WATER RESOURCES RESEARCH, VOL. 38, NO. 12, 1265, doi:10.1029/2001WR000611, 2002

Scope for further analytical solutions for constant flux infiltration into a semi-infinite soil profile or redistribution in a finite soil profile

D. A. Barry,[1] J.-Y. Parlange,[1,2] I. G. Lisle,[3] L. Li,[1,7] D.-S. Jeng,[4] F. Stagnitti,[5] and G. C. Sander[6]

Received 16 April 2001; revised 2 March 2002; accepted 13 March 2002; published 3 December 2002.

[1] We attempt to generate new solutions for the moisture content form of the one-dimensional *Richards*' [1931] equation using the *Lisle* [1992] equivalence mapping. This mapping is used as no more general set of transformations exists for mapping the one-dimensional Richards' equation into itself. Starting from a given solution, the mapping has the potential to generate an infinite number of new solutions for a series of nonlinear diffusivity and hydraulic conductivity functions. We first seek new analytical solutions satisfying Richards' equation subject to a constant flux surface boundary condition for a semi-infinite dry soil, starting with the Burgers model. The first iteration produces an existing solution, while subsequent iterations are shown to endlessly reproduce this same solution. Next, we briefly consider the problem of redistribution in a finite-length soil. In this case, Lisle's equivalence mapping is generalized to account for arbitrary initial conditions. As was the case for infiltration, however, it is found that new analytical solutions are not generated using the equivalence mapping, although existing solutions are recovered. *INDEX TERMS:* 1866 Hydrology: Soil moisture; 1875 Hydrology: Unsaturated zone; 3210 Mathematical Geophysics: Modeling

Citation: Barry, D. A., J.-Y. Parlange, I. G. Lisle, L. Li, D.-S. Jeng, F. Stagnitti, and G. C. Sander, Scope for further analytical solutions for constant flux infiltration into a semi-infinite soil profile or redistribution in a finite soil profile, *Water Resour. Res.*, *38*(12), 1265, doi:10.1029/2001WR000611, 2002.

1. Introduction

[2] Richards' equation [*Richards*, 1931] has stimulated much analytical and numerical analysis over several decades. Its strong nonlinearity and seemingly near-pathological behavior engender continuing interest in its underlying structure and symmetries [e.g., *Yung et al.*, 1994; *Sophocleous*, 1996; *Vijayakumar*, 1997].

[3] *Lisle* [1992] discovered an extended set of equivalence transformations for Richards' equation. This set was initiated by writing Richards' equation in potential form

$$\frac{\partial I}{\partial z} = -\theta, \tag{1}$$

and

$$\frac{\partial I}{\partial t} = q, \tag{2}$$

where $I(z, t)$ is the cumulative volume of water that has passed location z (positive downward) at time t, and θ is the volumetric content. The Darcy flux q is given by

$$q = K(\theta) - D(\theta)\frac{\partial \theta}{\partial z}, \tag{3}$$

where K is the hydraulic conductivity and D is the capillary diffusivity.

[4] *Lisle*'s [1992] equivalence transformations were used recently by *Barry et al.* [2001] in the context of body force scaling of unsaturated flow experimental data. They defined the new class of similar soils, called Lisle-similar soils, for which mapping of experimental results is permitted. Miller-similar soils [*Miller and Miller*, 1955a, 1955b, 1956; *Miller*, 1980] were shown to be a special case of the new class. On the basis of the mildly nonlinear Burgers equation solution, we show in some detail how the set of equivalence transformations can be used to solve Richards' equation for constant flux infiltration at the surface into a uniformly moist unsaturated soil but with a sequence of diffusivity and hydraulic conductivity functions. However, this sequence is shown to reduce to an existing solution. Next, we seek new solutions for redistribution in a finite-length profile. Again, the analysis yields solutions that are known.

[1]Contaminated Land Assessment and Remediation Research Centre, Institute for Infrastructure and Environment, School of Engineering and Electronics, University of Edinburg, Edinburg, UK.

[2]Permanently at Department of Biological and Environmental Engineering, Cornell University, Ithaca, New York, USA.

[3]School of Mathematics and Statistics, University of Canberra, Canberra, ACT, Australia.

[4]School of Engineering, Griffith University, Gold Coast, Australia.

[5]School of Ecology and Environment, Deakin University, Warrnambool, Australia.

[6]Department of Civil and Building Engineering, Loughborough University, Loughborough, Leicestershire, UK.

[7]Now at Department of Civil Engineering, The University of Queensland, St. Lucia, Australia.

0043-1397/02/2001WR000611

2. Infiltration

[5] The infiltration model we wish to solve is given by *Richards*' [1931] equation and associated boundary and initial conditions, namely,

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial\theta}{\partial z}\right] - \frac{dK(\theta)}{d\theta}\frac{\partial\theta}{\partial z}, \qquad 0 < z, 0 < t, \theta_r \le \theta \le \theta_s \tag{4}$$

$$q = q_0, \qquad z = 0, 0 < t \tag{5}$$

$$\theta(z,0) = \theta_i, \qquad 0 < z,\ \theta_r \le \theta_i \le \theta_s \tag{6}$$

and

$$\lim_{z\to\infty} \theta(z,t) = \theta_i \qquad 0 < t \tag{7}$$

where q_0 is the constant surface flux, while θ_r and θ_s are the residual and saturated moisture contents, respectively. The soil is uniformly moist initially, with moisture content θ_i. Note that Richards' equation is written with moisture content as the dependent variable. Thus there are some limitations on its applicability [*LaBolle and Clausnitzer*, 1999], e.g., it should not normally be used for soil-water pressure greater than atmospheric pressure.

[6] For convenience, equations (4)–(7) are normalized usingz

$$Z = \frac{z[K(\theta_s) - K(\theta_i)]}{D(\theta_s)\Delta\theta,} \tag{8}$$

$$\tau = \frac{t[K(\theta_s) - K(\theta_i)]^2}{D(\theta_s)\Delta\theta^2,} \tag{9}$$

$$Q = \frac{q - K(\mathrm{q}_i)}{K(\theta_s) - K(\theta_i),} \tag{10}$$

$$\theta(Z,\tau) = \frac{\theta(z,t) - \mathrm{q}_i}{\Delta\theta,} \tag{11}$$

$$\mathcal{D}(\Theta) = \frac{D(\theta)}{D(\theta_s)} \tag{12}$$

and

$$\mathcal{K}(\Theta) = \frac{K(\theta) - K(\theta_i)}{K(\theta_s) - K(\theta_i),} \tag{13}$$

where $\Delta\theta = \theta_s - \theta_i$. The scaled diffusivity, $\mathcal{D}$, satisfies $0 < \mathcal{D} \le 1$, while the hydraulic conductivity, $\mathcal{K}$, is bounded according to $0 \le \mathcal{K} \le 1$.

[7] The various transformations, equations (8)–(13), give dimensionless forms of equations (4)–(7). These are, respectively,

$$\frac{\partial\Theta}{\partial\tau} = \frac{\partial}{\partial Z}\left[D(\Theta)\frac{\partial\Theta}{\partial Z}\right] - \frac{dK(\Theta)}{d\Theta}\frac{\partial\Theta}{\partial Z}, \qquad 0 < Z, 0 < \tau, 0 \le \Theta \le 1 \tag{14}$$

$$Q = Q_0 = \frac{q_0 - K(\theta_i)}{K(\theta_s)}, \qquad Z = 0, 0 < \tau \tag{15}$$

$$\Theta(Z,0) = 0, \qquad 0 < Z \tag{16}$$

and

$$\lim_{Z\to\infty} \Theta(Z,\tau) = 0. \qquad 0 < \tau \tag{17}$$

Below, we present a procedure with the potential for producing new analytical solutions satisfying equations (14)–(17). The hydraulic functions, $\mathcal{D}$ and $\mathcal{K}$, for which the solutions apply, will emerge as part of the solution process.

2.1. Lisle's Equivalence Transformations

[8] *Lisle* [1992] presented the set of transformations that map equations (1) and (2) into equations of exactly the same form. The set is the most general in existence. For the normalized *Richards*' [1931] equation given in equation (14), Lisle's equivalence transformations would produce an identical equation to (14), with different symbols. Below, we seek to generate new solutions to equations (14)–(17) from existing solutions. We denote the given solution using a superscripted asterisk: It is mapped to the system without an asterisk. The set of equivalence transformations is [*Lisle*, 1992]

$$I^* = \lambda(\alpha I - \beta Z)\zeta^{-1} - \vartheta\tau - I_0, \tag{18}$$

$$Z^* = \lambda(\delta Z - \gamma I)\zeta^{-1} + \nu\tau + Z_0, \tag{19}$$

$$\tau^* = \lambda^2\tau\zeta^{-1} + \tau_0, \tag{20}$$

$$\Theta^* = (\alpha\Theta + \beta)(\gamma\Theta + \delta)^{-1}, \tag{21}$$

$$Q^* = [\lambda Q + \zeta(\alpha\nu - \gamma\vartheta)\Theta + \zeta(\beta\nu - \delta\vartheta)]\lambda^{-2}(\gamma\Theta + \delta)^{-1}, \tag{22}$$

$$K^* = [\lambda K + \zeta(\alpha\nu - \gamma\vartheta)\Theta + \zeta(\beta\nu - \delta\vartheta)]\lambda^{-2}(\gamma\Theta + \delta)^{-1}, \tag{23}$$

and

$$\mathcal{D}^* = (\gamma\Theta + \delta)^2\zeta^{-1}\mathcal{D}, \tag{24}$$

where $\alpha = (1 + \beta\gamma)\delta^{-1}$. The 10 constants appearing in equations (18)–(24), λ, γ, ζ, β, δ, ν, ϑ, I_0, z_0, and t_0, will be specified to match the problem to be solved.

[9] As already stated, this set of mappings will be used below to derive exact solutions to equations (14)–(17). We will start with a known solution, indicated by the asterisk in equations (18)–(24), and map it to another constant flux infiltration problem of the same form. That is, the equivalence mapping ensures that the new solution will satisfy *Richards*' [1931] equation, and if the constants are suitably chosen, then the new boundary conditions will also be appropriate for constant flux infiltration into a uniformly moist soil.

2.2. Choice of Constants

[10] The constants in equations (18)–(24) are determined using the correspondences shown in Table 1. In addition to the 10 constants listed in section 2.1 a new constant, Q_o, appears (assuming that the asterisked solution is known), giving 11 constants in all. There are 8 conditions in Table 1, which can be shown to determine 10 of the 11 constants (the independent parameter is δ), giving the values shown in

Table 1. Conditions to Be Satisfied in the Equivalence Transformation Mapping Between the Asterisked and Nonasterisked Systems

Class of Condition	Asterisked System Condition	Nonasterisked System Condition
Moisture content	$\Theta^* = 0$	$\Theta = 0$
	$\Theta^* = 1$	$\Theta = 1$
Initial time and position	$\tau^* = 0$	$\tau = 0$
	$Z^* = 0$	$Z = 0$
Constant flux	$I^*(0, \tau^*) = Q_o^* \tau^*$	$I(0, \tau) = Q_o \tau$
	$\mathcal{K}^* = 0$	$\mathcal{K} = 0$
Hydraulic function limits	$\mathcal{K}^* = 1$	$\mathcal{K} = 1$
	$\mathcal{D}^* = 1$	$\mathcal{D} = 1$

Table 2. Taking these parameter values ensures that the mapped infiltration solution satisfies the same form of problem in both the asterisked and nonasterisked domains. That is, if the solution $\Theta^*(Z^*, \tau^*)$ is known, it can be mapped to a solution, $\Theta(Z, \tau)$, of a constant flux infiltration problem, albeit with diffusivity and hydraulic conductivity functions given by

$$\mathcal{K}(\Theta) = \left\{ \mathcal{K}^* \left[\frac{\Theta}{\delta^2 + (1-\delta^2)\Theta} \right] \left[\delta^2 + \left(1 - \delta^2\right)\Theta\right] - \left(1 - \delta^2\right)\Theta Q_0^* \right\} \cdot \left[1 - \left(1 - \delta^2\right) Q_0^*\right]^{-1} \quad (25)$$

and

$$\mathcal{D}(\Theta) = \mathcal{D}^* \left\{ \Theta \left[\delta^2 + \left(1 - \delta^2\right)\Theta\right]^{-1} \right\} \left[\delta^2 + \left(1 - \delta^2\right)\Theta\right]^{-2}, \quad (26)$$

respectively. In equations (25) and (26) the parameter δ can be used to define new hydraulic functions in the solution to the transformed problem.

2.3. Exact Solution

2.3.1. Solution to Burgers' Equation

[11] The simplest nonlinear model for infiltration is given by Burgers' equation [e.g., *Pearson*, 1990b]. To be specific, equations (14)–(17) are written as

$$\frac{\partial \Theta^*}{\partial \tau^*} = \frac{\partial^2 \Theta^*}{\partial Z^{*2}} - \left(1 - \frac{B}{2} + B\theta^*\right) \frac{\partial \Theta^*}{\partial Z^*}, \quad 0 < Z^*, 0 < \tau^*, 0 \le \Theta^* \le 1 \quad (27)$$

$$\left(1 - \frac{B}{2}\right)\Theta^* + B\frac{\Theta^{*2}}{2} - \frac{\partial \Theta^*}{\partial Z^*} = Q_0^*, \qquad Z^* = 0, 0 < \tau^* \quad (28)$$

$$\Theta^*(Z^*, 0) = 0, \qquad 0 < Z^* \quad (29)$$

$$\lim_{Z^* \to \infty} \Theta^*(Z^*, \tau^*) = 0, \qquad 0 < \tau^* \quad (30)$$

where B is an arbitrary constant. Note that equation (27) is Burgers' equation with an additional advection term. It can be solved using the Hopf-Cole transformation [*Hopf*, 1950; *Cole*, 1951]. In equation (27) the diffusivity is

$$\mathcal{D}^* = 1, \quad (31)$$

while the hydraulic conductivity is

$$K^* = \left(1 - 2^{-1}B\right)\Theta^* + 2^{-1}B\Theta^{*2}, \quad (32)$$

where B is a soil-dependent parameter.

[12] The solution to equations (27)–(30) is [e.g., *Clothier et al.*, 1981; *Barry and Sander*, 1991]

$$\Theta^*(Z^*, \tau^*) = -\frac{2}{B} \frac{w_{Z^*}(Z^*, \tau^*)}{w(Z^*, \tau^*)}, \quad (33)$$

where [*van Genuchten and Alves*, 1982, p. 19]

$$2w(Z^*, \tau^*) = 2 - M(Z^*, \tau^*) + \exp\left(2^{-1}\tau^* B Q_0^*\right) N(Z^*, \tau^*), \quad (34)$$

with

$$M(Z^*, \tau^*) = \operatorname{erfc}\left(\frac{Z^* - A\tau^*}{\sqrt{4\tau^*}}\right) + \exp(AZ^*)\operatorname{erfc}\left(\frac{Z^* + A\tau^*}{\sqrt{4\tau^*}}\right), \quad (35)$$

$$N(Z^*, \tau^*) = \exp\left(\frac{A - X}{2} Z^*\right) \operatorname{erfc}\left(\frac{Z^* - X\tau^*}{\sqrt{4\tau^*}}\right) + \exp\left(\frac{A + X}{2} Z^*\right) \operatorname{erfc}\left(\frac{Z^* + X\tau^*}{\sqrt{4\tau^*}}\right), \quad (36)$$

$$X = A\sqrt{1 + 2Q_0^* B A^{-1}}, \quad (37)$$

and $A = 1 - 2^{-1}B$. The function $w_{Z^*}(Z^*, \tau^*)$ is just $\partial w(Z^*, \tau^*)/\partial Z^*$, or

$$w_{Z^*}(Z^*, \tau^*) = \frac{1}{2}\left[\exp\left(\frac{\tau^* B Q_0^*}{2}\right) \frac{\partial N(Z^*, \tau^*)}{\partial Z^*} - \frac{\partial M(Z^*, \tau^*)}{\partial Z^*}\right], \quad (38)$$

where

$$\frac{\partial M(Z^*, \tau^*)}{\partial Z^*} = A\exp(AZ^*)\operatorname{erfc}\left(\frac{Z^* + A\tau^*}{\sqrt{4\tau^*}}\right) - \frac{2}{\sqrt{\pi\tau^*}} \exp\left[\frac{-(Z^* - A\tau^*)^2}{4\tau^*}\right], \quad (39)$$

Table 2. Parameter Values Determined by Application of Conditions in Table 1[a]

Equivalence Transformation Constant	Value
γ	$\delta^{-1} - \delta$
λ	$\delta + (\delta^{-1} - \delta)Q_0$
ζ	δ^{-2}
β	0
ν	0
ϑ	$[\delta^2 + (1 - \delta^2)Q_0](1 - \delta^2)Q_0$
I_0	0
z_0	0
t_0	0
$Q_0[\delta^2 + (1 - \delta^2)Q_0]^{-1}$	Q_0^*
Q_0	$\delta^2 Q_0^*[1-(1 - \delta^2)Q_0^*]^{-1}$

[a] The last two entries are equivalent (presented for convenience).

and

$$\frac{\partial N(Z^*,\tau^*)}{\partial Z^*} = \frac{A-X}{2}\exp\left(\frac{A-X}{2}Z^*\right)\operatorname{erfc}\left(\frac{Z^*-X\tau^*}{\sqrt{4\tau^*}}\right) + \frac{A+X}{2}\exp\left(\frac{A+X}{2}Z^*\right)\operatorname{erfc}\left(\frac{Z^*+X\tau^*}{\sqrt{4\tau^*}}\right) - \frac{2}{\sqrt{\pi\tau^*}}\exp\left(\frac{AZ^*}{2}-\frac{Z^{*2}+X^2\tau^2}{4\tau^*}\right). \quad (40)$$

2.3.2. First Iteration: Solution Generated From the Burgers' Equation Model

[13] We now seek the functional forms of $\mathcal{D}$ and $\mathcal{K}$ that allow equations (14)–(17) to be mapped to equations (27)–(30) using equations (18)–(24). From equations (26) and (31) the diffusivity generated is given by

$$\mathcal{D} = \left[\left(1-\delta^2\right)\Theta + \delta^2\right]^{-2}. \quad (41)$$

Similarly, the hydraulic conductivity function generated from equations (25) and (32) is

$$\mathcal{K}(\Theta) = \Theta\left\{1+2^{-1}B\left[Q_0 - \left(\Theta\delta^2-\delta^2-Q_0\right)\left(\Theta\delta^2-\delta^2-\Theta\right)^{-1}\right]\right\}. \quad (42)$$

[14] For the hydraulic functions defined by equations (41) and (42) the solution to equations (14)–(17) is given by

$$\Theta(Z,\tau) = \delta^2\left[\delta^2 - 1 + \frac{1}{\Theta^*(Z^*,\tau^*)}\right]^{-1}, \quad (43)$$

where, as noted above, δ is an independent parameter. It can be fixed by choice of the capillary diffusivity, equation (41). To determine $\Theta^*(Z^*, \tau^*)$, equation (33) is used. To obtain Z, we need to find Z and I simultaneously from Z^* and I^*. For any given Z^* and τ^*, the corresponding value of $I^*(Z^*, \tau^*)$ is

$$I^*(Z^*,\tau^*) = 2B^{-1}\ln[w(Z^*,\tau^*)]. \quad (44)$$

With $I^*(Z^*, \tau^*)$ known, Z is given by

$$Z = \delta^{-2}\left\{\left[Z^* + \left(1-\delta^2\right)I^*\right]\left[\delta^2 + \left(1-\delta^2\right)Q_0\right]^{-1} - \left(1-\delta^2\right)Q_0\tau\right\}. \quad (45)$$

The solution to equations (14)–(17) presented here appears similar to that presented by *Sander et al.* [1988a], albeit obtained by a much more straightforward method. Indeed, setting $\delta^2 = (1-\varepsilon)^{-1}$ and $B = 2(\varepsilon + P)(1-\varepsilon Q_o)^{-1}$ allows equations (41) and (42) to be written as

$$\mathcal{D} = (1-\varepsilon)^2(1-\varepsilon\Theta)^{-2} \quad (46)$$

and

$$\mathcal{K} = \left[(1-\varepsilon-P)\Theta + P\Theta^2\right](1-\varepsilon\Theta)^{-1}. \quad (47)$$

respectively. With trivial notation and scaling changes, equations (46) and (47) are identical to the hydraulic functions for which the analytical solution of *Sander et al.* [1988a] applies.

2.3.3. Second Iteration: Solution Generated From the *Sander et al.* [1988a] Model

[15] We take as hydraulic functions equations (41) and (42), written in the form

$$\mathcal{D}^*(\Theta^*) = \left[\left(1-\delta_1^2\right)\Theta^* + \delta_1^2\right]^{-2} \quad (48)$$

and

$$\mathcal{K}^*(\Theta^*) = \Theta^*\left[1 + P\delta_1^2(\Theta^*-1)\right]\left[\left(1-\delta_1^2\right)\Theta^* + \delta_1^2\right]^{-1}, \quad (49)$$

where δ_1 and P are now considered as free parameters to be determined by soil type. The asterisked infiltration solution is simply that of *Sander et al.* [1988a]. As before, we seek to map this to a new solution using equations (18)–(24) and the parameters listed in Table 2. The functions in equations (48) and (49) map, using equations (25) and (26), to

$$\mathcal{D}(\Theta) = \left[\left(1-\delta^2\delta_1^2\right)\Theta + \delta^2\delta_1^2\right]^{-2} \quad (50)$$

and

$$\mathcal{K}(\Theta) = \left\{\mathcal{K}^*(\Theta)\left[\delta^2 + \left(1-\delta^2\right)\Theta\right] - Q_0^*\Theta\left(1-\delta^2\right)\right\}\left[1 - Q_0^*\left(1-\delta^2\right)\right]^{-1}, \quad (51)$$

where

$$\mathcal{K}^*(\Theta) = \frac{\Theta\left[\left(P\delta_1^2\delta^2 + 1 - \delta^2\right)\Theta + \delta^2\left(1-P\delta_1^2\right)\right]}{\left[\delta^2 + \left(1-\delta^2\right)\Theta\right]\left[\delta_1^2\delta^2 + \left(1-\delta_1^2\delta^2\right)\Theta\right].} \quad (52)$$

Clearly, equations (48) and (50) are identical in form if we let $\delta_2^2 = \delta^2\delta_1^2$ in equation (50). Similarly, taking this same relationship and $P_1 = [\delta_2^2 - \delta_1^2 + Q_o^*(\delta_2^4 - \delta_2^2 + \delta_1^2 - \delta_1^2\delta_2^2)]\delta_1^{-2}\delta_2^{-2} + P(2 - \delta_1^2\delta_2^{-2})$ reduces equation (51) to the form of equation (49). Thus no new solution has been generated, merely a more algebraically complicated form of the starting solution.

3. Redistribution in a Finite-Length Soil Profile

[16] We treat this case briefly as the details closely follow that for infiltration presented above in section 2. To be precise, equation (14) is considered to apply in the spatial domain $0 < Z < l$. Boundary conditions to be applied are those of constant and equal flux at the top and bottom boundaries (the case of zero flux at the boundaries was solved by *Sander et al.* [1991]),

$$Q = Q_0,\ Z = 0 \qquad l, 0 < \tau \quad (53)$$

The initial condition is

$$\Theta(Z,0) = \Theta_i(Z). \qquad 0 < Z < l \quad (54)$$

As a first step, we extend the equivalence transformations presented in section 2.1 to account for the case when the normalized initial moisture content in the soil profile is nonuniform. Details are presented in Appendix A. Then, we proceed as in section 2 and determine the parameter set such that the mapping produces another redistribution problem to be solved on a fixed spatial domain. This step results in the same parameter set as given in Table 2, with the additional constraints $A = \int_0^l \Theta_i(Z)dZ$ and $l^* = [\delta^2 + Q_o(1 - \delta^2)]\ [\delta^2 l + A(1 - \delta^2)]$, where A is proportional to the total mass of water in the profile and $l^* = Z^*(l)$. The key point is that the relationships already presented relating the asterisked and nonasterisked hydraulic functions, equations (25) and (26), are unchanged. Thus starting the solution generation iteration using the solution to Burgers' equation on a finite domain [*Sander*, 1992] will produce an existing solution for the mapped hydraulic functions as presented in equations (41) and (42) or, equivalently, equations (46) and (47). This solution is detailed by *Sander* [1992] and so will not be presented here. Thus carrying out an additional iteration, in the manner of section 2.3.3, does not produce a new solution.

4. Discussion and Conclusions

[17] The diffusivity for which most of the above results pertain was first presented by *Fujita* [1952], who provided an exact solution for the nonlinear diffusion equation without gravity. A variety of solutions with the gravity term included have been presented [*Fokas and Yortsos*, 1982; *Rosen*, 1982; *Rogers et al.*, 1983; *Broadbridge and White*, 1988; *Broadbridge et al.*, 1988; *Rogers*, 1988; *Sander et al.*, 1988a, 1988b, 1991; *Warrick et al.*, 1990, 1991; *Kuhnel et al.*, 1990; *Barry and Sander*, 1991; *Parkin et al.*, 1992, 1995; *Sander*, 1992]. These solutions are all members of the same family in that the capillary diffusivity and hydraulic conductivity are closely related. In essence, the equivalence mapping approach used here allows these solutions to be obtained in a more transparent fashion. However, it has been shown that for the problems considered here, that is, constant flux infiltration into a semi-infinite medium or redistribution in a finite medium with constant flux boundary conditions, the generation of new solutions from existing ones is not feasible because the new solutions are algebraically identical to known solutions. That is, the mapping converges to a single endpoint, beyond which the same endpoint is endlessly repeated.

[18] Our results do not mean that new solutions do not exist, rather that the equivalence mapping method has been proved incapable of producing new solutions. There are, we suggest, two different avenues for systematically developing new analytical results for the constant flux infiltration and redistribution problems considered here. First, if a new analytical solution for either problem were discovered, the equivalence mapping could immediately be used to investigate the potential for generating new solutions. Second, if a more general mapping of *Richards*' [1931] equation into itself was discovered, then the existing solutions could be used to produce new results so long as the boundary and initial conditions mapped appropriately.

Appendix A: Mapping From *Richards*' [1931] Equation Onto Itself for the Case of a Nonuniform Initial Moisture Content

[19] *Richards*' [1931] equation is written as

$$\frac{\partial \Theta}{\partial \tau} = -\frac{\partial Q}{\partial Z}, \qquad \tau > 0, 0 < Z < l. \tag{A1}$$

Darcy's law, equation (3), is, in normalized form,

$$Q = \mathcal{K}(\Theta) - \mathcal{D}(\Theta)\frac{\partial \Theta}{\partial Z}. \tag{A2}$$

Rewrite equation (A2) in terms of Z,

$$Z = \int_{\Theta}^{\Theta(0,\tau)} \mathcal{D}(Q - \mathcal{K})^{-1} d\overline{\Theta}. \tag{A3}$$

In potential form, equation (A1) can be written as the pair of equations

$$\frac{\partial I}{\partial \tau} = Q \tag{A4}$$

and

$$\frac{\partial I}{\partial Z} = -(\Theta - \Theta_i), \tag{A5}$$

where $\Theta_i(Z)$ is the nonuniform, initial normalized moisture content. Integration of equation (A4) gives

$$I = \int_0^{\tau} Q(Z, \overline{\tau}) d\overline{\tau}, \tag{A6}$$

while the corresponding operation on equation (A5) results in

$$I = \int_Z^l \Theta(\overline{Z}, t) - \Theta_i(\overline{Z}) d\overline{Z} + \int_0^{\tau} Q_l(\overline{\tau}) d\overline{\tau}, \tag{A7}$$

where Q_l is the flux at $Z = l$.

[20] We wish to map equations (A1) and (A3) to equivalent expressions written with a superscripted asterisk, i.e.,

$$\frac{\partial \Theta^*}{\partial \tau^*} = -\frac{\partial Q^*}{\partial Z^*}, Z_l^*(\tau^*) > Z^* > Z_i^*(\tau^*), \tau^* > \tau_i^*, \tag{A8}$$

and

$$Z^* - Z_i^*(\tau^*) = \int_{\Theta^*}^{\Theta^*(0,\tau^*)} \mathcal{D}^*(Q^* - \mathcal{K}^*)^{-1} d\overline{\Theta^*}, \tag{A9}$$

where $Z_i^*(\tau^*)$ is the value that Z^* takes for $Z = 0$, $Z_l^*(\tau^*)$ is the corresponding value for $Z = l$, and maps to $\tau_i^* = 0$.

[21] Consider the transformation consisting of equation (20),

$$d\Theta^* = g(\Theta) d\Theta, \tag{A10}$$

$$\mathcal{D}^*(\Theta^*) d\Theta^* = \frac{\mathcal{D}(\Theta)}{f(\Theta)} d\Theta, \tag{A11}$$

$$\mathcal{K}^*(\Theta^*) = \frac{\mathcal{K}(\Theta) + h(\Theta)}{u(\Theta)}, \tag{A12}$$

and

$$Q^* = \frac{Q + h(\Theta)}{u(\Theta)}, \tag{A13}$$

where *f, g, h* and *u* are functions to be determined.

[22] Using equations (A11)–(A13), equation (A9) becomes

$$Z^* - Z_i^*(\tau^*) = \int_0^Z \frac{u}{f} d\overline{Z}. \tag{A14}$$

Similarly, each side of equation (A8) is transformed. First,

$$\frac{\partial \Theta^*}{\partial \tau^*} = \frac{\partial(\Theta^*, Z^*)}{\partial(\tau^*, Z)} \bigg/ \frac{\partial(\tau^*, Z^*)}{\partial(\tau^*, Z)} = \frac{\zeta g}{\lambda^2} \left\{ \frac{\partial \Theta}{\partial \tau} - \frac{f}{u} \frac{\partial \Theta}{\partial Z} \left[\frac{dZ_i^*}{d\tau} + \int_0^Z \frac{d(uf^{-1})}{d\Theta} \frac{\partial \Theta}{\partial \tau} d\overline{Z} \right] \right\}, \tag{A15}$$

where the middle term denotes the Jacobian [*Pearson*, 1990a]. Next,

$$-\frac{\partial Q^*}{\partial Z^*} = -\frac{f}{u} \frac{\partial}{\partial Z} \left(\frac{Q + h}{u} \right). \tag{A16}$$

The right-hand sides of equations (A15) and (A16) are equated and forced to reduce to equation (A8), in which case we must have

$$\zeta \lambda^{-2} g = f u^{-2}, \tag{A17}$$

$$\frac{d(hu^{-1})}{d\Theta} = g \zeta \nu \lambda^{-2}, \tag{A18}$$

$$\frac{d(uf^{-1})}{d\Theta} = \lambda \gamma \zeta^{-1}, \tag{A19}$$

and

$$\frac{d(u^{-1})}{d\Theta} = -g \gamma \lambda^{-2}. \tag{A20}$$

The new constants, ν and γ, appearing in equations (A18) and (A19) permit the reduction to equation (A8) so long as they are finite. In equation (A15) the integral in brackets on the right-hand side is simplified using equation (A19):

$$\frac{dZ_i^*}{d\tau} + \lambda \gamma Q_0 \zeta^{-1} = \nu. \tag{A21}$$

Integration of equation (A21) is immediate:

$$Z_i^* = \nu \tau - \lambda \gamma \zeta^{-1} \int_0^\tau Q_0(\overline{\tau}) d\overline{\tau} + Z_0. \tag{A22}$$

From equation (A19) we find

$$uf^{-1} = \lambda(\gamma \Theta + \delta) \zeta^{-1}. \tag{A23}$$

[23] An expression for Z^* can now be derived from equations (A7), (A14), (A22) and (A23):

$$Z^* = \lambda \zeta^{-1} \left\{ \delta Z - \gamma \left[I - \int_0^Z \Theta_i(\overline{Z}) d\overline{Z} \right] \right\} + \nu \tau + Z_0. \tag{A24}$$

Using equations (A17) and (A23), equation (A20) can be solved, yielding

$$u = \lambda(\gamma \Theta + \delta). \tag{A25}$$

Now, equation (A17) can be solved for *g* and *f*, i.e.,

$$g = (\gamma \Theta + \delta)^{-2} \tag{A26}$$

and

$$f = \zeta, \tag{A27}$$

where equations (A23) and (A25) were utilized.

[24] In equation (A18) the only unknown left is *h*, and so

$$h = \zeta[(\gamma \Theta + \delta)(\alpha \nu - \gamma \vartheta) - \nu](\lambda \gamma)^{-1}, \tag{A28}$$

where, as in section 2.1, $\alpha = (1 + \beta\gamma)\delta^{-1}$ and, from equation (A10),

$$\Theta^* = (\alpha \Theta + \beta)(\gamma \Theta + \delta)^{-1}. \tag{A29}$$

The transformed diffusivity is, from equations (A11), (A27), and (A29),

$$\mathcal{D}^*(\Theta^*) = (\gamma \Theta + \delta)^2 \zeta^{-1} \mathcal{D}(\Theta). \tag{A30}$$

From equations (A12), (A25), and (A28), the transformed hydraulic conductivity is

$$\mathcal{K}^*(\Theta^*) = \{\lambda \mathcal{K}(\Theta) + \zeta[\Theta(\alpha \nu - \gamma \vartheta) + \beta \nu - \vartheta \delta]\} \lambda^{-2} (\gamma \Theta + \delta)^{-1}, \tag{A31}$$

with a corresponding expression for Q^*. Finally, integration gives

$$I^* - \int_{Z^*(Z=0)}^{Z^*} \Theta_i^*\left(\overline{\Theta^*}\right) d\overline{\Theta^*} = \lambda \zeta^{-1} \left\{ \alpha \left[I - \int_0^Z \Theta_i(\overline{Z}) d\overline{Z} \right] - \beta Z \right\} - \vartheta \tau - I_0. \tag{A32}$$

References

Barry, D. A., and G. C. Sander, Exact solutions for water infiltration with an arbitrary surface flux or nonlinear solute adsorption, *Water Resour. Res.*, *27*, 2667–2680, 1991.

Barry, D. A., I. G. Lisle, L. Li, H. Prommer, J.-Y. Parlange, G. C. Sander, and J. W. Griffioen, Similitude applied to centrifugal scaling of unsaturated flow, *Water Resour. Res.*, *37*, 2471–2479, 2001.

Broadbridge, P., and I. White, Constant rate rainfall infiltration—A versatile nonlinear model, 1, Analytic solution, *Water Resour. Res.*, *24*, 145–154, 1988.

Broadbridge, P., J. H. Knight, and C. Rogers, Constant rate rainfall infiltration in a bounded profile—Solutions of a nonlinear model, *Soil Sci. Soc. Am. J.*, *52*, 1526–1533, 1988.

Clothier, B. E., J. H. Knight, and I. White, Burgers' equation: Application to field constant-flux infiltration, *Soil Sci.*, *132*, 255–261, 1981.

Cole, J. D., On a quasi-linear parabolic equation occurring in aerodynamics, *Q. Appl. Math.*, *9*, 225–236, 1951.

Fokas, A. S., and Y. C. Yortsos, On the exactly solvable equation $S_t = [(\beta S + \gamma)^{-2} S_x]_x + \alpha(\beta S + \gamma^{-2}) S_x$ ocurring in two-phase flow in porous media, *SIAM J. Appl. Math.*, *42*, 318–322, 1982.

Fujita, H., The exact pattern of a concentration-dependent diffusion in a semi-infinite medium, II, *Textile Res. J.*, *22*, 823–827, 1952.

Hopf, E., The partial differential equation $u_t + uu_x = \mu u_x$, *Comm. Pure Appl. Math.*, *3*, 201–230, 1950.

Kühnel, V., J. C. I. Dooge, G. C. Sander, and J. P. J. O'Kane, Duration of atmosphere-controlled and of soil-controlled phases of infiltration for constant rainfall at a soil surface, *Ann. Geophys.*, *18*, 11–20, 1990.

LaBolle, E. M., and V. Clausnitzer, Comment on *Russo* [1991], *Serrano* [1990, 1998], and other applications of the water-content-based form of Richards' equation to heterogeneous soils, *Water Resour. Res.*, *35*, 605–607, 1999.

Lisle, I. G., *Equivalence Transformations for Classes of Differential Equations*, Ph.D. thesis, Dept. of Math., Univ. of British Columbia, Vancouver, Canada, 1992.

Miller, E. E., Similitude and scaling of soil-water phenomena, in *Applications of Soil Physics*, edited by D. Hillel, pp. 300–318, Academic, San Diego, Calif., 1980.

Miller, E. E., and R. D. Miller, Theory of capillary flow, I, Practical implications, *Soil Sci. Soc. Am. Proc.*, *19*, 267–271, 1955a.

Miller, E. E., and R. D. Miller, Theory of capillary flow, II, Experimental information, *Soil Sci. Soc. Am. Proc.*, *19*, 271–275, 1955b.

Miller, E. E., and R. D. Miller, Physical theory for capillary flow phenomena, *J. Appl. Physics*, *27*, 324–332, 1956.

Parkin, G. W., D. E. Elrick, and R. G. Kachanoski, Cumulative storage of water under constant flux infiltration: Analytical solution, *Water Resour. Res.*, *28*, 2811–2818, 1992.

Parkin, G. W., A. W. Warrick, D. E. Elrick, and R. G. Kachanoski, Analytical solution for one-dimensional drainage, Water stored in a fixed depth, *Water Resour. Res.*, *31*, 1267–1271, 1995.

Pearson, C. E., Partial differential equations of second and higher order, in *Handbook of Applied Mathematics, Selected Results and Methods*, 2nd ed., edited by C. E. Pearson, pp. 448–511, Van Nostrand Reinhold, New York, 1990a.

Pearson, H. L.. Elements of analysis, in *Handbook of Applied Mathematics, Selected Results and Methods*, 2nd edition, edited by C. E. Pearson, pp. 83–128, Van Nostrand Reinhold, New York, 1990b.

Richards, L. A., Capillary conduction of liquids through porous mediums, *Physics*, *1*, 318–333, 1931.

Rogers, C., On a nonlinear moving boundary problem with heterogeneity: Application of a reciprocal transformation, *ZAMP*, *39*, 122–128, 1988.

Rogers, C., M. P. Stallybrass, and D. L. Clements, On two phase flow filtration under gravity and with boundary infiltration: Application of a Bäcklund transformation, *Nonlinear Anal. Theory Meth. Appl.*, *7*, 785–799, 1983.

Rosen, G., Method of the exact solution of a nonlinear diffusion-convection equation, *Phys. Rev. Lett.*, *49*, 1844–1846, 1982.

Sander, G. C., Exact solutions to nonlinear diffusion-convection problems on finite domains, *J. Austral. Math. Soc., Ser. B*, *33*, 384–401, 1992.

Sander, G. C., J.-Y. Parlange, V. Kühnel, W. L. Hogarth, D. Lockington, and J. P. J. O'Kane, Exact nonlinear solution for constant flux infiltration, *J. Hydrol.*, *97*, 341–346, 1988a.

Sander, G. C., J.-Y. Parlange, V. Kühnel, W. L. Hogarth, and J. P. J. O'Kane, Comment on "Constant rate rainfall infiltration: A versatile nonlinear model, 1, Analytic solution" by P. Broadbridge and I. White, *Water Resour. Res.*, *24*, 2107–2108, 1988b.

Sander, G. C., I. F. Cunning, W. L. Hogarth, and J.-Y. Parlange, Exact solution for nonlinear, nonhysteretic redistribution in vertical soil of finite depth, *Water Resour. Res.*, *27*, 1529–1536, 1991.

Sophocleous, C., Potential symmetries of nonlinear diffusion-convection equations, *J. Phys. A Math. Gen.*, *29*, 6951–6959, 1996.

van Genuchten, M. T., and W. J. Alves. Analytical solutions of the one-dimensional convective-dispersive solute transport equation, *Tech. Bull. 1661*, U. S. Dept. of Agric., Washington, D. C., 1982.

Vijayakumar, K., Isogroup classification and group-invariant solutions of the nonlinear diffusion-convection equation $T_t = (D_1(T)T_x)_x - (T)T_x$, *Int. J. Eng. Sci.*, *35*, 1–14, 1997.

Warrick, A. W., D. O. Lomen, and A. Islas, An analytical solution to Richards' equation for a draining soil profile, *Water Resour. Res.*, *26*, 253–258, 1990.

Warrick, A. W., A. Islas, and D. O. Lomen, An analytical solution to Richards' equation for time-varying infiltration, *Water Resour. Res.*, *27*, 763–766, 1991.

Yung, C. M., K. Verberg, and P. Baveye, Group classification and symmetry reductions of the non-linear diffusion-convection equation $u_t = (D(u)u_x)_x - K'(u)u_x$, *Int. J. Non-Lin. Mech.*, *29*, 273–278, 1994.

D. A. Barry, L. Li, and J.-Y. Parlange, Contaminated Land Assessment and Remediation Research Centre, Institute for Infrastructure and Environment, School of Engineering and Electronics, University of Edinburgh, Edinburgh EH9 3JN, UK. (d.a.barry@ed.ac.uk; jp58@cornell.edu)

D.-S. Jeng, School of Engineering, Griffith University, Gold Coast 9276, Australia. (d.jeng@mailbox.gu.edu.au)

L. Li, Department of Civil Engineering, The University of Queensland, St. Lucia 4072, Australia. (l.li@uq.edu.au)

I. G. Lisle, School of Mathematics and Statistics, University of Canberra, P.O. Box 1, Belconnen 2616, Australia. (ianl@ise.canberra.edu.au)

G. C. Sander, Department of Civil and Building Engineering, Loughborough University, Loughborough LE11 3TU, UK. (g.sander@lboro.ac.uk)

F. Stagnitti, School of Ecology and Environment, Deakin University, Warrnambool 3280, Australia. (frankst@deakin.edu.au)