Investigating Students' Mathematical Difficulties with Quadratic Equations

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Investigating Students' Mathematical Difficulties with Quadratic Equations

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This paper examines the factors that hinder students' success in working with and understanding the mathematics of quadratic equations using a case study analysis of student error patterns. Twenty-five Year 11 students were administered a written test to examine their understanding of concepts and procedures associated with this topic. The findings indicate that students' success was inhibited by errors that arose from a lack of procedural understanding regarding fractions, algebraic processes, and conceptual understanding regarding algebraic conventions. Without this prerequisite knowledge, working with and understanding the nature of quadratics was hindered.

The study of quadratic equations acts as a gateway to more advanced study of algebra and is a topic area that challenges many students (Bosse & Nandakumar, 2005; Vaiyavutjamai & Clements, 2006; Vaiyavutjamai, Ellerton, & Clements, 2005; Zakaria, Ibrahim, & Maat, 2010). Failure at working with quadratic equations virtually precludes students from accessing the powerful mathematics that is necessary to enrol in courses involving the study of sciences at tertiary levels (Watt, 2005). Despite the importance of this topic area there has been little research to inform the reform of pedagogy associated with quadratics.

Literature Review

The resounding theme in mathematics education research is that students' performance in the domain of quadratic equations is exceptionally poor and does not significantly increase even after instruction (Chaysuwan, 1996; Vaiyavutjamai et al., 2005). Students have been found to struggle particularly solving for x in the form $x^2 = k$ (k>0) and $(x - r)(x - s) = 0$ where r and s are any real numbers (Vaiyavutjamai et al., 2005). The most concerning of all the data was that, out of a subsample of 29 second-year university students in the United States who were preservice middle-school mathematics specialist teachers, only 37% and 78% respectively could answer the two questions correctly (Vaiyavutjamai et al., 2005). Other than studies by the researchers noted above, there is a deficit in research and empirical evidence regarding students' performance with respect to solving quadratic equations.

It is also important to consider the impact and current evidence relating to teaching methods and the learning of quadratic equations. Kotsopoulos (2007) reported that students need to develop procedural and conceptual knowledge through various learning experiences in an integrated manner. The Australian Academy of Science (AAS) also recognises the intertwined relationship between conceptual understanding, procedural fluency, and problem solving and reasoning due to the hierarchical nature of mathematics (AAS, 2015, p. 17). The cognitive load work by Kirschner, Sweller, and Clark (2006) gives an explanaton for the necessity of fluency with prerequisite knowledge. Without prerequisite fluency, short-term memory becomes overloaded and unable to effectively process the new concepts being learned. Hattie (2009) noted that fluency with prerequisite knowledge, even at a very early stage, was highly predictive of latter success.The key prerequisite concepts and

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processes necessary to engage meaningfully with quadratics include basic whole number fluency, fraction computation, linear algebraic procudures, and coordinate geometry.

A key process in working with quadratics is solving or finding the x intercepts, should there be any. In most curricula this has involved factorisation, the square root method, completing the square, and the use of the quadratic formula. Each of these techniques has its own advantages and disadvantages when it comes to teaching, learning, and applying. Research has shown that students and teachers shy away from some techniques and favour factorisation, generally using coefficients that are easy to factorise since students' ability to perform fractional and radical arithmetic has been reported as low (Bosse & Nandakumar, 2005). Overemphasis on relatively simple factorisation is concerning as many quadratic equations cannot be factorised. Further, other methods that are more efficient or that develop conceptualisation may be neglected in teaching (Bosse & Nandakumar, 2005). For example, factorisation with algebra tiles links quadratics with basic multiplication and division concepts via the area model of rectangles and squares (Howden 2001). Geometric models are useful in adding understanding in developing the quadratic formula via completing the square procedure (Norton, 2015). Barnes (1991) suggested using graphing calculators to plot quadratics with no roots, one root, or two roots and linking this to the discriminate values. Research suggests that teachers tend to avoid teaching alternative methods due to high instances of process skill errors with techniques such as the quadratic formula and completing the square (Zakaria et al., 2010). From this literature review, it is clear that there is a need for further research into the sources of students' difficulties with quadratic equations.

Method

Overview of Methodology

The method is essentially a case study, an in-depth investigation of one class in one school. It is qualitative in that the intention is to promote greater understanding of not just the way things are, but why (Gay & Airasian, 1992). There is an attempt to add to theory on the basis of data collected in real-world settings and we suggest tentative hypotheses to explain the observations, but not prior to the commencement of data collection, a position consistent with grounded theory (Gay & Airasian, 1992; Strauss & Corbin, 1994).

The Sample: Study and Curriculum Context

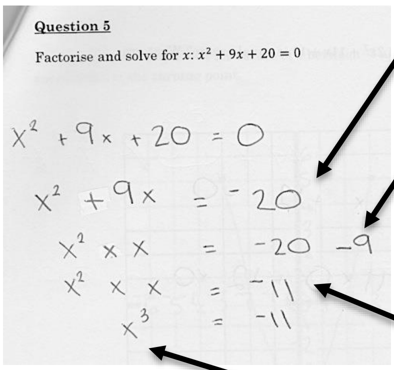
The sample school in this research project was a coeducational high school in south-east Queensland in a community of mixed socioeconomic index. The school is typical of outer-suburban schools according to MySchool data from the Australian Curriculum, Assessment and Reporting Authority (ACARA) (ACARA, 2012a). The sample included a Year 11 Mathematics B class of 25 students. In Queensland, Mathematics B is a calculus-oriented, advanced senior mathematics class that qualifies the students to study science-oriented subjects at university. All the students had studied quadratics in Year 10 as consistent with the state and national curriculum (ACARA, 2012b; Department of Education, Training and Employment [DETE], 2013; Queensland Studies Authority, 2004).

Testing Instruments

Students were administered a written test aimed at examining their attempts in working with the processes and concepts of quadratics. The authors constructed the test so that it reflected the expectations of the relevant syllabus documents. Calculators were not permitted in order to determine whether fundamental mathematics was a factor limiting student achievement. This paper examines the results of student responses to eight questions. A successful student was defined as one who obtained the correct answer from correctly employing any process (e.g., methods of solving) unless a particular process was required (e.g., factorise to solve). An unsuccessful student was defined as a student who had made some written attempt at the question but did not obtain a correct solution, and a student who made no attempt was defined as a student who did not make any written markings at all for the given question. The test questions and success rates are presented in Table 1 of the results and analysis section. The results of this analysis were reported as a percentage of the sample that achieved or did not achieve success in a particular form of question. During this evaluation, qualitative points of interest and errors were noted and categorised.

Analysis of Written Scripts

Each student solution was analysed for procedural or conceptual errors as illustrated in Figure 1 and these were categorised. Consistent with grounded theory methodology, the data were coded or classified as they were collected and categories emerged from the data. An example of the analysis and coding processes is modelled below in Figure 1.



Student does not understand to factorise, rather applies linear algebra thinking to solve for one possible *x*. This is a lack of conceptual understanding regarding quadratics.

Student takes 9 from both sides and also changes the addition to a multiplication sign indicating lack of conceptual understanding regarding basic algebraic processes.

Student incorrectly applies integer procedures.

Student now “correctly” applies index laws on this mistake. The student has assumed that x^3 is equal to -11, but cannot proceed. This dead end is a result of the conceptual and procedural errors previously made. The occurrence of this error was three out of 24 students who attempted the question.

Figure 1. Sample of analysis of student solution.

In the example above, four critical errors of either a *conceptual* or *procedural* nature were made, and each linked to prerequisite mathematics content. This process of analysis was carried out for each student solution. The sum of all error patterns was synthesised into categories that evolved out of the data. The bringing together of these categories enables new ways of understanding students’ difficulties with quadratics. The totalling of such error patterns enables the prevalence of such misconceptions to be documented and has the potential to inform practice and curriculum theory.

Results and Analysis

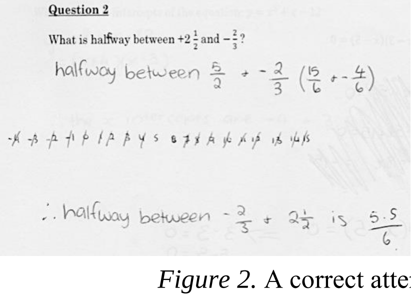
Data are presented first in terms of success, or otherwise, and later the data are examined. Table 1 presents the proportions of the sample of 25 Mathematics B students that were successful, unsuccessful, or did not attempt the given questions.

Table 1

Percentage Proportions for Successful, Unsuccessful, and No Attempt for each Question, with Key Concepts Identified

Questions	Successful students	Unsuccessful students	No attempt
1. What is halfway between -3 and 4? (Operating with whole numbers and fractions)	100%	0%	0%
2. What is halfway between $+2\frac{1}{2}$ and $-\frac{2}{3}$? (Operating with mixed numbers)	28%	56%	16%
3. Solve for x: $x^2 = 9$. (Recognising two solutions)	92%	4%	4%
4. Solve for x: $(x-3)(x-5) = 0$. (Solving in factorised form)	56%	40%	4%
5. Factorise and solve for x: $x^2 + 9x + 20 = 0$. (Factorising and solving where a =1)	48%	48%	4%
6. What are the x -intercepts of the equation: $y = x^2 + x - 12$? (Factorising and solving with a negative y intercept)	32%	32%	36%
7. What are the values of x such that $2x^2 + 11x + 12$ is equal to zero? (Factorising and solving when a≠1)	20%	56%	24%
8. Using any method, find the x -intercepts of $3x^2 = -4x - 1$. (Algebraic conventions, factorising involving negative integers and solving)	8%	40%	52%

Success on the Question 1 was high, but interestingly 21 out of 24 students answered 0.5 rather than ½ suggesting a preference to work with decimals. For Question 2, only two students recorded the desired outcome of $\frac{11}{12}$, as illustrated in Figure 2.

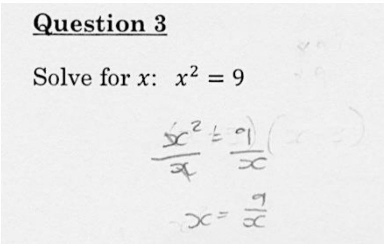


Students tended to use number lines, a method that saves on more abstract fraction calculations. Students such as this one had a preference for decimal solutions. Four students responded with $\frac{5.5}{6}$.

Figure 2. A correct attempt at using fraction lines to obtain the answer $\frac{5.5}{6}$.

In cross-tabulation of the results, all students who were successful at solving Question 2 were able to solve Question 6. When comparing the results of Question 2 and Question 7, similar relationships were again observed. All students who failed Question 2 were unable to complete Question 7 and only one student who was successful at Question 2 was not able to complete Question 7. All students who failed Question 2 also failed Question 8. While this is hardly surprising, there currently has been little empirical evidence to support these links.

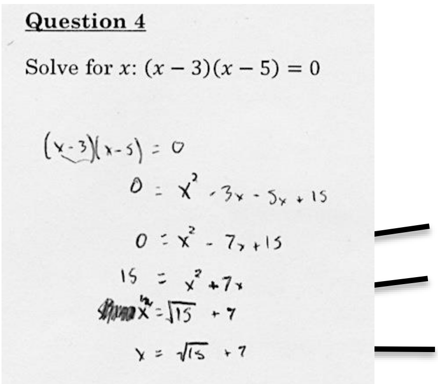
Students also demonstrated errors of algebraic convention as illustrated in Figures 3 and 4 below.



92% of students recognised the square root of 9 was 3, but none recognised it could have two solutions, -3 or 3. This student has divided both sides by x to obtain $\frac{9}{x}$. This suggests conceptual misunderstanding in regard to working with roots.

Rather than finding the square root of x and 9, two students attempted to solve for x as if it were working with a linear equation.

Figure 3. Incorrect application of "cancelling" demonstrated.



24% of students did not recognise that the equation was already factorised and they simply needed to apply the null factor law. Most of the students set about attempting to apply linear processes and in doing so made fundamental algebraic errors.

Incorrectly combines terms -3 x and -5 x.

Switches x^2 and 15, neglecting signs.

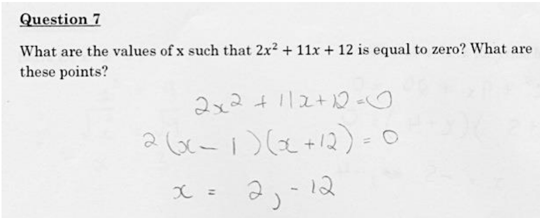
Confounds taking the square root of both sides, suggesting conceptual and procedural misconceptions in dealing with roots.

Figure 4. Example of student incorrectly attempting to rearrange the equation in order to solve.

Lateral analysis of the data also demonstrated that all students who were unable to answer Question 4 were also unable to answer Questions 6, 7, and 8, which regarded solving quadratics in standard form. This is hardly surprising since, if students struggle to solve equations that are already factorised, it is expected that they will struggle on ones that they have to factorise themselves before solving.

Questions 5 to 8 of the written test tested students' abilities to solve quadratic equations in the form $ax^2 + bx + c = 0$, and $ax^2 + bx - c = 0$ both where a=1 and a≠1. The four questions were all phrased uniquely; however, many of the error patterns observed were consistent across all of the questions.

A common error observed across Questions 5 to 8 was that students were able to factorise the equations but either did not know how to apply the null factor law or did so incorrectly. For Question 6, 7, and 8, all students attempted to factorise. Though some students were successful at factorising, many of the factorisation attempts suggested that students struggled with the procedures of factorisation, as seen in Figure 5.



This student does not seek the correct factors (multiply to ac=24 and add to b=11). This indicates procedural error in factorisation and application of the null factor law with $2(x-1)$. Three students from the sample exhibited similar difficulties with factorising.

Figure 5. Student inability to factorise where a≠1.

The high proportions of the sample that were unable to obtain correct solutions using factorisation illustrate that factorisation was not a technique that allowed students to solve the questions successfully. Unfortunately they did not have alternative strategies.

Discussion

The data revealed some concerning findings including high levels of both procedural and conceptual misconceptions. It was found that many students were lacking in fluency in dealing with integers, fractions, and algebraic conventions. Students had a preference for using number lines to find half way between two values and tended to avoid the use of fractions, preferring decimal operations. This may be an unintended consequence of the prevalent use of calculators in Queensland schools, consistent with syllabus recommendations (ACARA, 2012a; Queensland Studies Authority [QSA], 2004, 2008). The lack of fraction computation fluency was associated with subsequent failure on quadratic solving. While this is not necessarily surprising, there currently has been little empirical evidence to support these links.

Large proportions of the sample attempted to rearrange quadratic equations as if they were linear equations. These attempts suggest that students had misconceptions regarding the nature of quadratic equations and, in the attempts to rearrange the expressions, students also demonstrated various misconceptions regarding algebraic conventions. Observed errors included adding unlike terms, and incorrect applications of index notation and radical arithmetic. These results support earlier findings (Vaiyavutjamai et al., 2005; Vaiyavutjamai & Clements, 2006). The results also indicate that there was an overemphasis on factorisation processes and that lack of learning of alternative methods disadvantaged students, a finding earlier reported (Bosse & Nandakumar, 2005; Vaiyavutjamai et al., 2005). In addition, some students from the sample demonstrated that they could factorise a quadratic equation but could not obtain the final solutions due to a lack of conceptual understanding regarding the null factor law. These findings suggest an overemphasis on mechanical processes of factorisation at the expense of conceptual understanding. The findings have implications for Queensland curriculum planning and implementation. For example, a commonly used curriculum provided by Education Queensland is Curriculum into the Classroom (C2C). Throughout the 12 lessons allocated by C2C on patterns and algebra and linear and non-linear relationships, factorisation is heavily emphasised (4 out of 12 lessons) with alternative methods of solving only allocated 2 lessons at the very end of the unit. The findings of the study can be summarised in the model shown in Figure 6.

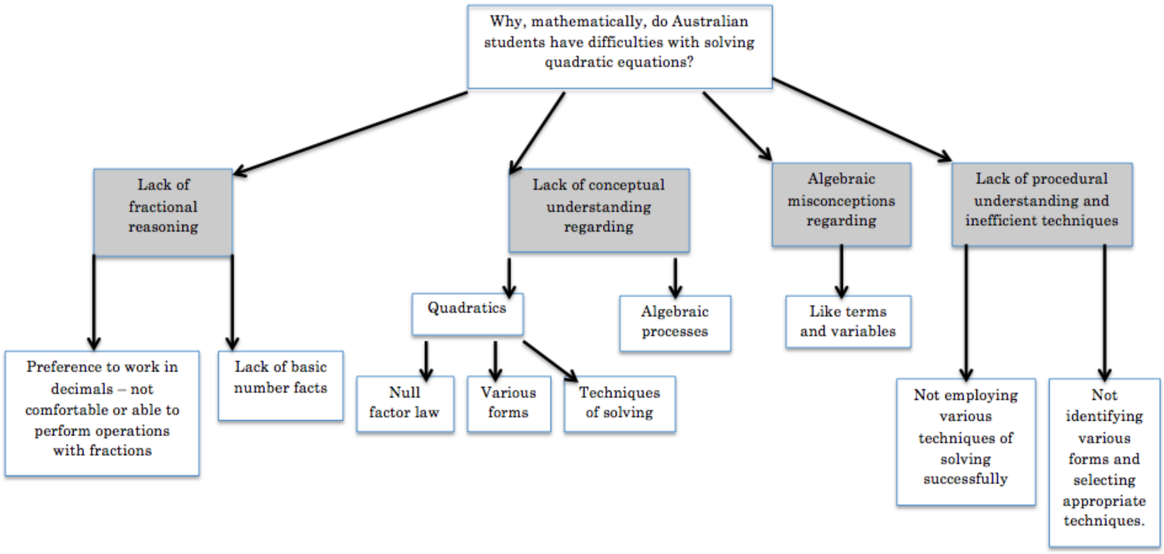


Figure 6. Model of student error patterns and relationships between them.

Conclusion

This study adds to the literature by supporting the findings of previous researchers who have documented that student understanding of quadratics equations is a problem area (e.g., Bosse & Nandakumar, 2005; Vaiyavutjamai & Clements, 2006; Vaiyavutjamai et al., 2005; Zakaria et al., 2010). It adds to the literature by helping to explain why this was the case, at least in one school. Students who struggled did so due to a combination of factors that became critical. Some of these included a lack of prerequisite concepts and processes associated with fractions and algebra conventions (e.g., index law conventions and understanding the meaning of solve). Cognitive load theorists (e.g., Kirschner et al., 2006) provide an explanation as to why these deficits become a critical hindrance to engagement with quadratics. Many students in this study did not have the tools to factorise, and many of those who could did not understand the implications of the factorised form for finding the roots. Conceptual errors were invariably preceeded by procedural errors. Sometimes the two were intertwined.

There was evidence that teacher interpretation of curriculum guidelines (DETE, 2013) resulted in overemphasis of symbolic factorisation processes. The results suggest a lack of alternative pedagogies such as links to geometric models recommended by some authors (e.g., Howden, 2001; Norton, 2015) and the integrated use of graphs in contexual settings (e.g., Barnes, 1991). This deficit was most obvious in lack of understanding of null factor law and various forms of quadratics. These results add to the findings of previous authors regarding too narrow a focus on factorisation (e.g., Bosse & Nandakumar, 2005).

The findings have implications for the study school, and potentially for wider teaching practices. The data suggest that attempting the teaching of quadratics without prerequisite fluency is unwise. This questions the validity of progressing students when they do not have mastery and a deep conceptual understanding of integers, fractions, and algebraic procedures. Clearly, greater focus on these procedures is required. The data indicate that a more systematic and multi-representiational approach to teaching the concepts underpinning quadratics is warranted. This topic area is an opportunity to consolidate all earlier algebraic learning and lead into the study of calculus; it deserves significant focus and the development of mastery prior to engagement with calculus.

References

Australian Academy of Science. (2015). *Desktop review of mathematics school education.* Canberra: Author.

Australian Curriculum, Assessment and Reporting Authority. (2012a). *My School*. Retrieved from http://www.myschool.edu.au

Australian Curriculum, Assessment and Reporting Authority. (2012b). *Mathematics F-10.* Retrieved from http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10#level=10

Barnes, M. (1991). *Investigating change: An introduction to calculus for Australian schools.* Carlton, VIC: Curriculum Corporation.

Bosse, M. J., & Nandakumar, N. R. (2005). Section A, factorability of quadratics: Motivation for more techniques. *Teaching Mathematics and Its Applications*, *24*(4), 143-153.

Chaysuwan, S. (1996). *The construction of a diagnostic test on basic knowledge of algebra for Mathayom Suksa three students in the Bangkok Metropolis.* Unpublished M.Ed thesis, Kasetsart University, Thailand.

Department of Education, Training and Employment. (2013). *Curriculum into the classroom (C2C).* Retrieved from education.qld.gov.au/c2c

Gay, L., Airasian, P. (1992). *Educational Research: Competencies for analysis and application*. Englewood Cliffs: Merril.

Hattie, J. (2009). *Visible learning: A synthesis of over 800 meta-analyses relating to achievement*. London: Routledge.

Howden, H. (2001). *Algebra tiles for the overhead projector*. Veron Hills: Learning Resources.

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologist*, *41*(2), 75-86.

Kotsopoulos, D. (2007). Unravelling student challenges with quadratics: A cognitive approach. *Australian Mathematics Teacher*, *63*(2), 19-24.

Norton, S. (2015). *Teaching and learning fundamental mathematics: Quadratic equations*. Producer Freddy Komp. Brisbane.

Queensland Studies Authority. (2004). *Mathematics Years 1-10 syllabus.* Retrieved from http://www.qsa.qld.edu.au/downloads/early_middle/kla_maths_syll.pdf

Queensland Studies Authority. (2008). *Senior syllabus: Mathematics B.* Retrieved from https://www.qcaa.qld.edu.au/downloads/senior/snr_maths_b_08_syll.pdf

Strauss, A., & Corbin, J. (1994). Grounded theory methodology. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 217-285). Thousand Oaks, CA: Sage Publications.

Vaiyavutjamai, P., & Clements, M. A. (2006). Effects of classroom instruction on students' understanding of quadratic equations. *Mathematics Education Research Journal*, *18*(1), 47-77.

Vaiyavutjamai, P., Ellerton, N. F., & Clements, M. A. (2005). Students' attempts to solve two elementary quadratic equations: A study in three nations. *Building connections: Theory, research and practice: Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia*. Sydney: MERGA.

Watt, H. (2005). Exploring adolescent motivations for pursuing maths-related careers. *Australian Journal for Educational Psychology, 5*, 107-116.

Zakaria, E., Ibrahim, & Maat, S. (2010). Analysis of students' error in learning of quadratic equations. *International Education Studies*, *3*(3), 105-110.