Leveraging Prior Known Vector Green Functions in Solving Perturbed Dirac Equation in Clifford Algebra

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Abstract. Solving boundary value problems with boundary element methods requires specific Green functions suited to the boundary conditions of the problem. Using vector algebra, one often needs to use a Green function for the Helmholtz equation whereas it is a solution of the perturbed Dirac equation that is required for solving electromagnetic problems using Clifford algebra. A wealth of different Green functions of the Helmholtz equation are already documented in the literature. However, perturbed Dirac equation is only solved for the generic case and only its fundamental solution is reported. In this paper, we present a simple framework to use known Green functions of Helmholtz equation to construct the corresponding Green functions of perturbed Dirac equation which are essential in finding the appropriate kernels for integral equations of electromagnetic problems. The procedure is further demonstrated in a few examples.

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1. Introduction

Electromagnetism is well developed using vector algebra and a great variety of problems are treated in-depth. Boundary value problems in electromagnetic theory are often formulated by the help of fundamental solutions and Green functions which can be written easily for the unbounded homogeneous medium (e.g. classical source in free space). However, if a problem has additional boundary conditions like infinite planar layers (stratified medium) or

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a wedge that is extended to infinity, we have to also incorporate those into the Green function [1–3].

Another algebraic tool chain to solve electromagnetic problems is the geometric algebra which is often also called Clifford algebra to honour *William Kingdon Clifford* [4]. Clifford generated his geometric algebra using two sets of mutually commutative quaternions producing what today would be called a four-dimensional Clifford algebra $Cl(0,4)$, supporting bivectors and trivectors in addition to vectors. Due to its flexibility, Clifford algebra has been used with different conventions and assumptions in the literature. A good review of early developments of using Clifford algebra for electromagnetic problems is provided in [5]. Development of new techniques in quaternion-valued functions and their associated boundary value problems (e.g. [6–9]) in around 1990s opened new horizons and made it possible to attempt solve new class of problems. For instance in [9], Laplace operator was extended to Helmoltz operator with complex quaternionic wave numbers and fundamental solutions for different types of wavenumbers (scalar only, vector only, mixed quaternion) for Helmholtz operator were developed.

In Clifford algebra it is convenient to combine four Maxwell's equations using a single first order Dirac operator in the transient domain while perturbed Dirac operator is used for time harmonic electromagnetic waves [10]. Some studied Dirac operator and massive Dirac operator mostly for quantum applications [11–13] while some others studied scattering problems on unbounded domains [14]. Meanwhile, radiation conditions were also specifically developed for Clifford boundary value problems by McIntosh & Mitrea [15] and Kravchenko [16].

Aforementioned boundary conditions are often solved using some integral equation formulations. More notably, the theory of singular integral operators were used in [14, 17] to formulate the wave problem using some modified Cauchy integral formula. An iterative approach to calculate the fields using Teodorescu transform was proposed in [18] and elaborated in [5].

Seagar & Chantaveerod [10, 19] followed some of the ideas and arguements of McIntosh, Axelsson, Grognard, Mitrea and Hogan [15,20] to develop a computational algorithm to solve scattering from the arbitrary objects in up to three dimensions [10, 19]. The method is known as the Clifford-Cauchy-Dirac (CCD) method and its competency is demonstrated with numerical examples [21, 22] in one and two dimensions.

To the best of our knowledge, so far, only the fundamental solution of perturbed Dirac equation [23, 24] is known and utilised in electromagnetism. As a result, methods like CCD can only solve for the isolated objects in free space. Here, we propose a simple method to use results from vector calculus and their already known Green functions in that context to find Green function of Dirac equation in the context of Clifford algebra. Similar to fundamental solutions used in the Cauchy integral formulas of previous works, Green functions found in this paper are essential to construct the Cauchy kernels for problems with boundary conditions.

In section 2, we review the conventions of Clifford algebra. Section 3 briefly discusses how Maxwell's equations in time and frequency domains are considered in the Clifford algebra. We try to make clear illustration of the five boundary value problems in section 4. The first three problems are formulated in vector algebra as Maxwell problem ($\mathcal{MP}$), Helmholtz problem for electric ($\mathcal{HP}-\mathcal{E}$) and magnetic fields ($\mathcal{HP}-\mathcal{H}$). The next two problems ($\mathcal{DP}-\mathcal{F}$) and ($\mathcal{DP}-\mathcal{G}$) are formulated as perturbed Dirac operator on $\mathcal{F}$ and $\boldsymbol{G}_D$, respectively. Section 5 is the main result of the paper that provides a powerful and yet simple procedure to find the Green function of Dirac equation. The use of the method is further illustrated by three examples in Section 6. The first two examples are the classical cases that reproduce the fundamental solutions, reassuring the validity of the proposed procedure. The last example is for a rectangular waveguide, that although popular with many engineering applications, but it has not been solved in Clifford framework, while a vector based solution has been available for more than half a century.

2. Conventions

Throughout the paper, a four-dimensional Clifford algebra $Cl(0,4)$ is used. We use different notations for the vector elements and Clifford elements. An arbitrary vector $\boldsymbol{A}$ is reported with $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$:

$$\boldsymbol{A} = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\boldsymbol{z}} \tag{2.1}$$

while a Clifford number in grade one is denoted by e_1, e_2 and e_3:

$$\boldsymbol{A} = A_x e_1 + A_y e_2 + A_z e_3. \tag{2.2}$$

It should be noted that a vector and a grade one Clifford number can both be used to represent the same kinds of geometric entities. However, we show them with different unit elements to emphasise that the Clifford operations and rules are applied on the latter.

The Clifford product of any two vectors is:

$$\boldsymbol{AB} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_i B_j e_i e_j \tag{2.3}$$

The product of two Clifford units e_i and e_j is defined [1] as:

$$e_i e_j = \begin{cases} -e_j e_i & i \neq j \\ -1 & i = j \end{cases} \tag{2.4}$$

[1]It is possible to use an alternative convention with $+1$ instead of -1 for the product when $i=j$. Here, the convention adopted by Clifford and Grassmann is used to align the algebra as they did with the quaternions adopted by Maxwell for his electromagnetic equations, and with the vector calculus extracted later by Gibbs from the quaternion algebra.

In the other words, Clifford units are anti-commutative and constructed as a imaginary units $e_i^2 = -1$. This leads to:

$$\boldsymbol{A}^2 = -|\boldsymbol{A}|^2 = -\sum_{i=1}^{3} A_i^2 \tag{2.5}$$

where the $|\cdot|$ operator returns the norm of the quantity.

It is often helpful to imagine a Clifford product through a relationship with dot and cross products of vector algebra. Using (2.4), we have

$$\boldsymbol{AB} = (\boldsymbol{A} \times \boldsymbol{B})\sigma - \boldsymbol{A} \cdot \boldsymbol{B} \tag{2.6}$$

where $\sigma = -e_1 e_2 e_3$ is the unit volume in three dimensions. It should be noted that (2.6) is only valid if both $\boldsymbol{A}$ and $\boldsymbol{B}$ are vectors, and it fails if one of them is constructed as a Clifford number of any grade other than one. In that case, the general formula (2.3) should be used.

It is possible to divide to some geometric quantities in Clifford algebra. For example, the inverse of a Clifford vector with unit product as in (2.4) is:

$$\boldsymbol{A}^{-1} = -\frac{\boldsymbol{A}}{|\boldsymbol{A}|^2}. \tag{2.7}$$

A projection operator Q^m is defined by:

$$Q^m \boldsymbol{A} = (\boldsymbol{A} - \boldsymbol{nAn})/2 \tag{2.8}$$

and splitting operators S and T are considered as [10, Ch.5]

$$S\boldsymbol{A} = (\boldsymbol{A} + \sigma \boldsymbol{A} \sigma)/2 \tag{2.9}$$

$$T\boldsymbol{A} = (\boldsymbol{A} - \sigma \boldsymbol{A} \sigma)/2 \tag{2.10}$$

A time harmonic convention of $\exp(i\omega t)$ is assumed throughout the paper.

3. Clifford Formalism for Maxwell's equations

3.1. Time Domain

Maxwell's equations of electromagnetism for a region containing a material of spatially uniform and temporally constant properties as written in the time domain using vector calculus:

$$\begin{cases} \epsilon \nabla \cdot \boldsymbol{E} = \rho \\ \nabla \times \boldsymbol{H} - \epsilon \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{J} \\ \nabla \times \boldsymbol{E} + \mu \frac{\partial \boldsymbol{H}}{\partial t} = 0 \\ \mu \nabla \cdot \boldsymbol{H} = 0 \end{cases} \tag{3.1}$$

require three spatial dimensions and one temporal dimension. Here $\boldsymbol{E} = E_x \hat{\boldsymbol{x}} + E_y \hat{\boldsymbol{y}} + E_z \hat{\boldsymbol{z}}$ and $\boldsymbol{H} = H_x \hat{\boldsymbol{x}} + H_y \hat{\boldsymbol{y}} + H_z \hat{\boldsymbol{z}}$ are the electric and magnetic fields respectively in Cartesian components, μ and ϵ are the (uniform and constant) values of magnetic permeability and electric permittivity, $\boldsymbol{J} = J_x \hat{\boldsymbol{x}} + J_y \hat{\boldsymbol{y}} + J_z \hat{\boldsymbol{z}}$ is the electric conduction current density and ρ is the (scalar) electric charge density. The curl and divergence partial differential operators are constructed by subjecting the entities on their right sides to the spatial gradient $\nabla = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial}{\partial z}\hat{\boldsymbol{z}}$ under the three-dimensional vector cross ($\times$) and dot ($\cdot$) multiplications respectively .

For a Clifford formalism of Maxwell's equations the unit vectors $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}, \hat{\boldsymbol{t}}$ for spatial and temporal dimensions are identified with Clifford units:

$$\begin{cases} \hat{\boldsymbol{t}} & \leftrightarrow & ie_0 \\ \hat{\boldsymbol{x}} & \leftrightarrow & e_1 \\ \hat{\boldsymbol{y}} & \leftrightarrow & e_2 \\ \hat{\boldsymbol{z}} & \leftrightarrow & e_3 \end{cases} \tag{3.2}$$

where i is the imaginary unit and the signature $e_p e_p = -1$ for all four values of p.

The Clifford valued equation:

$$\mathcal{DF} = \mathcal{S} \tag{3.3}$$

with four-dimensional k-Dirac operator $\mathcal{D} = e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z} - \frac{i}{c}e_0\frac{\partial}{\partial t}$ (where $c = 1/(\sqrt{\mu}\sqrt{\epsilon})$ is the speed of propagation), electromagnetic field $\mathcal{F} = \sqrt{\mu}H\sigma - i\sqrt{\epsilon}Ee_0$ (where $H = H_x e_1 + H_y e_2 + H_z e_3$, $E = E_x e_1 + E_y e_2 + E_z e_3$ and $\sigma = -e_1 e_2 e_3$), and source $\mathcal{S} = \sqrt{\mu}J + \frac{i}{\sqrt{\epsilon}}\rho e_0$ (where $J = J_x e_1 + J_y e_2 + J_z e_3$) expands using the identities $a \cdot b = -\frac{1}{2}(ab + ba)$ and $a \times b = +\frac{1}{2}(ab - ba)\sigma$ for three-dimensional Clifford valued spatial vectors a and b, as:

$$\begin{aligned} \mathcal{S} &= \mathcal{DF} \\ \sqrt{\mu}J + \frac{i}{\sqrt{\epsilon}}\rho e_0 &= (\nabla - \frac{i}{c}e_0\frac{\partial}{\partial t})(\sqrt{\mu}H\sigma - i\sqrt{\epsilon}Ee_0) \\ &= \sqrt{\mu}(-\nabla\cdot H + \nabla\times H\sigma)\sigma \; - i\sqrt{\epsilon}(-\nabla\cdot E + \nabla\times E\sigma)e_0 \\ &\qquad - \frac{i}{c}e_0\sqrt{\mu}\frac{\partial H}{\partial t}\sigma - \frac{1}{c}e_0\sqrt{\epsilon}\frac{\partial E}{\partial t}e_0 \\ &= -\sqrt{\mu}\nabla\cdot H\sigma + \sqrt{\mu}\nabla\times H \; + i\sqrt{\epsilon}\nabla\cdot Ee_0 + i\sqrt{\epsilon}\nabla\times Ee_0\sigma \\ &\qquad + i\sqrt{\epsilon}\mu\frac{\partial H}{\partial t}e_0\sigma - \sqrt{\mu}\epsilon\frac{\partial E}{\partial t} \end{aligned} \tag{3.4}$$

where $\nabla = e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}$ is the three-dimensional Clifford valued gradient (k-Dirac) operator. The equation then separates according to the quaternion units (1, $\boldsymbol{I} = e_0$, $\boldsymbol{J} = -i\sigma$, $\boldsymbol{K} = -ie_0\sigma$) as a result of their linear independence, giving:

$$\begin{cases} \frac{i}{\sqrt{\epsilon}}(\epsilon\nabla\cdot E)e_0 &= \frac{i}{\sqrt{\epsilon}}(\rho)\, e_0 \\ \sqrt{\mu}(\nabla\times H - \epsilon\frac{\partial E}{\partial t}) &= \sqrt{\mu}(J) \\ -\sqrt{\epsilon}(\nabla\times E + \mu\frac{\partial H}{\partial t})(-ie_0\sigma) &= 0 \\ -\frac{i}{\sqrt{\mu}}(\mu\nabla\cdot H)(-i\sigma) &= 0 \end{cases} \tag{3.5}$$

Reference to equation 3.1 verifies that each of these equations is a copy of one of Maxwell's equations in its vector form, scaled outside the parentheses on the left by a constant and on the right by a quaternion unit (1, $\boldsymbol{I} = e_0$, $\boldsymbol{J} = -i\sigma$, $\boldsymbol{K} = -ie_0\sigma$).

As a consequence, any electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ which solve Maxwell's equations in the form of equation 3.1 when cast in the form of the Clifford valued electromagnetic field $\mathcal{F}$ also solve Maxwell equations in the form of equation 3.3. Furthermore, the electric and magnetic components E and H of the electromagnetic field $\mathcal{F}$ which solve Maxwell equations in the form of equation 3.3 also solve Maxwell's equations in the form of equation 3.1.

3.2. Frequency Domain

If an electromagnetic field $\mathcal{F}(\boldsymbol{r}, t)$ is periodic in time then it can be composed of a Fourier series, a sum of individual monochromatic sinusoidal components:

$$\mathcal{F}(\boldsymbol{r}, t) = \sum_k \mathcal{F}_k(\boldsymbol{r}) e^{i\omega_k t} \tag{3.6}$$

where ω_k is the angular frequency for each component and $k = \omega_k / c$ is the corresponding wavenumber. The spectral components $\mathcal{F}_k(\boldsymbol{r})$ are complex scalars.

Applying the differential operator $\mathcal{D}$ to a monochromatic field in the time domain is equivalent to first applying the differential operator $\mathcal{D}_k = \nabla + k e_0$ to the corresponding spectral component in the frequency domain and then multiplying by the temporal factor $e^{i\omega_k t}$:

$$\begin{aligned} \mathcal{D}\mathcal{F}(\boldsymbol{r}, t) &= \mathcal{D}\left[\mathcal{F}_k(\boldsymbol{r}) e^{i\omega_k t}\right] &&= (\nabla - \tfrac{i}{c} e_0 \tfrac{\partial}{\partial t})[\mathcal{F}_k(\boldsymbol{r}) e^{i\omega_k t}] \\ &= [(\nabla + k e_0)\mathcal{F}_k(\boldsymbol{r})] e^{i\omega_k t} &&= [D_k \mathcal{F}_k(\boldsymbol{r})] e^{i\omega_k t} \end{aligned} \tag{3.7}$$

In the frequency domain the temporal factor appears on both sides of all equations. For simplicity it can be eliminated, leaving only the spectral components, and restored at any later time if required. In the frequency domain the four-dimensional Dirac operator $\mathcal{D}$ takes spectral form as a three-dimensional k-Dirac operator perturbed by the spectral parameter, wavenumber[2] k.

4. Equivalent boundary value problems

We assume an infinitesimal current $\boldsymbol{J} = \delta(\boldsymbol{R} - \boldsymbol{R}')\boldsymbol{u}_0$, where $\boldsymbol{u}_0$ is an arbitrary unit vector and δ is the Dirac delta function within an isotropic homogeneous medium with permittivity ϵ and permeability μ. We take $\boldsymbol{G}_e$ and $\boldsymbol{G}_m$ to represent the appropriate Green functions for electric and magnetic fields respectively.

Let $\partial\Omega$ be a surface boundary on a bounded domain $\Omega \subset \mathbb{R}^3$ and unit normal vector $\boldsymbol{n}$ pointing from the interior region Ω^+ into exterior region Ω^-.

[2]Care must be taken to avoid confusion since the symbol k represents, by two separate conventions, in one case the wavenumber and the other case the dimension of the Dirac operator.

One can use the Maxwell's equations to build a boundary value problem using the vector algebra that we refer to as Maxwell problem $(\mathcal{MP})$:

$$(\mathcal{MP})\begin{cases}\nabla\times\boldsymbol{G}_e+i\omega\mu\boldsymbol{G}_m=0 \quad \text{in } \Omega^\pm\\ \nabla\times\boldsymbol{G}_m-i\omega\epsilon\boldsymbol{G}_e=\delta(\boldsymbol{R}-\boldsymbol{R'})\boldsymbol{u}_0 \quad \text{in } \Omega^\pm,\ R'\notin\partial\Omega\\ \boldsymbol{n}\times\boldsymbol{G}_e^+-\boldsymbol{n}\times\boldsymbol{G}_e^-=0\\ \boldsymbol{n}\cdot\mu^+\boldsymbol{G}_m^+-\boldsymbol{n}\cdot\mu^-\boldsymbol{G}_m^-=0\\ \boldsymbol{G}_e^-,\boldsymbol{G}_m^- \quad \text{satisfy} \quad \text{SMRC}\end{cases} \tag{4.1}$$

where boundary conditions on the tangential component of $\boldsymbol{G}_e$ and normal component of $\boldsymbol{G}_m$ are general boundary conditions on electric and magnetic vectors. The $\pm$ superscripts denote different side of the boundary $\partial\Omega$. Also, fields in $\mathbb{R}^3\backslash\Omega$ should satify Silver-Müller Radiation Condition [25,26] at large distances e.g.

$$\lim_{R\to\infty} R\left(\eta\boldsymbol{G}_m^- - \hat{R}\times\boldsymbol{G}_e^-\right)=0 \tag{4.2}$$

where $\eta\triangleq\sqrt{\mu}/\sqrt{\epsilon}$ is the characteristic impedance of the medium.

It should be noted that radiation conditions are only to be fulfilled for problems with unbounded Ω^-. However, we can assume Ω^- as the perfect conductors for bounded problems leading to $\boldsymbol{E}^-=\boldsymbol{H}^-=0$.

Another class of boundary value problem in the vector algebra is the Helmholtz equation which is obtained from the Maxwell equation. When applied on electric field, we call it the Helmholtz equation on electric field $(\mathcal{HP}-\mathcal{E})$

$$(\mathcal{HP}-\mathcal{E})\begin{cases}\nabla\times\nabla\times\boldsymbol{G}_e-k^2\boldsymbol{G}_e=\delta(\boldsymbol{R}-\boldsymbol{R'})\boldsymbol{u}_0, \quad \text{in } \Omega^\pm,\ R'\notin\partial\Omega\\ \boldsymbol{n}\times\boldsymbol{G}_e^+-\boldsymbol{n}\times\boldsymbol{G}_e^-=0\\ \boldsymbol{n}\cdot\epsilon^+\boldsymbol{G}_e^+-\boldsymbol{n}\cdot\epsilon^-\boldsymbol{G}_e^-=f\\ \boldsymbol{G}_e^- \quad \text{satisfy} \quad \text{SRC}\end{cases} \tag{4.3}$$

where $k\triangleq\omega\sqrt{\mu}\sqrt{\epsilon}$ is the wave number. The function f is the surface electric charge density on the boundary $\partial\Omega$ which should be at least square integrable in the Lebesgue sense $f\in L^2(\partial\Omega)$.

Similarly, one can write a Helmholtz problem on magnetic field $(\mathcal{HP}-\mathcal{H})$:

$$(\mathcal{HP}-\mathcal{H})\begin{cases}\nabla\times\nabla\times\boldsymbol{G}_m-k^2\boldsymbol{G}_m=\nabla\times\delta(\boldsymbol{R}-\boldsymbol{R'})\boldsymbol{u}_0, \quad \text{in } \Omega^\pm,\ R'\notin\partial\Omega\\ \boldsymbol{n}\times\boldsymbol{G}_m^+-\boldsymbol{n}\times\boldsymbol{G}_m^-=g\\ \boldsymbol{n}\cdot\mu^+\boldsymbol{G}_m^+-\boldsymbol{n}\cdot\mu^-\boldsymbol{G}_m^-=0\\ \boldsymbol{G}_m^- \quad \text{satisfy} \quad \text{SRC}\end{cases} \tag{4.4}$$

where g is the surface current density on the boundary $g\in L^2(\partial\Omega)$. One should note that in $(\mathcal{HP}-\mathcal{E})$ and $(\mathcal{HP}-\mathcal{H})$, the solutions $\boldsymbol{G}_e^-$ and $\boldsymbol{G}_m^-$ should satisfy the Sommerfeld radiation conditions which is stated for a time convention of $\mathrm{e}^{i\omega t}$ as [27,28]:

$$\lim_{R\to\infty} R\left(\frac{\partial\boldsymbol{G}^-}{\partial R}+ik\boldsymbol{G}^-\right)=0 \tag{4.5}$$

One can also formulate an electromagnetic problem using the Clifford algebra and the perturbed Dirac operator as Dirac problem $(\mathcal{DP}-\mathcal{F})$:

$$(\mathcal{DP}-\mathcal{F})\begin{cases}\nabla\mathcal{F}+ke_0\mathcal{F}=\sqrt{\mu}\delta(\boldsymbol{R}-\boldsymbol{R}')\boldsymbol{u}_0 & \text{in } \Omega^{\pm},\ R'\notin\partial\Omega\\ TQ^m\mathcal{F}^+-TQ^m\mathcal{F}^-=0\\ SQ^m\mathcal{F}^+-SQ^m\mathcal{F}^-=0\\ \mathcal{F}^- \quad \text{satisfy} \quad \text{KRC}\end{cases} \tag{4.6}$$

where the projection Q^m and splitting operators S and T are defined in section 2. An analogue of the Sommerfeld radiation condition is proposed by Kravchenko and Castillo [16] for the Dirac operator that is stated for the electromagnetic problems as [24]:

$$\lim_{R\to\infty}\left(kf(R)+\frac{iR}{|R|}f(R)\boldsymbol{k}\right)=0 \tag{4.7}$$

Lastly, we define another problem $(\mathcal{DP}-\mathcal{G})$ on the perturbed Dirac operator where the problem is formulated as a grade one on $\boldsymbol{G}_D$ rather grade two problem on $\mathcal{F}$.

$$(\mathcal{DP}-\mathcal{G})\begin{cases}\nabla\boldsymbol{G}_D+ke_0\boldsymbol{G}_D=\sqrt{\mu}\delta(\boldsymbol{R}-\boldsymbol{R}')\boldsymbol{u}_0 & \text{in } \Omega^{\pm},\ R'\notin\partial\Omega\\ TQ^m\boldsymbol{G}_D^+\boldsymbol{u}_0-TQ^m\boldsymbol{G}_D^-\boldsymbol{u}_0=0\\ SQ^m\boldsymbol{G}_D^+\boldsymbol{u}_0-SQ^m\boldsymbol{G}_D^-\boldsymbol{u}_0=0\\ \boldsymbol{G}_D^- \quad \text{satisfy} \quad \text{KRC}\end{cases} \tag{4.8}$$

It might seem rudimentary at the first glance why two similar problems $(\mathcal{DP}-\mathcal{F})$ and $(\mathcal{DP}-\mathcal{G})$ are defined above. It should be noted that we use (4.6) as an auxiliary problem that can be solved relatively easily (by the help of solutions from vector algebra) to solve (4.8).

While solving the scattering problems in the Clifford algebra, one often ends up with some sort of integral equation that is formulated similarly as the Lippmann-Schwinger [24, 29] or the Cauchy integral [10] or in the more general form as the Borel-Pompeiu [9, 11, 14, 30]. The common point in all of the above formulations is the need to use a fundamental solution of the problem in constructing the integral kernels. However, when the boundaries are present (e.g. waveguide, cavities), one needs to use the Green functions for the specific problem. As far as the authors are aware of, the Green function of the perturbed Dirac operator with boundary conditions of interest in electromagnetic applications are not reported yet. Here, a general method is presented to indirectly find Green function $\boldsymbol{G}_D$ using the already known solutions of electromagnetic problems in vector algebra.

5. Green function of the Dirac equation

Transmission boundary problems of electromagnetics are discussed in detail in [31]. Recently, the results are extended to higher dimensions in [32] and an equivalence between the solutions of Helmholtz equation and Dirac equation is proved rigorously.

One can also solve an electromagnetic problem using geometric algebra [33]. One of the remarkable properties of geometric algebra is that Maxwell four partial differential equations are simply cast into the single first order ordinary differential equation of Dirac [10] for linear homogenous isotropic time-invariant medium as

$$\mathcal{D}_k \mathcal{F} = \mathcal{S} \tag{5.1}$$

where the Dirac operator $\mathcal{D}_k$, bivector field $\mathcal{F}$ (which is constructed from both electric and magentic fields), and source vector $\mathcal{S}$ are defined [10] as:

$$\mathcal{D}_k = e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} + e_3 \frac{\partial}{\partial z} + k e_0 \tag{5.2}$$

$$\mathcal{F} = \sqrt{\mu} \boldsymbol{H} \sigma - j \sqrt{\epsilon} \boldsymbol{E} e_0, \tag{5.3}$$

$$\mathcal{S} = \sqrt{\mu} \boldsymbol{J} + \frac{j}{\sqrt{\epsilon}} \rho e_0, \tag{5.4}$$

and where the Clifford unit e_0 is introduced to accommodate the dimension of time.

We reduce the excitation $\mathcal{S}$ to $\delta(\boldsymbol{R} - \boldsymbol{R'}) \boldsymbol{u}_0$, therefore, making a Dirac problem $(\mathcal{DP} - \mathcal{F})$ as defined in previous section. Therefore, we have a physical phenomenon of the electromagnetism that fits into vector algebra with (4.1)-(4.4) as well as the geometric algebra with (4.6). One can solve $(\mathcal{DP} - \mathcal{F})$ rather easily by constructing the bivector field as $\mathcal{F}$ that incorporates known Green functions for electric and magnetic fields:

$$\mathcal{F} = \sqrt{\mu} \boldsymbol{G}_m \sigma - j \sqrt{\epsilon} \boldsymbol{G}_e e_0 \tag{5.5}$$

We can rewrite (5.1) and replace the impulse function by (4.8)

$$\mathcal{D}_k \mathcal{F} = \sqrt{\mu} \delta(\boldsymbol{R} - \boldsymbol{R'}) \boldsymbol{u}_0 = \sqrt{\mu} \mathcal{D}_k \boldsymbol{G}_D \boldsymbol{u}_0 \tag{5.6}$$

We assume $\nabla = e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} + e_3 \frac{\partial}{\partial z} = \mathcal{D}_k - k e_0$ and rewrite (5.6):

$$\nabla \mathcal{F} + k e_0 \mathcal{F} = \sqrt{\mu} \left(\nabla + k e_0 \right) \boldsymbol{G}_D \boldsymbol{u}_0 \tag{5.7}$$

$$\nabla \mathcal{F} - \sqrt{\mu} \nabla \boldsymbol{G}_D \boldsymbol{u}_0 + k e_0 \mathcal{F} - k \sqrt{\mu} e_0 \boldsymbol{G}_D \boldsymbol{u}_0 = 0 \tag{5.8}$$

We construct the following relations from (5.8):

$$\nabla \mathcal{F} - \sqrt{\mu} \nabla \boldsymbol{G}_D \boldsymbol{u}_0 = 0 \tag{5.9}$$

$$k e_0 \mathcal{F} - k e_0 \sqrt{\mu} \boldsymbol{G}_D \boldsymbol{u}_0 = 0 \tag{5.10}$$

Since (5.9) is the differential form of (5.10) scaled by a constant Clifford vector, we argue a solution to (5.10) also satisfies (5.9) and ultimately (5.8). It should be noted that the inverse of the above argument is not true as (5.9) contains extra spurious solutions. From (5.10), we have

$$k e_0 \mathcal{F} = k \sqrt{\mu} e_0 \boldsymbol{G}_D \boldsymbol{u}_0 \tag{5.11}$$

Since the Clifford product is not commutative and the result is sensitive to the order of the appearance, extra care must be taken to simplify (5.11). We

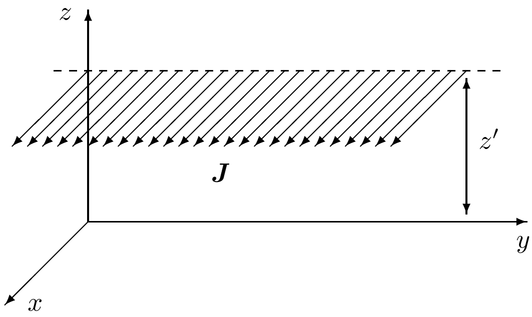

FIGURE 1. Uniform current $\boldsymbol{J} = \delta(z - z')\hat{\boldsymbol{x}}$ over a plane $z = z'$

multiply both sides by $\frac{1}{k\sqrt{\mu}}e_0$ from the left:

$$\frac{1}{\sqrt{\mu}}e_0 e_0 \mathcal{F} = e_0 e_0 \boldsymbol{G}_D \boldsymbol{u}_0 \tag{5.12}$$

One can further simplify (5.12) by multiplying both sides of (5.12) by $\boldsymbol{u}_0$ from the right and using (2.4) for unit vectors e_0 and $\boldsymbol{u}_0$. Therefore, the Green function for the Dirac equation is simply found as:

$$\boldsymbol{G}_D = -\frac{1}{\sqrt{\mu}}\mathcal{F}\boldsymbol{u}_0. \tag{5.13}$$

where $\mathcal{F}$ is compose of the Green functions for electric and magnetic fields using (5.5).

6. Examples

6.1. One-dimensional case

As a first example (see Fig. 1), we consider a one-dimensional problem with an infinite plane of current sheet $\boldsymbol{J} = \delta(z - z')\hat{\boldsymbol{x}}$. For instance, this problem can be formulated as $(\mathcal{MP})$ while the only boundary condition is the Silver-Müller radiation condition (4.2). The corresponding $\boldsymbol{E}$ and $\boldsymbol{H}$ fields radiated from the sheet are [3, 34]:

$$\boldsymbol{E} = -\frac{\eta}{2}\mathrm{e}^{-jk|z-z'|}\hat{\boldsymbol{x}} \tag{6.1}$$

$$\boldsymbol{H} = -\frac{1}{2}\mathrm{e}^{-jk|z-z'|}\hat{\boldsymbol{y}} \tag{6.2}$$

It should be noted that fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are the corresponding Green electric and magnetic functions, since their generating function has the shape of an impulse.

Following the method of section 5, we assume $\mathcal{F}$ as in (5.5).

$$\mathcal{F} = \frac{\sqrt{\mu}}{2}\mathrm{e}^{-jk|z-z'|}e_1 e_3 + j\frac{\sqrt{\mu}}{2}\mathrm{e}^{-jk|z-z'|}e_1 e_0 \tag{6.3}$$

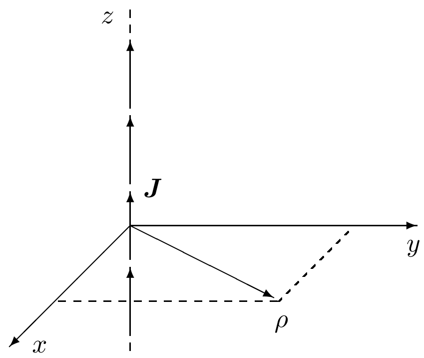

FIGURE 2. An infinite current line on z-axis $\boldsymbol{J} = \delta(\rho)\hat{\boldsymbol{z}}$

We emphasise that even for this one-dimensional case σ is defined as $-e_1e_2e_3$ which is due to the fact that we have three-dimensional fields, although they are only dependent on one dimension (z coordinate).

Using (5.13), we find the Green function of perturbed Dirac equation with Kravchenko radiation condition as:

$$\boldsymbol{G}_D = -\frac{1}{2}\mathrm{e}^{-jk|z-z'|}(e_3 + je_0) \tag{6.4}$$

It should be noted that (6.4) is identical to the fundamental solution of the Dirac equation for 1D problems. Since in this problem, boundary conditions are not specified at any points, the Green function is reduced to the fundamental solution [35]. The key point to observe is that the Clifford valued solution has been obtained directly from the vector-valued solution, without any need to perform any manipulations using Clifford algebra.

6.2. Two-dimensional case

As a second example, we consider an infinite line of current $\boldsymbol{J} = \delta(\rho)\hat{\boldsymbol{z}}$ on the z axis which is illustrated in Fig. 2. This problem can be formulated as $(\mathcal{HP} - \mathcal{H})$ and solved for Sommerfeld radiation conditions. Using the traditional methods, one easily finds the following vector potentials and fields [34]:

$$\boldsymbol{A} = \frac{1}{4j}H_0^2(k\rho)\hat{\boldsymbol{z}} \tag{6.5}$$

$$\boldsymbol{E} = -\frac{k\eta}{4}H_0^2(k\rho)\hat{\boldsymbol{z}} \tag{6.6}$$

$$\boldsymbol{H} = -\frac{jk}{4}H_1^2(k\rho)\hat{\boldsymbol{\phi}} \tag{6.7}$$

Using a similar approach and directly from (5.5), we have:

$$\mathcal{F} = \frac{jk\sqrt{\mu}}{4}\left[H_1^2(k\rho)\hat{\boldsymbol{\rho}}e_3 + H_0^2(k\rho)e_3e_0\right] \tag{6.8}$$

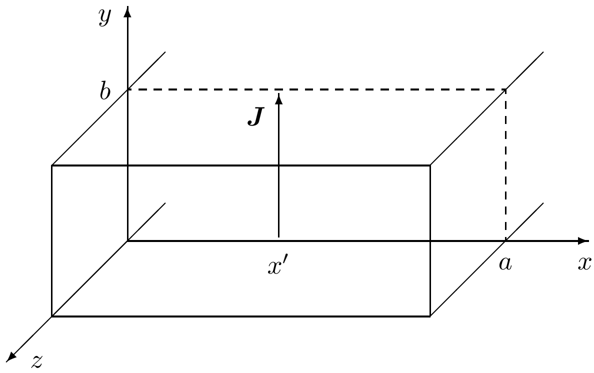

FIGURE 3. Rectangular waveguide with a current $\boldsymbol{J}$ stretched between waveguide plates.

where

$$\begin{aligned}\hat{\boldsymbol{\phi}}\sigma &= (-\sin\phi e_1 + \cos\phi e_2)(-e_1 e_2 e_3) \\ &= -\sin\phi\, e_2 e_3 - \cos\phi\, e_1 e_3 \\ &= -\hat{\boldsymbol{\rho}} e_3, \end{aligned} \tag{6.9}$$

and we then find the Green function of the Dirac equation directly from (5.13) as:

$$\boldsymbol{G}_D = \frac{jk}{4}\left[H_1^2(k\rho)\hat{\boldsymbol{\rho}} - H_0^2(k\rho)e_0\right]. \tag{6.10}$$

In Clifford algebra, one finds the fundamental solution of the Dirac equation for two-dimensional problems by differentiating the two-dimensional Bessel potential [10, Ch.7]. That also leads to (6.10). However again, the Clifford-valued solution can be obtained directly without recourse to Clifford algebra.

6.3. Rectangular Waveguide

As the third example, a more general but extensively used geometry of the rectangular waveguide is examined. The cross section of the waveguide is assumed to be on $x - y$ plane and the long side of the waveguide is aligned with x axis. Waveguide walls are assumed to be made of perfect electric conductors (PEC), therefore, $\boldsymbol{E}$ and $\boldsymbol{H}$ fields can not penetrate into it. This dictates the boundary conditions for this problem. A typical source for our problem is a line current $\boldsymbol{J} = \delta(x-x')\delta(z-z')\hat{\boldsymbol{y}}$. Figure 3 shows the geometry of the waveguide and source. It is evident that only transverse electric modes of TE are excited by the source since the $\boldsymbol{E}$ should be in the same direction as the current $\boldsymbol{J}$. Furthermore, since there is no dependency on the y coordinate, only TE_{n0} modes are to be excited in the waveguide.

The problem can be formulated as the Helmholtz problem ($\mathcal{HP}-\mathcal{E}$):

$$\begin{cases} \nabla\times\nabla\times\boldsymbol{E}-k^2\boldsymbol{E}=\delta(x-x')\,\delta(z-z')\hat{\boldsymbol{y}}, \\ \hat{\boldsymbol{x}}\times\boldsymbol{E}=0 \quad \text{on} \quad x=0,\,a \\ \hat{\boldsymbol{y}}\times\boldsymbol{E}=0 \quad \text{on} \quad y=0,\,b \end{cases} \tag{6.11}$$

The complex propagation constant for mode n is determined by the waveguide dimensions and the operating frequency.

$$\gamma_n=\sqrt{\frac{n\pi}{a}^2-\omega^2\mu\epsilon} \tag{6.12}$$

Green function for the geometry illustrated in Fig. 3 is given in [36, Sec.2.7] for the vector potential $\boldsymbol{A}$ as the summation of the eigenfunctions. We deduce the electric Green function using $\boldsymbol{E}=i\omega\boldsymbol{A}$

$$\boldsymbol{E}=-\frac{1}{a}\sum_{n=1}^{\infty}Z_n\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\mathrm{e}^{-\gamma_n|z-z'|}\hat{\boldsymbol{y}}, \tag{6.13}$$

where $Z_n\triangleq i\omega\mu/\gamma_n$ is the characteristic impedance of the TE_{n0} mode. Using the first relation in (4.1), one simply finds the excited $\boldsymbol{H}$ field as:

$$\begin{aligned}\boldsymbol{H}=&\pm\frac{1}{a}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\mathrm{e}^{-\gamma_n|z-z'|}\,\hat{\boldsymbol{x}}\\&-\frac{1}{a^2}\sum_{n=1}^{\infty}\frac{n\pi}{\gamma_n}\cos\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\mathrm{e}^{-i\gamma_n|z-z'|}\hat{\boldsymbol{z}}.\end{aligned} \tag{6.14}$$

In this section, in the terms with double signs (e.g.$\pm$) the top sign is for $z>z'$ and bottom sign is related to $z<z'$.

Alternatively, one can arrive at (6.13) and (6.14) by following the method of [3, Sec 5.2] by dropping the dependencies on y coordinates and reducing the dyads to vectors for the excitation pointing towards $\hat{\boldsymbol{y}}$.

The equivalent problem in Clifford form can be constructed as:

$$\begin{cases} \nabla\boldsymbol{G}_D+ke_0\boldsymbol{G}_D=\sqrt{\mu}\delta(x-x')\delta(z-z') \\ TQ_1^m\boldsymbol{G}_De_2=0 \quad \text{on} \quad x=0,\,a \\ TQ_2^m\boldsymbol{G}_De_2=0 \quad \text{on} \quad y=0,\,b \end{cases} \tag{6.15}$$

where Q_1^m and Q_2^m are defined by setting $\boldsymbol{n}$ in (2.8) to e_1 and e_2, respectively.

Bivector $\mathcal{F}$ is found as:

$$\begin{aligned}\mathcal{F}=&\pm\frac{\sqrt{\mu}}{a}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\mathrm{e}^{-\gamma_n|z-z'|}\,e_2e_3\\&-\frac{\sqrt{\mu}}{a^2}\sum_{n=1}^{\infty}\frac{n\pi}{\gamma_n}\cos\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\mathrm{e}^{-i\gamma_n|z-z'|}e_1e_2\\&+\frac{\mathrm{j}\sqrt{\epsilon}}{a}\sum_{n=1}^{\infty}Z_n\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\mathrm{e}^{-\gamma_n|z-z'|}e_2e_0.\end{aligned} \tag{6.16}$$

Using the (5.13) and setting $\boldsymbol{u}_0 = e_2$, we find the Green function for the Dirac equation in the following form:

$$\begin{aligned}\boldsymbol{G}_D = &\mp \frac{1}{a}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{a} \sin\frac{n\pi x'}{a} \mathrm{e}^{-\gamma_n|z-z'|}\, e_3 \\ &+ \frac{1}{a^2}\sum_{n=1}^{\infty} \frac{n\pi}{\gamma_n} \cos\frac{n\pi x}{a} \sin\frac{n\pi x'}{a} \mathrm{e}^{-i\gamma_n|z-z'|} e_1 \\ &- \frac{i}{a}\sqrt{\frac{\epsilon}{\mu}}\sum_{n=1}^{\infty} Z_n \sin\frac{n\pi x}{a} \sin\frac{n\pi x'}{a} \mathrm{e}^{-\gamma_n|z-z'|} e_0. \end{aligned} \tag{6.17}$$

As far as the authors are aware of, it is the first time a Green function is provided for the Dirac equation with the boundary conditions of a rectangular waveguide. A direct solution in Clifford algebra remains an unsolved problem. Therefore, (6.17) cannot be verified with direct comparison to the results of others. A quick check on the validity of (6.17) is to verify if the $\boldsymbol{G}_D$ satisfies the boundary conditions.

7. Conclusion

In this paper, we demonstrated a simple method to find the Green function of Dirac equation for electromagnetic problems. Particularly, we use prior known Green functions of the Helmholtz equation with the same boundary conditions. The method is further illustrated by three different examples for one and two dimensions. The value here is in being able to obtain the Clifford-valued Green functions corresponding to all known vector-valued Green functions without needing to perform any Clifford algebra or repeat and derivations already known for vectors.

Occasionally, Green function is reported for various quantities $\boldsymbol{A}$ rather $\boldsymbol{E}$ and $\boldsymbol{H}$. For example, Green function of the rectangular waveguide in 6.3 is extracted from [36] that actually provides a Green function for the magnetic potential $\boldsymbol{A}$. While constructing $\mathcal{F}$, one should take extra care to use appropriate Green functions to ensure the calculated Green function of the Dirac equation is a proper answer.

Green function has different forms that are equivalent to each other (e.g. spectral form, spatial form) [1]. Here we showed eigenfunction expansion form in Section 6.3, but the method should also work for other forms of Green function (e.g. spectral form).

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