the aquifer is D measured from the bottom of the aquifer to the MSL, and A is the amplitude of the tide.
the 2-D transient head distribution $\phi(x, z, t)$ is [Neuman and Witherspoon, 1970; Batu, 1998]

$$K_x \frac{\partial^2 \phi}{\partial x^2} + K_z \frac{\partial^2 \phi}{\partial z^2} = S_y \frac{\partial \phi}{\partial t},$$

where $K_x$ and $K_z$ are horizontal and vertical hydraulic conductivities, respectively, and $S_y$ is specific storage.

[6] The boundary condition for the bottom of the aquifer is

$$\frac{\partial \phi}{\partial z}(x, -D, t) = 0,$$

and the remote boundary condition is

$$\lim_{z \to \infty} \frac{\partial \phi}{\partial z} = 0.$$

The seepage face is neglected if the coastal aquifer is highly permeable. The tidal boundary is then expressed as

$$\phi(x, z, 0) = A \sin(\omega t),$$

where $\omega$ is tide frequency.

[7] The 2-D equation describing the free surface for the unconfined aquifer without a surface recharge can be written as [Batu, 1998, p. 107]

$$S_y \frac{\partial \phi}{\partial t} = K_x \left( \frac{\partial \phi}{\partial x} \right)^2 + K_z \left( \frac{\partial \phi}{\partial z} \right)^2 - K_z \frac{\partial \phi}{\partial z} \text{ at } z = \phi.$$  \(5\)

Consider the case that the slope of the water table is small such that the second-order terms of the hydraulic gradient on the right-hand side (RHS) of equation (5) are negligible. Thus, equation (5) may be simplified as

$$S_y \frac{\partial \phi}{\partial t} = -K_z \frac{\partial \phi}{\partial z} \text{ at } z = \phi.$$  \(6\)

Compared to the aquifer thickness ($D$), the tidal amplitude may be small and the actual saturated thickness $\phi$ can therefore be approximated by the initial aquifer thickness ($D$). The head at the free surface ($z = \phi$) is hence replaced by the head at the MSL ($z = 0$), implying that the head distribution at the MSL is the water table. The domain in $z$ direction ranges from 0 to $-D$. Equation (6) is then expressed as [Neuman, 1972]

$$S_y \frac{\partial \phi}{\partial t} = -K_z \frac{\partial \phi}{\partial z} \text{ at } z = 0.$$  \(7\)

It is worth noting that equation (7) gives good approximation when $\alpha = A/D$ (amplitude parameter) is small.

[8] Consider that the initial condition is equal to the MSL, thus

$$\phi(x, z, 0) = 0.$$  \(8\)

It is common to express the governing equation and boundary conditions in dimensionless form. Define the following dimensionless variables:

$$X = \frac{x}{L}, \quad Z = \frac{z}{D}, \quad \Phi = \frac{\phi}{D}, \quad \text{and } T = \omega t,$$

where $L$ is the decay length defined as [Nielsen, 1990]

$$L = \sqrt{\frac{2K_zD}{n_w}}.$$  \(10\)

With equation (9), the governing equation (1) leads to

$$\varepsilon^2 \frac{\partial^2 \Phi}{\partial X^2} + K_D \frac{\partial^2 \Phi}{\partial Z^2} = a_s \frac{\partial \Phi}{\partial T},$$  \(11\)

where $\varepsilon = D/L$ (perturbation parameter), $K_D = K_z/K_x$, and $a_s = SwD/K_x$ with $S = S_y D$. The boundary conditions (2)–(4) and (7)–(8) then become

$$\frac{\partial \Phi}{\partial Z}(X, -1, T) = 0,$$

$$\lim_{X \to \infty} \frac{\partial \Phi}{\partial X} = 0,$$

$$\Phi(0, Z, T) = \alpha \sin(T),$$

$$a_s \frac{\partial \Phi}{\partial T}(X, 0, T) = -\sigma K_D \frac{\partial \Phi}{\partial Z}(X, 0, T),$$

$$\Phi(X, Z, 0) = 0,$$

where $\sigma = S/S_y$. 

Figure 1. Schematic diagram of a tidal unconfined aquifer with a vertical beach.
2.2. Previous Perturbation Solution

[8] Teo et al. [2003] presented a perturbation solution for 2-D Laplace equation with a tidal boundary at the sloping beach and free surface boundary defined by equation (5) for isotropic unconfined aquifers. Their head solution for aquifers with a vertical beach can be expanded in powers of $\varepsilon$ and $\alpha$. Using the MSL as the reference datum, the zero-order ($\varepsilon^0$) solution can be expressed as

$$
\Phi_0 = \alpha e^{-X} \cos(\theta_1) + \alpha^2 \left\{ \frac{1}{2} \left( 1 - e^{-2X} \right) + \frac{1}{2} \left[ e^{-2X} \cos(\theta_2) - e^{-2X} \cos(2\theta_1) \right] \right\},
$$

(17)

where $\theta_1 = T - X$ and $\theta_2 = 2T - \sqrt{2}X$. Owing to the vertical beach, the first-order ($\varepsilon^1$) solution is equal to the zero-order solution, i.e.,

$$
\Phi_1 = \Phi_0.
$$

(18)

Their second-order ($\varepsilon^2$) solution can be expressed as

$$
\Phi_2 = \Phi_0 - \frac{\sqrt{2}}{3} \alpha \varepsilon^2 X e^{-X} \cos\left( \theta_1 - \frac{\pi}{4} \right) + \frac{1}{3} \alpha^2 \varepsilon^2 \left\{ 1 + \left( 1 + \frac{X}{2} \right) \right\} e^{-2X} - 2X e^{-\sqrt{2}X} \cos\left( \theta_1 - \frac{\pi}{4} \right) + e^{-\sqrt{2}X} \sin(\theta_2) + \sqrt{2} X e^{-2X} \cos\left( 2\theta_1 - \frac{\pi}{4} \right) - e^{-2X} \sin(2\theta_1).
$$

(19)

Note that $\Phi_0$, $\Phi_1$, and $\Phi_2$ are expanded in powers of $\alpha$ and $\varepsilon$, and thus, equation (19) gives accurate approximation only when both $\varepsilon$ and $\alpha$ are small.

2.3. New Analytical Solution

[10] Applying Laplace and Fourier sine transforms to equations (11)-(16) and inverting the result leads to the following solution:

$$
\Phi(X, Z, T) = \frac{4\alpha e^{-2}}{\pi} \int_0^\infty \left\{ w_0(\zeta) + \sum_{n=1}^{\infty} w_n(\zeta) \right\} \sin(X\zeta)d\zeta.
$$

(20)

with

$$
w_0(\zeta) = \frac{c_{01}(\zeta)c_{02}(\zeta)\{- \exp[c_{02}(\zeta) T] + \cos(T) + c_{02}(\zeta) \sin(T)\}}{1 + c_{02}(\zeta)^2},
$$

(21)

$$
w_n(\zeta) = \frac{c_{01}(\zeta)c_{02}(\zeta)\{ \exp[-c_{02}(\zeta) T] - \cos(T) + c_{02}(\zeta) \sin(T)\}}{1 + c_{02}(\zeta)^2},
$$

(22)

where

$$
c_{01}(\zeta) = \frac{\zeta [1 + \exp[2(1 + Z)\beta_0]]}{\beta_0 \exp[Z\beta_0] \{- K_D(\sigma - 2\beta_0) + \exp(2/\beta_0) \{ K_D[\sigma + 2\sigma\beta_0 + 2\beta_0(1 + \beta_0)] - 2\epsilon^2\zeta^2 \} \}},
$$

(23)

For the detailed development of equation (20), readers are referred to section A1.

[11] The $\beta_0$ in equations (23) and (25) and $\beta_n$ in equations (24) and (26) are the roots of following two equations:

$$
\tan(2\beta_n) = \frac{2K_D^2 \beta_n \sigma \mu c_{02}(\zeta)}{K_D^2 (\sigma - 2\beta_n + \beta_n^2) + 2K_D^2 \epsilon^2 \beta_n^2 + \epsilon^4 \zeta^2},
$$

(28)

respectively. Note that the variables $\Phi$, $X$, $Z$, and $T$ have been defined as the dimensionless hydraulic head, inland distance, elevation, and time, respectively, in equation (9). In addition, $\zeta$ is a variable in Fourier sine transform domain defined in section A1.

[12] Equation (27) can be rearranged as

$$
e^\frac{\sigma^2 \zeta^2}{2} = \frac{K_D^2 \beta_0^2 (e^{2\beta_0} - 1) + K_D^2 \beta_0^2 (e^{2\beta_0} + 1)}{(e^{2\beta_0} + 1)}.
$$

(29)

Substituting equation (29) into equation (25) results in

$$
c_{02}(\beta_0) = \frac{-K_D^2 \beta_0^2 (e^{2\beta_0} - 1)}{a_c (e^{2\beta_0} + 1)}.
$$

(30)

From equation (30), since the parameters $K_D$, $\sigma$, and $a_c$ are all positive, the term $e^{2\beta_0}$ is less than one for a negative $\beta_0$. On the other hand, $e^{2\beta_0}$ is greater than one for a positive $\beta_0$. This indicates that $c_{02}(\beta_0)$ is always negative for any value of $\beta_0$.

[13] A large-time solution can be obtained by neglecting the exponential term in equations (21) and (22) and replacing $T$ by $T'$ which denotes the large-time variable since $c_{02}(\zeta)$ is always less than zero. The result is

$$
\Phi(X, Z, T') = \alpha' \sin(T' + \phi),
$$

(31)

with

$$
\alpha' = \sqrt{a^2 + b^2},
$$

(32)
the superscript \( n+1 \) denotes the time level of the head which is one step later than the time level \( n \).

[15] The boundaries of (2)–(4) are approximated as

\[
\phi_{i+1,j}^{n+1} = \phi_{i,j}^{n+1},
\]

\[
\phi_{n+1,j} = \phi_{n+1,j+1},
\]

\[
\phi_{n+1,j} = A \sin[w(t^n + \Delta t)],
\]

where \( nx \) and \( nz \) represent the total nodal numbers in \( x \) and \( z \) axis of the problem domain, respectively.

[16] The boundary describing the free surface at the MSL can be approximated as

\[
S_i \frac{\phi_{i+1}^{n+1} - \phi_i^{n+1}}{\Delta t} = K_x \left( \frac{\phi_{i+1}^{n+1} - \phi_i^{n+1}}{\Delta x} \right) + K_z \left( \frac{\phi_{i+1}^{n+1} - \phi_i^{n+1}}{\Delta z} \right) - \frac{1}{2} \frac{\phi_i^{n+1} - \phi_{i+1}^{n+1}}{2\Delta x}.
\]

(40)

Furthermore, equation (40) is rearranged as

\[
\phi_{i+1}^{n+1} + \frac{2\Delta x}{K_x} S_i \frac{\phi_{i+1}^{n+1} - \phi_i^{n+1}}{\Delta t} - \frac{2\Delta z}{K_z} \left( \frac{\phi_{i+1}^{n+1} - \phi_i^{n+1}}{\Delta z} \right)^2 = \frac{1}{2} \left( \frac{\phi_i^{n+1} - \phi_{i+1}^{n+1}}{2\Delta x} \right)^2.
\]

(41)

The \( \Delta x \) and \( \Delta z \) are chosen as 1 m, and \( \Delta t \) is set as 0.01 min in the numerical simulations. The dynamic fluctuation is assumed to propagate to a distance of 300 m in \( x \) direction, and this distance is chosen as the infinite boundary. The convergence of the finite difference solution is discussed in section B1.

3. Results and Discussion

[17] The following sections give the comparisons of the present solution with the results of equation (19), Teo et al.’s [2003] perturbation solution, and finite difference solution, i.e., equation (41). The effect of the anisotropy on the head distributions is investigated for \( \sigma = 5 \times 10^{-3}, \ Z = 0, \ \alpha_x = 8.73 \times 10^{-4}, \ \alpha = 0.05, \ \varepsilon = 0.3 \). The parameter \( \alpha_x \) is defined as \( S_w D/K_x \), and its value can be obtained with following parameters: \( S = 10^{-3}, \ w = 2\pi(12 \times 60) \ \text{min}^{-1}, \ D = 10 \ \text{m}, \ K_x = 0.1 \ \text{m} \ \text{min}^{-1} \).

3.1. Comparison With the Finite Difference Solution

[18] Figure 2 shows the spatial tidal fluctuations at \( T = \pi/2, \ 3\pi/2, \ \text{and} \ 2\pi \) graphically based on the present solution, equation (20), and the finite difference solution. As shown in Figure 2, the present solution has a good agreement with the finite difference solution. This implies that the neglect of the second-order terms in equation (5) has no significant effect on the aquifer head distribution, and the motion of the water table is mainly vertical. Define the relative difference as

\[
D(X) = \left| \frac{\Phi_{\text{Num}}(X) - \Phi_{\text{Ana}}(X)}{\Phi_{\text{Num}}(X)} \right| \times 100%.
\]
where $\Phi_{\text{Num}}(X)$ and $\Phi_{\text{Ana}}(X)$ are the finite difference solution and present solution, respectively, at inland distance $X$. Then, the maximum relative difference between the present solution and finite difference solution is $D(0.83) \approx 5\%$ at the largest tide period, i.e., $T = 0.5\pi$. This discrepancy comes from the fact that the horizontal hydraulic gradient near the free surface is larger at large-tide period than those at other tide periods, and therefore, the neglect of second-order terms of the hydraulic gradient in equation (5) results in small error.

### 3.2. Application of the Large-Time Solution

Consider that the threshold of large-time $T'$ is a dimensionless time when the sum of the exponential terms in equation (20) reaches a very small value; say, e.g., $10^{-4}$. Then those exponential terms are negligible. Figure 3 shows the threshold of large-time $T'$ versus dimensionless inland distance, indicating that the threshold time increases with dimensionless inland distance. Figure 4 exhibits the head fluctuations versus dimensionless time at different dimensionless inland distances. Figure 4 indicates that storage coefficient plays the role of dumping effect at early time, and the head fluctuation has a tendency toward a harmonic motion because of the decrease in exponential terms. When the dimensionless time exceeds the threshold of large time, the motion of the water table is almost a harmonic motion, and the effect of storage coefficient on the head fluctuation is negligible.

### 3.3. Comparison With Teo et al.’s [2003] Perturbation Solution

The large-time solution, equation (31), is compared with Teo et al.’s [2003] perturbation solution for a vertical beach. Note that the present solution gives good accuracy when $\alpha$ is not very large while their solution is accurate only for small $\varepsilon$ and $\alpha$. Figures 5–8 illustrate the present solution for head distribution versus horizontal distance from the sea shore, the first-order and second-order solutions of Teo et al. [2003] for two different tidal periods. Figures 5–8 demonstrate that the difference between Teo et al.’s [2003] first-order and second-order solutions increases with the perturbation parameter $\varepsilon$, indicating that the high-order terms have the significant influence on the groundwater flow in a coastal aquifer.

The finite difference solutions are included in Figures 5–8 to compare the present solution with Teo et al.’s [2003] solution for $\alpha = 0.1$ and $\alpha = 0.4$, respectively, with various $\varepsilon$. In Figures 5a and 5b, the difference between the first-order and second-order solutions of Teo et al. [2003] is very small when $\varepsilon = 0.1$ and 0.2 for $\alpha = 0.1$. This indicates that their solution gives accurate results when both $\varepsilon$ and $\alpha$ are small. In addition, the present solution is of a good agreement with Teo et al.’s [2003] solution when $\alpha = 0.1$ as shown in Figures 5a and 5b for $\varepsilon = 0.1$ and 0.2, respectively. The difference between first-order and second-order solutions by Teo et al. [2003] increases with $\varepsilon$ when $\alpha = 0.1$. On the other hand, the present solution gives accurate results as shown in Figures 5c and 5d even $\varepsilon$ is very large, say $\varepsilon \geq 0.5$, when $\alpha = 0.1$.

Compared Figures 5c with 6c and 5d with 6d, since Teo et al.’s [2003] solutions are expanded in power of $\alpha$, the difference between the first-order and second-order solutions increases with $\alpha$ for a fixed $\varepsilon$. Therefore, their results are in poor accuracy when $\alpha$ is large. However, compared Figures 5a with 6a and 5b with 6b, there is no obvious difference between the first-order and second-order solutions of Teo et al. [2003] for various $\alpha$ when $\varepsilon$ is fixed. This is because $\varepsilon$ gives a second-order effect in equation (19), and thus, the difference between first-order and second-order solutions by Teo et al. [2003] is small for small $\varepsilon$ even...
when $\alpha$ is large. This reflects that $\varepsilon$ has a larger effect on the accuracy of their solution than that of $\alpha$.

Define the maximum absolute difference between the present solution and Teo et al.'s solution as a function of $\varepsilon$ and $\alpha$

$$D_{\text{max}}(\varepsilon, \alpha) = \max|\Phi_{\text{Ana}}(\varepsilon, \alpha) - \Phi_{\text{Teo}}(\varepsilon, \alpha)|,$$

where $\Phi_{\text{Ana}}(\varepsilon, \alpha)$ and $\Phi_{\text{Teo}}(\varepsilon, \alpha)$ are the hydraulic heads predicted by the present solution and Teo et al.'s [2003] solution, respectively. The maximum absolute difference increases with $\varepsilon$ and $\alpha$ as shown in Figure 9. Obviously, the present model can provide more accurate predictions than Teo et al. [2003], especially when both $\varepsilon$ and $\alpha$ are large.

### 3.4. Effect of Aquifer Anisotropy

The effect of aquifer anisotropy on the head fluctuation is examined in Figure 10. The long-time sediment depositional process may result in an aquifer with the smaller vertical hydraulic conductivity in comparison with the horizontal hydraulic conductivity. The ratio of the vertical to horizontal hydraulic conductivity (conductivity ratio) may range from 10% to 33% [Freeze and Cherry, 1979], and therefore, the tidal aquifer is very likely anisotropic. Figure 10 shows that the anisotropy of the hydraulic conductivity has obvious influences on the head fluctuation near the sea shore when the conductivity ratio is significantly smaller than one and the head increases slowly in response to the rise of the sea level for small $K_D$. It reflects that the head has a slow response to the tide when $K_D$ is small.

### 3.5. Potential Applications of the Proposed Model

The proposed solution provides better predictions of tide-induced groundwater fluctuations in coastal aquifers and wider range of applications, compared with the previous perturbation approximation proposed by Teo et al. [2003]. The present analytical solutions can be applied to several practical engineering problems. For example, storm surge induced groundwater fluctuations, which normally associates with large-wave amplitude related to the thickness of aquifers. In this case, the previous perturbation solution cannot provide good predictions, but the present solution will do. Since the prediction of groundwater fluctuations in coastal aquifers has significant effects on the understanding of processes of saltwater intrusion and biologic activities in coastal regions, the present model will provide an effective
tool for environmental scientists who involve in the assessment of environmental impacts in coastal regions.

4. Concluding Remarks

[26] A two-dimensional mathematical model for describing dynamic head response in a tidal unconfined aquifer with a vertical beach has been developed. The aquifer is anisotropic, homogenous and of an impermeable bottom. In addition, a first-order free surface equation is chosen as the upper boundary condition and the MSL as the initial condition for the aquifer. The analytical solution of the model is developed by using Fourier sine transforms and Laplace transforms. In addition, a large-time solution is obtained from neglecting the exponential terms in the solution. The large-time solution is compared with the finite difference solution and Teo et al.’s perturbation solution, where both solutions include second-order hydraulic gradient terms at the free surface boundary. The following major conclusions can be drawn: (1) The neglect of the second-order terms of the hydraulic gradient in the free surface equation has no obvious effect on the predicted head distribution in tidal unconfined aquifers because the motion of the water table is mainly vertical. (2) Unlike the previous approximation [Teo et al., 2003], the present model does not have any restriction on the value of $\varepsilon$. (3) The influence of the storage coefficient on dynamic response of the tidal aquifer is very small and thus negligible. (4) The anisotropic hydraulic conductivity has significant influence on the predicted head distribution if the ratio of the vertical to horizontal hydraulic conductivity is smaller than one.

Appendix A: Derivation of Equation (20)

[27] Applying Fourier sine transforms and Laplace transforms to the equations (11)–(16) results in an ordinary differential equation and boundary conditions in terms of $Z$. Solving the differential equation, one can obtain

\[
\Phi_1(\zeta, Z, p) = \Phi_{11}(\zeta, Z, p) + \Phi_{12}(\zeta, p),
\]

\[
\Phi_{11}(\zeta, Z, p) = \sqrt{\frac{2}{\pi}} \frac{\alpha^{1.5}}{K_D p^{1.5}} \left[ (2+Z)\eta + e^{-2\eta} \right],
\]

\[
\Phi_{12}(\zeta, Z, p) = \sqrt{\frac{2}{\pi}} \frac{\alpha^{1.5}}{K_D p^{1.5}} \left( p^2 + 1 \right)^{-0.5},
\]

\[
\eta = \sqrt{\frac{\alpha^{2.5} + a_p p}{K_D}}.
\]
where $p$ and $\zeta$ are the variables of the Laplace and Fourier sine transforms, respectively.

[28] Taking inverse Laplace transform of equation (A1) results in

$$f(p) = \left( \frac{K \sigma}{a_s} - p \right) e^{2p} - \frac{K \sigma}{a_s} e^{2p} + p , \quad (A5)$$

where $p$ and $\zeta$ are the variables of the Laplace and Fourier sine transforms, respectively.

[29] Applying Cauchy's residue theorem to $G(p)$ and $L^{-1}[\Phi_2]$ obtains

$$g(T) = L^{-1}[G(p)] = \sin(T) , \quad (A9)$$

respectively.

[30] Let $h(T)$ be the inverse Laplace transforms of $H(p)$ and the result after applying the Bromwich integral is

$$h(T) = \frac{1}{2\pi i} \int_{C}^{C+r+i\infty} H(p) e^{rt} \, dp , \quad (A11)$$

where $i$ is an imaginary unit and $r$ is a real constant, which is so large that all of the real parts of the poles are smaller than it. The graph of the Bromwich integral contains a close contour with a straight line parallel to the imaginary axis and two semicircles. According to Jordan's Lemma, the values of the integration for the semicircles tend to zero when radius $R$ approaches infinity. Consequently, one can obtain

$$h(T) = \sum_{n=1}^{\infty} \text{Res}_{p_n} , \quad (A12)$$

where $p_n$ are the singularities in complex plane.
There are infinite singularities in $H(p)$ and obviously one pole occurs at $p = -\frac{\varepsilon^2 \zeta^2}{\alpha}$. In order to determine other singularities, we have $f(p) = 0$ for equation (A5). There are two ways to satisfy equation (A5): one is $\eta = \beta_0$ or $p = c_{02}(\zeta)$ and the other is $\eta = i\beta_n$ or $p = -c_{n2}(\zeta)$, where both $\beta_0$ and $\beta_n$ are real variables. Note that both $c_{02}(\zeta)$ and $-c_{n2}(\zeta)$ are the singularities in terms of $\beta_0$ and $\beta_n$ respectively. Substituting $\eta = \beta_0$ and $p = c_{02}(\zeta)$ into equation (A5) results in equation (27). Similarly, substituting $\eta = i\beta_n$ and $p = -c_{n2}(\zeta)$ into equation (A5) yields equation (28). Observe that there are infinite roots $\beta_n$ in equation (28). The residues of the singularities can be determined from the following formula [Kreyszig, 1999]:

$$\text{Re } s_{p=p_n} = \lim_{p \to p_n} H(p) e^{\rho T} (p - p_n). \quad (A13)$$

Substituting equation (A8) into equation (A13), one obtain

$$\text{Re } s_{p=p_n} = \lim_{p \to p_n} \frac{\sqrt{2 - \alpha \varepsilon^2 \zeta^2 p [e^{(2+\alpha)\pi} + e^{-(2+\alpha)\pi}]} \right \rho T (p - p_n). \quad (A14)$$
By L’Hopital’s rule, equation (A14) can be reduced to

\[
\text{Re } s_{|p-p_n}| = \lim_{p \to p_n} \frac{-2\pi \alpha e^{2} \left( 1 / a_n \right) \exp \left( -\frac{e^{2} \zeta^2}{a_n} T \right) \rho^T}{\pi^2 (1 + e^{2n}) K_n \eta + 2 e^{2n} (1 + \eta^2) + 2 \alpha_n (2 + e^{2n}(2 + \eta + \sigma))},
\]

respectively. Note that there are infinite singularities in \(-c_{n2}(\zeta)\) due to the infinite roots of \(\beta_n\). Therefore,

\[
h(T) = \left( \text{Re } s_{|p-c_{n2}(\zeta)} + \sum_{k=1}^{\infty} \text{Re } s_{|p-c_{n2}(\zeta)} \right)
\]

By the convolution theorem,

\[
\Phi_n = \int_0^T g(\tau) h(T - \tau) d\tau,
\]

Figure 10. The head fluctuation versus the inland distance with various vertical hydraulic conductivities at various times.
Figure B1. The convergence of the finite difference solution.

The convergence of the finite difference solution

$T = 0.5\pi, T = \pi, T = 1.5\pi$

\[ F(\zeta, Z, T) = \sqrt{\frac{2}{\pi}} a_n T \left[ w_0(\zeta) + \sum_{n=1}^{\infty} w_n(\zeta) \right] - \sqrt{\frac{2}{\pi}} a_n \zeta^2 \cos(T) \cos(T) \sin(T). \]  

\[ \Phi_\zeta(\zeta, Z, T) = \frac{2}{\pi} \alpha e^2 \left[ w_0(\zeta) + \sum_{n=1}^{\infty} w_n(\zeta) \right] . \]  

The final solution given as equation (20) can be obtained after taking inverse Fourier sine transform of equation (A22).

Appendix B: Convergence of the Finite Difference Solution

The validity of the numerical solution can be checked by comparing with an analytical solution. Unfortunately, the analytical solution is not available because the problem has a nonlinear free surface boundary (i.e., equation (5)). Therefore, the convergence of the numerical solution is checked via the use of successively smaller nodal spacing. Figure B1 presents the head fluctuations graphically at times $0.5\pi, \pi, 1.5\pi$, and $2\pi$ with different spatial and temporal increments. It demonstrates that the result of the simulation

\[ \Delta \tau = 0.02 \text{ min} \text{ is close to that with} \Delta \tau = \Delta \zeta = 1 \text{ m and} \Delta \tau = 0.01 \text{ min. Furthermore, the result is almost unchanged even when the spatial and temporal increments are reduced to} \Delta \tau = \Delta \zeta = 0.5 \text{ m and} \Delta \tau = 0.005 \text{ min, respectively, indicating that the good accuracy in the numerical solution has been reached.} \]

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