Quantum Mechanics and the Time Travel Paradox

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Abstract

The closed causal chains arising from backward time travel do not lead to paradoxes if they are self consistent. This raises the question as to how physics ensures that only self-consistent loops are possible. We show that, for one particular case at least, the condition of self consistency is ensured by the interference of quantum mechanical amplitudes associated with the loop. If this can be applied to all loops then we have a mechanism by which inconsistent loops eliminate themselves.
I. INTRODUCTION

This paper is based on a talk given at a conference in Naples [1]. It is well known that if backward time travel could be implemented, or if the present could shape the past by some other means, then closed causal chains, or causal loops, could be formed. The possibility of causal loops must arise because changing, or rewriting, the past is not particularly meaningful. If these loops are not self consistent, paradoxes arise [2]. The best known of these is the grandfather paradox: a man travels into the past to kill his grandfather before his grandfather meets his grandmother. If he succeeds then we have the paradox that the man is never born, thus the grandfather is not killed, thus the man is born and so on. A simpler and more direct example is autoinfanticide, whereby a man travels into the past and shoots himself as a baby. A different example is where a man travels back and introduces his parents to each other [3]. Here we have a paradox if he fails in his task. A more direct version of this latter type of paradox involves a robot, let us say, which enters a time machine in the present. When it emerges from the machine in the past it is sent through a rejuvenator which removes its memory and programs it to walk around until it enters the time machine in the present. A paradox occurs if something prevents the robot from entering the time machine in the present.

Sometimes the possibility of such paradoxes is taken as an argument against the possibility of backward time travel. It can, however, be argued that, because of continuity in nature, self-consistent loops or cycles in these situations always exist and it is only these cycles that nature allows [4]. Somehow the physics sorts things out so that the probability of inconsistent cycles is zero. We shall refer to this argument as the principle of self consistency [5]. An example of a self-consistent solution to the autoinfanticide cycle is the scenario in which the man raises his rifle to shoot himself as a baby but, because of his bad shoulder, misses the baby’s heart and hits it in the shoulder [6].

If we accept that self-consistent loops are reasonable, the question arises as to the mechanism by which physics disallows inconsistent loops and ensures self consistency. In this paper we show that quantum mechanics may provide an answer to this question.
II. QUANTUM MECHANICS

Quantum mechanics is essentially concerned with the preparation of a system and its subsequent measurement. The measurement device can be described mathematically by a probability operator measure (POM) \([7]\). Each element \(\hat{\Pi}_j\) of the POM corresponds to a possible outcome event \(j\) of the measurement. All the POM elements are non-negative definite, that is, their expectation values for any state of the system are greater than or equal to zero. The POM elements sum to the unit operator, ensuring that the probability that there is some outcome of the measurement is unity. The POM elements themselves indicate the states the system can be “found in”. The preparation device can be described mathematically by a collection of preparation device operators (PDO) \(\hat{\Lambda}_i\) associated with possible preparation outcomes \([8]\). These non-negative definite operators indicate the possible states the system can be “prepared in”. The \textit{a priori} probability \(P(i)\) for a preparation event \(i\) is given by the trace \(\text{Tr}\hat{\Lambda}_i\). The trace of the sum of all the \(\hat{\Lambda}_i\) is unity, ensuring that the probability that there is some outcome of the preparation is unity. An essential feature of quantum mechanics, as opposed to classical mechanics, is that, even if there is no evolution of the system between the preparation and measurement events, the state the system is found in is not necessarily the state the system is prepared in.

A. Predictive formalism

The usual formalism of quantum mechanics is predictive. Between preparation and measurement we assign to the system a state on the basis of our knowledge of the preparation event. If we have no knowledge of the outcome of the preparation, the best we can do is to assign a density operator immediately after preparation equal to the sum of the PDO’s. If, however, we know the outcome was the preparation event \(i\), we assign the normalized density operator \(\hat{\rho}_i^{\text{pre}} = \hat{\Lambda}_i / \text{Tr}\hat{\Lambda}_i\) immediately after preparation. This state evolves unitarily forward in time to become \(\hat{U}\hat{\rho}_i^{\text{pre}}\hat{U}^\dagger\) at the time of measurement. \(\hat{U}\) is the usual forward time unitary evolution operator. Upon measurement there is a discontinuous change, or collapse, of the state to that corresponding to the POM element associated with the particular measurement outcome \(j\). The probability for the measurement outcome \(j\), given the
preparation outcome \( i \) is given by
\[
P(j|i) = \text{Tr}[\hat{\mathcal{U}} \hat{\rho}_i^{\text{pre}} \hat{\mathcal{U}}^\dagger \hat{\Pi}_j].
\] (1)

As the name indicates, the predictive formalism is particularly useful for predicting outcomes of measurement events based on knowledge of preparation events. We can use the formalism, in conjunction with Bayes’ theorem, to retrodict outcomes of preparation events given measurement events[9]. From Bayes’ theorem we have
\[
P(i|j) = \frac{P(j|i)P(i)}{P(j)} = \frac{P(j|i)P(i)}{\sum_i P(j|i)P(i)}.
\] (2)

When the known outcome \( i \) of the preparation device corresponds to a pure state \( |p_i\rangle \), both \( \hat{\Lambda}_i \) and \( \hat{\Lambda}_i/\text{Tr}\hat{\Lambda}_i \) are just \( |p_i\rangle \langle p_i| \). Also when the known outcome \( j \) of the measurement device corresponds to a pure state \( |m_j\rangle \), both \( \hat{\Pi}_j \) and \( \hat{\Pi}_j/\text{Tr}\hat{\Pi}_j \) are just \( |m_j\rangle \langle m_j| \). It is simpler in such cases to work with the associated pure states instead of the PDO’s and POM elements. Expression (1) then becomes simply \( \left| \langle m_j| \hat{\mathcal{U}} |p_i\rangle \right|^2 \).

An interesting situation arises when we have prepared two systems \( a \) and \( b \) in an entangled pure state, which evolves to, say,
\[
2^{-1/2} (|a_1\rangle_a |b_1\rangle_b + |a_2\rangle_a |b_2\rangle_b)
\]
at the time of the measurement of system \( b \), which is measured before system \( a \). If the measurement outcome is the POM element corresponding to state \( |b_1\rangle_b \), then, in the predictive formalism, the measurement projects this state onto the entangled state, giving the state \( |a_1\rangle_a \) after normalization. This collapsed state continues to evolve forwards in time until system \( a \) is measured.

B. Retrodictive formalism

A less usual, but equally valid, quantum mechanical formalism is the retrodictive formalism [10]. Here the state of a system between preparation and measurement is assigned on the basis of our knowledge of the \textit{measurement event}. If we know the outcome was the measurement event \( j \), we assign the normalized density operator \( \hat{\rho}_j^{\text{ret}} = \hat{\Pi}_j/\text{Tr}\hat{\Pi}_j \) to the system immediately before measurement [9]. This state evolves unitarily \textit{backwards} in time to become \( \hat{\mathcal{U}}^\dagger \hat{\rho}_j^{\text{ret}} \hat{\mathcal{U}} \) at the time of preparation. Upon preparation there is a discontinuous change, or collapse, of the state to that corresponding to the PDO associated with the
particular preparation outcome \( i \). The probability for the preparation outcome \( i \) given the measurement outcome \( j \) is given by

\[
P(i|j) = \frac{\text{Tr}(\hat{U}_i^{\dagger} \hat{\rho}_j^{\text{ext}} \hat{U}_i)}{\sum_i \text{Tr}(\hat{U}_i^{\dagger} \hat{\rho}_j^{\text{ext}} \hat{U}_i)}.
\]

(3)

The denominator ensures the probabilities for all possible preparation events sum to unity. It is not difficult to show, using the cyclic property of the trace, that this is the same as the formula for \( P(i|j) \) in (2) obtained from the predictive formalism plus Bayes’ theorem. Use of the retrodictive formalism for this purpose is more direct, however, which is why it is becoming useful for solving the basic quantum communication problem [11], in which Bob measures the state of a quantum system sent by Alice and has to retrodict the state she selected to send. Our interest now, however, is the application of the retrodictive formalism to entangled systems.

Suppose the result of a measurement on two systems \( a \) and \( b \) corresponds to the POM element associated with the pure retrodictive entangled state which evolves backwards in time to

\[
2^{-1/2}(|a_1\rangle_a |b_1\rangle_b + |a_2\rangle_a |b_2\rangle_b)
\]

at the time of the preparation of system \( b \), which is prepared after system \( a \) is prepared. If the preparation outcome is the PDO corresponding to state \( |b_1\rangle_b \), then we project this state onto the entangled state, renormalize, and obtain the state \( |a_1\rangle_a \). This collapsed retrodictive state continues to evolve backwards in time until the time of the preparation of system \( a \).

An even more interesting case arises when a retrodictive state from a measurement of system \( c \) at time \( t_m \) evolves backwards to a state \( |c_1\rangle_c \) at the time \( t_p \) of a preparation outcome that corresponds to the entangled state

\[
2^{-1/2}(|c_1\rangle_c |d_1\rangle_d + |c_2\rangle_c |d_2\rangle_d).
\]

The projection of \( |c_1\rangle_c \) onto this state collapses it to \( |d_1\rangle_d \), which can be associated with a predictive state evolving forwards in time from \( t_p \).

III. BEAM SPLITTERS

A method of making predictive entangled states is to prepare the light in the input modes \( b \) and \( c \) of a beam splitter in the vacuum and one-photon states with preparation devices P0
FIG. 1: Beam splitters. In (a) preparation devices P0 and P1 prepare vacuum and one-photon states in input modes \( b \) and \( c \). In (b) photodetectors D1 and D0 detect one and zero photons in output modes \( b \) and \( a \).

and P1 respectively. We can then write the predictive input state as \( |0\rangle_b |1\rangle_c \). To be specific, let the beam splitter, as shown in Fig. 1(a), be symmetric and reflect as much light as it transmits. The action of the beam splitter can be represented by a unitary operator \( \hat{R} \). It is possible to show that for this 50/50 beam splitter, the action of the beam splitter on the input state is to transform it to the entangled state [12]

\[
\hat{R} |0\rangle_b |1\rangle_c = 2^{-1/2}( |0\rangle_b |1\rangle_c + i |1\rangle_b |0\rangle_c )
\]

in the output of the beam splitter. This shows that there is an equal chance for the photon to remain in mode \( c \), that is to be transmitted, and for it to be reflected into mode \( b \). It is sometimes convenient to regard P0, P1 and the beam splitter as a combined preparation device that generates the predictive state \( 2^{-1/2}( |0\rangle_b |1\rangle_c + i |1\rangle_b |0\rangle_c ) \) if the outcomes of P0 and P1 correspond to zero and one photons. This entangled state then propagates forward in time. If, at a later time \( t_m \), a measurement on the field in mode \( c \) shows that the photon is in mode \( c \) then, according to the usual description of the measurement process in the predictive formalism, the field instantaneously collapses at \( t_m \) to \( \varepsilon |1\rangle ( |0\rangle_b |1\rangle_c + i |1\rangle_b |0\rangle_c ) = |0\rangle_b \) after normalization. That is, the field in mode \( b \) changes suddenly to the vacuum simultaneously with the measurement of the field in mode \( c \), even though this measurement event can take place an appreciable distance away. This, of course, is just an illustration of the well-known Einstein-Podolsky-Rosen paradox involving instantaneous collapse over a large distance. The vacuum state \( |0\rangle_b \), being a predictive state, continues to propagate forwards in time.

An entangled retrodictive state can be made as follows. Photon detectors D1 and D0 are in the output ports \( b \) and \( a \) of a 50/50 beam splitter as shown in Fig. 1(b). If D1 detects one photon and D0 detects zero photons, the output field of the beam splitter is in
the retrodictive state $|1\rangle_a |0\rangle_b$. This state propagates backwards in time through the beam splitter. The unitary operator for this evolution is $\hat{R}^\dagger$. We can show that

$$\hat{R}^\dagger |1\rangle_b |0\rangle_a = 2^{-1/2}(|1\rangle_b |0\rangle_a - i |0\rangle_b |1\rangle_a)$$

which can be interpreted that the photon has an equal chance of being reflected in its journey back in time into mode $a$ or of staying in mode $b$. It is sometimes convenient to regard D1, D0 and the beam splitter as a combined measurement device that generates the retrodictive state $2^{-1/2}(|1\rangle_b |0\rangle_a - i |0\rangle_b |1\rangle_a)$ corresponding to the combined event of D1 detecting one photon and D0 detecting zero photons. If there is a preparation device with a known output state acting on input mode $a$, then we can project this state onto the retrodictive entangled state, resulting in a retrodictive state in input mode $b$. This retrodictive state continues to evolve backwards in time.

Let us examine, in terms of the retrodictive formalism, the case where a combined preparation device generates the predictive state $2^{-1/2}(|0\rangle_b |1\rangle_c + i |1\rangle_b |0\rangle_c)$ at time $t_p$, for example as described above, and where a later measurement on the field in mode $c$ at time $t_m$ shows the presence of one photon in this mode. We attach a retrodictive state $|1\rangle_c$ to the field at the time of measurement. This state evolves backwards in time and becomes $\hat{U}^\dagger |1\rangle_c$ at the output of the combined preparation device, that is at the output port of the beam splitter, at time $t_p$. Here $\hat{U}^\dagger$ is the hermitian conjugate of the free-space forward time evolution operator from $t_p$ to $t_m$. The effect of $\hat{U}^\dagger$ on $|1\rangle_c$ is to leave this state unchanged, so at the preparation time $t_p$ we project $|1\rangle_c$ onto the state $2^{-1/2}(|0\rangle_b |1\rangle_c + i |1\rangle_b |0\rangle_c)$, leaving us after normalization with the predictive state $|0\rangle_b$ which evolves forward in time. This is the same state that we obtained in the predictive description of the same situation but there is an important difference. In the retrodictive formalism the collapse to this state takes place at $t_p$, the time of preparation, that is as the field is leaving the beam splitter, whereas in the predictive formalism it occurs at $t_m$, the time of measurement, giving rise to the Einstein-Podolsky-Rosen paradox.

IV. UNCONTROLLABLE TIME MACHINE

We can combine the above beam splitter with preparation devices P0 and P1 in its input modes, which we shall refer to as BSL, and the above beam splitter with photon detectors
D0 and D1 in its output modes, which we shall refer to as BSU, to form a double beam splitter arrangement that shares the common mode \( b \), as shown in Fig. 2 [13]. BSL and BSU are the lower and upper beam splitters in this figure. The output mode \( b \) of beam splitter BSL becomes the input mode \( b \) of BSU. In the other input mode of BSU, that is input mode \( a \), we put a preparation device whose single possible outcome event corresponds to the known predictive state

\[
|in\rangle_a = a_0 |0\rangle_a + a_1 |1\rangle_a
\]  

(6)
at the immediate entry to BSU. We assume also that we can adjust the preparation device to control the ratio of the coefficients \( a_0/a_1 \). We also know that P0 and P1 prepare vacuum and one-photon states respectively. We can interpret the superposition state (6) as showing that one or zero photons might be in input mode \( a \). Thus, including the photon input from P1, detectors D0 and D1 might detect a total of either one or two photons. There is a sizeable probability that D0 and D1 will detect zero and one photons respectively. We can calculate this probability, which depends on \( a_0/a_1 \), but we do not need the precise result here. Other possible measurement outcomes are D0 and D1 detecting one and zero photons respectively, one and one photon respectively, zero and two photons respectively and two and zero photons respectively.

Let us study the case in which detectors D0 and D1 detect zero and one photons respectively. As seen in (5), this detection event generates an entangled retrodictive state

\[
2^{-1/2}(|1\rangle_b |0\rangle_a - i |0\rangle_b |1\rangle_a)
\]
at the input to BSU. As we know there is a preparation event corresponding to a predictive state \(|in\rangle_a\) here, we project this state onto the entangled state and obtain the retrodictive state that, after normalization, is

\[
a \langle in | (|1\rangle_b |0\rangle_a - i |0\rangle_b |1\rangle_a) = a_0^* |1\rangle_b - ia_1^* |0\rangle_b.
\]  

(7)
This is the state of the field at the time \( t_m \). This state propagates backwards in time in mode \( b \) to become

\[
\hat{U}^\dagger(a_0^* |1\rangle_b - ia_1^* |0\rangle_b)
\]
at the output mode \( b \) port of the other beam splitter BSL at the earlier time \( t_p \). The action of the free space backward time evolution operator \( \hat{U}^\dagger \) is just to change the phase of the state \( a_0^* |1\rangle_b - ia_1^* |0\rangle_b \). To avoid unnecessary complications without losing the essential physics, we now specify that the optical distance between the two beam splitters is an integer number
FIG. 2: Double beam splitter arrangement. The upper and lower beam splitters BSU and BSL are an integer number of wavelengths apart. If detectors D0 and D1 detect zero and one photons respectively, the state $|\text{out}\rangle$ is the same as the state $|\text{in}\rangle$.

of wavelengths of the light. This means that the retrodictive state at the earlier time $t_p$ at the output of BSL becomes again $a_0^* |1\rangle_b - ia_1^* |0\rangle_b$. At this time this state is projected onto the known output $2^{-1/2}(|0\rangle_b |1\rangle_c + i |1\rangle_b |0\rangle_c)$ of the combined P0, P1 and BSL preparation device to give

$$2^{-1/2}(a_0 |1\rangle_b + i a_1 |0\rangle_b) = 2^{-1/2}i(a_0 |0\rangle_c + a_1 |1\rangle_c).$$

After normalization and removal of the undetectable phase factor $i$, we see that this predictive state in mode $c$, which we label $|\text{out}\rangle_c$, is identical to the predictive state $|\text{in}\rangle_a$ in (6) which we inject into BSU at a later time.

We see therefore that, by means of the double beam splitter device, we can send a state of light $|\text{in}\rangle = a_0 |0\rangle + a_1 |1\rangle$, for which we can choose the ratio $a_0/a_1$, backwards in time from when we prepare it at time $t_m$ to an earlier time $t_p$. In principle the beam splitters can be separated by, say, one light day so the state $|\text{in}\rangle$ which we choose today can be sent back to appear yesterday. Can we use this device to send a message to our earlier selves? Unfortunately, we cannot. The state $|\text{in}\rangle$ appears yesterday only if the detectors D0 and
D1 detect zero and one photons respectively and we have no control over what they will
detect. If they detect one and zero photons respectively, which is just as likely, then the
state \(|\text{out}\rangle\) that appears yesterday is \(a_0 |0\rangle - a_1 |1\rangle\). If they both detect zero photons then
\(|\text{out}\rangle = |1\rangle\). If they detect a total of two photons then \(|\text{out}\rangle = |0\rangle\). Thus in the absence of
preknowledge of what will be detected the best we can do is to send a mixed state back. This
mixed state is the sum of the density matrices representing the four possible states \(|\text{out}\rangle\)
weighted by the probabilities of them occurring. A formal calculation of this mixed state
and of the quantum information it can carry shows that zero information can be transmitted
to the past. This may seem a little surprising because, if we could measure the state \(|\text{out}\rangle\),
sometimes we might find \(|1\rangle\) or \(|0\rangle\), which we could ignore, but on a sizeable number of
occasions we would find \(a_0 |0\rangle + a_1 |1\rangle\) or \(a_0 |0\rangle - a_1 |1\rangle\) from which we could calculate the
ratio \(a_0/a_1\), which could be the number of the winner of a horse race. Unfortunately, to
determine the state \(|\text{out}\rangle\) with any reasonable precision, we need a large of identical copies
of it. This cannot be done by cloning and, as we have noted above, our lack of control over
the outcomes of D0 and D1 prevents us from sending identical copies back.

V. A CLOSED CYCLE

What use is an uncontrollable time machine if the lack of control is such as to prevent us
sending any information back in time? Even though we cannot control it we do, however,
send something back. Further, we do know what we have sent back. Thus it might be possible
to use such a device to examine a causal loop associated with a time travel paradox.

Specifically, let us consider the robot cycle described in the introduction. The robot enters
a time machine today in a particular state \(S\) and is sent back in time to yesterday. Upon
emerging from the time machine yesterday it enters a rejuvenator which adjusts its state
and programs it to walk around for a day and then to enter the time machine in precisely
the state \(S\). Although the robot exists only during the cycle, the cycle is not isolated from
the rest of the world. As well as there being a need to build and adjust the time machine
and rejuvenator, the robot itself will leave footprints as it enters the time machine. These
footprints will provide evidence of the existence of the robot as well as providing information
about its properties, for example its size. The action of the rejuvenator on the robot may
depend on properties of the robot itself. If the rejuvenator fails to program the robot to be
in the state $S$ today, either because of bad adjustment or because of some property of the robot, then we have an inconsistency. According to the principle of self consistency only self-consistent loops are possible, so in this case there would be no footprints associated with this particular type of robot entering the time machine. The adjustment of the rejuvenator might, however, be suitable to provide a self-consistent cycle for a different robot, say a small robot. In this case, the need for consistency would not eliminate this cycle and any footprints left would be small.

Our aim is to use the double beam splitter device to examine closed cycles. Classically if the rejuvenator programs the robot to be in a state today even slightly different from $S$, an inconsistency arises because we regard different classical states as effectively orthogonal. This is justifiable to the extent that, in quantum mechanical terms, even if one particle of the robot is in an orthogonal state to what it should be then the complete state of the robot is orthogonal to what it should be. For simpler quantum mechanical systems, however, the distinction is not as clear cut. Here we have the possibility of complete self-consistency if the two states under consideration are identical and of complete inconsistency if these states are orthogonal. In between we have states that are not identical but not orthogonal. We shall see the effect of this later. To create a consistent cycle we wish to allow the field in state $|\text{out}\rangle$ at $t_p$ to evolve forward in time until, at $t_m$, it is in the state $|\text{in}\rangle$ entering the input port of BSU in mode $a$. To do this we create a feedback loop with fully reflecting mirrors while keeping the two path lengths between BSL and BSU equal, as shown in Fig. 3. Again, for simplicity, we assume both these path lengths are an integer number of wavelengths. In order to experiment with consistent and inconsistent cycles we also insert a phase shifter into the feedback part of the cycle. This acts as our adjustable rejuvenator. Introducing the other part of the loop in Fig. 3 removes the controllable preparation device which prepares $|\text{in}\rangle$ in Fig. 2 so we no longer have the control we had to choose the ratio $a_0/a_1$. The only preparation devices are now P0 and P1. From conservation of energy, this means that a total of only one photon can be detected by D0 and D1. Also, from symmetry, the photon should have an equal probability of being found in mode $b$ and in the mode containing the phase shifter if we were to try to detect photons in these modes. That is, there should be a probability of $1/2$ of finding the photon in either mode. We can ensure this by letting $|a_0|^2 = |a_1|^2 = 1/2$.

The effect of applying a phase shift $\varphi$ to the field in state $|\text{out}\rangle$ is to change it from
FIG. 3: Double beam splitter arrangement with feedback. The output light in state $|out\rangle$ from the lower beam splitter is directed through a phase shifter PS and via a fully reflecting mirror to become the input light in state $|in\rangle$ at the input of the upper beam splitter.

$a_0 |0\rangle + a_1 |1\rangle$ to $a_0 |0\rangle + \exp(i\varphi)a_1 |1\rangle$, so $|out\rangle$ will evolve to the state

$$|\tilde{m}\rangle = a_0 |0\rangle + a_1 \exp(i\varphi) |1\rangle$$

at time $t_m$. Thus by adjusting the value of $\varphi$ to zero or an integer multiple of $2\pi$, we can make the rejuvenator work perfectly and obtain the consistency requirement $|\tilde{m}\rangle = |in\rangle$. For a general value of $\varphi$, we obtain the projection

$$\langle in | \tilde{m}\rangle = |a_0|^2 + \exp(i\varphi) |a_1|^2 = [1 + \exp(i\varphi)]/2.$$  \hspace{1cm} (10)

By setting $\varphi = \pi$ we see that (10) vanishes, so $|\tilde{m}\rangle$ is orthogonal to $|in\rangle$. In this case we can say that the rejuvenated state is definitely not the state required for a self-consistent cycle. The principle of self consistency would then imply that this closed cycle is impossible. The outcome or “footprints” associated with this particular cycle, that is, D0 and D1 measuring zero and one photons respectively, should thus never be observed.

Let us now keep $\varphi = \pi$ and examine the cycle we would obtain associated with the other outcome that D0 and D1 measure one and zero photons respectively. This is the only other possible outcome for the closed cycle because, as noted above, a total of only one photon can be registered by D0 and D1. We then find, in place of (5) that

$$\hat{H}^t |0\rangle_b |1\rangle_a = 2^{-1/2}(|0\rangle_b |1\rangle_a - i |1\rangle_b |0\rangle_a).$$  \hspace{1cm} (11)
We find that projecting $|in\rangle$ onto (11) gives us the retrodictive state $-i\alpha_0^* |1\rangle_b + \alpha_1^* |0\rangle_b$ at $t_m$. This state evolves backwards in time to $-i\alpha_0^* |1\rangle_b + \alpha_1^* |0\rangle_b$ at $t_p$ because the mode $b$ path is an integer number of wavelengths. Projecting this onto the prepared entangled state $2^{-1/2}(|0\rangle_b |1\rangle_c + i |1\rangle_b |0\rangle_c)$ gives, after normalization and removal of the undetectable factor, the state

$$|out\rangle = a_0 |0\rangle_c - a_1 |1\rangle_c.$$  

(12)

This leads to

$$|in\rangle = a_0 |0\rangle - a_1 \exp(i\varphi) |1\rangle = a_0 |0\rangle + a_1 |1\rangle.$$  

(13)

We see that for this case we do have a self-consistent cycle, with associated “footprints” being D0 and D1 measuring one and zero photons respectively. Thus there is no reason on the basis of the principle of self consistency to eliminate this cycle. On the other hand if we set $\varphi = 0$ we find for this case that $|in\rangle = a_0 |0\rangle - a_1 |1\rangle$ so we have an inconsistent cycle so there should be no chance of D0 and D1 measuring one and zero photons respectively for this setting of $\varphi$.

We have seen that if we use the only output measure we have, that is what D0 and D1 detect, as a “probability meter” for a cycle occurring, then the principle of self consistency implies that we can adjust the setting of $\varphi$ to make one or other of the two possible measurement outcomes impossible. While this is really all we can talk about classically, in quantum mechanics we also have intermediate cases. For example, what happens if we choose $0 < \varphi < \pi$? In this case there would be some incomplete overlap of $|in\rangle$ and $|in\rangle$. We might expect that we could extend the principle of self consistency such that the probability for the cycle occurring is $\left| \langle in |in \rangle \right|^2$. Thus on repeating the experiment a large number of times, in some of cases the footprints for one cycle would be found, for example D0 and D1 measuring one and zero photons respectively, and for the remaining cases the alternative footprints would be found.

VI. INTERFERING AMPLITUDES

The next question is whether or not we should build the uncontrollable time machine with the feedback loop to investigate experimentally the principle of self consistency, that is, to check if we do indeed find it impossible for D0 and D1 to detect zero and one photons respectively if $\varphi = \pi$. Fortunately there is no need actually to do the experiment because
what we have shown in Fig. 3 is essentially a Mach-Zehnder interferometer for which we know the results. As shown below, these confirm that D0 and D1 do not detect zero and one photons respectively if \( \varphi = \pi \) and do not detect one and zero photons respectively if \( \varphi = 0 \). Even more fortunately, we can explain how the Mach-Zehnder interferometer works in terms of interference of quantum mechanical amplitudes in the usual predictive formalism of quantum mechanics. Essentially there are two paths by which a photon from P1 in Fig. 3 can be detected by D1, that is, via the mode \( b \) path and via the phase shifter path. These two paths are the two components of the cycle. We might think of the photon being reflected by BSI, and then transmitted by BSU or as being transmitted by BSL and reflected by BSU. The amplitude for reflection \( A(r) \) by BSL into mode \( b \) is the coefficient of \( |1\>_b \) \( |0\>_e \) in (4), that is \( A(r) = i2^{-1/2} \). Likewise the transmission amplitude for BSU is the coefficient of \( |0\>_e \) \( |1\>_b \) on (4), that is, \( A(t) = 2^{-1/2} \). Thus the compound amplitude for reaching D1 via mode \( b \) is \( A(r)A(t) = i/2 \). The transmission amplitude for BSL is also \( A(t) \) and phase shifter introduces another factor \( \exp(i\varphi) \), so the amplitude of being transmitted and reaching BSU is \( A(t, \varphi) = 2^{-1/2}\exp(i\varphi) \). The reflection amplitude for BSU is also \( A(r) \). The compound amplitude for reaching D1 via the mode containing the phase shifter is \( A(t, \varphi)A(r) \). The total amplitude for detection of the photon by D1 is then

\[
A(r)A(t) + A(t, \varphi)A(r) = i[1 + \exp(i\varphi)]/2.
\]  

(14)

If \( \varphi = \pi \) the amplitude for one path is the negative of the amplitude for the other path and the total amplitude, and therefore the probability for D1 to detect a photon, is zero. If \( \varphi = 0 \), however, the amplitudes for these two paths add constructively, giving a unit probability for detection. We note that the probability obtained by squaring the modulus of (14) is equal to \( \left| \langle in|m \rangle \right|^2 \). These results confirm the principle of self consistency for this particular case and its extension that the probability for footprints associated with the cycle occurring is \( \left| \langle in|m \rangle \right|^2 \).

These results indicate that we should look to quantum mechanical \textit{amplitudes}, rather than probabilities, in seeking the physical mechanism underlying the impossibility of inconsistent cycles, that is, in understanding how physics sorts things out and arrives at a consistent cycle. For a causal closed cycle, we can choose the earliest event and the latest event, which is associated with the footprints for example, and then regard the two parts of the cycle as two different ways of reaching the latest event, or footprints, from the earliest event.
Indeed, we could also regard the cycle as two different ways of reaching the earliest event from the latest event. If the amplitudes associated with the two parts completely interfere then the total amplitude associated with the cycle is zero and the probability for the cycle occurring is zero, that is, it is impossible. Constructive interference, however, renders the cycle possible. For simple quantum systems, there are intermediate cases between completely destructive and completely constructive interference, so the probability for some cycles will be somewhere between zero and unity. There only needs to be a slight difference in two states of a macroscopic object, for example if the two states are identical except for the states of just one particle of the object which are orthogonal, to make the macroscopic states orthogonal themselves. Consequently we usually consider classical cycles as being either consistent or inconsistent.

VII. CONCLUSION

The possibility of inconsistent closed causal cycles has been used as an argument against the possibility of backward time travel. Against this, the principle of self consistency has been proposed which states that physics sorts things out so that inconsistent cycles are impossible anyway, so any closed cycles must be consistent. If this principle is correct then the possibility of closed cycles is not a valid argument against backward time travel. This principle, however, opens up another question - how does physics sort out things so that only consistent cycles occur? In this paper we have examined this question in the light of a device which can be interpreted, in terms of the retrodictive formalism of quantum mechanics, as an uncontrollable time machine. A known state of light \( |in\rangle \) at the input of a beam splitter at time \( t_m \) is sent into the past and then allowed to evolve into the future so that at time \( t_m \) it is in state \( |\overline{n}\rangle \) at the input of the beam splitter. Because the device is interpretable in usual predictive quantum theory as an interferometer, we can calculate the probability of the cycle occurring and leaving its particular footprints to be \( \left| \langle \overline{n} | in \rangle \right|^2 \). This means that the probability of the cycle occurring, as observed by its footprints, is unity if \( |\overline{n}\rangle \) is the same state as \( |in\rangle \) and is zero if \( |\overline{n}\rangle \) is orthogonal to \( |in\rangle \). The former case is a self consistent cycle and the latter is an inconsistent cycle. These results confirm the principle of self consistency for this particular case. Furthermore, the results allow a possible extension of the principle, which is essentially classical, to quantum mechanics to say that
the probability of a cycle occurring can be found by calculating the quantum evolution of the state around the cycle, which involves evolution both backwards and forwards in time, to find the evolved state at the starting point. The probability of the cycle is then given by the square of the modulus of the projection of the evolved state onto the original state.

If what we have found for the cycle we have studied can be applied to all cycles, then we have an underlying quantum mechanical explanation of the principle of self consistency. Essentially, if we work in terms of quantum mechanical amplitudes rather than probabilities, we can select the latest and earliest events on the cycle and then say that the amplitude for reaching the latest event from the earliest event has two terms, corresponding to the two different pathways to the later event. For inconsistent cycles these two amplitudes cancel each other when added. For consistent cycles, they interfere constructively. The probability of the cycle is the square of the modulus of the total amplitude and thus fully inconsistent cycles have a probability of zero of occurring, that is, they are impossible. Even slightly different classical states, that is quantum states of macroscopic objects, can be orthogonal so these slight differences induced by evolution around the cycle can render the cycle impossible. Classical cycles are thus usually regarded as consistent or inconsistent.

It is not totally surprising that a principle applying to classical physics has a quantum mechanical basis. The classical principle of least action can be explained in terms of the addition of amplitudes associated with all possible paths. The amplitudes for all paths except for those in the region of the path of least action cancel, so the probability for finding that the system has taken a path not near the path of least action is zero [14]. This explains how the system “knows” to take the path of least action. In this paper we suggest that closed causal cycles are sorted out by a similar mechanism. Only those cycles with a net non-zero amplitude have a non-zero probability of occurring and these are the consistent cycles. In conclusion, rather than just being invoked to save the possibility of the present shaping the past, it now seems that the principle of self consistency could well have a solid physical basis in quantum mechanics.
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[6] I have heard this scenario attributed to Feynman but do not know the reference.


