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Author

Frembs, Markus, Cavalcanti, Eric G

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Variations on the Choi-Jamiołkowski isomorphism

Markus Frembs[1,*] and Eric G. Cavalcanti[1,†]

[1] *Centre for Quantum Dynamics, Griffith University, Yugambeh Country, Gold Coast, QLD 4222, Australia*

Abstract

The Choi-Jamiołkowski isomorphism is an essential component in every quantum information theorist's toolkit: it allows to identify linear maps between two quantum systems with linear operators on the composite system. Here, we analyse this widely used gadget from a new perspective. Namely, we explicitly distinguish between its kinematical and dynamical properties, that is, we study the isomorphism on two different levels: Jordan algebras and the different C^*-algebras they arise from, which are distinguished by their order of composition.

A number of important and novel insights stem from our analysis. We find that Choi's theorem, which asserts that Choi's version of the isomorphism [M.-D. Choi, Lin. Alg. Appl., 10, 285 (1975)] further maps the positive cone of completely positive linear maps (such as quantum channels) to the cone of positive linear operators (such as quantum states) on the composite system, crucially depends on the dynamical structure in C^*-algebras. We explain in detail how this dependence gives rise to the mismatch between the basis-dependence of Choi's version of the isomorphism, and the basis-independent version by Jamiołkowski [A. Jamiołkowski, Rep. Math. Phys., 3, 275 (1972)]. We then overcome this subtle but pervasive issue in a number of ways: first, we prove a version of Choi's theorem for Jamiołkowski's isomorphism, second, we define a basis-independent variant of Choi's isomorphism and, third, by making explicit the dynamical distinction between Jordan and C^*-algebras, we combine the different variants of the isomorphism into a unified description, that subsumes their individual features. We also embed and interpret our results in the graphical calculus of categorical quantum mechanics.

* m.frembs@griffith.edu.au

† e.cavalcanti@griffith.edu.au

I. INTRODUCTION

Let $\mathcal{A} = \mathcal{L}(\mathcal{H}_\mathcal{A}) = M_n(\mathbb{C})$ and $\mathcal{B} = \mathcal{L}(\mathcal{H}_\mathcal{B}) = M_m(\mathbb{C})$ be two finite-dimensional matrix algebras over the complex numbers. The *Choi-Jamiołkowski isomorphism* [18, 22, 44] refers to an identification of linear operators in the tensor product algebra $\mathcal{A} \otimes \mathcal{B} \cong \mathcal{L}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B})$ and linear maps $\mathcal{L}(\mathcal{A}, \mathcal{B}) \coloneqq \{\phi : \mathcal{A} \to \mathcal{B} \mid \phi \text{ linear}\}$ between these algebras. This identification is not unique, a number of different versions exist, each with different properties. Given this generality, it is not surprising that the Choi-Jamiołkowski isomorphism is a widely used tool in quantum information theory and beyond.[1]

Out of the many research areas influenced by the Choi-Jamiołkowski isomorphism, we pick two in order to highlight two different and seemingly conflicting properties between the two main versions of the isomorphism—as introduced by Jamiołkowski in Ref. [44] (see Sec. II A) and Choi in Ref. [18] (see Sec. II B). On the one hand, Jamiołkowski's version assumes a clear and simple form in the graphical calculus for quantum theory [1, 5, 19, 40, 62] (see App. 2), where it is also referred to as 'process-state duality' [19]. In this setting, it is natural to require the isomorphism to be represented in a basis-independent way. On the other hand, Choi's theorem (see Thm. 3 below) extends the isomorphism to the respective positive cones of bipartite operators and completely positive maps. Since quantum states are represented by normalised, positive operators and quantum channels (or 'processes') by completely positive, trace-preserving maps, it is Choi's version of the isomorphism that expresses the duality between states and processes in its strongest form, and that has found most use in quantum information [49, 68].

Unfortunately, basis-independence is in conflict with Choi's theorem, since Choi's isomorphism is basis-dependent and since Choi's theorem does not apply to the basis-independent version by Jamiołkowski.[2] The result of this unresolved tension is an awkward compromise in the use of a basis-dependent representation of process-state duality, leading to clumsy appearances of transposes in various formulas.

A key contribution of our work is to alleviate this tension in a number of ways:

[1] See the recent online symposium "Celebrating the Choi-Jamiołkowski isomorphism" [49] for an overview of the various subjects it has impacted, as well as for more recent developments in some of those subjects.

[2] Note, however, that Choi's theorem exhibits a restricted from of basis-independence: it holds for all operator bases $\{a_{ij}\}_{ij} \in \mathrm{ONB}(\mathcal{L}(\mathcal{H}_\mathcal{A}))$ of the form $a_{ij} = |i\rangle\langle j|$ for $\{|i\rangle\}_i$ a basis in $\mathcal{H}_\mathcal{A}$ (cf. Ref. [53, 59].)

(i) We highlight a crucial distinction between kinematical and dynamical aspects inherent to quantum systems, as captured by Jordan and C^*-algebras, respectively. Based on this distinction, we analyse various properties of the Choi-Jamiołkowski isomorphism that hold already on the (kinematical) level of Jordan algebras (Thm. 4 and Thm. 5).

(ii) We prove a novel, basis-independent variant of Choi's theorem for Jamiolkowski's isomorphism, which explicitly refers to the (dynamical) notion of operator orderings in C^*-algebras (Thm. 6). In this regard, our result complements recent results in Ref. [35, 53, 59] which study Choi's theorem under basis-change.

(iii) We provide a unified description of different variants of the isomorphism in terms of different operator orderings on Jordan algebras in Eq. (15).

(iv) Finally, we give a graphical representation of these variants on 'process-state dualities' in the language of dagger, compact closed categories (see Fig. 6 in App. 2).

The rest of the paper is organised as follows: in Sec. II, we present two variants of the isomorphism, due to Jamiołkowski [44] and Choi [18]. Our main conceptual contribution is the identification of the physical significance of the relative operator ordering (equivalently, relative time orientation) between C^*-algebras in the Choi-Jamiołkowski isomorphism. To this end, we give a brief formal background on the distinction between Jordan and C^*-algebras in Sec. III. In particular, we emphasise that while Jordan algebras encode the kinematics of quantum theory, they lack a key dynamical component: the ordering in the (associative) compositional structure of C^*-algebras. In Sec.IV we show that Choi's theorem depends on the operator ordering in the respective C^*-algebras and thus relates to this dynamical aspect of quantum theory. Taking the operator ordering into account explicitly, we prove a version of Choi's theorem for Jamiołkowski's version of the isomorphism in Sec. IV A, give a basis-independent variant of Choi's isomorphism and provide a unified description of Choi's and Jamiołkowski's variants of the isomorphism in Sec. IV B. Moreover, we relate those to the graphical calculus of dagger, compact closed categories in App. 2.

II. JAMIOŁKOWSKI AND CHOI - TWO ISOMORPHISMS, TWO THEOREMS

In this section, we review and compare the isomorphisms defined in Ref. [44] and Ref. [18].

A. Jamiołkowski's isomorphism and theorem

Ref. [44] defines a linear isomorphism $J : \mathcal{L}(\mathcal{A},\mathcal{B}) \to \mathcal{A}\otimes\mathcal{B}$ by

$$\begin{aligned} \rho_J^\phi = J(\phi) &:= \sum_k e_k^* \otimes \phi(e_k)\ , \\ \phi_J^\rho(a) = J^{-1}(\rho)(a) &:= \mathrm{tr}_\mathcal{A}[\rho(a\otimes \mathbb{1}_\mathcal{B})]\ , \end{aligned} \tag{1}$$

for all $a \in \mathcal{A}$, where $\{e_k\}_k \in \mathrm{ONB}(\mathcal{A})$ is any orthonormal basis in $\mathcal{A}$ with respect to the Hilbert-Schmidt inner product (see Eq. (8) below).[3] In particular, J is *basis-independent with respect to* $\mathcal{A}$: let $\{f_l\}_l \in \mathrm{ONB}(\mathcal{A})$ be another orthonormal basis in $\mathcal{A}$, then

$$\begin{aligned} \rho_J^\phi = \sum_k e_k^* \otimes \phi(e_k) &= \sum_k \Big(\sum_l f_l \mathrm{tr}_\mathcal{A}[f_l^* e_k]\Big)^* \otimes \phi(e_k) \\ &= \sum_k \sum_l \overline{(e_k, f_l)}_\mathcal{A} f_l^* \otimes \phi(e_k) \\ &= \sum_l \sum_k f_l^* \otimes (f_l, e_k)_\mathcal{A} \phi(e_k) \\ &= \sum_l f_l^* \otimes \phi\Big(\big(\sum_k e_k \mathrm{tr}_\mathcal{A}[e_k^* f_l]\Big) = \sum_l f_l^* \otimes \phi(f_l)\ , \end{aligned} \tag{2}$$

where we used that $\sum_l f_l \mathrm{tr}_\mathcal{A}[f_l^* \cdot]$ and $\sum_k e_k \mathrm{tr}_\mathcal{A}[e_k^* \cdot]$ define resolutions of the identity in the inner product space $\mathcal{A}$. Note that the argument depends on ϕ being linear; in contrast, for antilinear maps Eq. (1) yields a basis-dependent isomorphism. In Sec. IV B, we will see that this is the root cause of the basis-dependence of Choi's isomorphism.

The following theorems hold for J.

Theorem 1 (de Pillis [22]). *Let $\phi \in \mathcal{L}(\mathcal{A},\mathcal{B})$. Then ϕ is Hermiticity-preserving, that is, $(*\circ\phi)(a) = \phi^*(a) = \phi(a^*) = (\phi\circ *)(a)$ for all $a \in \mathcal{A}$, if and only if ρ_J^ϕ is Hermitian.*

Theorem 2 (Jamiołkowski [44]). *Let $\phi \in \mathcal{L}(\mathcal{A},\mathcal{B})$.*

(a) ϕ is positive if and only if ρ_J^ϕ is positive on pure tensors (POPT) (*or* block positive, cf. [53, 59]), *that is,* $\mathrm{tr}[\rho_J^\phi(a\otimes b)] \geq 0$ *for all $a \in \mathcal{A}_+$, $b \in \mathcal{B}_+$.*

(b) Let $*\circ\phi = \phi\circ *$. Then ϕ is trace-preserving if and only if $\mathrm{tr}_\mathcal{B}[\rho_J^\phi] = \mathbb{1}_\mathcal{A}$.

We remark that even though positivity is strictly weaker than complete positivity of quantum channels, positive maps play an important role in entanglement theory, where they are used to characterise entanglement witnesses [34, 41, 43, 60]. Basis-independence of the isomorphism in Eq. (1) is a natural requirement in such applications.

[3] Following notation in Ref. [44] and related work [27–30], we use $*$ instead of $\dagger$ for the (Hermitian) adjoint.

B. Choi's isomorphism and theorem

The most commonly used version of the Choi-Jamiołkowski isomorphism is motivated by a theorem due to Choi [18] (Thm. 3 below), who defines a map $C : \mathcal{L}(\mathcal{A},\mathcal{B}) \to \mathcal{A} \otimes \mathcal{B}$ by

$$\begin{aligned} \rho_C^\phi = C(\phi) &:= \sum_{ij} |i\rangle\langle j| \otimes \phi(|i\rangle\langle j|) , \\ \phi_C^\rho(a) = C^{-1}(\rho)(a) &:= \mathrm{tr}_\mathcal{A}[\rho(a^T \otimes \mathbb{1}_\mathcal{B})] , \end{aligned} \tag{3}$$

for all $a \in \mathcal{A}$, where $\{|i\rangle\langle j|\}_{ij}$ denotes the orthonormal basis in $\mathcal{A}$ built from an orthonormal basis $\{|i\rangle\}_i \in \mathrm{ONB}(\mathcal{H}_\mathcal{A})$, and T denotes transposition in that basis.[4]

As a special case of Stinespring's theorem [63], Choi's theorem yields the following useful characterisation of completely positive maps.

Theorem 3 (Choi [18]). *Let $\phi \in \mathcal{L}(\mathcal{A},\mathcal{B})$. Then ϕ is completely positive (CP) if and only if ρ_C^ϕ is positive. That is, C restricts to a map $C|_{\mathcal{L}(\mathcal{A},\mathcal{B})_{\mathrm{CP}}} : \mathcal{L}(\mathcal{A},\mathcal{B})_{\mathrm{CP}} \to (\mathcal{A} \otimes \mathcal{B})_+$.*

Unlike Eq. (1), Eq. (3) depends on the choice of basis, since transposition is a basis-dependent operation. Despite this fact, Choi's theorem holds for all bases $\{a_{ij} = |i\rangle\langle j|\}_{ij} \in \mathrm{ONB}(\mathcal{L}(\mathcal{H}_\mathcal{A}))$ with $\{|i\rangle\}_i$ any basis in $\mathcal{H}_\mathcal{A}$ by Thm. 2 in Ref. [53] (see also Ref. [35, 59]).

Importantly, since quantum states are represented by (normalised) positive operators ('density matrices') and quantum channels by (trace-preserving) completely positive maps, Choi's version of the isomorphism in Eq. (3) relates these two (up to normalisation); it is therefore also referred to as 'channel-state duality' (cf. Refs. [45, 69]). This duality enables the translation between properties of quantum states and those of quantum channels. As a consequence, Choi's isomorphism has found various applications in quantum information theory [49, 68]. For example, separable states correspond with measure-prepare (also known as entanglement-breaking) channels under the isomorphism [42], and this correspondence is at the center of the resource theory of entanglement [11, 14, 61] and the ongoing pursuit to quantify entanglement - a problem with applications in all of quantum physics. What is more, in the field of quantum computation Choi's isomorphism underlies the widely used protocol of gate teleportation [31], which in conjunction with magic state distillation [12, 50] forms one of the main architectures for fault-tolerant universal quantum computation [55].

[4] This is sometimes also written as $\rho_C^\phi := (\mathrm{id} \otimes \phi)(|\Phi\rangle\langle\Phi|)$, where $|\Phi\rangle = \sum_i |i\rangle \otimes |i\rangle$ is an (unnormalised) maximally entangled state. Note that if ϕ is trace-preserving, then $\mathrm{tr}[\rho_C^\phi] = \mathrm{tr}[\rho_J^\phi] = d_A$ (cf. Thm. 2).

III. BACKGROUND

A. Jordan algebras, complex structure and inner product

The different versions and properties of the Choi-Jamiołkowski isomorphism can be traced back to a subtle distinction between kinematical and dynamical aspects, as expressed by Jordan and C^*-algebras, respectively. We will only need the bare essentials of the latters' structure theory here, and liberally refer to Refs. [2, 3, 37] for many more details.

Jordan algebras. The operator product on $\mathcal{L}(\mathcal{H})$ admits the decomposition,

$$ab = \frac{1}{2}\{a,b\} + \frac{1}{2}[a,b]\,, \tag{4}$$

where $[a,b] = ab - ba$ denotes the commutator and $\{a,b\} \coloneqq ab + ba$ denotes the anti-commutator. $(\mathcal{L}(\mathcal{H}),[\cdot,\cdot])$ defines a Lie algebra while $\mathcal{J}(\mathcal{H}) \coloneqq (\mathcal{L}(\mathcal{H}),\{\cdot,\cdot\})$ defines a (complex) Jordan algebra.[5][6] Importantly, the anti-commutator closes on (the real-linear space of) self-adjoint (Hermitian) elements $\mathcal{L}(\mathcal{H})_{\text{sa}} = \{a \in \mathcal{L}(\mathcal{H}) \mid a^* = a\}$—unlike the operator product in $\mathcal{L}(\mathcal{H})$. Consequently, already $\mathcal{J}(\mathcal{H})_{\text{sa}} \coloneqq (\mathcal{L}(\mathcal{H})_{\text{sa}},\{\cdot,\cdot\})$ defines a real Jordan algebra. Indeed, this was the original motivation to study Jordan algebras in Refs. [46, 47], viz. as generalisations of observable algebras in quantum mechanics.

Not every real Jordan algebra $\mathcal{J}$ arises as the self-adjoint part (closed under the anti-commutator) from a C^*-algebra $\mathcal{A}$, i.e., $\mathcal{J} \notin (\mathcal{A}_{\text{sa}},\{\cdot,\cdot\})$ in general.[6] If it does, $\mathcal{J}$ is called *special*, otherwise it is called an *exceptional* Jordan algebra. The key to identify special Jordan algebras is the double role played by self-adjoint operators in $\mathcal{A}$: as observables and as generators of symmetries [3, 32].[7] In the case of $\mathcal{J}(\mathcal{H})_{\text{sa}}$, there are exactly two such associative products that extend the anti-commutator in $\mathcal{J}(\mathcal{H})_{\text{sa}}$ (cf. Ref. [2]),

$$a \cdot_+ b \coloneqq \frac{1}{2}\{a,b\} + \frac{1}{2}[a,b] = ab \qquad a \cdot_- b \coloneqq \frac{1}{2}\{a,b\} - \frac{1}{2}[a,b] = ba\,, \tag{5}$$

for all $a,b \in \mathcal{J}(\mathcal{H})$. In other words, $\mathcal{J}(\mathcal{H})_{\text{sa}}$ arises as the self-adjoint part of both $\mathcal{L}(\mathcal{H}) = (\mathcal{L}(\mathcal{H}),\cdot_+)$ and its opposite algebra $\mathcal{L}(\mathcal{H})^{\text{op}} = (\mathcal{L}(\mathcal{H}),\cdot_-)$, for which the order of

[5] A Jordan algebra $(\mathcal{J},\circ)$ is commutative (but generally non-associative) and satisfies the Jordan identity: $(a\circ b)\circ(a\circ a) = a\circ(b\circ(a\circ a))$ for all $a,b\in\mathcal{J}$. For a classification of Jordan algebras, see Refs. [37, 47, 56].

[6] Overloading notation slightly, we write $\mathcal{L}(\mathcal{H})$ for the linear space and for the algebra of operators on $\mathcal{H}$.

[7] Jordan algebras whose elements mimic this dichotomy are characterised by the existence of a *dynamical correspondence* [2, 3] (see also Ref. [7]). In turn, for every dynamical correspondence on $\mathcal{J}$, there is an associative C^*-algebra $\mathcal{A}$ from which $\mathcal{J}$ arises as a self-adjoint part $\mathcal{J} \subseteq (\mathcal{A}_{\text{sa}},\{\cdot,\cdot\})$ (Thm. 23 in Ref. [2]).

composition is reversed with respect to that in $\mathcal{L}(\mathcal{H})$.[6] Since transposition is an order-reversing involution on $\mathcal{L}(\mathcal{H})$, it maps between $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})^{\mathrm{op}}$. We will see (cf. Eq. (15) below) that the distinction between different C^*-algebras is at the heart of the different versions of the Choi-Jamiołkowski isomorphism. Moreover, we will show that this distinction is crucial for formulating Choi's theorem (cf. Thm. 6 below), where $\mathcal{A} = \mathcal{L}(\mathcal{H}_\mathcal{A})$ and $\mathcal{B} = \mathcal{L}(\mathcal{H}_\mathcal{B})$ enter not just as linear spaces or Jordan algebras, but as C^*-algebras. Before making precise these assertions, we briefly expand on the physical meaning of this distinction.

Time orientations and complex structure. The two (associative) products in Eq. (5) are readily distinguished from one another by the sign in front of the commutator. This sign obtains a physical interpretation as the *direction of time* as follows (cf. Refs. [2, 23, 27–29]): in $\mathcal{L}(\mathcal{H})$, the evolution of the system (in the Heisenberg picture) is given by the map

$$\Psi(t,a)(b) = e^{ita} b e^{-ita} \qquad \forall a,b \in \mathcal{J}(\mathcal{H})_{\mathrm{sa}}, t \in \mathbb{R} ,$$

where a assumes the role of a Hamiltonian operator, b that of an observable and t that of a time parameter. Ψ describes the unitary evolution in $\mathcal{L}(\mathcal{H})$ (as opposed to anti-unitary evolution [8, 67]). With Refs. [27–29], we therefore call Ψ the *canonical time orientation on* $\mathcal{L}(\mathcal{H})$, and $\Psi^*(t,a) := \Psi(-t,a)$ for all $a \in \mathcal{J}(\mathcal{H})_{\mathrm{sa}}$ and $t \in \mathbb{R}$ the *reverse time orientation on* $\mathcal{L}(\mathcal{H})$. The relation with the choice of sign in Eq. (5) follows by differentiation:

$$\frac{d}{dt}\Big|_{t=0} \Psi(t,a) = i[a,\cdot] \qquad \frac{d}{dt}\Big|_{t=0} \Psi^*(t,a) = -i[a,\cdot] , \tag{6}$$

for all $a \in \mathcal{J}(\mathcal{H})_{\mathrm{sa}}$. This suggests a slight reformulation of the decomposition in Eq. (4) into

$$ab := a \circ b - ia \star b , \tag{7}$$

where $a \circ b := \frac{1}{2}\{a,b\}$ and $a \star b := \frac{i}{2}[a,b]$ for all $a,b \in \mathcal{J}(\mathcal{H})_{\mathrm{sa}}$.[8] Let $\delta_d(x) := \frac{1}{2}(dx + xd^*)$ with $d \in \mathcal{J}(\mathcal{H})$, then $a \circ b = \delta_a(b)$ and $a \star b = \delta_{ia}(b)$ for $a \in \mathcal{J}(\mathcal{H})_{\mathrm{sa}}$. δ_d is called an *order derivation*, and is called *skew* if $\delta(1) = 0$.[9] Every skew order derivation δ_{ia} for $a \in \mathcal{J}(\mathcal{H})_{\mathrm{sa}}$ induces Jordan homomorphisms of the form $\Psi(t,a) = e^{2t\delta_{ia}}$ (Lm. 9 and Lm. 14 in Ref. [2]).

[8] Note that while the operator product in $\mathcal{L}(\mathcal{H})$ does not close on self-adjoint elements, since the commutator of two Hermitian operators is skew-Hermitian, the product $\star$ *does* close on self-adjoint elements.

[9] An order derivation δ on a JB algebra $\mathcal{J}$ (see Ref. [2]) is a bounded linear operator such that $e^{t\delta}(b) \in \mathcal{J}_+$ for all $b \in \mathcal{J}_+$ and $t \in \mathbb{R}$. An order derivation δ is called *self-adjoint* if $\delta(b) = \delta_a(b) = a \circ b$ for some $a \in \mathcal{J}(\mathcal{H})_{\mathrm{sa}}$, and *skew* if $\delta(1) = 0$. Every order derivation on a unital JB algebra decomposes into a self-adjoint and *skew* part (Lm. 11 in Ref. [2]). Moreover, if $\mathcal{J}$ is the self-adjoint part of a von Neumann algebra $\mathcal{A}$, then $\delta = \delta_d$ for $d \in \mathcal{A}$ and δ is self-adjoint (skew) if and only if $\delta = \delta_d$ ($\delta = \delta_{id}$) for $d \in \mathcal{A}_{\mathrm{sa}}$ (Prop. 15 in Ref. [2]).

Importantly, Eq. (7) defines a complex structure on $\mathcal{J}(\mathcal{H})$—viewed as the space of order derivations on $\mathcal{J}(\mathcal{H})_{\text{sa}}$ [20][10]—and it follows with Eq. (5) and Eq. (6) that such a complex structure promotes the Jordan algebra $\mathcal{J}(\mathcal{H})$ to the C^*-algebra $\mathcal{L}(\mathcal{H})$. Furthermore, we find that under this identification 'complex conjugation' $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H})_{\text{sa}} + i\mathcal{L}(\mathcal{H})_{\text{sa}} \leftrightarrow \mathcal{L}(\mathcal{H})_{\text{sa}} - i\mathcal{L}(\mathcal{H})_{\text{sa}} =: \mathcal{L}(\mathcal{H})^*$ is given by the Hermitian adjoint $*$ on $\mathcal{L}(\mathcal{H})$ (cf. Prop. 15 in Ref. [2]). In particular, this captures that $*$ is an order-reversing involution on $\mathcal{L}(\mathcal{H})$: by comparing Eq. (5), Eq. (6) and Eq. (7) we find that $*$ induces a change between time orientations $\Psi \longleftrightarrow \Psi^*$ on $\mathcal{L}(\mathcal{H})$.[11] To emphasise the relation with (skew) order derivations and time orientations, below we will use the Hermitian adjoint as an order-reversing map between C^*-algebras $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})^* \cong \mathcal{L}(\mathcal{H})^{\text{op}}$.

Inner product and Born rule. The Hilbert-Schmidt inner product on $\mathcal{L}(\mathcal{H})$,

$$(a,b)_{\mathcal{L}(\mathcal{H})} \coloneqq \text{tr}_{\mathcal{L}(\mathcal{H})}[b^* a] \,, \tag{8}$$

can be used to encode Eq. (1) as follows [44]: for all $a \in \mathcal{A} = \mathcal{L}(\mathcal{H}_\mathcal{A})$ and $b \in \mathcal{B} = \mathcal{L}(\mathcal{H}_\mathcal{B})$,

$$(\rho^\phi_J, a^* \otimes b)_{\mathcal{A}\otimes\mathcal{B}} = \text{tr}_{\mathcal{A}\otimes\mathcal{B}}[\rho^\phi_J(a \otimes b^*)] = \text{tr}_\mathcal{B}[\phi(a)b^*] = (\phi(a), b)_\mathcal{B} \,. \tag{9}$$

Note that *for a single system* with Hilbert space $\mathcal{H}$ Eq. (8) is a property of $\mathcal{J}(\mathcal{H})$ only,

$$(a,b)_{\mathcal{L}(\mathcal{H})} = \text{tr}_{\mathcal{L}(\mathcal{H})}[b^* a] = \frac{1}{2}\text{tr}_{\mathcal{L}(\mathcal{H})}[\{b^*, a\}] \,, \tag{10}$$

since $\text{tr}[ab - ba] = 0$ for all $a, b \in \mathcal{L}(\mathcal{H})$. The Hilbert-Schmidt inner product therefore restricts to the (complexified) Jordan algebra $\mathcal{J}(\mathcal{H})$. What is more, restricting to the real part $\mathcal{J}(\mathcal{H})_{\text{sa}}$, Eq. (10) is readily identified with the Born rule of the system described by $\mathcal{J}(\mathcal{H})_{\text{sa}}$.[12] In particular, we obtain the same Born rule from the inner product on $\mathcal{L}(\mathcal{H})^* \cong \mathcal{L}(\mathcal{H})^{\text{op}}$,

$$(a,b)_{\mathcal{L}(\mathcal{H})^{\text{op}}} \coloneqq \text{tr}_{\mathcal{L}(\mathcal{H})}[b \cdot_- a^*] = \frac{1}{2}\text{tr}_{\mathcal{L}(\mathcal{H})}[\{b, a^*\}] = (\text{tr}_{\mathcal{L}(\mathcal{H})}[b^* \cdot_+ a])^* = \overline{(a,b)}_{\mathcal{L}(\mathcal{H})} \,. \tag{11}$$

Together with Eq. (9) this suggest to study the Choi-Jamiołkowski isomorphism under the restriction to Jordan algebras.[13] In the next section, we will show that Thm. 1 and

[10] A complex structure on the order derivations of a Jordan algebra is also called a 'Connes orientation' [20].

[11] This central observation is at the heart of Thm. 3 in Ref. [29] and Thm. 2 in Ref. [27].

[12] By the 'state-observable duality' [6, 52], a quantum state corresponds to a normalised self-adjoint operator ('density matrix') $\rho \in \mathcal{J}(\mathcal{H})_{\text{sa}}$, $\text{tr}[\rho] = 1$, and the Born rule reads $\langle a \rangle_\rho \coloneqq \text{tr}[a\rho]$ for all $a \in \mathcal{L}(\mathcal{H})_{\text{sa}}$.

[13] While we do not pursue this here, we remark that this raises the question of defining a tensor product of (special) Jordan algebras. This is naturally done in reference to their ambient complex C^*-algebras [9, 36], which requires a choice of dynamical correspondence [2] (equivalently, a choice of time orientation) in addition to the mere Jordan algebra structure. Importantly, this choice is not arbitrary, i.e., the Born rule of a composite quantum system *does* depend on the relative choice of C^*-algebras [27, 29, 30].

Thm. 2 obtain under the restriction to Jordan algebras. In contrast, we will see that Thm. 3 *does* depend on the choice of complex structure [20] (equivalently, dynamical correspondence [2] or time orientation [27, 29]) and thus on the choice of C^*-algebra $\mathcal{L}(\mathcal{H})$ or $\mathcal{L}(\mathcal{H})^{\mathrm{op}}$.

B. Hermiticity-preserving, positive, decomposable and completely positive maps

In order to expose the significance of the choice of operator ordering on $\mathcal{J}(\mathcal{A})$ and $\mathcal{J}(\mathcal{B})$ in Thm. 1, Thm. 2 and Thm. 3, we first study the (linear) isomorphism J in Eq. (1) (equivalently, Eq. (9)) under the restriction to maps between Jordan algebras $\phi : \mathcal{J}(\mathcal{A}) \to \mathcal{J}(\mathcal{B})$. Note that $\mathcal{A} \cong \mathcal{J}(\mathcal{A})$ ($\mathcal{A}_{\mathrm{sa}} \cong \mathcal{J}(\mathcal{A})_{\mathrm{sa}}$) as complex (real) linear spaces (similarly for $\mathcal{B}$).

De Pillis's theorem and Jamiołkowski's theorem revisited. We start with Hermiticity-preserving maps, that is, linear maps $\phi : \mathcal{J}(\mathcal{A}) \to \mathcal{J}(\mathcal{B})$ such that $* \circ \phi = \phi \circ *$.[14] This is equivalent to the condition that ϕ preserves the self-adjoint parts of $\mathcal{J}(\mathcal{A})$ and $\mathcal{J}(\mathcal{B})$, i.e., that ϕ restricts to a linear map $\phi|_{\mathcal{J}(\mathcal{A})_{\mathrm{sa}}} : \mathcal{J}(\mathcal{A})_{\mathrm{sa}} \to \mathcal{J}(\mathcal{B})_{\mathrm{sa}}$.[15]

It follows that the set of operators ρ_J^ϕ corresponding to Hermiticity-preserving maps $\phi : \mathcal{A} \to \mathcal{B}$ under Eq. (1) is the same as the set of operators ρ_J^ϕ corresponding to linear maps $\phi : \mathcal{J}(\mathcal{A})_{\mathrm{sa}} \to \mathcal{J}(\mathcal{B})_{\mathrm{sa}}$. Thm. 1 thus applies to (complex) Jordan algebras:

Theorem 4 (de Pillis [22], for Jordan algebras)**.** *Let $\phi \in \mathcal{L}(\mathcal{J}(\mathcal{A}), \mathcal{J}(\mathcal{B}))$. Then ϕ restricts to a map $\phi|_{\mathcal{J}(\mathcal{A})_{\mathrm{sa}}} : \mathcal{J}(\mathcal{A})_{\mathrm{sa}} \to \mathcal{J}(\mathcal{B})_{\mathrm{sa}}$ if and only if ρ_J^ϕ is Hermitian.*

Proof. Indeed, this holds on the level of the complex vector spaces $\mathcal{A} \cong \mathcal{J}(\mathcal{A})$, $\mathcal{B} \cong \mathcal{J}(\mathcal{B})$. □

Next, recall that a map $\phi : \mathcal{A} \to \mathcal{B}$ is positive if and only if $\phi(a) \in \mathcal{B}_+$ whenever $a \in \mathcal{A}_+$, and that a positive operator is in particular self-adjoint. Hence, $\mathcal{A}_+ \subset \mathcal{J}(\mathcal{A})_{\mathrm{sa}}$ (as real linear spaces) such that the identification between operators $\rho_J^\phi \in \mathcal{A} \otimes \mathcal{B}$ that are positive on pure tensors and positive maps $\phi \in \mathcal{L}(\mathcal{A}, \mathcal{B})_+$ also holds on the level of Jordan algebras:

Theorem 5 (Jamiołkowski [44], for Jordan algebras)**.** *Let $\phi \in \mathcal{L}(\mathcal{J}(\mathcal{A}), \mathcal{J}(\mathcal{B}))$.*

(a) ϕ is positive if and only if ρ_J^ϕ is positive on pure tensors (POPT).

[14] Here, '$\circ$' denotes composition of maps, not the Jordan product of operators. Both notations are standard, their respective meaning will be clear from context.

[15] Conversely, every linear map $\phi : \mathcal{J}(\mathcal{A})_{\mathrm{sa}} \to \mathcal{J}(\mathcal{B})_{\mathrm{sa}}$ has a unique linear extension to the complexification $\mathcal{J}(\mathcal{A}) := \mathcal{J}(\mathcal{A})_{\mathrm{sa}} + i\mathcal{J}(\mathcal{A})_{\mathrm{sa}}$, given by $\phi(a + ib) := \phi(a) + i\phi(b)$. The same is true for Hermiticity-preserving maps with additional structure (see Thm. 4 and Thm. 5).

(b) Let $* \circ \phi = \phi \circ *$. Then ϕ is trace-preserving if and only if $\mathrm{tr}_{\mathcal{B}}[\rho_J^\phi] = \mathbb{1}_{\mathcal{A}}$.

Proof. As with Thm. 4, (b) is a statement about the complex vector spaces $\mathcal{A} \cong \mathcal{J}(\mathcal{A})$, $\mathcal{B} \cong \mathcal{J}(\mathcal{B})$, while the positive cones in (a) are readily defined on the level of Jordan algebras.[16] □

Decomposable and completely positive maps. We can further study the Choi-Jamiołkowski isomorphism under the restriction to maps between Jordan algebras with more structure. Recall that a Jordan $*$-homomorphism $\Phi : \mathcal{J}(\mathcal{A}) \to \mathcal{J}(\mathcal{B})$ is a Hermiticity-preserving map ($* \circ \Phi = \Phi \circ *$) that preserves the anti-commutator in the respective algebras, i.e., $\Phi(\{a, a'\}) = \{\Phi(a), \Phi(a')\}$ for all $a, a' \in \mathcal{J}(\mathcal{A})$. More generally, a map $\phi : \mathcal{J}(\mathcal{A}) \to \mathcal{J}(\mathcal{B})$ is called decomposable, denoted by $\phi \in \mathcal{L}(\mathcal{A}, \mathcal{B})_{\mathrm{D}}$, if and only if it is of the form $\phi = v^* \Phi v$, where $v : \mathcal{H}_{\mathcal{B}} \to \mathcal{K}$ is linear and $\Phi : \mathcal{J}(\mathcal{A}) \to \mathcal{J}(\mathcal{L}(\mathcal{K}))$ is a Jordan $*$-homomorphism [64]. Since decomposable maps are by definition maps between Jordan algebras, we can define the corresponding set of operators ρ_J^ϕ (under J in Eq. (1)), denoted by $(\mathcal{A} \otimes \mathcal{B})_{\mathrm{D}}$.

What about completely positive (CP) maps? Can we capture the set of CP maps (and thus the set of operators under Eq. (3)) by studying their restriction to real Jordan algebras? This is not the case! To see this, it is helpful to consider the classification of completely positive maps due to Stinespring [63]: every completely positive map $\phi : \mathcal{A} \to \mathcal{B}$ is of the form $\phi = v^* \Phi v$, where $v : \mathcal{H}_{\mathcal{B}} \to \mathcal{K}$ is a linear map and $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ is a C^*-algebra homomorphism, i.e., $* \circ \Phi = \Phi \circ *$ and $\Phi(aa') = \Phi(a)\Phi(a')$ for all $a, a' \in \mathcal{A}$. Of course, every C^*-algebra homomorphism $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ restricts to a Jordan $*$-homomorphism $\Phi|_{\mathcal{J}(\mathcal{A})} : \mathcal{J}(\mathcal{A}) \to \mathcal{J}(\mathcal{L}(\mathcal{K}))$, however, the converse is generally not true: a Jordan $*$-homomorphism $\Phi : \mathcal{J}(\mathcal{A}) \to \mathcal{J}(\mathcal{L}(\mathcal{K}))$ only preserves commutators up to sign [2]. More precisely, either $\Phi(aa') = \Phi(a)\Phi(a')$ or $\Phi(aa') = \Phi(a' \cdot_- a) = \Phi(a')\Phi(a)$, equivalently either $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ or $\Phi : \mathcal{A}^{\mathrm{op}} \to \mathcal{L}(\mathcal{K})$ is a C^*-algebra homomorphism [48] (see also Lm. 1 below).

Recall (from Eq. (5)) that both $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ reduce to the same Jordan algebra $\mathcal{J}(\mathcal{A})_{\mathrm{sa}}$. It thus follows from Stinespring's theorem that restricting the Choi-Jamiołkowski isomorphism to maps between Jordan algebras involves both CP maps $\mathcal{A} \to \mathcal{B}$ and CP maps $\mathcal{A}^{\mathrm{op}} \to \mathcal{B}$. Evidently, a CP map $\phi : \mathcal{A} \to \mathcal{B}$ is generally not also completely positive as a map $\phi : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ (see Lm. 1 below).[17] As a consequence, the set of operators $\rho_J^\phi \in (\mathcal{A} \otimes \mathcal{B})_{\mathrm{D}}$ (corresponding to decomposable maps) is strictly larger than the set of operators ρ_J^ϕ corresponding to CP

[16] In fact, Jordan algebras can be characterised in terms of positive cones [51, 52, 66] (see also Ref. [6]).

[17] Analysing when this is the case is closely related to Peres' separability criterion [41, 60] (see also Ref. [27]).

maps, since $\mathcal{L}(\mathcal{A},\mathcal{B})_{\mathrm{CP}} \subsetneq \mathcal{L}(\mathcal{A},\mathcal{B})_{\mathrm{D}}$. This argument shows that the isomorphism between positive operators and completely positive maps in Choi's theorem cannot be restricted to Jordan algebras as in the case of Thm. 4 and Thm. 5: *complete positivity for maps* ϕ^ρ_C *(equivalently by Thm. 3, positivity of (quantum) states* $\rho \in (\mathcal{A}\otimes\mathcal{B})_+ \cong \mathcal{L}(\mathcal{H}_\mathcal{A}\otimes\mathcal{H}_\mathcal{B})_+$*) depends on the (relative) operator ordering between the algebras* $\mathcal{A}$ *and* $\mathcal{B}$ (cf. Refs. [27, 29, 30]).

IV. A UNIFIED VERSION OF THE CHOI-JAMIOŁKOWSKI ISOMORPHISM

A. Choi's theorem revisited

The distinction between C^*-algebras in Sec. III A exposes why Choi's theorem (Thm. 3) does not hold for the isomorphism J in Eq. (1):[18] *J and C are implicitly defined with respect to different C^*-algebras.* Explicitly, since transposition changes the order of composition, ϕ^ρ_C in Eq. (3) is more naturally identified with a map of signature $\mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ rather than with a map of signature $\mathcal{A} \to \mathcal{B}$—the latter being the natural identification for ϕ^ρ_J. Taking this into account will lead to a reformulation of Thm. 3 in Thm. 6 below.

In order to distinguish between the different operator orderings in $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})^{\mathrm{op}}$ in a basis-independent way, we replace the basis-dependent operation of transposition with the order-reversing involution on the complex structure in Eq. (7), given by the Hermitian adjoint $*$ (cf. Sec. III A). Note from above that $\mathcal{L}(\mathcal{H})^* \cong \mathcal{L}(\mathcal{H})^{\mathrm{op}}$.

We will need the following straightforward result, which is implicit in Refs. [27, 29, 30].[19]

Lemma 1. *Let $\phi \in \mathcal{L}(\mathcal{A},\mathcal{B})$. Then $\phi : \mathcal{A} \to \mathcal{B}$ is completely positive if and only if $\phi \circ * : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ is completely positive.*

Proof. By Stinespring's theorem [63], every completely positive map $\phi : \mathcal{A} \to \mathcal{B}$ is of the form $\phi = v^*\Phi v$ for a linear map $v : \mathcal{H}_\mathcal{B} \to \mathcal{K}$ and $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ a C^*-algebra homomorphism. It follows that $\Phi \circ * : \mathcal{A}^{\mathrm{op}} \to \mathcal{L}(\mathcal{K})$ is also a C^*-algebra homomorphism,

$$(\Phi \circ *)(a \cdot_- a') = \Phi((a'a)^*) = \Phi(a^*a'^*) = \Phi(a^*)\Phi(a'^*) = (\Phi \circ *)(a)(\Phi \circ *)(a') \,.$$

Hence, $\phi \circ * = v^*(\Phi \circ *)v : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ is completely positive [63]. □

[18] That is, given a completely positive map $\phi : \mathcal{A} \to \mathcal{B}$, $\rho^\phi_J \in \mathcal{A}\otimes\mathcal{B}$ is generally not positive.

[19] Lm. 1 becomes Lm. 4 in Ref. [27], when expressed in terms of time orientations (see also Ref. [29]).

A similar result holds for $*$ replaced by any order-reversing involution on $\mathcal{A}$, in particular, it also holds for T (cf. Ref. [17]). We thus obtain the following reformulation of Thm. 3.

Theorem 6 (Choi (reformulated, basis-independent variant) [18]). *Let* $\phi^* \in \mathcal{L}(\mathcal{A},\mathcal{B})$. *Then* $\rho_J^{\phi^*} := \sum_k e_k^* \otimes \phi^*(e_k) \in \mathcal{A}^{\rm op} \otimes \mathcal{B}$ *is positive if and only if* $\phi^* : \mathcal{A} \to \mathcal{B}$ *is completely positive.*

Proof. By direct computation,

$$\rho_J^{\phi^*} = \sum_k e_k^* \otimes \phi^*(e_k) = \sum_{ij} |i\rangle\langle j| \otimes (\phi \circ *)(|j\rangle\langle i|) = \sum_{ij} |i\rangle\langle j| \otimes \phi(|i\rangle\langle j|) = \rho_C^\phi ,$$

where we used $* \circ \phi = \phi \circ *$, and the fact that J is basis-independent, to fix a choice of basis by $\{e_k\}_k = \{|i\rangle\langle j|\}_{ij} \in \mathrm{ONB}(\mathcal{A})$ in the second step. By Choi's theorem (Thm. 3, with $\mathcal{A} \longleftrightarrow \mathcal{A}^* \cong \mathcal{A}^{\rm op}$ exchanged), $\rho_J^{\phi^*} = \rho_C^\phi \in \mathcal{A}^{\rm op} \otimes \mathcal{B}$ is positive if and only if $\phi : \mathcal{A}^{\rm op} \to \mathcal{B}$ is completely positive, if and only if $\phi^* : \mathcal{A} \to \mathcal{B}$ is completely positive by Lm. 1. □

We emphasise that for a linear map $\phi : \mathcal{A} \to \mathcal{B}$ being Hermiticity-preserving, positive or decomposable are properties that are readily evaluated with respect to the Jordan algebras $\mathcal{J}(\mathcal{A})$ and $\mathcal{J}(\mathcal{B})$ (see Thm. 4 and Thm. 5). In contrast, complete positivity refers to the C^*-structure in $\mathcal{A}$ and $\mathcal{B}$: a completely positive map ϕ and its corresponding operator ρ_J^ϕ depend on the relative operator ordering between the respective algebras[20]. Since quantum states correspond with completely positive maps under Choi's theorem, they too are generally sensitive to the relative operator ordering in subalgebras [29].[21]

Thm. 6 achieves what was promised: it combines the correspondence between completely positive maps and positive operators in Choi's theorem with the basis-independence of J. Indeed, Thm. 6 gives rise to several (basis-independent) variants of the isomorphism.

B. Variations on the Choi-Jamiołkowski isomorphism

We begin with a variant on the isomorphism in Eq. (1) (Eq. (9)): for all $a \in \mathcal{A}$, $b \in \mathcal{B}$,

$$(\rho^\phi_{\mathcal{A}^{\rm op}\otimes\mathcal{B}}, a^* \otimes b)_{\mathcal{A}\otimes\mathcal{B}} := (\phi_{\mathcal{A}\to\mathcal{B}}(a), b)_{\mathcal{B}} , \qquad (12)$$

where for added emphasis here we explicitly indicate (and name the isomorphism by) a choice of inclusion for ρ into a C^*-algebra as well as a choice of signature for maps $\phi : \mathcal{A} \to \mathcal{B}$

[20] Equivalently, ρ_J^ϕ depends on the (relative) time orientation between the respective algebras [27, 29].
[21] In fact, a quantum state is sensitive to the relative operator ordering if and only if it is entangled [27].

in subscript. Importantly, note that Eq. (12) differs from J in Eq. (9) merely by a change of identification from $\rho^\phi_J \in \mathcal{A}\otimes\mathcal{B} \cong \mathcal{L}(\mathcal{H}_\mathcal{A}\otimes\mathcal{H}_\mathcal{B})$ to $\rho^\phi_{\mathcal{A}^{\rm op}\otimes\mathcal{B}} \in \mathcal{A}^{\rm op}\otimes\mathcal{B} \cong \mathcal{L}(\mathcal{H}^*_A\otimes\mathcal{H}_\mathcal{B})$. Eq. (12) thus inherits basis-independence with respect to $\mathcal{A}$ from J (cf. Sec. II A), and both Thm. 4 and Thm. 5 apply. Moreover, by Thm. 6 (replace ϕ^* with ϕ), $\phi : \mathcal{A} \to \mathcal{B}$ is completely positive if and only if $\rho^\phi_{\mathcal{A}^{\rm op}\otimes\mathcal{B}} \in \mathcal{A}^{\rm op}\otimes\mathcal{B} \cong \mathcal{L}(\mathcal{H}^*_A\otimes\mathcal{H}_\mathcal{B})$ is positive. Our nomenclature of the isomorphism in Eq. (12) is chosen such as to reflect this correspondence. Finally, we remark that Eq. (12) recovers the isomorphism proposed in Ref. [4] (see Eq. (16) in App. 1 for details).

Next, we define another variant of the isomorphism for maps with signature $\phi_{\mathcal{A}^{\rm op}\to\mathcal{B}}$ by

$$\begin{aligned} \rho^\phi_{\mathcal{A}\otimes\mathcal{B}} &:= \sum_k e_k \otimes \phi_{\mathcal{A}^{\rm op}\to\mathcal{B}}(e_k)\,, \\ \phi^\rho_{\mathcal{A}^{\rm op}\to\mathcal{B}}(a) &:= \mathrm{tr}_\mathcal{A}[\rho_{\mathcal{A}\otimes\mathcal{B}}(a^*\otimes \mathbb{1}_\mathcal{B})]\,. \end{aligned} \tag{13}$$

Note the similarity with Choi's isomorphism in Eq. (3). Yet, unlike the latter, Eq. (13) is basis-independent. This follows with Eq. (2) under the assumption that $\phi_{\mathcal{A}^{\rm op}\to\mathcal{B}} : \mathcal{A} \to \mathcal{B}$ is antilinear. Moreover, comparing with Eq. (1), we deduce $\rho^\phi_{\mathcal{A}\otimes\mathcal{B}} = (\rho^\phi_J)^{*_\mathcal{A}} = (\rho^\phi_{\mathcal{A}^{\rm op}\otimes\mathcal{B}})^{*_\mathcal{A}}$, where $*_\mathcal{A}$ denotes the Hermitian adjoint on $\mathcal{A}$. In analogy with Jamiołkowski's isomorphism in Eq. (9), we can thus express Eq. (13) using inner products (see Eq. (8) and Eq. (11)) as,

$$(\rho^\phi_{\mathcal{A}\otimes\mathcal{B}}, a\otimes b)_{\mathcal{A}\otimes\mathcal{B}} = (\rho^\phi_{\mathcal{A}^{\rm op}\otimes\mathcal{B}}, a^*\otimes b)_{\mathcal{A}^{\rm op}\otimes\mathcal{B}} = (\phi_{\mathcal{A}^{\rm op}\to\mathcal{B}}(a), b)_\mathcal{B}\,, \tag{14}$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. In fact, Eq. (14) simply arises from Eq. (12) under a change of C^*-algebras $\mathcal{A} \leftrightarrow \mathcal{A}^* \cong \mathcal{A}^{\rm op}$. Consequently, Thm. 4 and Thm. 5 apply since Eq. (14) agrees with Eq. (12) under restriction to Jordan algebras. Finally, $\rho^\phi_{\mathcal{A}\otimes\mathcal{B}} \in \mathcal{A}\otimes\mathcal{B} \cong \mathcal{L}(\mathcal{H}_\mathcal{A}\otimes\mathcal{H}_\mathcal{B})$ is positive if and only if $\rho^\phi_{\mathcal{A}^{\rm op}\otimes\mathcal{B}} = (\rho^\phi_{\mathcal{A}\otimes\mathcal{B}})^{*_\mathcal{A}} \in \mathcal{A}^{\rm op}\otimes\mathcal{B} \cong \mathcal{L}(\mathcal{H}^*_\mathcal{A}\otimes\mathcal{H}_\mathcal{B})$ is positive if and only if $\phi : \mathcal{A}^{\rm op} \to \mathcal{B}$ is completely positive by Thm. 6. As with our convention in Eq. (12), this correspondence is reflected in the nomenclature of the isomorphism.

Finally, note that $\phi : \mathcal{A} \to \mathcal{B}$ is completely positive if and only if $\phi : \mathcal{A}^{\rm op} \to \mathcal{B}^{\rm op}$ is completely positive; similarly, $\phi : \mathcal{A}^{\rm op} \to \mathcal{B}$ is completely positive if and only if $\phi : \mathcal{A} \to \mathcal{B}^{\rm op}$ is completely positive. It follows that Eq. (12) and Eq. (14) are invariant under an overall change: $\mathcal{A} \leftrightarrow \mathcal{A}^{\rm op}$ and $\mathcal{B} \leftrightarrow \mathcal{B}^{\rm op}$, which corresponds to the symmetry of those equations under complex conjugation (cf. Eq. (11)). In other words, $\rho^\phi_{\mathcal{A}\otimes\mathcal{B}^{\rm op}} := (\rho^\phi_{\mathcal{A}^{\rm op}\otimes\mathcal{B}})^* = \rho^\phi_{\mathcal{A}^{\rm op}\otimes\mathcal{B}}$ is positive on both $\mathcal{A}^{\rm op}\otimes\mathcal{B}$ and $\mathcal{A}\otimes\mathcal{B}^{\rm op}$; similarly, $\rho^\phi_{\mathcal{A}^{\rm op}\otimes\mathcal{B}^{\rm op}} := (\rho^\phi_{\mathcal{A}\otimes\mathcal{B}})^* = \rho^\phi_{\mathcal{A}\otimes\mathcal{B}}$ is positive on both $\mathcal{A}\otimes\mathcal{B}$ and $\mathcal{A}^{\rm op}\otimes\mathcal{B}^{\rm op}$. Consequently, positivity of $\rho^\phi_{\mathcal{A}^{\rm op}\otimes\mathcal{B}}$ (similarly $\rho^\phi_{\mathcal{A}\otimes\mathcal{B}}$) encodes a *relative choice of operator ordering (time orientation) between the two systems.* In summary, we record our *variations on the Choi-Jamiołkowski isomorphism*: for all $a \in \mathcal{A} = \mathcal{L}(\mathcal{H}_\mathcal{A})$, $b \in \mathcal{B} = \mathcal{L}(\mathcal{H}_\mathcal{B})$,

$$\begin{aligned}
& (\rho^\phi_{\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}}, a\otimes b^*)_{\mathcal{A}\otimes\mathcal{B}} &&= (b, \phi_{\mathcal{A}^{\mathrm{op}}\to\mathcal{B}^{\mathrm{op}}}(a))_\mathcal{B} &&= (a^*\otimes b, \rho^\phi_{\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B}})_{\mathcal{A}\otimes\mathcal{B}} \\
\overset{\substack{\mathcal{A}\leftrightarrow\mathcal{A}^{\mathrm{op}}\\ \mathcal{B}\leftrightarrow\mathcal{B}^{\mathrm{op}}}}{\longleftrightarrow}\; & (\rho^\phi_{\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B}}, a^*\otimes b)_{\mathcal{A}\otimes\mathcal{B}} &&= (\phi_{\mathcal{A}\to\mathcal{B}}(a), b)_\mathcal{B} &&= (a\otimes b^*, \rho^\phi_{\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}})_{\mathcal{A}\otimes\mathcal{B}} \\
\overset{\mathcal{A}\leftrightarrow\mathcal{A}^{\mathrm{op}}}{\longleftrightarrow}\; & (\rho^\phi_{\mathcal{A}\otimes\mathcal{B}}, a\otimes b)_{\mathcal{A}\otimes\mathcal{B}} &&= (\phi_{\mathcal{A}^{\mathrm{op}}\to\mathcal{B}}(a), b)_\mathcal{B} &&= (a^*\otimes b^*, \rho^\phi_{\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B}^{\mathrm{op}}})_{\mathcal{A}\otimes\mathcal{B}} \\
\overset{\substack{\mathcal{A}\leftrightarrow\mathcal{A}^{\mathrm{op}}\\ \mathcal{B}\leftrightarrow\mathcal{B}^{\mathrm{op}}}}{\longleftrightarrow}\; & (\rho^\phi_{\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B}^{\mathrm{op}}}, a^*\otimes b^*)_{\mathcal{A}\otimes\mathcal{B}} &&= (b, \phi_{\mathcal{A}\to\mathcal{B}^{\mathrm{op}}}(a))_\mathcal{B} &&= (a\otimes b, \rho^\phi_{\mathcal{A}\otimes\mathcal{B}})_{\mathcal{A}\otimes\mathcal{B}}\,.
\end{aligned} \tag{15}$$

For a graphical representation of these isomorphisms, see Fig. 6 in App. 2.

V. CONCLUSION

The Choi-Jamiołkowski isomorphism is an essential component in every quantum information theorist's toolkit: it allows to identify linear maps between two quantum systems with linear operators on the composite system. After a brief review of both Jamiołkowski's and Choi's version of the isomorphism (cf. Refs. [18, 22, 44]), *we gave physical meaning to their distinction in terms of different choices of the (relative) dynamics between the two systems involved, as expressed by the operator orderings on the respective C^*-algebras.* In contrast to other properties of the isomorphism that already hold on the kinematical level of Jordan algebras, making explicit this additional dynamical property—which is only present at and defining of the C^*-algebra level—we obtained a unified description of these isomorphisms in Eq. (15). In particular, we extended Choi's theorem (Thm. 3) to Jamiołkowski's (basis-independent) version of the isomorphism [44] in Thm. 6, and we obtained a basis-independent variant of Choi's isomorphism [18] in Eq. (14). Our work also complements recent results in Refs. [53, 59] on the extent of the invariance of Choi's theorem under basis-change with the basis-independence of the Choi-Jamiołkowski isomorphism.

The transition from kinematics in the form of Jordan algebras to dynamics in the form of C^*-algebras is formally captured by a choice of time orientation (or dynamical correspondence) [3, 23, 32, 33]. Time orientations have proven to play a crucial role in related work by the first author: on the classification of bipartite quantum states from more general non-signalling correlations [29, 30], and on the classification of bipartite entanglement [27].

Since the Choi-Jamiołkowski isomorphism is a basic tool across quantum information theory, we expect the techniques used here to be useful for and to shed light on various other results in the field. For instance, the change from a kinematical to a dynamical perspective is leveraged in the protocol of gate teleportation, a central ingredient to fault-tolerant quantum computation [31, 55]. What is more, in relativity theory the distinction between kinematics and dynamics becomes even less clear-cut, and it has been suggested that the Choi-Jamiołkowski isomorphism might be more than a useful mathematical gadget, but rather that it hints at a deep symmetry between kinematics and dynamics in fundamental physics [54]. Indeed, the Choi-Jamiołkowski isomorphism is of central importance in novel approaches to quantum causality [13] and quantum causal models [4, 10, 21, 65], which build on the process matrix framework [21, 57]. We therefore suspect that an analysis parallel to the one in this work might hold missing clues to a better physical understanding of time (reversal symmetry) [15, 17, 38] and causality [13] in quantum theory.

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1. The basis-independent variant of the Choi-Jamiołkowski isomorphism in Ref. [4]

The basis-dependence of Choi's isomorphism in Eq. (3) is unappealing and has prompted alternative definitions of the isomorphism. Of course, one can just resort to Eq. (1), yet Thm. 3 does not hold in this case: given a completely positive map ϕ, ρ_J^ϕ is generally not positive (cf. Refs. [53, 54, 59]). A formal trick to bypass this issue has been suggested in Refs. [4, 10] (see also Ref. [58]), which define an isomorphism $\mathcal{L}(\mathcal{A},\mathcal{B}) \to \mathcal{L}(\mathcal{H}_\mathcal{A}^* \otimes \mathcal{H}_\mathcal{B})$ by[22]

$$\begin{aligned} \rho^\mathcal{E}_{\mathcal{B}|\mathcal{A}} &:= \sum_{ij} |i\rangle_{\mathcal{A}^*}\langle j| \otimes \mathcal{E}(|i\rangle_\mathcal{A}\langle j|)\ , \\ \mathcal{E}^\rho(a) &:= \mathrm{tr}[\rho(a_{\mathcal{A}^*} \otimes \mathbb{1}_\mathcal{B})]\ , \quad a_{\mathcal{A}^*} := a^T \qquad \forall a \in \mathcal{A}\ . \end{aligned} \tag{16}$$

Here, $\{|i\rangle_{\mathcal{A}^*} \in \mathcal{H}^*_\mathcal{A}\}_i$ denotes the dual basis of $\{|i\rangle_\mathcal{A} \in \mathcal{H}_\mathcal{A}\}_i$,[23] yet the elements $|i\rangle_{\mathcal{A}*}$ are thought of as vectors (rather than dual vectors) such that $\langle j|i\rangle_{\mathcal{A}^*} = \langle i|j\rangle_\mathcal{A} = \overline{\langle j|i\rangle}_\mathcal{A}$, and thus

$$\begin{aligned} \mathrm{tr}[\rho^\mathcal{E}_{\mathcal{B}|\mathcal{A}}(a_{\mathcal{A}^*} \otimes \mathbb{1}_\mathcal{B})] &= \mathrm{tr}[(\sum_{ij} |i\rangle_{\mathcal{A}^*}\langle j| \otimes \mathcal{E}(|i\rangle_\mathcal{A}\langle j|))(\sum_{mn} |m\rangle(a_{\mathcal{A}^*})_{mn}\langle n| \otimes \mathbb{1}_\mathcal{B})] \\ &= \sum_{ijkmn} \langle k|i\rangle_{\mathcal{A}^*}\langle j|m\rangle_{\mathcal{A}^*}(a_{\mathcal{A}^*})_{mn}\langle n|k\rangle_{\mathcal{A}^*}\ \mathcal{E}(|i\rangle_\mathcal{A}\langle j|) \\ &= \sum_{ijkmn} \langle k|n\rangle_\mathcal{A} a_{nm}\langle m|j\rangle_\mathcal{A}\langle i|k\rangle_\mathcal{A}\ \mathcal{E}(|i\rangle_\mathcal{A}\langle j|) \\ &= \mathrm{tr}[(\sum_{ij} |j\rangle_\mathcal{A}\langle i| \otimes \mathcal{E}(|i\rangle_\mathcal{A}\langle j|))(\sum_{mn} |m\rangle a_{mn}\langle n| \otimes \mathbb{1}_\mathcal{B})] = \mathrm{tr}[\rho^\mathcal{E}_J(a \otimes \mathbb{1}_\mathcal{B})]\ . \end{aligned}$$

Consequently, the isomorphism defined in Eq. (16) is basis-independent by Eq. (2) and, by comparison with Eq. (3), restricts to a map from completely positive maps $\mathcal{E} : \mathcal{A} \to \mathcal{B}$ to positive matrices $\rho^\mathcal{E}_{\mathcal{B}|\mathcal{A}} \in \mathcal{L}(\mathcal{H}^*_\mathcal{A} \otimes \mathcal{H}_\mathcal{B})_+$. Indeed, Eq. (16) follows immediately from Eq. (12); in particular, the identification of the dual Hilbert space in Eq. (16) simply reflects the fact that $\mathcal{A}^* \cong \mathcal{A}^{\mathrm{op}}$ naturally acts on $\mathcal{H}^*_\mathcal{A}$ rather than on $\mathcal{H}_\mathcal{A}$.

2. Graphical calculus and process-state duality

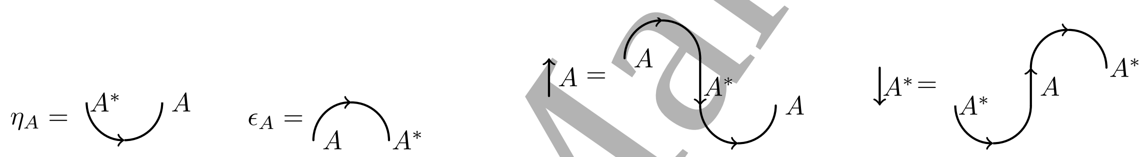
In Sec. III, we made a distinction between the kinematics and dynamics in quantum theory, in terms of Jordan and C^*-algebras. A more radical shift from kinematical to dynamical primitives forms the backbone of operational reformulations of quantum theory [16], as well

[22] Ref. [4] defines $a_{\mathcal{A}^*} := \mathrm{tr}_\mathcal{A}[\tau^{\mathrm{id}}_\mathcal{A} a]$ for $\tau^{\mathrm{id}}_\mathcal{A} := \sum_{ij} |i\rangle_{\mathcal{A}^*}\langle j| \otimes |i\rangle_\mathcal{A}\langle j|$; hence, $a_{mn} = (a_\mathcal{A})_{mn} = (a_{\mathcal{A}^*})_{nm}$ as matrices, with respect to an (arbitrary choice of) basis $\{|i\rangle_\mathcal{A} \in \mathcal{H}_\mathcal{A}\}_i$ and the corresponding basis $\{|i\rangle_{\mathcal{A}^*} \in \mathcal{H}^*_\mathcal{A}\}_i$.

[23] Recall that the dual basis $\{\langle j|_\mathcal{A}\}_j$ of $\{|i\rangle_\mathcal{A}\}_i$ is uniquely defined by the condition $\langle j|i\rangle_\mathcal{A} = \delta_{ij}$.

as of category-theoretic axiomatisations of quantum theory [39][24]. A key advantage of such reformulations is that they possess a rich graphical calculus [5, 19, 62]. In this final section, we will cast our results into this graphical calculus, adopting the notation of Ref. [19].

We will refer to those references for many more details on categorical approaches to quantum theory and physics more generally. Here, we merely highlight two concepts necessary to discuss the different (kinematical and dynamical) levels of the Choi-Jamiołkowski isomorphism within the setting of symmetric monoidal categories. The first concept is the duality of vector spaces and their duals which is captured by 'compact closed' (symmetric monoidal) categories [1, 19, 40, 62]. Compact closed categories assign every object A a 'dual object' A^* such that $(A^*)^* = A$, as well as two special morphisms, the unit or 'cup' $\eta_A : I \to A \otimes A^*$ and counit or 'cap' $\epsilon_A : A \otimes A^* \to I$ (see Fig, 1 (a)), where I denotes the unit under the tensor product $\otimes$, and such that the so-called 'zig-zag' relations in Fig. 1 (b) hold.



(a) unit η_A and counit ϵ_A of A and A^* (b) zig-zag relations

FIG. 1: (a) In a compact closed category, every object A comes with a dual object A^*, as well as a unit η_A and co-unit ϵ_A, which satisfy the so-called 'zig-zag relations' in (b).

The Choi-Jamiołkowski isomorphism can be stated at this level: $\phi_{A\to B} \mapsto \rho^{\phi}_{A^*\otimes B} = \phi_{A\to B} \circ \eta_A$ with inverse $\rho_{A^*\otimes B} \mapsto \phi^{\rho}_{A\to B} = \rho_{A^*\otimes B} \circ \epsilon_A$ (see Fig. 2). Since finite-dimensional quantum

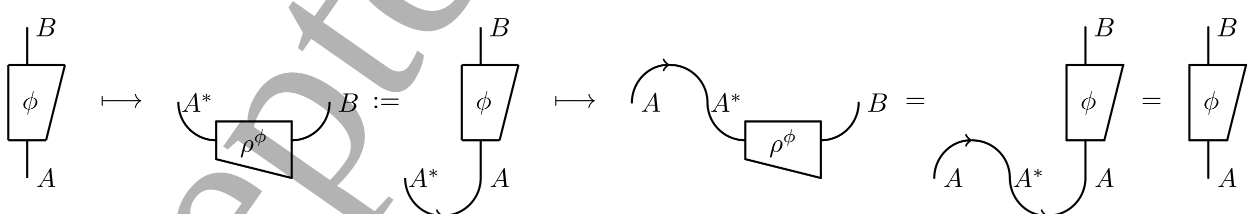


FIG. 2: A variant of the Choi-Jamiołkowski isomorphism (cf. Eq. (12)) in compact closed categories (cf. Ref. [19, 62]). The isomorphism holds by the 'zig-zag relations' in Fig. 1 (b).

theory can be viewed as a compact closed category [1, 39, 62], it is evident from Fig. 2 that

[24] Both build on a paradigm shift in the underlying philosophy of mathematics: from the static primitive of set-theoretic elements or 'points' to the dynamical primitive of 'morphisms'. The transformational impact this has had on quantum theory and physics more generally has only started to unfold [19, 24–26, 40].

the signatures in Eq. (1) as well as Eq. (3) are at odds with the structure of duals. Instead, the isomorphism naturally corresponds with the (basis-independent) variant in Eq. (12), for which $\eta_A : I \to \sum_k e_k^* \otimes e_k$ with $\{e_k\}_k$ a basis on $\mathcal{A}$ as in Eq. (1), yet under the change of identification of operators $\rho^{\phi}_{\mathcal{A}^{\rm op}\otimes\mathcal{B}} \in \mathcal{A}^* \otimes \mathcal{B} \cong \mathcal{A}^{\rm op} \otimes \mathcal{B}$.

Note further that changing operator orderings $\mathcal{A} \longleftrightarrow \mathcal{A}^* \cong \mathcal{A}^{\rm op}$ on both systems in Eq. (15) corresponds to applying cups and caps on either side, or pictorially to a rotation of 180 degrees, as shown in Fig. 3. Together, this captures half of Eq. (15) (see also Fig. 6 below).

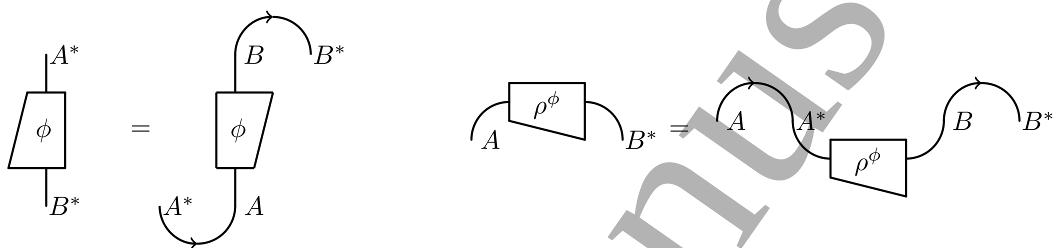


FIG. 3: Applying units and counits encodes dualities between morphisms $\phi_{A\to B} : A \to B$ and $\epsilon_B \circ \phi_{A\to B} \circ \eta_A : A^* \to B^*$, and between 'states' $\rho^{\phi}_{A^*\otimes B}$ and 'effects' $(\epsilon_B \circ \rho^{\phi}_{A^*\otimes B} \circ \epsilon_A)_{A\otimes B^*}$.

What about the other half? It belongs with the variant in Eq. (14) which is concerned with maps of signature $\phi : \mathcal{A}^{\rm op} \to \mathcal{B}$. Naively, we obtain a graphical representation for this variant of the isomorphism simply by changing $\mathcal{A} \longleftrightarrow \mathcal{A}^* \cong \mathcal{A}^{\rm op}$, as shown in Fig. 4. Notably, the convention in Fig. 4 is at odds with the directionality of wires. Indeed, in order

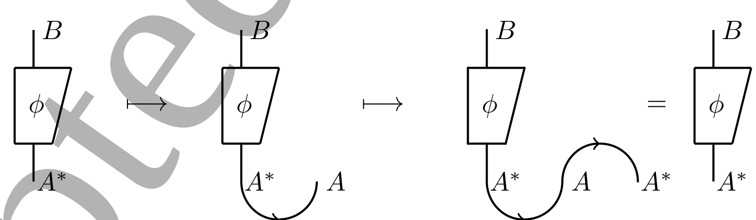
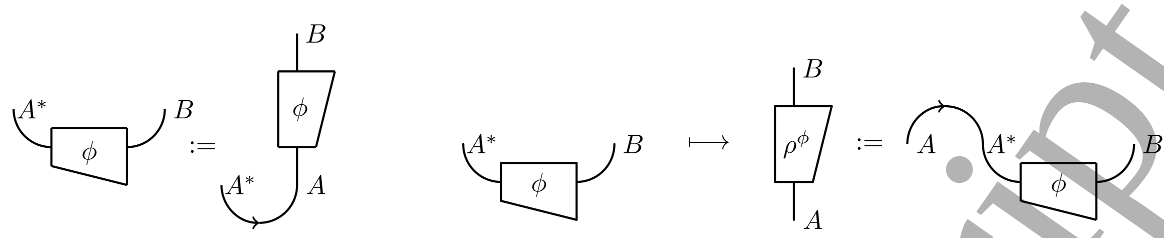


FIG. 4: A variant of the Choi-Jamiołkowski isomorphism (cf. Eq. (14)) in compact closed categories (cf. Ref. [19, 62]). The isomorphism holds by the 'zig-zag relations' in Fig. 1 (b).

to compose maps $\phi_{A^*\to B}$ and $\phi'_{B^*\to C}$ one needs to compose with (co-)units, thus effectively reducing to usual composition of maps, $\phi'_{B^*\to C} \circ \epsilon_B \circ \phi_{A^*\to B} = (\phi' \circ \epsilon_B)_{B\to C} \circ (\phi \circ \epsilon_A)_{A\to B} \circ \eta_A$.

Still, we may express Eq. (14) in the graphical calculus of compact closed categories using rotations by 90 degrees, as shown in Fig. 5. Comparing Fig. 5 (a) with the isomorphism in Fig. 2, we find that our convention merely replaces the (abstract) roles of 'process' and 'state'.



(a) processes as states (b) variant of Choi-Jamiołkowski isomorphism

FIG. 5: (a) We introduce a graphical notation for morphisms $\phi : A^* \to B$, in terms of a rotation by 90 degrees. (b) Using this notation, the isomorphism in Eq. (14) is readily expressed as $\phi_{A^*\to B} \mapsto \rho^{\phi}_{A\otimes B} = \phi_{A^*\to B} \circ \epsilon_A$ with inverse $\rho_{A\otimes B} \mapsto \phi^{\rho}_{A^*\to B} = \rho_{A\otimes B} \circ \eta_A$.

Indeed, unless we impose additional (normalisation) constraints, this choice of convention reflects a symmetry between states and morphisms at this abstract level.[25]

Finally, we remark that compact closed categories are still lacking of linear structure, as well as of notions of positivity (as required for Thm. 2 and Thm. 3). As it turns out, (complete) positivity on the level of symmetric monoidal categories is captured in the form of an involutive, identity-on-objects, contravariant 'dagger' functor.[26] Indeed, finite-dimensional quantum mechanics forms a dagger compact closed category [1, 39, 62], where the dagger operator is given by the order-reversing Hermitian adjoint. The dependence on the relative operator ordering between C^*-algebras in Thm. 6 thus translates to the categorical setting as follows: it asserts that the dependence on dual objects in the Choi-Jamiołkowski isomorphism must be extended to dagger structure in order to lift the Choi-Jamiołkowski isomorphism to the relevant positive cones in Choi's theorem.[27]

In summary, the categorical perspective thus unveils the Choi-Jamiołkowski isomorphism as various ways of sequentially evaluating maps ('currying'). These different conventions and resulting dependencies on dual objects take the form of different variants of the isomorphism, as expressed algebraically in Eq. (15) and represented graphically in Fig. 6.

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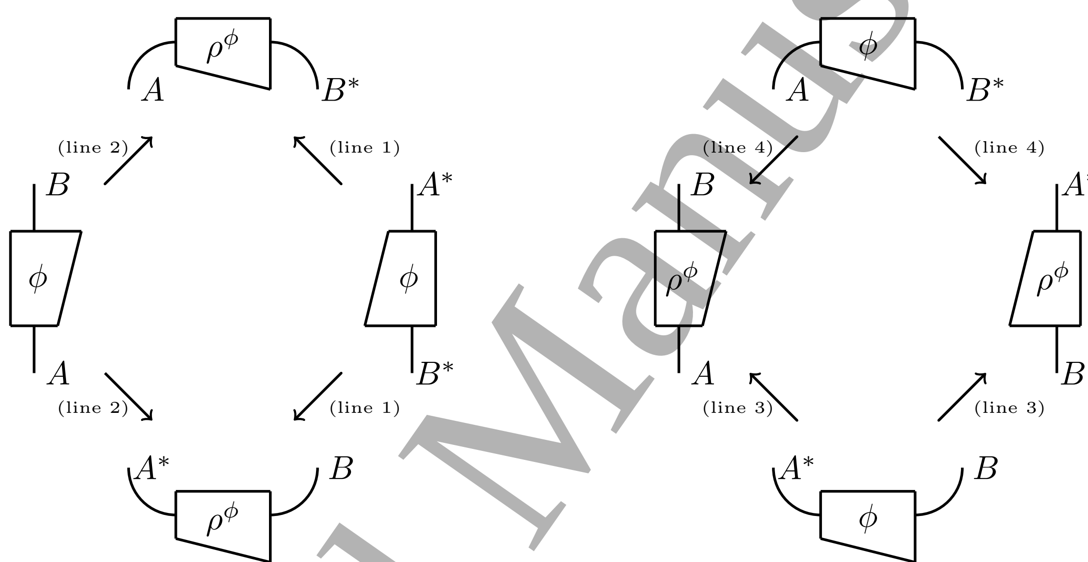
[25] Recall that the normalisation conditions for quantum states (positive operators of unit trace) and quantum processes (trace-preserving CP maps) differ under the Choi-Jamiołkowski isomorphism. In particular, Choi's theorem (Thm. 3, Thm. 6) is only concerned with (unnormalised) positive cones.

[26] A morphism $f : A \to A$ in a dagger category is called *positive* if there exists an object B and a morphism $g : A \to B$ such that $f = g^\dagger \circ g$. For the definition of completely positive morphisms, see e.g. Ref. [62].

[27] Indeed, a version of Choi's theorem holds in dagger compact closed categories, see Lm. 4.12 in Ref. [62].

$$
\begin{aligned}
& (\rho^{\phi}_{\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}}, a\otimes b^*)_{\mathcal{A}\otimes\mathcal{B}} &&= (b, \phi_{\mathcal{A}^{\mathrm{op}}\to\mathcal{B}^{\mathrm{op}}}(a))_{\mathcal{B}} &&= (a^*\otimes b, \rho^{\phi}_{\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B}})_{\mathcal{A}\otimes\mathcal{B}} \\
\overset{\substack{\mathcal{A}\leftrightarrow\mathcal{A}^{\mathrm{op}}\\ \mathcal{B}\leftrightarrow\mathcal{B}^{\mathrm{op}}}}{\longleftrightarrow}\quad & (\rho^{\phi}_{\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B}}, a^*\otimes b)_{\mathcal{A}\otimes\mathcal{B}} &&= (\phi_{\mathcal{A}\to\mathcal{B}}(a), b)_{\mathcal{B}} &&= (a\otimes b^*, \rho^{\phi}_{\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}})_{\mathcal{A}\otimes\mathcal{B}} \\
\overset{\mathcal{A}\leftrightarrow\mathcal{A}^{\mathrm{op}}}{\longleftrightarrow}\quad & (\rho^{\phi}_{\mathcal{A}\otimes\mathcal{B}}, a\otimes b)_{\mathcal{A}\otimes\mathcal{B}} &&= (\phi_{\mathcal{A}^{\mathrm{op}}\to\mathcal{B}}(a), b)_{\mathcal{B}} &&= (a^*\otimes b^*, \rho^{\phi}_{\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B}^{\mathrm{op}}})_{\mathcal{A}\otimes\mathcal{B}} \\
\overset{\substack{\mathcal{A}\leftrightarrow\mathcal{A}^{\mathrm{op}}\\ \mathcal{B}\leftrightarrow\mathcal{B}^{\mathrm{op}}}}{\longleftrightarrow}\quad & (\rho^{\phi}_{\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B}^{\mathrm{op}}}, a^*\otimes b^*)_{\mathcal{A}\otimes\mathcal{B}} &&= (b, \phi_{\mathcal{A}\to\mathcal{B}^{\mathrm{op}}}(a))_{\mathcal{B}} &&= (a\otimes b, \rho^{\phi}_{\mathcal{A}\otimes\mathcal{B}})_{\mathcal{A}\otimes\mathcal{B}}\,.
\end{aligned}
\tag{17}
$$

(a) Variations on the Choi-Jamiołkowski isomorphism in Eq. (15), reprinted for convenience.



(b) graphical representation of the isomorphisms in Eq. (17), lines 1, 2

(c) graphical representation of the isomorphisms in Eq. (17), lines 3, 4

FIG. 6: (a) The variations on the Choi-Jamiołkowski isomorphism in Eq. (17) can be graphically represented, by various applications of units and counits in compact closed categories (cf. Ref. [19, 62]): while the variants in (b) correspond to the canonical identification of morphisms with signature $\phi_{A\to B} : A \to B$ and states $\rho^{\phi}_{A^*\otimes B}$ (Eq. (17), lines 1,2), in (c) the roles of states and morphisms are reversed, as by Fig. 5.