# Exact Solution of a Stefan Problem in a Nonhomogeneous Cylinder

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**Abstract-** The exact solution is found for the Stefan problem of inwards freezing of a nonhomogeneous cylinder whose specific heat and latent heat depend upon the inverse square of the radial distance. The freezing time for the corresponding annular cylinder is found exactly. The validity of the pseudo-steady-state approximation for the solvable cylinder problem is verified explicitly. It is argued that the exact solution for the solvable cylinder problem may be used as a small-time start-up solution for numerical procedures for general axi-symmetric cylinder problems and for general Stefan numbers.

**Keywords-** Stefan problem, Moving boundary, Freezing, Cylinder, Annulus, Pseudo-steady-state, Small-time.

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# 1. INTRODUCTION

The Stefan problem of inwards freezing (solidification) of a homogeneous cylinder does not admit of an analytical solution, although it has been the subject of several numerical investigations (e.g. [1,II]-[3] are early works). We adopt a slightly different approach here. We find a nonhomogeneous cylindrical Stefan problem (including the case of an annular cylinder), with radial-dependent, viz. inverse-square, specific and latent heats, which does have an explicit analytical solution.

Solutions of the heat conduction equation for systems with spatial variation of thermal properties have been investigated previously by several authors (e.g. [4],[5]). Such examples, although they may be not of immediate physical importance, act as testing grounds for solution techniques. Where exact solutions are found, as in the present paper, they provide for benchmarking of the accuracy of numerical procedures, which may then be used on other problems which do not have exact solutions. The analysis of the explicit solutions in this paper also clarifies the relationships between pseudo-steady-state (PSS) solutions and small-time solutions and exact solutions.

Since numerical solutions for Stefan problems are generally started off with a small-time approximation (c.f. [6]), a suitable such approximation needs to be determined explicitly. It is argued that the solution to the problem of this paper may be used as a start-off for any (axi-symmetric) cylinder Stefan problem involving any Stefan number.

We first summarize the formulation of a Stefan problem of inwards freezing of an axially-independent, angle-independent cylindrical system of radius $R_0$ . If the thermal conductivity k and the density $\rho$ are constant but the specific heat C is a function of r, the temperature U(r,t) satisfies the p.d.e.

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = \frac{\rho C(r)}{k}\frac{\partial U}{\partial t} \quad . \tag{1.1}$$

The thermal diffusivity $\kappa$ is here given by $k/(\rho C)$ . The Stefan condition for the moving liquid/solid interface radial location R(t), with constant k and $\rho$ and with latent heat L a function of r is, for inwards freezing,

$$\frac{\partial U[R(t)+0,t]}{\partial r} = \frac{\rho\, L[R(t)]}{k}\frac{dR(t)}{dt} \quad . \tag{1.2}$$

The cylindrical region is initially liquid at its freezing point (set to zero) and its surface is then kept at a constant negative temperature. The temperature U satisfies (1.1) in the region $R_0 > r > R(t)$ , and is zero for $r < R(t)$ . The boundary conditions for this problem are

$$U(R_0,t) = -U_1 < 0 \quad ; \qquad U[R(t),t] = 0 \quad (t>0) \quad , \tag{1.3a,b}$$

and the initial conditions are

$$U(r,0) = 0 \; , \; 0 \le r < R_0 \quad ; \qquad R(0) = R_0 \quad . \tag{1.4a,b}$$

## 2. A SOLVABLE NONHOMOGENEOUS PROBLEM

In cylindrical systems with radial symmetry, it is often fruitful to make a logarithmic transformation of the radial coordinate. For convenience and dimensional reasons, r is scaled with respect to $R_0$ and the transformed variable x is here scaled with respect to some arbitrary length quantity $X_0$ . Thus we set (c.f. [3], [7])

$u(x,t) \equiv U(r,t)$ with

$$x/X_0 = -\,ln(r/R_0) \quad , \quad X(t)/X_0 = -\,ln[R(t)/R_0] \quad . \tag{2.1a,b}$$

Then, for inwards freezing of the cylinder, the boundary conditions (1.3a,b) become

$$u(x=0, t) = -U_1 \quad ; \quad u[x=X(t), t] = 0 \quad , \qquad (2.2a,b)$$

and the initial conditions (1.4a,b) become

$$u(x,0) = 0 \;,\; 0 < x \le \infty \quad ; \quad X(0) = 0 \quad . \qquad (2.3a,b)$$

Thus whilst $U(r,t)$ refers to inwards freezing, $0 \le R(t) \le r \le R_0$ with $R(t=0) = R_0$ , $u(x,t)$ then corresponds to a one-dimensional, semi-infinite Stefan problem for outwards freezing with interface location $X(t)$: $0 \le x \le X(t) \le \infty$ with $X(t=0) = 0$ , and $u = 0$ for $x > X(t)$ .

A solvable Stefan problem may now be obtained as follows. With the above transformation, i.e. $r = R_0 \exp(-x/X_0)$ , the heat p.d.e. (1.1) becomes $(kX_0^2/r^2)\, \partial^2u/\partial x^2 = \rho\, C(r)\, \partial u/\partial t$ . If now

$$C(r) = \frac{X_0^{\,2}}{r^2} C_0 \quad , \qquad (2.4)$$

with $C_0$ constant, then, for U, $\partial^2U/\partial r^2 + (1/r)\partial U/\partial r = (1/\kappa_0)(X_0^2/r^2)\partial U/\partial t$ , where $\kappa_0 = k/(\rho C_0)$ ; and, for u, $\partial^2u/\partial x^2 = (1/\kappa_0)\, \partial u/\partial t$ . Thus we obtain the one-dimensional linear heat equation for temperature u, with constant thermal diffusivity coefficient $\kappa_0$, in the semi-infinite region $x \ge 0$ if $0 \le r \le R_0$ .

With the logarithmic transformation of radial interface function, i.e. $R(t) = R_0 \exp[-X(t)/X_0]$ , the Stefan condition (1.2) becomes $k\, \{ X_0^2/[R(t)]^2 \}\, \partial u[X(t)-0, t]/\partial x = \rho\, L[R(t)]\, dX/dt$ . If now

$$L(r) = L_0 \frac{X_0^{\,2}}{r^2} \quad , \qquad (2.5)$$

with $L_0$ constant, then $L[R(t)] = L_0 X_0^2 / [R(t)]^2$ and, for $R(t)$,

$$\frac{\partial U[R(t)+0, t]}{\partial r} = \gamma_0 \frac{X_0^{\,2}}{[R(t)]^2} \frac{d\,R(t)}{d\,t} \quad , \quad \gamma_0 = \rho L_0/k \quad . \qquad (2.6a,b)$$

For the transformed interface function X(t) , $\partial u[X(t)-0, t]/\partial x = \gamma_0\, dX/dt$ . Thus we get the one-dimensional Stefan condition for interface X(t), with constant coefficient $\gamma_0$ , for freezing outwards from origin x=0 . (Note that $\gamma_0\kappa_0 = L_0/C_0 = L(r)/C(r)$ .)

The solutions to the one-dimensional (Cartesian) Stefan problem for u(x,t) and X(t) are well-known ([8], p.286; [9], p.103). From this solution, we transform back to get the explicit, exact solution in $R(t) \le r < R_0$ to our cylindrical Stefan problem for the inwards freezing of a liquid, initially at its freezing temperature, with constant density ρ and constant thermal conductivity k, and with inverse-square radially-dependent specific heat (2.4) and radially-dependent latent heat (2.5) :

$$R(t) = R_0 \exp[-\, 2m_0\sqrt{\kappa_0}\sqrt{t}\, /\, X_0 ] \quad ; \tag{2.7}$$

$$U(r,t) = -\,U_1 \left\{ 1 - \frac{1}{erf(m_0)}\, erf\left[ \frac{X_0}{2\sqrt{\kappa_0\, t}} \ln\left( {R_0}/{r} \right) \right] \right\} \tag{2.8a}$$

$$= -\,U_1 \left\{ 1 - \frac{1}{erf(m_0)}\, erf\left[ m_0\, \frac{ln(R_0\, /\, r)}{ln[R_0\, /\, R(t)]} \right] \right\} \tag{2.8b}$$

where erf is the Error Function. The second expression for the solution U contains t only implicitly through R(t). The interface R(t) takes an indefinitely long time to reach the origin, i.e. the centre of the cylinder. This is reasonable, since in this problem the latent heat is inversely proportional to $r^2$ , and so increases indefinitely towards the origin.

From the standard one-dimensional solution, the constant $m_0$ appearing in this solution here satisfies

$$m_0\, e^{m_0^2}\, erf\,(m_0) = \frac{U_1}{\sqrt{\pi}\,\gamma_0\,\kappa_0} \equiv \frac{U_1}{\sqrt{\pi}}\frac{C_0}{L_0} \equiv \frac{U_1}{\sqrt{\pi}}\frac{C(r)}{L(r)} \quad . \tag{2.9}$$

Thus the $m_0$ value does not depend on r. If $m_0$ is small,

$$m_0 \approx \frac{\sqrt{U_1}}{\sqrt{2\gamma_0 \kappa_0}} \equiv \sqrt{\frac{U_1}{2}}\sqrt{\frac{C_0}{L_0}} \quad , \tag{2.10}$$

so this corresponds to large Stefan number $L/(CU_1)$, for instance to large $L_0$ , or to small $C_0$ .

## 2.1 EXACT FREEZING TIME FOR A NONHOMOGENEOUS ANNULUS

The singularities of the material properties (2.4) and (2.5) at the radial origin, inherent in the preceding exactly solvable cylindrical model, may be avoided by considering the inwards freezing of an *annular* cylinder $0 < R_1 \leq r \leq R_0$ with $U = 0$ for $R_1 \leq r \leq R(t)$ .

The previous formalism and solutions hold, up until the time $T_F$ when $R(T_F) = R_1$ . From (2.7), the finite time of complete freezing is therefore given exactly by

$$T_F = \frac{X_0{}^2}{4m_0{}^2\kappa_0}\left[ln\left(\frac{R_0}{R_1}\right)\right]^2 \quad . \tag{2.11}$$

This would provide a precise test for any numerical method implemented to solve this problem. The general symbol $X_0$ has been retained during the derivation for dimensional reasons and mathematical form. It could conveniently be set equal to $R_0$ so that $C(R_0) = C_0$ and $L(R_0) = L_0$ when numerical values are assigned to these surface quantities. It is worth emphasizing that equation (2.11) holds for any value of the surface thermal diffusivity $\kappa_0$ and for any Stefan number.

## 3. PSEUDO-STEADY-STATE SOLUTIONS

Consider first the cylindrical Stefan problem, with *constant* C and L (as well as constant k and ρ). An approximate solution to this analytically intractable problem may be obtained by making the "pseudo-steady-state" (PSS) approximation: $\partial^2 U/\partial r^2 + (1/r)\, \partial U/\partial r \approx 0$ . That is (c.f. [1,II]), whilst time-dependence is retained for the moving interface in the Stefan condition (1.2) (L constant), the approximate steady state (Laplace) equation is adopted in place of the governing heat p.d.e. (1.1). Thus this corresponds to the large thermal diffusivity κ , i.e. the small (constant) specific heat C approximation, i.e. the case of large Stefan number $L/(CU_{1)}$ . The solution to the PSS equation subject to (1.3a,b) is

$$U \quad \approx \quad -U_1 \left[ 1 - \frac{\ln(r / R_0)}{\ln(R / R_0)} \right] \quad . \tag{3.1}$$

For subsequent developments, it is important to note that, in terms of R, this is also the pseudo-steady-state solution to the heat p.d.e. (1.1) even in the nonhomogeneous case of C=C(r) and/or ρ = ρ(r), and/or L=L(r), since then its right-hand side is still zero, as in the above.

To solve the homogeneous cylinder Stefan condition (1.2) (with constant L, k, ρ) with (1.4b), the above approximate solution (3.1) for U is substituted into the resulting o.d.e. for R(t) . Then integration yields the *implicit* PSS relation for R(t), where $\gamma = \rho L/k$ (c.f. [1,II], eqs.(23)-(24)) :

$$(U_1/\gamma)\, t \;\approx\; -(1/2)\, R^2\, ln(R_0/R) \;+\; (1/4)\, (R_0^{\,2} - R^2) \quad . \tag{3.2}$$

The pseudo-steady-state relation (3.2) for R(t) evidently also holds for the pseudo-steady-state Stefan problem with C = C(r) (but constant γ), since the Stefan condition

(1.2) does not depend on the specific heat C. The PSS solution for the homogeneous cylinder problem cannot be directly compared with the exact analytical solution since the latter is unknown.

We now discuss the PSS solution of the *solvable* cylindrical problem of this paper. It may be found directly from the approximate PSS d.e.s themselves, or from the known exact solution to the full problem. In general, if thermal conductivity k and density ρ are constant (actually, even if only k is constant) the PSS differential equation for U obtained from (1.1) is just as for the PSS constant-coefficient cylindrical problem, whatever the functional dependence of C(r). The boundary conditions (1.3a,b) then yield (3.1) as for the constant-coefficient case, as mentioned above. Furthermore, the PSS solution (3.1) can be shown to agree with the exact solution for the solvable problem (second form for U(r,t), (2.8b)) for small $m_0$ (and hence large Stefan number: see (2.10)), by use of the approximation

$erf(\varepsilon) \approx (2/\sqrt{\pi})\varepsilon$ for ε small.

The Stefan condition here is (2.6a). Then solving this Stefan condition differential equation with the PSS U of (3.1) gives, upon integration, an explicit form:

$$R(t) \approx R_0 \exp\left[-\frac{\sqrt{2U_1}}{X_0\sqrt{\gamma_0}}\sqrt{t}\right] \quad . \tag{3.3}$$

Furthermore, the exact interface solution (2.7) for R(t) to the solvable problem agrees with the PSS solution (3.3) for small $m_0$, since then, by (2.10), $2m_0\sqrt{\kappa_0} \approx \sqrt{2U_1/\gamma_0}$ . The availability of the exact solutions for U and R for this solvable cylinder Stefan problem and their relationship to the PSS solutions thus allows us to conclude that for this problem, for large Stefan number, the pseudo-steady-state approximation is truly justified.

## 4. SMALL-TIME SOLUTIONS

Existence and uniqueness proofs for small times in rather general formulations of Stefan-type problems are included in treatises by Rubinstein [10] (via conversion to a system of integral equations) and Meirmanov [11]. In a series of papers on various versions of the one-dimensional Stefan problem, Tao ([12]; and see references therein) investigated the analyticity of solutions, expressing them as series involving functions of the similarity variable ($x/t^{1/2}$) and fractional powers of t .

It is often important, for analytical or numerical work, to have available approximate expressions for the solutions for small time, for instance to start off numerical computations ([1], [6]). In the present case, the best option for the solvable cylinder problem is simply to evaluate the known exact solution (2.7), (2.8), which is valid for all parameter values, at (small) time t.

Often in problems without available exact solutions, such as the homogeneous cylinder, the pseudo-steady-state (PSS) solution is used for the start-off, based on the physical argument of the thinness of the solid phase for small times ([1,I],[6]). However, this should strictly only be used when the specific heat C is small, i.e. for large Stefan number (c.f. Section 3 above), although this is often masked by a transformation to dimensionless variables [6].

For large Stefan number, for both homogeneous and nonhomogeneous cylinders, the approximate analytical PSS solution, (3.1), is available for U in terms of R(t). For *small* t, to order t, it can be verified that the implicit homogeneous cylinder PSS relation (3.2) for R(t) is satisfied by the approximate analytical expressions

$$R(t) \approx R_0 \exp[-\frac{\sqrt{2U_1}}{R_0\sqrt{\gamma}}\sqrt{t}] \approx R_0 - \frac{\sqrt{2U_1}}{\sqrt{\gamma}}\sqrt{t} + \frac{U_1}{R_0\gamma}t - \cdots \quad . \quad (4.1a,b)$$

But (4.1a) agrees with the solvable problem PSS R(t), (3.3), in terms of surface-measured quantities at $r = R_0$ , via $X_0\sqrt{\gamma_0} = R_0\sqrt{\gamma}$ . Thus one verifies *explicitly* that the *small time* solution for the homogeneous cylinder case and the small time solution for the *exactly solvable* cylinder problem agree *in the pseudo-steady-state regime*.

Such a result is to be expected: for very small times, the interface moves only a small distance (R is near $R_0$), and the thermal properties may be regarded as approximately constant and equal to their surface values over the range, provided that their functional form does not have rapid variation with position near $r = R_0$ . For small times, therefore, this same physical situation and quantitative agreement are expected to hold even when the PSS approximation is not valid, i.e. independently of the Stefan number value.

Furthermore, this should remain so *in general* for any radially nonhomogeneous cylinder Stefan problem. Thus the explicit analytical formulae (2.7) and (2.8) with (2.9), of the *solvable* cylinder problem, may be used for small times for any nonhomogeneous (or homogeneous) Stefan cylinder problem for which the thermal parameters do not vary rapidly with r near the surface. Pseudo-steady-state solutions, although they may be more direct and natural, need not be used, especially if the Stefan number is not large. This suggests that our exactly solved problem may thus provide a small-time start-up paradigm for numerical schemes for all (angle-independent) cylinder Stefan problems for any Stefan number.

## 5. COMPUTATIONAL IMPLICATIONS

Whilst it was not the intention here to pursue numerical techniques, it is emphasized that these may use the above exact solution, for small times, to start off the computations. In this way, one could examine the accuracy of various competing computational algorithms first of all for the above solvable nonhomogeneous cylinder Stefan problem whose exact solution is known, for various Stefan number values. Then the most accurate procedure thereby found, together with the above start-off solution, could be applied to some physically more important but analytically unsolvable cylinder Stefan problems, including the homogeneous case, for any Stefan number.

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