Non-Markovian Stochastic Schrodinger Equations and Interpretations of Quantum Mechanics

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NON-MARKOVIAN STOCHASTIC SCHRÖDINGER EQUATIONS AND INTERPRETATIONS OF QUANTUM MECHANICS

By

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Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Jay M. Gambetta

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List of Publications

The following list is in chronological order.

1. Jay Gambetta and H.M. Wiseman, "State and dynamical parameter estimation for open quantum systems," Phys. Rev. A **64**, 041205 (2001).
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2. Jay Gambetta and H. M. Wiseman, "Non-Markovian stochastic Schrödinger equations: Generalized to real-valued noise using quantum measurement theory," Phys. Rev. A **66**, 012108 (2002).
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3. Jay Gambetta and H. M. Wiseman, "Perturbative approach to non-Markovian stochastic Schrödinger equations," Phys. Rev. A **66**, 052105 (2002).
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5. Jay Gambetta and H. M. Wiseman, "A non-Markovian stochastic Schrödinger equation developed from a hidden variable interpretation," in *Proceedings of SPIE* **5111** *Fluctuation and Noise in Photonics and Quantum Optics*, edited by D. Abbott, J. H. Shapiro, and Y. Yamamoto, (SPIE, Bellingham, 2003), pp. 313-324.
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Abstract

It has been almost eighty years since quantum mechanics emerged as a complete theory, yet debates about "how should quantum mechanics be interpreted" still occur. Interpretations are many and varied, some taking us as fundamental in determining reality (orthodox interpretation), while others proposing that reality exists outside of us, but it is a lot more complicated than that implied by classical mechanics. In this thesis I am going to try to provide new light on this debate by investigating dynamics under both the orthodox and modal interpretation. In particular I will answer the question what is the interpretation of non-Markovian stochastic Schrödinger equations? I conclude that under the orthodox view these equations have only a numerical interpretation. They provide a rule for calculating the state of the system at time t *if we made a measurement on the bath (a collection of oscillators* $\{\hat{a}_k\}$*) at that time, yielding results* $\{z_k\}$*. However in the modal view they have a meaning: non-Markovian stochastic Schrödinger equations represent the evolution of the system part of the property state of the universe (bath + system).*

List of Symbols

Mathematical notation

$\hbar$	Planck's constant.
∞	Infinity.
$:=$	Defined as
$=$	Equal to
$\approx$	Approximately equal to
$\neq$	Not equal to
$\equiv$	Equivalent to
$\Rightarrow$	Implies
$\rightarrow$	Goes to
$:$	Such that
dx	An infinitesimal change in x
δx	A small change in x.
$\in$	Belongs to
$\{\}$	A set.
$\otimes$	Tensor multiplication.
$\oplus$	Tensor addition.

Functions

d_t...	Total derivative of ... wrt t.
∂_t...	Partial derivative of ... wrt t.
$\delta_{z(t)}$...	Function derivative of ... wrt the function $z(t)$.
Tr[...]	Trace of ...
Im[...]	Imaginary part of ...
Re[...]	Real part of ...
...*	The complex conjugate of ...
E[...]	Average value of ...

Observables or Properties

$Z(t)$	A physical property/observable.
$v(Z(t), t)$	The value of property $Z(t)$ at time t.
$r(Z(t), t)$	The result of measuring observable $Z(t)$ at time t.
z_n	The eigenvalues $Z(t)$ when the possibilities are discrete.
z	The eigenvalues for $Z(t)$ when the possibilities are continuous.
$\mathbf{I}_{[t_0,t)}$	A current, continuous-in-time sequence of results for $Z(t)$.
$\Pr(z_n, t)$	Probability of getting/being z_n at time t.
$P(z, t)$	Probability density of getting/being z at time t.
$\Pr([z, z'], t)$	Probability of getting/being in the range $[z, z']$ at time t.
$\Pr([z], t)$	Probability of getting/being in an infinitesimal range centered on z at time t.
$\Pr(z_n, t' \mid z_m, t)$	The probability of z_n at time t' given z_m at time t.
$\langle Z(t) \rangle$	Mean (expectation) value of $Z(t)$.
$\Delta Z(t)$	Standard deviation in $Z(t)$.
$\Lambda\mathrm{r}(z_n, t)$	An ostensible probability.
$\Lambda(z, t)$	An ostensible probability density.

States and spaces

$\mathcal{H}$	A Hilbert space.
$\lvert\psi(t)\rangle$	A quantum state, element of $\mathcal{H}$.
$\lvert\bar{\psi}(t)\rangle$	A quantum state normalized to an ostensible distribution.
$\lvert\tilde{\psi}(t)\rangle$	An unnormalized quantum state.
$\lvert\psi_n(t)\rangle$	A conditioned system state or a property system-state
$\lvert\Psi(t)\rangle$	A composite (2 or more systems) quantum state or a guiding state.
$\lvert\Psi_n(t)\rangle$	A conditioned composite quantum state or a property state.
$\lvert\psi_{\mathbf{I}}(t)\rangle$	A system state conditioned on the current $\mathbf{I}_{[t_0,t)}$.
$\mathcal{B}$	The space containing all possible $\rho(t)$.
$\rho(t)$	A state matrix.
$\bar{\rho}(t)$	A state matrix normalized to an ostensible distribution.
$\tilde{\rho}(t)$	A unnormalized state matrix.
$\rho_{\mathrm{red}}(t)$	The reduced state for the system.
$\rho_n(t)$	A conditioned state matrix for the system.
$W(t)$	A composite state matrix.
$W_n(t)$	A conditioned composite state matrix.
$\rho_{\mathbf{I}}(t)$	A state matrix conditioned on the current $\mathbf{I}_{[t_0,t)}$.
$p(t)$	The purity of $\rho(t)$.

Operators

$\hat{A}(t)$	An arbitrary operator that acts in a Hilbert space.
$\hat{R}(t)$	A Hermitian operator.
$\hat{Z}(t)$	A normal operator.
$\hat{1}$	The identity operator.
$\hat{H}(t)$	A Hamiltonian operator.
$\hat{U}(t,t_0)$	An unitary operator.
$\hat{Q}$	The position operator.
$\hat{P}$	The momentum operator.
$\hat{X}$	The dimensionless position operator.
$\hat{Y}$	The dimensionless momentum operator.
$\hat{a}$	The annihilation operator.
$\hat{a}^\dagger$	The creation operator.
$\hat{L}$	A general system lowering operator.
$\hat{L}^\dagger$	A general system raising operator.
$\hat{\pi}_n(t)$	A projector.
$\hat{\pi}(z_n,t)dz$	An infinitesimal projector.
$\hat{F}_n(t)$	A POM element.
$\hat{F}(z_n,t)dz$	An infinitesimal POM element.
$\hat{M}_n(t)$	A measurement operator.
$\hat{\mathcal{A}}(t)$	An arbitrary superoperator that acts in $\mathcal{B}$.
$\hat{\mathcal{D}}(t)$	The damping superoperator for a Markovian bath.
$\hat{\mathcal{K}}(t,t')$	The two time memory superoperator for a non-Markovian bath.

List of Abbreviations

CSL	Continuous Spontaneous Localization
EPR	Einstien, Podolsky, Rosen
FPE	Fokker-Planck Equation
LHS	Left Hand Side
POM	Positive Operator Measure
PDE	Partial Differential Equation
QMT	Quantum Measurement Theory
QND	Quantum Non-Demolition
RHS	Right Hand Side
RWA	Rotating Wave Approximation
SDE	Stochastic Differential Equation
SME	Stochastic Master Equations
SSE	Stochastic Schrödinger Equation
TLA	Two Level Atom
YDGS	Yu, Diósi, Gisin, Strunz

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Chapter 1

Introduction

The purpose of this introduction is to outline the two central topics of this thesis, non-Markovian stochastic Schrödinger equations (SSE) and interpretations of quantum mechanics. In particular in this thesis I will (or try to) answer the question, which interpretation of quantum mechanics provides the best understanding of non-Markovian SSEs?

A non-Markovian SSE is a non-linear, differential equation for a quantum state $|\psi_z(t)\rangle$ conditioned on some noise function $z(t,s)$ for $s \leq t$. For example, what I call the quadrature non-Markovian SSE (first derived by myself and Wiseman in 2002 [59]) is

$$\begin{aligned} d_t|\psi_z(t)\rangle &= \Big[-\frac{i}{\hbar}\hat{H}_{\rm int}(t) - (\hat{L}+\hat{L}^\dagger - \langle\hat{L}+\hat{L}^\dagger\rangle_t)\hat{Q}_z(t) + \Big\langle(\hat{L}+\hat{L}^\dagger - \langle\hat{L}+\hat{L}^\dagger\rangle_t)\hat{Q}_z(t)\Big\rangle_t \\ &\quad + z(t,t)(\hat{L}-\langle\hat{L}\rangle_t)\Big]|\psi_z(t)\rangle \end{aligned} \tag{1.1}$$

where

$$z(t,s) = z(t_0,s) + \int_{t_0}^{t}\beta(s-t')\langle\hat{L}+\hat{L}^\dagger\rangle_{t'}dt' \tag{1.2}$$

and $z(t_0,s)$ obeys the statistics

$$E[z(t_0,s)] = 0, \tag{1.3}$$
$$E[z(t_0,s),z(t_0,s')] = \beta(s-s'). \tag{1.4}$$

Thus it generates the evolution of a quantum state through time, but what quantum state? As we are all aware universal dynamics in a quantum system is usually governed by a linear, Markovian and deterministic Schrödinger equation,

$$d_t|\Psi(t)\rangle = -\frac{i}{\hbar}\hat{H}_{\rm uni}(t)|\Psi(t)\rangle, \tag{1.5}$$

so what dynamics do non-Markovian SSE describe? Do they represent a new interpretation of quantum mechanics (this is the view of Bassi and Ghirardi [5, 6, 7]) or can they be explained within the current interpretations of quantum mechanics and if so do they shed any new light onto the measurement problem?

To answer these question I am going to break my thesis into two parts. The first part will be concerned with interpretations of quantum mechanics, while the second part will deal with SSEs and their interpretation.

1.1 Interpretations of quantum mechanics

Presently I believe there are only four interpretations of quantum mechanics worth considering. These are the orthodox [13, 57, 82, 101, 131], modal [4, 11, 23, 38, 124, 62, 80, 114, 126], many worlds [36] (or relative state [54, 55, 130]) and the dynamical reduction interpretation [5, 7, 44, 72, 71, 73, 76, 99]. These interpretations all try to solve the measurement problem. This problem concerns how is the reality of a property (for example the position, momentum, etc of a system) defined in the quantum regime. In quantum mechanics it is well known that two non-commuting properties can not be measured simultaneously. That is we can not give definite status to all properties, with out a contradiction (see chapter 3). What I believe the measurement problem refers to is how do we choose which properties will be given definite values, and given this choice can reality be ascribed independently of us. Thus I believe the measurement problem is a two-fold problem, the problem of choice and the problem of reality.

In the orthodox interpretation (see chapter 2 for more detail) to answer the measurement problem we say that the universe is split into two worlds, a quantum (described by $|\Psi(t)\rangle$) and a classical world (described by Lagrangian mechanics). We live in the classical world and reality is only ascribed to the quantum world when we choose to observe it (hence why properties are commonly referred to as observables). Depending on how we intend to observe the system (arrangement of the experimental apparatus) determines what property we will be observing. For example a quantum system can not be ascribed both a position and momentum, until we decide we are going to be observing (say) position, then and only then position receives a definite value while momentum stays undefined.

In the modal interpretation (see chapter 4 for more detail), we introduce an extra quantum state (or hidden variable) to the universe. Thus the universe is now describe by two states, $|\Psi(t)\rangle$ (the guiding wave) and $|\Psi_n(t)\rangle$ (the property state). The property state is stochastic in nature [jumps between different n's with a weighting factor determined by $|\Psi(t)\rangle$] and determines which value of the property is definite. In this interpretation reality is defined independently of us, it exists even when we are not looking at it. That is, reality is objectively real. However, in this interpretation the problem of choice still remains. That is, presently there is no satisfactory answer to the question: what chooses the set of property states? Some variants of the modal theory try to answer choice by proposing different constraints [4, 80, 22, 38, 114, 126] [based on the universal wavefunction $|\Psi(t)\rangle$] which determine what projectors can be given definite status. However, as shown by Wiseman and myself choice is actually a lot bigger problem then previously thought [62].

The next interpretation which I believe is a valid interpretation is the many worlds (or relative state interpretation). In this interpretation the universe is actually a multi-universe. For a measurement with D different results the universe splits into D worlds, with each results occurring in one of these worlds. Thus we can ascribe reality to the property independently of us (we see only one result depending on which world we are in). However like the modal interpretation (which is similar to this view, except instead of having D different worlds, there is D possible property states) choice presently can only be answered partially.

The last interpretation which I believe is a valid interpretation is that based on dynamical reduction models. In this interpretation the wavefunction is considerd to be the basic element of (an objective) reality and to solve the measurement problem the linear Schrödinger equation has to be replaced by a modified stochastic non-linear Schrödinger equation. The three most presented and discussed models are the spontaneous localization [7, 72], the continuous spontaneous localization [7, 44, 71, 73, 76, 99] (CSL), and the gaussian dynamical reduction model [5, 7]. The CSL models result in Markovian SSEs while the gaussian dynamical reduction models result in non-Markovian SSEs.

I am aware that by saying that there are only four interpretations of quantum mechanics is a rather bold statement given the success of Bohmian mechanics [14, 15], but I hope by the end of this thesis I will convince you of this. For example, in chapter 4 I will show that Bohmian mechanics is the continuous limit of modal dynamics.

To summarize, in part I, I will review the orthodox (chapter 2), modal (chapter 4), many worlds (chapter 5), and the dynamical reduction (chapter 5) interpretations as well as briefly describing the contextual and non-local nature of quantum mechanics (chapter 3). Secondly I will generalize the modal theory to include non-orthogonal decompositions, and apply this generalization to a simple system: a harmonic oscillator (chapter 4).

1.2 Stochastic Schrödinger Equations

The second part of my thesis will be devoted to both Markovian and non-Markovian SSEs. Before my work, satisfying interpretations of non-Markovian SSEs did not exist. Diósi, Gisin and Strunz [46, 47, 48, 116, 117] who were the first to propose a diffusive non-Markovian SSE (with complex noise) presented it as a numerical tool for finding the master equation (reduced state evolution equation) of a non-Markovian system. But at the same time they indicated that it should have an interpretation [48]. They have also applied it to complicated systems such as Brownian motion [119, 118]. Cresser has re-derived it from a Heisenberg equation of motion [33] and others have presented non-Markovian SSE (but I believe these should be referred to as post-Markovian SSE), as numerical tools for evaluating perturbative non-Markovian master equations [69, 18]. In 2002 Wiseman and myself [59] generalized non-Markovian SSE to include real noise by illustrating how they could be derived under the orthodox interpretation of quantum mechanics (see chapter 9). At the same time Bassi and Ghirardi [5] simultaneously published a non-Markovian SSE similar to ours, but under the view that it represented a new interpretation of quantum mechanics (a new dynamical reduction model).

Thus it appears, history is repeating itself. By this I mean around the late 80's to early 90's when Markovian SSE first emerged they had many different interpretations. Diósi [44], Pearle [99] and Gisin [73] each proposed that their Markovian SSE represented a new interpretation of quantum mechanics, a dynamical reduction model for continuous spontaneous localization, CSL for short (for a more complete version, with real noise, see Ref. [71] and for complex noise see Refs. [76, 77]). In Ref. [76, 77] Gisin and Percival said that their Markovian SSE (complex noise) represented "the evolution of a quantum system in interaction with its environment"(for a Markovian environment). Ghirardi, Pearle, and Rimini [71] believed their (real noise) Markovian SSE represented the pure state evolution for the same master equation as Gisin and Percival. Thus we already have a problem: how can two different evolution equations represent the true objective quantum evolution for the same master equation.

In 1992 Dalibard, Castin, and Mølmer [34, 95] presented a new type of Markovian SSE (jump-like), which when averaged gives the same average dynamics (Markovian master equation) as Ghirardi, Pearle, and Rimini real noise and Gisin and Percival complex noise Markovian SSE. However they took the view that their method (which they label the Monte-Carlo wave function (MCWF) method) is really just a numerical tool for finding the reduced state of a Markovian system (although they do indicate in Ref. [95] that this method may have an interpretation under quantum measurement theory).

In 1993 Carmichael [28] showed that by introducing a Markovian bath (a collection of harmonic oscillators) the solutions of the real-noise diffusive and jump-like Markovian SSE can be interpreted as trajectories for the system state conditioned on continuous-in-time measurements of the bath. He termed this quantum trajectory theory. He showed that the real noise Markovian SSE correspond to continuous homodyne detection, while the jump-like Markovian SSE corresponds to direct detection. Later Wiseman [137] showed that the Gisin and Percival Markovian SSE corresponds to continuous heterodyne detection of the bath. For a nice review of quantum trajectory and quantum measurement theory see Ref. [134] and for a complete parameterization of diffusive Markovian SSE see Ref. [135].

The success of quantum trajectory theory is that it provides one theory which explains all Markovian SSE, whereas other interpretations, like the dynamical reduction models, cannot provide reasons for preferring one SSE over another. Thus one would have to conclude that quantum trajectory theory provides the best understanding of Markovian SSEs and since this is just an application of the orthodox interpretation to continuous monitoring, we must conclude that the orthodox view provides the best interpretation of Markovian SSEs. In this thesis I will present quantum trajectory theory in more detail in chapter 7.

Due to the success of quantum trajectory theory in explaining Markovian SSEs, in 2002 Wiseman and myself sought a continuous measurement interpretation of non-Markovian SSEs. While we came to the conclusion that non-Markovian SSEs can be generalized and given an interpretation under the orthodox theory, it was not a quantum trajectory type interpretation. We showed that they provide a method for calculating what the system state at time t would be, given a measurement on the bath (a collection of harmonic oscillators) at that time which yielded results $\{r(Z_k,t)=z_k\}$ (in Eq. (1.1) $z(t,t)$ is a function of $\{r(Z_k,t)\}$). The choice of Z_k is determined by what type of bath measurement is performed [59] (see chapter 9).

If we want to give an interpretation to the non-Markovian SSE, rather than a solution at one (and only one) time, we have to consider the modal view of quantum mechanics. Doing this Wiseman and myself [61, 63] have shown that we can give an objective interpretation to non-Markovian SSEs, which is not along the lines of Bassi and Ghirardi dynamical reduction model [5]. Non-Markovian SSEs represent the evolution equation for the system part of the property state of the universe, they also act as a guiding field for the objective bath values $\{\upsilon(Z_k,t)=z_k\}$. This is presented in chapter 9. This result implies that the current interpretations of Markovian SSEs also have to be extended to include this type of objective trajectory.

To summarize this section, chapter 6 is devoted to defining open quantum systems and deriving the master equation for both non-Markovian and Markovian baths. In chapter 7 quantum trajectory theory will be presented in more detail and in chapter 8 I will show a nice application of quantum trajectory theory to quantum state and dynamical parameter estimation. This application will show how under continuous-in-time measurements information concerning both the state of the system and an unknown dynamical parameter can be obtained for five different detection schemes.

In chapter 9 diffusive non-Markovian SSEs will be derived for three different unravelings, the coherent-state, quadrature and position-state. I will also show in this chapter that the modal theory provides the best interpretation of non-Markovian SSEs. In chapter 10 I will conclude part II by presenting the current perturbative techniques used to evaluate non-Markovian SSEs [60, 139, 140]. I will show that in general it turns out that non-Markovian SSEs are not as useful as originally thought [48] in calculating the reduced state (master equation). There are more effective methods, namely the enlarged system method of Imamoḡlu [88, 115].

Finally in chapter 11 I will conclude this thesis and introduce some topics for future work.

Part I

INTERPRETATIONS OF QUANTUM MECHANICS

Chapter 2

The Orthodox Interpretation of Quantum Mechanics

The central aim of this thesis is to investigate the interpretation of non-Markovian SSEs. However, before we can begin down this path, we must first of all provide a general overview of the different interpretations of quantum mechanics. This will be the aim of part I of this thesis. To do this we need to firstly review the orthodox interpretation of quantum mechanics and define the measurement problem within this interpretation.

2.1 Review of quantum mechanics

2.1.1 States and Hilbert spaces

The standard claim of quantum mechanics is that all the information about the system is contained in something called the wave function or quantum state. We represent this by $|\psi(t)\rangle$ (Dirac notation), which is a vector belonging to some Hilbert space ($\mathcal{H}$). Here I am not going to go into all the mathematical detail about this space, except to say it is a complex vector space (satisfies all the requirements of a vector space) and has a scalar (or inner) product between vectors that can have complex numbers as values. That is, it is the complex equivalent of a Euclidean space. For two vectors in $\mathcal{H}$, $|\psi(t)\rangle$ and $|\phi(t)\rangle$ the scalar product is denoted by $\langle\phi(t)|\psi(t)\rangle$, where $\langle\phi(t)|$ is referred to as the dual vector of $|\phi(t)\rangle$. The conditions this scalar product must obey are

$$\langle\phi(t)|\psi(t)\rangle = \langle\psi(t)|\phi(t)\rangle^*, \tag{2.1}$$

$$\langle\phi(t)|[c_1|\psi(t)\rangle + c_2|\chi(t)\rangle] = c_1\langle\phi(t)|\psi(t)\rangle + c_2\langle\phi(t)|\chi(t)\rangle, \tag{2.2}$$

where c_1 and c_2 are complex numbers and $|\chi(t)\rangle$ is a third vector in $\mathcal{H}$. If the quantum states are normalized to one, then the scalar product must also obey

$$1 \geq |\langle\phi(t)|\psi(t)\rangle|^2 \geq 0. \tag{2.3}$$

The scalar product can be though of as a measure of how much the vector $|\psi(t)\rangle$ overlaps (projection onto) the vector $|\phi(t)\rangle$. If this is zero then these vectors are said to be orthogonal and when equal to one they are the same vector (provided our vectors are normalized).

The dimensions D of this space is defined as the maximum number of linearly independent (orthogonal) vectors in $\mathcal{H}$. If there is no maximum number; that is, if there are arbitrarily many linearly independent vectors, then $\mathcal{H}$ is infinite-dimensional. The set of linearly independent vectors (denote $\{|\phi_j(t)\rangle\}$) is referred to as a basis set for the Hilbert space. As in a Euclidean space there are infinite ways we can define this linear set, but they will always contain D basis vectors. When this set is normalized it forms an orthonormal basis, which means

$$\langle\phi_j(t)|\phi_k(t)\rangle = \delta_{j,k}, \tag{2.4}$$

where $\delta_{j,k}$ is a Kronecker delta function. In terms of this orthonormal basis any state $|\psi(t)\rangle$ can be written as

$$|\psi(t)\rangle = \sum_j^D c_j(t)|\phi_j(t)\rangle, \tag{2.5}$$

where $c_j(t) = \langle\phi_j(t)|\psi(t)\rangle$. This defines the identity operator (the operator that maps any vector in $\mathcal{H}$ to itself) as

$$\hat{1} = \sum_j^D |\phi_j(t)\rangle\langle\phi_j(t)|. \tag{2.6}$$

The above is for discrete basis sets, but in quantum mechanics we can also consider continuous basis sets. I will define $\{|\zeta(t)\rangle\}$ to label an arbitrary continuous basis set. Here the time dependence implies a basis set changing in time. In the continuous case (which has $D = \infty$), the basis states are no longer normalized to one, instead we define the orthonormal condition as

$$\langle\zeta(t)|\zeta'(t)\rangle = \delta(\zeta - \zeta'), \tag{2.7}$$

where $\delta(\zeta - \zeta')$ is a Dirac delta function. The identity operator is

$$\hat{1} = \int d\zeta|\zeta(t)\rangle\langle\zeta(t)|, \tag{2.8}$$

which in turn allows us to define any state in terms on this continuous basis as

$$|\psi(t)\rangle = \int d\zeta\ \psi(\zeta, t)|\zeta(t)\rangle, \tag{2.9}$$

where $\psi(\zeta, t) = \langle\zeta(t)|\psi(t)\rangle$. The fact that these basis state are normalized to infinity can sometimes lead to problems, but for the purposed of this thesis this does not occur.

2.1.2 Operators

To obtain the physics from this abstract mathematical space, and hence an understanding of $|\psi(t)\rangle$, we have to define something which we can give a physical meaning to; these are operators. An operator is defined to act in $\mathcal{H}$, it takes a state $|\psi(t)\rangle$ to $|\psi'(t)\rangle$, therefore mathematically speaking it is a map which maps $\mathcal{H}$ to itself. That is, an operator $\hat{A}$ is mathematically defined as

$$\hat{A} : \mathcal{H} \rightarrow \mathcal{H}. \tag{2.10}$$

In general for a discrete basis set, by using the above identity operator, Eq. (2.6), we can rewrite $\hat{A}$ as

$$\hat{A} = \hat{1}\hat{A}\hat{1} = \sum_{j,k}^D A_{j,k}(t)|\phi_j(t)\rangle\langle\phi_k(t)|, \tag{2.11}$$

where $A_{j,k}(t) = \langle\phi_j(t)|\hat{A}|\phi_k(t)\rangle$ is a complex number. For a continuous basis set, Eq. (2.8) allows us to rewrite $\hat{A}$ as

$$\hat{A} = \hat{1}\hat{A}\hat{1} = \int d\zeta \int d\zeta' A(\zeta,\zeta',t)|\zeta(t)\rangle\langle\zeta'(t)|, \tag{2.12}$$

where $A(\zeta,\zeta',t) = \langle\zeta(t)|\hat{A}|\zeta'(t)\rangle$ is a complex function of ζ and ζ'.

Traditionally the set of operators which have real eigenvalues (Hermitian) we assume represent physical observables (or properties) such as position, momentum, and energy. These observables obey $\hat{R} = \hat{R}^\dagger$ and take the diagonal form

$$\hat{R} = \sum_n^D r_n|r_n\rangle\langle r_n|, \tag{2.13}$$

where the set $\{r_n\}$ are the eigenvalues of $\hat{R}$ and $\{|r_n\rangle\}$ are the corresponding eigenstates (which form an orthogonal basis set). This is where the connection between the physical world and this abstract Hilbert space occurs. We say that if the state $|\psi(t)\rangle$ is $|r_n\rangle$ then the physical observable $\hat{R}$ has a definite value, this value being r_n.

However when the state is in a superposition of at least two of these eigenstates, what value does it take? Does it take both or is the value undefined? This question forms what I like to call problem one of the measurement problem. In quantum mechanics the standard answer to this question is to say that both the values were undefined prior to the measurement and upon measurement one of these values is promoted to becoming definite. This is known as collapse of the wavefunction. Thus, it is our observation (measurement) which determines the reality of the observable (hence the use of observable rather than property).

For example, if the quantum state is

$$|\psi(t)\rangle = c_1(t)|r_1\rangle + c_2(t)|r_2\rangle, \tag{2.14}$$

then we postulate a mechanism which collapses this state to $|r_1\rangle$ (or $|r_2\rangle$) with probability $|c_1(t)|^2$ (or $|c_2(t)|^2$) upon measurement of $\hat{R}$. Once in this state the physical observable becomes defined and has the appropriate value r_1 (or r_2). For the general case (arbitrary state) the probability of a collapse occurring from this state to $|r_n\rangle$ is determined by the Born rule [16]; the probability of getting result r_n upon measurement of observable $\hat{R}$ at time t is

$$\Pr(r_n,t) = \langle\psi(t)|r_n\rangle\langle r_n|\psi(t)\rangle. \tag{2.15}$$

For observables that have a continuous basis set as their eigenset, we can write $\hat{R}$ as,

$$\hat{R} = \int r|r\rangle\langle r|dr, \tag{2.16}$$

where r labels the eigenvalue and $|r\rangle$ labels the eigenstate. These have the same physical meaning as in the discrete case, except probabilities are restricted to a range. That is, the probability of getting a results in the range r to $r+\delta r$ upon measurement of observable $\hat{R}$ at time t is

$$\Pr([r,r+\delta r],t) = \int_r^{r+\delta r} P(r',t)dr', \tag{2.17}$$

where $P(r',t)$ is called the probability density and is defined as

$$P(r,t) = \langle\psi(t)|r\rangle\langle r|\psi(t)\rangle. \tag{2.18}$$

In the above argument we have only considered time independent observables. To explain time changing observables we simply place a time dependence on the basis states. That is,

$$\hat{R}(t) = \sum_{n}^{D} r_n |r_n(t)\rangle\langle r_n(t)| \tag{2.19}$$

or

$$\hat{R}(t) = \int dr \; r|r(t)\rangle\langle r(t)|. \tag{2.20}$$

for the continuous case.

What I refer to as the second problem in the measurement problem, occurs when we consider different physical observables. It turns out that only when two physical observables commute

$$[\hat{R}, \hat{R}'] = \hat{R}\hat{R}' - \hat{R}'\hat{R} = 0, \tag{2.21}$$

it is possible to diagonalise both observables in terms of the same set of eigenvectors $\{|r_n\rangle\}$. This means that even if $|\psi(t)\rangle = |r_n\rangle$, not all physical observables have definite status, only those which have $\{|r_n\rangle\}$ as an eigenset do. This is a problem as it says that depending on the observable which we wish to describe one has to choose the basis set in which the collapse occurs. The standard example of this is position ($\hat{Q}$) and momentum ($\hat{P}$). Since $[\hat{Q}, \hat{P}] = i\hbar$, it is impossible to assign both position and momentum a definite status (for any quantum state), and if we chose to measure position (give it a definite status) then momentum becomes undefined. For example if we take an arbitrary quantum state $|\psi(t)\rangle$, in terms of a position basis this can be written as

$$|\psi(t)\rangle = \int c(q,t)|q\rangle dq. \tag{2.22}$$

Then a perfect measurement of position will collapse $|\psi(t)\rangle \to |q\rangle$, thereby yielding a result q with probability $\Pr([q], t) = |c(q,t)|^2 dq$. (Here $[q] \equiv \lim_{\delta q \to dq}[q, q+\delta q]$). It is well known that $|q\rangle$ in terms of a momentum basis is

$$|q\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp \; \exp(-ipq/\hbar)|p\rangle \tag{2.23}$$

which is completely flat in momentum space (each p-state has equal weight). Thus we can't assign $\hat{P}$ a definite value when a perfect measurement of position is made. If now we were to measure momentum we would cause a collapse in momentum state, thereby giving momentum a definite value and making position undefined.

Bohr's [12, 13] resolution to this was to say that position and momentum are complementary properties, and in an experiment one can only arrange a classical apparatus to measure either position or momentum. That is we have to define a classical system which exists outside of the quantum regime that has the ability of measuring the quantum system. He defines this idea by introducing the word phenomenon to describe measurement. "No elementary phenomenon is a phenomenon until it is a registered (observed) phenomenon" [12]. In terms of position and momentum example, this means that until the apparatus registers a result the position and momentum of the system are undefined and when the classical apparatus, for example, is designed to measure position, upon measurement, position receives a definite value, whilst momentum stays undefined.

Bohr's colleague, Heisenberg, however, I believe took a slightly different approach. He believed that it is meaningless to assign a position and momentum to the system, as to acquire information about the position of the system, the apparatus disturbs the momentum (and vice versa) [82, 83]. Nowadays these two belief have been grouped as one and labeled the Copenhagen interpretation of quantum mechanics. In either case a collapse postulate and a mechanism (classical apparatus) which chooses which set of eigenstates this collapse occurs into has to be introduced.

The above is not intended to be a complete review on the Copenhagen interpretation, but just to outline the key points which I will focus on in the rest of this thesis. One important point which I neglected above and should be presented in any review of quantum mechanics is Heisenberg's uncertainty principle. This principle [107] states that for two Hermitian operators $\hat{R}$ and $\hat{R}'$ we can define two parameters ΔR and $\Delta R'$, such that

$$\Delta R \Delta R' \geq \frac{1}{2}|\langle[\hat{R}, \hat{R}']\rangle|, \tag{2.24}$$

where

$$\Delta R^2 = \langle\hat{R}^2\rangle - \langle\hat{R}\rangle^2. \tag{2.25}$$

Here $\langle\hat{R}\rangle$ is defined as the expectation value

$$\langle\hat{R}\rangle = \langle\psi(t)|\hat{R}|\psi(t)\rangle = \sum_n r_n \mathrm{Pr}(r_n, t), \tag{2.26}$$

or

$$\langle\hat{R}\rangle = \int dr\ rP(r,t), \tag{2.27}$$

for a continuous basis. This obviously represents the average result obtained when measuring observable $\hat{R}$. Thus the parameter ΔR^2 represents the variance in this observable. Thus the uncertainty principle can be interpreted as: if we prepared an ensemble of identical states $|\psi(t)\rangle$ and on each element we measured either observable $\hat{R}$ or $\hat{R}'$. Then our uncertainty in these observables is physically restricted via this relation. For example when the commutator is a complex number (like position and momentum) this uncertainty principle defines the lower bound in which an ensemble of measurements can determine the uncertainty in both $\hat{R}$ and $\hat{R}'$. Thus emphasizing that for any state $|\psi(t)\rangle$ two non-commuting variables can never be given definite status.

2.1.3 Time evolution (unitary)

So far we have only considered evolution under measurement, in 1926 Schrödinger proposed a wave equation which determines the evolution of $|\psi(t)\rangle$ [108]. This equation is,

$$d_t|\psi(t)\rangle = -\frac{i}{\hbar}\hat{H}(t)|\psi(t)\rangle, \tag{2.28}$$

where $\hat{H}(t)$ is a Hermitian operator called the Hamiltonian, which, unlike in classical mechanics, is always equal to the total energy. The solution of this equation is

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle \tag{2.29}$$

where

$$\hat{U}(t, t_0) = \hat{1} + \sum_{n=1}^{\infty}(\frac{-i}{\hbar})^n \int_{t_0}^{t} ds_n \hat{H}(s_n) \int_{t_0}^{s_n} ds_{n-1}\hat{H}(s_{n-1}) ... \int_{t_0}^{s_2} ds_1 \hat{H}(s_1) \tag{2.30}$$

is the unitary time-evolution operator (that is, $\hat{U}^\dagger = \hat{U}^{-1}$) and $|\psi(t_0)\rangle$ is the initial state. If $\hat{H}(t)$ is time independent, the above reduces to $\hat{U}(t, t_0) = \exp(-i\hat{H}(t - t_0)/\hbar)$, which is obviously unitary.

Here I would like to point out that in quantum mechanics we can define different pictures to describe evolution. The above picture is referred to the Schrödinger picture. The 2nd picture, the

Heisenberg picture is defined by placing all the time evolution onto the operators. That is, the quantum state in the Heisenberg's picture (labeled with a subscript H) is a constant of motion and equals

$$|\psi_{\rm Hei}(t)\rangle = \hat{U}^\dagger(t,t_0)|\psi(t)\rangle = |\psi(t_0)\rangle. \tag{2.31}$$

Whereas, an Arbitrary operator $\hat{A}(t)$ becomes $\hat{A}_{\rm Hei}(t)$, and is defined as

$$\hat{A}_{\rm Hei}(t) = \hat{U}^\dagger(t,t_0)\hat{A}(t)\hat{U}(t,t_0). \tag{2.32}$$

With this definition we can derive Heisenberg's equation of motion

$$d_t\hat{A}_{\rm Hei}(t) = -\frac{i}{\hbar}[\hat{A}_{\rm Hei}(t),\hat{H}_{\rm Hei}(t)] + \partial_t\hat{A}_{\rm Hei}(t), \tag{2.33}$$

where $\partial_t\hat{A}_{\rm Hei}(t) = \hat{U}^\dagger(t,t_0)\partial_t\hat{A}(t)\hat{U}(t,t_0)$. It should be noted that both these pictures produce the same physical statistics, for example

$$\langle\psi_{\rm Hei}(t)|\hat{A}_{\rm Hei}(t)|\psi_{\rm Hei}(t)\rangle = \langle\psi(t)|\hat{U}(t,t_0)\hat{U}^\dagger(t,t_0)\hat{A}(t)\hat{U}(t,t_0)\hat{U}^\dagger(t,t_0)|\psi(t)\rangle = \langle\psi(t)|\hat{A}(t)|\psi(t)\rangle. \tag{2.34}$$

Another useful picture is the interaction picture. In this picture we split the Hamiltonian $\hat{H}(t)$ into $\hat{H}_0 + \hat{V}(t)$. This allows us to define the quantum state in this picture as

$$|\psi_{\rm int}(t)\rangle = \hat{U}_0^\dagger(t,t_0)|\psi(t)\rangle = \hat{U}_{\rm int}(t,t_0)|\psi(t_0)\rangle. \tag{2.35}$$

where $\hat{U}_0(t,t_0) = \exp[-i\hat{H}_0(t-t_0)/\hbar]$ and $\hat{U}_{\rm int}(t,t_0)$ is defined by

$$\hat{U}(t,t_0) = \hat{U}_0(t,t_0)\hat{U}_{\rm int}(t,t_0). \tag{2.36}$$

In this picture an arbitrary operator $\hat{A}(t)$ becomes $\hat{A}_{\rm int}(t) = \hat{U}_0^\dagger(t,t_0)\hat{A}(t)\hat{U}_0(t,t_0)$, whose time evolution is determined by

$$d_t\hat{A}_{\rm int}(t) = -\frac{i}{\hbar}[\hat{A}_{\rm int}(t),\hat{H}_0] + \partial_t\hat{A}_{\rm int}(t), \tag{2.37}$$

where $\partial_t\hat{A}_{\rm int}(t) = \hat{U}_0^\dagger(t,t_0)\partial_t\hat{A}(t)\hat{U}_0(t,t_0)$.

The evolution of the quantum state ($\hat{U}_{\rm int}(t,t_0)$) is now determined by the Schrödinger equation,

$$d_t|\psi_{\rm int}(t)\rangle = \frac{-i}{\hbar}\hat{V}_{\rm int}(t)|\psi_{\rm int}(t)\rangle, \tag{2.38}$$

where $\hat{V}_{\rm int}(t) = \hat{U}_0^\dagger(t,t_0)\hat{V}(t)\hat{U}_0(t,t_0)$. The general solution for $\hat{U}_{\rm int}(t,t_0)$ is

$$\hat{U}_{\rm int}(t,t_0) = \hat{1} + \sum_{n=1}^{\infty}(\frac{-i}{\hbar})^n\int_{t_0}^{t}ds_n\hat{V}_{\rm int}(s_n)\int_{t_0}^{s_n}ds_{n-1}\hat{V}_{\rm int}(s_{n-1})...\int_{t_0}^{s_2}ds_1\hat{V}_{\rm int}(s_1). \tag{2.39}$$

As in the Heisenberg picture the physical statistics found with this picture will agree with the Schrödinger picture, using the mean example we get

$$\langle\psi_{\rm int}(t)|\hat{A}_{\rm int}(t)|\psi_{\rm int}(t)\rangle = \langle\psi(t)|\hat{U}_0(t,t_0)\hat{U}_0^\dagger(t,t_0)\hat{A}(t)\hat{U}_0(t,t_0)\hat{U}_0^\dagger(t,t_0)|\psi(t)\rangle = \langle\psi(t)|\hat{A}(t)|\psi(t)\rangle. \tag{2.40}$$

As we will see this picture is very useful for investigating the evolution of coupled (entangled) quantum systems, as with this picture we can remove the free "boring" evolution ($\hat{H}_0$) from the Schrödinger equation.

2.1.4 Entanglement

As briefly mentioned above in quantum mechanics when there exists or has existed, a Hamiltonian which couples at least two quantum systems the resulting quantum state can be entangled. For example taking two systems "sys" and "env", where sys stands for the system of interest and env stands for environment (this notation will become clearer later), prior to the coupling Hamiltonian we can write the composite quantum state for the total system as

$$|\Psi(t)\rangle = |\psi(t)\rangle_{\text{sys}} \otimes |\psi'(t)\rangle_{\text{env}}, \tag{2.41}$$

a tensor product of the two quantum systems. That is $|\Psi(t)\rangle$ belongs to a Hilbert space $\mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{env}}$, whereas $|\psi(t)\rangle_{\text{sys}}$ and $|\psi'(t)\rangle_{\text{env}}$ belong to the Hilbert spaces $\mathcal{H}_{\text{sys}}$ and $\mathcal{H}_{\text{env}}$ respectively. Here we have introduced the notation, a capital $|\Psi(t)\rangle$ to refer to states that belong to a composite Hilbert space. In terms of a orthonormal basis Eq. (2.5) this can be written as

$$|\Psi(t)\rangle = \sum_{j}^{D_{\text{sys}}} \sum_{k}^{D_{\text{env}}} c_j(t) c'_k(t) |\phi_j(t)\rangle_{\text{sys}} \otimes |\phi'_k(t)\rangle_{\text{env}}. \tag{2.42}$$

Now if a coupling Hamiltonian exists between the system and bath then the above composite state will evolve to $|\Psi(t')\rangle$, which is no longer separable. In terms of the same orthogonal basis as in Eq. (2.42) this is

$$|\Psi(t')\rangle = \sum_{j}^{D_{\text{sys}}} \sum_{k}^{D_{\text{env}}} c_{j,k}(t') |\phi_j(t')\rangle_{\text{sys}} \otimes |\phi'_k(t')\rangle_{\text{env}}, \tag{2.43}$$

where the coefficients of the state $c_{j,k}(t')$ are no longer separable. For a continuous orthonormal basis the general form of an entangled state is

$$|\Psi(t')\rangle = \int d\zeta d\zeta' \;\psi(\zeta, \zeta', t') |\zeta(t')\rangle_{\text{sys}} |\zeta'(t')\rangle_{\text{env}}, \tag{2.44}$$

where $\psi(\zeta, \zeta', t')$ is non-separable.

An example of an entangled state is the passing of a single photon through a beam splitter. Prior to the introduction of a beam splitter the quantum state of this single photon is $|\Psi(t)\rangle = |1\rangle_{\text{A}}|0\rangle_{\text{B}}$ where $|1\rangle_{\text{A}}$ and $|0\rangle_{\text{B}}$ refer to the photon in mode A and mode B having no photon. After the introduction of the beam splitter the quantum state experience a Hamiltonian of the form,

$$\hat{H}_{\text{Bs}} = i\hbar\alpha(\hat{a}\hat{b}^\dagger - \hat{a}^\dagger\hat{b}) \tag{2.45}$$

where $\hat{a}$ ($\hat{a}^\dagger$) and $\hat{b}$ ($\hat{b}^\dagger$) are the annihilation (creation) operators form mode A and B respectively and α is determine by the reflective strength of the beam splitter. This has the effect of evolving the initial separable state to

$$|\Psi(t')\rangle = \cos[\alpha(t'-t)]|1\rangle_{\text{A}}|0\rangle_{\text{B}} + \sin[\alpha(t'-t)]|1\rangle_{\text{A}}|0\rangle_{\text{B}}. \tag{2.46}$$

For an appropriately chosen α and interaction time this becomes

$$|\Psi(t')\rangle = [|1\rangle_{\text{A}}|0\rangle_{\text{B}} + |1\rangle_{\text{A}}|0\rangle_{\text{B}}]/\sqrt{2}\,. \tag{2.47}$$

This state is obversely entangled as it shows after the introduction of the beam splitter the total state is a superposition of the photon being in both modes at once, and by performing a measurement on mode A, we can predict with certainty the results of a measurement on mode B. This type of state is referred to as an EPR state, after Einstein, Podolsky and Rosen who in 1935 presented the EPR paradox [51]. This paradox starts by defining the following criteria of reality: "If without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to 1) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity" [51]. Thus, the above example, under this criteria, implies we can give a reality to the occupation number of mode B prior to performing a measurement on system B (a measurement on A tells us what the occupation number of B is). EPR claimed this showed the incompleteness of quantum mechanics. To get around this paradox, nowadays most physicists accept that quantum mechanics is non-local, and that a measurement on one mode instantaneously measures the second mode (I am not sure if the founding farther's of quantum mechanics would have accepted these ideas). This might at first sound like we can send useful information faster than the speed of light, but for a second person to figure out what the value of B is (more correctly how to measure B to get this value) they need to know what measurement we performed on A. This classical information is restricted to the speed of light.

Entanglement is not only interesting in explaining non-classical correlations between quantum systems, but as pointed out by Von Neumann [129] it should be considered when explaining quantum measurements. The Schrödinger equation forbids us from mathematically proposing a difference between the system and the meter attached to it. Thus when describing a system and meter (measuring apparatus) in quantum mechanics we should first of all consider the composite state $|\Psi(t)\rangle = |\psi(t)\rangle_{\rm sys}|m_0\rangle_{\rm env}$ where $|m_0\rangle_{\rm env}$ labels the ready state of meter and $|\psi(t)\rangle_{\rm sys}$ labels the state of the system. We now say that during the measurement the different states of the system couple linearly to the different orthogonal states of the meter. This results in the entangled state

$$|\psi(t)\rangle_{\rm sys}|m_0\rangle_{\rm env} \to \sum_j^D c_j(t)|\phi_j(t)\rangle_{\rm sys}|m_j(t)\rangle_{\rm env} \tag{2.48}$$

which is commonly referred to as a bi-orthogonal (or Schmidt) decomposition [111]. Just before the end of the measurement a collapse occurs in the meter Hilbert space. This then causes

$$\sum_j^D c_j(t)|\phi_j(t)\rangle_{\rm sys}|m_j(t)\rangle_{\rm env} \to |\phi_j(t)\rangle_{\rm sys}|m_j(t)\rangle_{\rm env}, \tag{2.49}$$

Thus the system is promoted to having the $j^{\rm th}$ definite value with probability $|c_j(t)|^2$.

In actuality this argument can be extended up the von Neumann chain to include the person looking at the meter and so forth. That is, under the orthodox theory there is no physical equation which governs the line between a quantum system and a classical apparatus. But as we are all aware we don't see classical objects in superpositions. This is precisely what was implied by Schrödinger in his cat paradox [109]. This is actually an extension on what I call problem two of the measurement problem. Thus I will restate problem 2 as: under the orthodox theory, what determines the type of measurement being performed (arrangement of the classical apparatus) and where does the distinction between the classical world and the quantum world occur; where is the Heisenberg cut [83]?

Wigner in 1961 [132] proposed that it occurs when a conscience observer, enters the chain. I don't really believe in this view as I feel science is not ready to introduce the mind into physics. In this thesis when explaining things under the orthodox theory I am going to take the view similar to Wheeler "A phenomena in not yet a phenomenon until it has been brought to a close by an irreversible act of amplification" [131] (an example of this is the cascade effect of electron in a photo-detector). In this thesis I will generally assume that this irreversible act occurs when the bath (immediate environment) of a quantum system interacts with a classical detector. By this I mean we have a system of interest (for example a two level atom) immersed in a bath (for example the electromagnetic field) and we perform measurements on the bath (the collapse occurs here). This in turn results (by for example Eq. (2.48)) in a measurement on the system. In the next section I will discuss this measurement process in more detail and show that Eq. (2.48) is not general enough. But before I go on to talk about this theory in more detail, I feel in any review chapter it is necessary to talk about mixed states.

2.1.5 Mixed states

So far we have only considered pure states, but in quantum mechanics we can have states which are mixed. To explain these states we have to introduce the statistical operator, $\rho(t)$, (sometimes known as state matrix or density operator). This operator is the only operator I will not denote with a hat. This is because it is an alternative way of labeling a quantum state. The statistical operator is a positive operator (all its eigenvalues are non-negative) and it is normalized by

$$\mathrm{Tr}[\rho(t)] = 1. \tag{2.50}$$

For pure states, $|\psi(t)\rangle$, the statistical operator takes the form $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$. But the usefulness of this state matrix is fully displayed when we consider classical mixtures of pure states. If for example, we assume there is some preparation device which prepares pure quantum states $|\psi_\nu(t)\rangle$ with probability $\mathrm{Pr}(\nu, t)$, then the statistical operator is defined as the ensemble average of the pure states. That is,

$$\rho(t) = E[|\psi_{r(\nu,t)}(t)\rangle\langle\psi_{r(\nu,t)}(t)|], \tag{2.51}$$

where $r(\nu, t)$ refers to a random variable associated with the distribution $\mathrm{Pr}(\nu, t)$ and E refers to an ensemble average. This is equivalent to

$$\rho(t) = \sum_\nu \mathrm{Pr}(\nu, t)|\psi_\nu(t)\rangle\langle\psi_\nu(t)|. \tag{2.52}$$

The "mixedness" of $\rho(t)$ can be defined in many ways, in this thesis I will use the purity, $p(t)$, as a measure of mixedness. It is defined as

$$p(t) = \mathrm{Tr}[\rho^2(t)]. \tag{2.53}$$

The purity is bonded by the upper limit of 1, which represents a pure state (completely unmixed state), and a lower limit of $1/D$ for $\rho(t) = \hat{1}/D$ (a completely mixed state).

The statistical operator allows us to define an expectation value for an observable $\hat{R}$ as

$$\langle\hat{R}\rangle = \sum_\nu \mathrm{Pr}(\nu, t)\langle\psi_\nu(t)|\hat{R}|\psi_\nu(t)\rangle = \mathrm{Tr}[\hat{R}\rho(t)]. \tag{2.54}$$

The above is only one way mixed states can be defined. It is usually referred to as proper mixtures. Improper mixtures arises when we trace over a larger system. That is, an improper mixed state is defined as

$$\rho(t) = \mathrm{Tr}_{\mathrm{env}}[|\Psi(t)\rangle\langle\Psi(t)|] \tag{2.55}$$

where $|\Psi(t)\rangle$ is the composite state for the system and bath. This is commonly referred to as the reduced state and form now on I will label this by $\rho_{\mathrm{red}}(t)$. With this mixed state a system observable $\hat{R}_{\mathrm{sys}}$ has an expectation value given by

$$\langle\hat{R}\rangle = \langle\Psi(t)|\hat{R}_{\mathrm{sys}} \otimes \hat{1}_{\mathrm{env}}|\Psi(t)\rangle = \mathrm{Tr}[\hat{R}_{\mathrm{sys}}\rho_{\mathrm{red}}(t)]. \tag{2.56}$$

2.2 Quantum measurement theory

Above I have presented a very limited introduction to quantum measurement theory, by only introducing the concept of a collapse. Here I will present in much more detail quantum measurement theory. This will be done by firstly presenting von Neumann projective measures and then proceeding on to describe positive operator measures (POMs). After this I will outline linear quantum measurement theory.

2.2.1 Projector measures

Direct projective measurements

In the above review I only talked about measurement of Hermitian operators by introducing briefly the concept of wavefunction collapse. This measurement actually corresponds to projective measurements. To explain projective measurements we have to define the projector $\hat{\pi}_n(t)$. $\hat{\pi}_n(t)$ is Hermitian ($\hat{\pi}_n(t) = \hat{\pi}_n^\dagger(t)$) and only has eigenvalues of 0 and 1. The set of projectors is also defined to satisfy the orthogonality condition,

$$\hat{\pi}_n(t)\hat{\pi}_m(t) = \hat{\pi}_n(t)\delta_{n,m} \tag{2.57}$$

and the completeness condition

$$\sum_n \hat{\pi}_n(t) = \hat{1}. \tag{2.58}$$

In terms of this projector any physical observable $\hat{R}$ can be written as

$$\hat{R} = \sum_n r_n \hat{\pi}_n. \tag{2.59}$$

Comparing this with Eq. (2.13) we see that for observable $\hat{R}$, the projector is $\hat{\pi}_n = |r_n\rangle\langle r_n|$. This actuality represents a special case of projectors, rank one projectors. If for example our physical observable had some degenerate eigenvalues (eg $r_1 = r_2$) the projector notation still allows us to represent the observable by Eq. (2.59) (the sum would contain less elements), but the projectors would not be rank one. For example, if $r_1 = r_2$ the first projector would be rank 2 and of the form $\hat{\pi}_1 = |r_1\rangle\langle r_1| + |r_2\rangle\langle r_2|$. In general a projector has the form $\hat{\pi}_n(t) = \sum_j^{N_n} |r_n, j\rangle\langle r_n, j|$, where N_n represents the number of degenerate eigenvalues r_n.

Before going on to explain measurements, I would like to note that we can actually measure operators which are not Hermitian. These operators are called normal operators [97]. I will denote these by $\hat{Z}$. A normal operator is defined such that it satisfies $[\hat{Z}, \hat{Z}^\dagger] = 0$. That is,

$$\hat{Z}|z_n\rangle = z_n|z_n\rangle, \tag{2.60}$$

where $\{|z_n\rangle\}$ is a set of orthonormal basis states and $\{z_n\}$ is the set of complex eigenvalues. With this set we can write the normal operator as

$$\hat{Z} = \sum_n z_n|z_n\rangle\langle z_n| = \sum_n z_n\hat{\pi}_n, \tag{2.61}$$

which like above can be generalized, by including a second indies j, to include degenerate eigenvalues. It can be shown that all normal operators can be written in the form $\hat{Z} = \hat{R} + i\hat{R}'$, where $\hat{R}$ and $\hat{R}'$ are commuting Hermitian operators (can be measured simultaneously). Thus a normal operator represents two physical observables which can be measured simultaneously with an eigenvalue z_n which lies in a complex plane, rather than just the real plane.

We can also argue that the result of a measurement does not have to be a number (or complex number), it could be a string of numbers or even a statement, yes or no. Thus for completeness from now on I will label an arbitrary observable by Z. When this observable is described by a projective measurement it has the general form

$$Z(t) = \{(z_n, \hat{\pi}_n(t))\}. \tag{2.62}$$

That is, it is a set of paired elements containing both the results of the n measurements ($\{z_n\}$) and the projector operators for these results $\{\hat{\pi}_n(t)\}$. The reason for choosing this notation will become clearer when we consider POM measurements in the next section (Sec. 2.2.2). To find the moments of this observable one can use the following rule

$$\langle Z^m(t)\rangle = \sum_n (z_n)^m \langle\psi(t)|\hat{\pi}_n(t)|\psi(t)\rangle = \sum_n (z_n)^m \mathrm{Pr}(z_n, t). \tag{2.63}$$

Here I have defined the Born rule in terms of the projectors as

$$\mathrm{Pr}(z_n, t) = \langle\psi(t)|\hat{\pi}_n(t)|\psi(t)\rangle. \tag{2.64}$$

This reads as the probability of getting result z_n at time t, when a measurement described by the projective measure $\{\hat{\pi}_n(t)\}$ is performed. It is effectively a measure of the likelihood of outcome z_n. I would like to note here that the time dependence on the projectors physically corresponds to the classical apparatus changing in time what it measures (its experimental arrangement).

When we measure $Z(t)$ the state of the system after the measurement is

$$|\psi_n(t)\rangle = \frac{\hat{\pi}_n(t)|\psi(t)\rangle}{\sqrt{N}}, \tag{2.65}$$

where $|\psi(t)\rangle$ is the state before the measurement, and N is the normalization constant. Here we interpret $|\psi_n(t)\rangle$ as the state conditioned on the measurement, so we call it the conditioned state. The normalization constant by definition is

$$N = \langle\psi(t)|\hat{\pi}_n(t)|\psi(t)\rangle. \tag{2.66}$$

Thus $N = \mathrm{Pr}(z_n, t)$ by Eq. (2.64). In terms of the set of conditioned states we can write $|\psi(t)\rangle$ as

$$|\psi(t)\rangle = \sum_n \sqrt{\mathrm{Pr}(z_n, t)}\,|\psi_n(t)\rangle. \tag{2.67}$$

For measurement of a non degenerate Hermitian operator $\hat{R}$, the projector $\hat{\pi}_n$ will be rank one and equal to $|r_n\rangle\langle r_n|$. Thus the conditioned state for result r_n will be

$$|\psi_n(t)\rangle = |r_n\rangle, \tag{2.68}$$

which is equivalent to the collapse examples considered earlier. Thus by introducing the projector notation we can explain this simple rank one measurement as well as measurement of degenerate observables.

One interesting point to note about quantum measurements is that if we repeate a projective measurement then by Eq. (2.57) we always obtain the same result as the first measurement. This may not sound so interesting, but if there is some unitary evolution acting on the system, we would naively expect our results to change in time. However, if the repeated measurement time is faster than the unitary evolution time, the above shows that later measurement results will be the same as the first measurement result (as with high probability we keep collapsing the conditional state into the same state when we perform the next measurement). This to me clearly points in favour of the idea that the measurement creates the reality. This effect is known as the Zeno effect, and for a good review see chapter 12 of Ref. [101].

When the observables $Z(t)$ represents a continuous spectrum, rather than the above discrete version, we need to modify the above slightly. To represent the continuous spectrum, we define a projector density $\hat{\pi}(z,t)$. This satisfies the completeness condition,

$$\int dz\ \hat{\pi}(z,t) = \hat{1}, \tag{2.69}$$

and the orthogonal condition

$$\hat{\pi}(z,t)\hat{\pi}(z',t) = \hat{\pi}(z,t)\delta(z-z'). \tag{2.70}$$

The probability density Eq. (2.18), in terms of $\hat{\pi}(z,t)$ is

$$P(z,t) = \langle\psi(t)|\hat{\pi}(z,t)|\psi(t)\rangle. \tag{2.71}$$

To define a probability, as above we need to define a range $z_n + \delta z$. The probability of getting a result in this range (similar to Eq. (2.17)) is

$$\Pr([z_n, z_n+\delta z], t) = \int_{z_n}^{z_n+\delta z} dz\ P(z,t) = \langle\psi(t)|\hat{\pi}_n(t)|\psi(t)\rangle. \tag{2.72}$$

where

$$\hat{\pi}_n(t) = \int_{z_n}^{z_n+\delta z} dz\hat{\pi}(z,t) \tag{2.73}$$

forms a set of orthogonal projectors $\{\hat{\pi}_n(t)\}$. Taking the continuous limit $\delta z \to dz$ this becomes $\hat{\pi}_n(t) = \hat{\pi}(z_n,t)dz$, which we label the "infinitesimal" projector. This infinitesimal projector obeys the same conditions as the standard projector, Eqs. (2.57) and (2.58) and the conditioned state for this infinitesimal projector is given by Eq. (2.65).

So far we have only talked about projective measurements that assumed a pure initial state. What about mixed states? If we have an initial mixed state, the only difference will be when we measure $\hat{Z}(t)$ the state of the system after the measurement is

$$\rho_n(t) = \frac{\hat{\pi}_n(t)\rho(t)\hat{\pi}_n(t)}{\Pr(z_n,t)}, \tag{2.74}$$

where $\rho(t)$ is the mixed state before the measurement, $\rho_n(t)$ is the mixed state after the measurement (the conditioned state matrix) and the probability (normalization constant) is defined as

$$\Pr(z_n,t) = \mathrm{Tr}[\rho(t)\hat{\pi}_n(t)]. \tag{2.75}$$

For rank one projectors, $\hat{\pi}_n(t) = |z_n(t)\rangle\langle z_n(t)|$ the conditioned state matrix will be pure and equal to the projector. For higher ranks this will not be the case.

Indirect projective measurements

In the above I presented quantum measurement theory for systems in which we could effectively make the Heisenberg cut after the system. In this subsection I will consider the situation where the Heisenberg cut can not be placed here, it must be placed later. I will assume that it can be placed between the bath and the apparatus. That is, I will consider measurements on open quantum systems. An open quantum system refers to as a composite system comprising of a "system of interest" immersed in a bath (or environment). The typical example is a TLA immersed in the electromagnetic field. In this example the quantum measurement occurs on the electromagnetic field. The composite state, $|\Psi(t)\rangle$, for this system is described by Eq. (2.43) (or Eq. (2.44)).

Since measurement only occurs on the bath the projective measure is $\{\hat{\pi}_n(t)_{\rm env}\otimes\hat{1}_{\rm sys}\}$, where $\hat{\pi}_n(t)_{\rm env}$ acts only in the Hilbert space of the bath. For bath observable $Z(t)$ [described by Eq. (2.62)] the probability of measuring result z_n at time t is

$$\Pr(z_n,t)=\langle\Psi(t)|\hat{\pi}_n(t)_{\rm env}\otimes\hat{1}_{\rm sys}|\Psi(t)\rangle. \tag{2.76}$$

Upon measurement the conditioned composite state, $|\Psi_n(t)\rangle$, will be

$$|\Psi_n(t)\rangle=\frac{\hat{\pi}_n(t)_{\rm env}|\Psi(t)\rangle}{\sqrt{\Pr(z_n,t)}}, \tag{2.77}$$

where $|\Psi(t)\rangle$ is the state before the measurement, and because of the completeness condition for projectors is equal to

$$|\Psi(t)\rangle=\sum_n\sqrt{\Pr(z_n,t)}\,|\Psi_n(t)\rangle. \tag{2.78}$$

For rank one projectors, $\hat{\pi}_n(t)_{\rm env}=|z_n(t)\rangle_{\rm env}\langle z_n(t)|$, we can rewrite this conditioned composite state as

$$|\Psi_n(t)\rangle=|z_n(t)\rangle_{\rm env}\otimes|\psi_n(t)\rangle_{\rm sys}, \tag{2.79}$$

where $|\psi_n(t)\rangle_{\rm sys}$ is called the conditioned system state and is defined by

$$|\psi_n(t)\rangle_{\rm sys}=\frac{{}_{\rm env}\langle z_n(t)|\Psi(t)\rangle}{\sqrt{\Pr(z_n,t)}}. \tag{2.80}$$

Here we see that by performing a rank one projective measurement on the bath the system is projected into a pure state. For these projectors we can rewrite Eq. (2.78) as

$$|\Psi(t)\rangle=\sum_n\sqrt{\Pr(z_n,t)}\,|\psi_n(t)\rangle_{\rm sys}|z_n(t)\rangle_{\rm env}. \tag{2.81}$$

which is the more general form of the measurements I briefly talked about in section 2.1.4.

We can define a (proper) mixed state for the system as the average of this conditioned system states,

$$\rho(t)=\sum_n\Pr(z_n,t)|\psi_n(t)\rangle_{\rm sys}\langle\psi_n(t)|. \tag{2.82}$$

That is, this state represents an ensemble of conditioned system states, which we have obtained by ignoring the results of the bath measurements.

Since the basis set $|z_n(t)\rangle$ forms a complete basis set, we can rewrite the reduced state Eq. (2.55) (an improper mixed state) as

$$\rho_{\rm red}(t)={\rm Tr}_{\rm env}[|\Psi(t)\rangle\langle\Psi(t)|]=\sum_n\ {}_{\rm bath}\langle z_n(t)|\Psi(t)\rangle\langle\Psi(t)|z_n(t)\rangle_{\rm bath}, \tag{2.83}$$

which equals Eq. (2.82) by Eqs. (2.76) and (2.80). Thus one interpretation of the reduced state is, it represents the average state (ensemble of conditioned states) given that rank one measurements have been performed on the bath. It should be noted that since $Z(t)$ represents any arbitrary observable, all conditioned system states must average to $\rho_{\rm red}(t)$. That is, there are infinitely many ways (by choice of $\{\hat{\pi}_n(t)\}$) one reduced state can be obtained from conditioned states. These different ways are referred to as an unraveling. That is, an unraveling in the orthodox view corresponds to which bath observable is being measured.

In the continuous limit the bath projectors become the infinitesimal projectors

$$\hat{\pi}_n(t)_{\rm env} = \hat{\pi}(z_n, t)_{\rm env} dz = |z(t) = z_n\rangle_{\rm env}\langle z(t) = z_n| dz. \tag{2.84}$$

This only has a notational effect on the conditioned system states, Eq. (2.80) for the continuous case should read

$$|\psi_n(t)\rangle_{\rm sys} = \frac{{}_{\rm env}\langle z(t) = z_n|\Psi(t)\rangle}{\sqrt{\Pr([z_n], t)}}, \tag{2.85}$$

where $\Pr([z_n], t)$ now reads as the probability of measuring a bath result in the infinitesimal range $[z_n, z_n + dz]$ at time t.

2.2.2 Positive Operator Measures (POMs)

Direct POM measurements

Above we only considered projective measurements. Actually we can have measurement into bases, which are overcomplete (not orthogonal). Examples of these are informationally complete POMs [30] and the Husimi (or Q-function) POM [87]. Other examples (which are not rank one POM) of measurements described by POMs can be found in Refs. [84, 91]. All of these measurement represent a measurement of the observable

$$Z(t) = \{(z_n, \hat{F}_n(t))\} \tag{2.86}$$

where $\hat{F}_n(t)$ is referred to as a POM element [84] or effect [91]. The POM elements, like the projectors, must be non-negative and Hermitian. They must also be complete,

$$\sum_n \hat{F}_n(t) = \hat{1}. \tag{2.87}$$

However they are not necessarily orthogonal (a POM with orthogonal POM elements is a projective measure). Actually because they are not orthogonal we can't represent this observable by an equation similar to Eq. (2.61). That is, they represent observables which are not normal operators. Thus the importance of the notation presented in Eqs. (2.62) and (2.86) should be clearer.

If the system is in the quantum state $|\psi(t)\rangle$, then the moments of $Z(t)$ are given by

$$\langle Z^m(t)\rangle = \sum_n (z_n)^m \langle\Psi(t)|\hat{F}_n(t)|\psi(t)\rangle = \sum_n (z_n)^m \Pr(z_n, t), \tag{2.88}$$

where

$$\Pr(z_n, t) = \langle\psi(t)|\hat{F}_n(t)|\psi(t)\rangle. \tag{2.89}$$

As with projector measurements, the probability is defined as a measure, between 0 and 1, of how likely it is that the system will yield the result z_n upon measurement.

To calculate the conditional state for measurement of these observables, we have to decompose $\hat{F}_n(t)$ into measurement operators $\hat{M}_n(t)$ (Kraus Operators) [91]. This decomposition must satisfy

$$\hat{F}_n(t) = \hat{M}_n^\dagger(t)\hat{M}_n(t). \tag{2.90}$$

With this measurement operator the conditioned state is given by

$$|\psi_n(t)\rangle = \frac{\hat{M}_n(t)|\psi(t)\rangle}{\sqrt{\Pr(z_n, t)}}, \tag{2.91}$$

Here we see that in general (as $\hat{M}_n(t)$ doesn't necessarily obey a completeness condition) $|\psi(t)\rangle \neq \sum_n \sqrt{\Pr(z_n,t)}\,|\psi_n(t)\rangle$. I would also like to mention that this is not completely general as here I have only considered perfect measurements.

When describing measurement into an overcomplete basis the POM elements will take the general form

$$\hat{F}_n(t) = \frac{1}{N}|z_n(t)\rangle\langle z_n(t)|, \tag{2.92}$$

where N is a normalization constant (which in some circumstance could be time and n dependent) chosen such that $\hat{F}_n(t)$ obeys Eq. (2.87). With this effect the only decomposition for the measurement operators is

$$\hat{M}_n(t) = \frac{1}{\sqrt{N}}|\chi_n(t)\rangle\langle z_n(t)| \tag{2.93}$$

where $|\chi_n(t)\rangle$ labels a completely arbitrary state. Thus with the measurement operators we can explain not only measurements into non-orthogonal decomposition but also measurements which can completely change the state of the system. An example of this is measurement of a photon (a single mode harmonic oscillator). When we measure a single photon we usually say that during the measurement the photon is completely absorbed by the detector, thereby collapsing the quantum state to the vacuum state. Here $|\chi_n(t)\rangle = |0\rangle$ for all n.

For completeness of this section I will briefly outline POM measurements for mixed states. If we have an initial mixed state, then when we measure $Z(t)$ the conditioned state matrix after the measurement is

$$\rho_n(t) = \frac{\hat{M}_n(t)\rho(t)\hat{M}_n^\dagger(t)}{\Pr(z_n, t)}, \tag{2.94}$$

where the probability (normalization constant) is defined as

$$\Pr(z_n, t) = \mathrm{Tr}[\rho(t)\hat{F}_n(t)]. \tag{2.95}$$

For a POM element of the form displayed in Eq. (2.92) the conditional state will be pure and equal $|\chi_n(t)\rangle\langle\chi_n(t)|$.

When the observables $Z(t)$ has a continuous spectrum we can define POM element densities $\hat{F}(z,t)$, which satisfy an integral completeness condition. Taking the infinitesimal limit these densities can be written as POM elements of the form $\hat{F}_n(t) = \hat{F}(z_n,t)dz$, which satisfy all the above equations for POMs, with the only difference being the probability should be written as $\Pr([z],t)$ and interpreted as the probability of measuring a result z in the infinitesimal range $[z_n, z_n + dz]$ at time t.

Indirect POM measurements

When considering open quantum systems and performing POM measurements on the bath, we consider the POMs of the form $\{\hat{F}_n(t)_{\text{env}} \otimes \hat{1}_{\text{sys}}\}$, where $\hat{F}_n(t)_{\text{env}}$ acts only in the Hilbert space of the bath. For bath observable $Z(t)$ [described by Eq. (2.86)] the probability of measuring result z_n at time t is

$$\Pr(z_n, t) = \langle\Psi(t)|\hat{F}_n(t)_{\rm env} \otimes \hat{1}_{\rm sys}|\Psi(t)\rangle. \tag{2.96}$$

Upon measurement the conditioned composite state, $|\Psi_n(t)\rangle$, will obey

$$|\Psi_n(t)\rangle = \frac{\hat{M}_n(t)_{\rm env}|\Psi(t)\rangle}{\sqrt{\Pr(z_n, t)}}, \tag{2.97}$$

where $\hat{M}_n(t)_{\rm env}$ satisfies Eq. (2.90). For POM elements described by Eq. (2.92), which have measurement operators of the form depicted in Eq. (2.93), we can rewrite this conditioned composite state as

$$|\Psi_n(t)\rangle = |\chi_n(t)\rangle_{\rm env} \otimes |\psi_n(t)\rangle_{\rm sys}, \tag{2.98}$$

where $|\psi_n(t)\rangle_{\rm sys}$ is called the conditioned system state and obeys

$$|\psi_n(t)\rangle_{\rm sys} = \frac{{}_{\rm env}\langle z_n(t)|\Psi(t)\rangle}{\sqrt{N\ \Pr(z_n, t)}}, \tag{2.99}$$

which is similar to Eq. (2.80). The only difference is that $\{|z_n(t)\rangle\}$ can refer to both an orthonormal and overcomplete basis set. Because $\{|z_n(t)\rangle_{\rm env}\}$ still forms a basis set the reduced state (as in the projector case) can be written as the ensemble average of the conditioned system state $|\psi_n(t)\rangle_{\rm sys}$ (Eq. (2.82)).

For cases when $|\chi_n(t)\rangle_{\rm env} = |z_n(t)\rangle_{\rm env}$ (a quantum non-demolition or QND measurement) we can write

$$|\Psi(t)\rangle = \sum_n \frac{\sqrt{\Pr(z_n, t)}\,|\Psi_n(t)\rangle}{\sqrt{N}} = \sum_n \frac{\sqrt{\Pr(z_n, t)}\,|\psi_n(t)\rangle_{\rm sys}|z_n(t)\rangle_{\rm env}}{\sqrt{N}} \tag{2.100}$$

where both $\{|\psi_n(t)\rangle_{\rm sys}\}$ and $\{|z_n(t)\rangle_{\rm env}\}$ are not necessarily orthogonal basis states.

2.2.3 Linear quantum measurement theory

In this section I am going to briefly outline linear quantum measurement theory, as it is a useful technique for deriving linear stochastic Schrödinger equations. Linear quantum measurement theory uses the same principles as quantum measurement theory except we use an ostensible probability distribution, $\Lambda\mathrm{r}(z_n)$) (which can be chosen time independent), in place of the actual probability distribution [134, 79]. As its name suggests, the ostensible probability distribution need bear no relation to the actual probability distribution. However, it must be a proper probability distribution (non-negative, and sum to unity), and must be non-zero wherever the actual probability distribution is non-zero. Using the ostensible probability distribution, and the measurement operators defined in Eq. (2.93), the conditioned system state is

$$|\bar{\psi}_n(t)\rangle_{\rm sys} = \frac{{}_{\rm env}\langle z_n|\Psi(t)\rangle}{\sqrt{N\Lambda r(z_n)}}. \tag{2.101}$$

We will call this the linear conditioned system state, because it depends linearly on the pre-measurement state $|\Psi(t)\rangle$, unlike Eq. (2.99). Since $\Lambda\mathrm{r}(z_n)$ is not equal to the actual probability, $|\bar{\psi}_n(t)\rangle_{\rm sys}$ will not be normalized to 1 and to signify this we use a bar above the state. Here I have considered only the composite system because the application to the single system is straight forward.

Because the linear conditioned systems state is unnormalized, it does not have a clear physical interpretation. However, it still is useful as it is easier to calculate (involving only linear equations), and we can write

$$\begin{aligned} \rho_{\rm red}(t) &= \sum_n \frac{{}_{\rm env}\langle z_n(t)|\Psi(t)\rangle\langle\Psi(t)|z_n(t)\rangle_{\rm env}}{N} \\ &= \sum_n \Lambda{\rm r}(z_n)|\bar{\psi}_n(t)\rangle_{\rm sys}\langle\bar{\psi}_n(t)|. \end{aligned} \tag{2.102}$$

Thus the reduced state can also be found by averaging over an ensemble of linear conditioned system state.

The actual probability $\Pr(z_n, t)$ is related to the ostensible probability by the Girsanov transformation [70]. This is

$$\Pr(z_n, t) = {}_{\rm sys}\langle\bar{\psi}_n(t)|\bar{\psi}_n(t)\rangle_{\rm sys}\Lambda{\rm r}(z_n), \tag{2.103}$$

which follows from Eqs. (2.101) and (2.96).

2.3 Representing bath measurements by system POMs

In the above I have presented quantum measurement theory for both projectors and POMs when the Heisenberg cut can be placed after the system and after the bath. Here I am going to show that when considering the second case (after the bath) we can represent all rank one projective and rank one POM measurements (of the form depicted in Eqs. (2.92) and (2.93)) on the bath by POM measurements on the system. To do this we rewrite the conditioned composite state, Eq. (2.97) as

$$|\Psi_n(t)\rangle = \frac{\hat{M}_n(t)_{\rm env}\hat{U}(t,t_0)|m(t_o)\rangle_{\rm env}|\psi(t)\rangle_{\rm sys}}{\sqrt{\Pr(z_n,t)}}, \tag{2.104}$$

where $|\Psi(t)\rangle = \hat{U}(t,t_0)|m(t_0)\rangle_{\rm env}|\psi(t_0)\rangle_{\rm sys}$ (using Eq. (2.29)). Here $|m(t_0)\rangle_{\rm env}$ and $|\psi(t_0)\rangle_{\rm sys}$ represent the initial state of the bath and the system.

For the cases when $\hat{M}_n(t)_{\rm env} = |\chi_n(t)\rangle_{\rm env}\langle z_n(t)|/\sqrt{N}$ where $\{|z_n(t)\rangle\}$ is in general a non-orthogonal basis set we can define a measurement operator for the system as

$$\hat{M}_n(t,t_0)_{\rm sys} = \frac{{}_{\rm env}\langle z_n(t)|\hat{U}(t,t_0)|m(t_o)\rangle_{\rm env}}{\sqrt{N}}. \tag{2.105}$$

Because $\hat{U}(t,t_0)$ operates in the Hilbert space $\mathcal{H}_{\rm env}\otimes\mathcal{H}_{\rm sys}$, by including the inner product of two bath states the measurement operator will only act in $\mathcal{H}_{\rm sys}$. In this equation we see that this measurement is no-longer instantaneous, it has a time duration $T = t - t_0$.

With this system measurement operator we can rewrite Eq. (2.104) as

$$|\Psi_n(t)\rangle = |\chi_n(t)\rangle_{\rm env}|\psi_n(t)\rangle_{\rm sys}, \tag{2.106}$$

where

$$|\psi_n(t)\rangle_{\rm sys} = \frac{\hat{M}_n(t,t_0)_{\rm sys}|\psi(t_0)\rangle_{\rm sys}}{\sqrt{\Pr(z_n,t)}}. \tag{2.107}$$

This is the same as Eq. (2.99), the only difference is here we have defined a measurement operator which acts in the system space (and is not instantaneous). This operator takes the initial system state (prior to any entanglement with the bath) to the state after the measurement. By contrast in

Eq. (2.99) the measurement operator was defined to act only on the bath (instantaneously) which in turn collapses the system to a pure state.

With this notation we can rewrite the probability as

$$\begin{aligned}\Pr(z_n,t) &= \frac{{}_{\rm sys}\langle\psi(t_0)|\ {}_{\rm sys}\langle m(t_o)|\hat{U}^\dagger(t,t_0)|z_n(t)\rangle_{\rm env}\langle z_n(t)|\hat{U}(t,t_0)|m(t_o)\rangle_{\rm env}|\psi(t_0)\rangle_{\rm sys}}{N} \\ &= {}_{\rm sys}\langle\psi(t_0)|\hat{F}_n(t,t_0)_{\rm sys}|\psi(t_0)\rangle_{\rm sys}. \end{aligned} \tag{2.108}$$

where

$$\hat{F}_n(t,t_0)_{\rm sys} = \hat{M}_n^\dagger(t,t_0)_{\rm sys}\hat{M}_n(t,t_0)_{\rm sys}. \tag{2.109}$$

Thus the system POM elements must also satisfy the completeness condition

$$\sum_n \hat{F}_n(t,t_0)_{\rm sys} = \hat{1}_{\rm sys}. \tag{2.110}$$

If we assume that the initial state of the composite system is $W(t) = \rho(t_0)_{\rm sys}\otimes|m(t_0)\rangle_{\rm env}\langle m(t_0)|$, where $W(t)$ is the statistical operator for the composite system (for pure states it equals $|\Psi(t_0)\rangle\langle\Psi(t_0)|$), then the above system measurement operators and POM elements still hold. However, Eqs. (2.106) and (2.107) must be replaced by

$$W_n(t) = \rho_n(t)_{\rm sys}\otimes|\chi_n(t)\rangle_{\rm env}\langle\chi_n(t)|, \tag{2.111}$$

where $W_n(t)$ labels the conditioned composite state (which in general does not have to be pure) and

$$\rho_n(t)_{\rm sys} = \frac{\hat{M}_n(t,t_0)_{\rm sys}\rho(t_0)_{\rm sys}\hat{M}_n^\dagger(t,t_0)_{\rm sys}}{\Pr(z_n,t)}. \tag{2.112}$$

By introducing a non-instantaneous system POM we can represent all bath measurements by POM measurement on only the system. To me this method seems to hide the underlying dynamics, but if you are only interested in how the system state is going to by affected by a measurement then this method is fine. I would also like to note here, that we can go the other way, by which I mean we can keep enlarging the Hilbert space until all POM measurements can be described by projectors. This is obvious for system POMs, which arise from bath projectors ($\hat{M}_n(t)_{\rm env} = |z_n(t)\rangle_{\rm env}\langle z_n(t)|$, where $\{|z_n(t)\rangle\}$ forms an orthonormal basis set); it is just the reverse. For cases which arise from bath POMs ($\hat{M}_n(t)_{\rm env} = |\chi_n(t)\rangle_{\rm env}\langle z_n(t)|/\sqrt{N}$, where $\{|z_n(t)\rangle\}$ forms a non-orthogonal basis set) we can construct a Hilbert space $\mathcal{H}_{\rm sys}\otimes\mathcal{H}_{\rm env}\otimes\mathcal{H}_{\rm aux}$ where $\mathcal{H}_{\rm aux}$ is purely a mathematical extension to the Hilbert space which allows us to describe these measurements as projectors. This is Naimark's theorem [24, 84] which will be discussed in greater detail in Chapter 4.

2.4 Continuous measurement theory

For measurements which can be described by a projector on the system then continuous-in-time measurements are trivial; we get the Zeno effect [101]. But for measurements which are described by a POM acting on the system, this is not the case.

In continuous monitoring, repeated measurements of duration δt are performed on the state. This results in the state being conditioned on a record $\mathbf{I}_{[t_0,t)}$, which is a string containing the results z_{n_k} of each measurement. Here the subscript k refers to a measurement at time $t_k = k\delta t$, with $t_0 = 0$. From the record $\mathbf{I}_{[t_0,t)}$, the conditioned state at time t can be written as

$$|\psi_{\mathbf{I}}(t)\rangle = \frac{\hat{M}_{n_k}(t_k,t_{k-1})\hat{M}_{n_{k-1}}(t_{k-1},t_{k-2})\ldots\hat{M}_{n_1}(t_1,t_0)|\psi(t_0)\rangle}{\sqrt{\Pr(\mathbf{I}_{[t_0,t)})}}, \tag{2.113}$$

where $|\psi(t_0)\rangle$ is the initial state of the system before monitoring started and $\Pr(\mathbf{I}_{[t_0,t)})$ is the probability for obtaining the record.

To completely achieve continuous monitoring we let the time step between measurements, δt, tend towards the infinitesimal interval dt. Doing this we can, for suitable measurements, derive a differential equation for $|\psi_{\mathbf{I}}(t)\rangle$. This equation is equivalent to a Markovian stochastic Schrödinger equation (SSE) (a non-linear modification to the Schrödinger equation, see part two of this thesis). Thus Markovian SSEs have an interpretation under the orthodox theory.

In the above we considered initially pure state, for mixed initial states Eq. (2.113) needs to be replaced by

$$\rho_{\mathbf{I}}(t) = \frac{\tilde{\rho}_{\mathbf{I}}(t)}{\Pr(\mathbf{I}_{[t_0,t)})}, \tag{2.114}$$

where $\tilde{\rho}_{\mathbf{I}}(t)$ is an unnormalized state conditioned on $\mathbf{I}_{[t_0,t)}$ and is equal to

$$\begin{aligned} \tilde{\rho}_{\mathbf{I}}(t) \quad = \quad & \hat{M}_{n_k}(t_k, t_{k-1})\hat{M}_{n_{k-1}}(t_{k-1}, t_{k-2})\ldots\hat{M}_{n_1}(t_1, t_0)\rho(t_0)M^\dagger_{n_1}(t_1, t_0) \\ & \times\ldots\hat{M}^\dagger_{n_{k-1}}(t_{k-1}, t_{k-2})\hat{M}^\dagger_{n_k}(t_k, t_{k-1}). \end{aligned} \tag{2.115}$$

The probability here is

$$\Pr(\mathbf{I}_{[t_0,t)}) = \mathrm{Tr}[\tilde{\rho}_{\mathbf{I}}(t)]. \tag{2.116}$$

In the continuous limit this gives a Markovian stochastic master equation (SME).

2.5 Summary of chapter

In this chapter I have outlined the orthodox interpretation of quantum mechanics as well as presenting the measurement problem within this interpretation. I concluded that this problem is a two part problem. The first is that under the orthodox interpretation reality is not defined prior to a measurement. It is the measurement that determines the reality; a phenomenon is not a phenomenon until it is an observed phenomenon [12]. To explain why definite values occur when we measure a quantum system we have to propose an extra dynamical equation (not just Schrödinger evolution) which collapses the quantum state to a state which is consistent with the result of the measurement. However the form of this collapse equation depends on what observable we are measuring (this is what I call the second problem in the measurement problem).

The view which I have taken to explain this collapse is the world can be split into two, a classical world and quantum world (the Heisenberg cut). Inside the classical world a measurement apparatus exists which we can design to measure any observable $Z(t)$ (Eq. (2.86)). This answers the problem of choice, but at the same time creates the problem of where to place the Heisenberg cut. I have decided to choose the positioning of this cut by the place where an irreversible act of amplification occurs (eg the cascade effect of electron in a photo-detector). Doing this, the general form of the collapse equation for an arbitrary observable is Eq. (2.91) (when the cut can be placed after the system) or Eq. (2.97) (when the cut occurs later). I would also like to note that this collapse is non-local as suggested by Einstein, Podolsky and Rosen in the EPR paradox. Thus I feel if one wants to give meaning to $|\psi(t)\rangle$ then it represents a state of knowledge (it is an epistemic state) about the quantum system, and a collapse simply implies an update of this knowledge.

In recent times, with the success of quantum information theory many scientist have promoted and developed this idea further, by defining the information interpretation of quantum mechanics [20, 56, 57]. In this interpretation the questions concerning reality and the positioning of the Heisenberg cut does not enter into the debate as quantum mechanics is only about information. Because of this they take $\rho(t)$ as being fundamental (the closest quantum equivalent to probability). In this view the collapse simply refers to an updating of our information [Eq. (2.112) with no underlying meaning to system measurement operators]. This is very similar to Bayesian inference with conditional probabilities $\Pr([x],t|z_n,t)$,

$$\Pr([x],t|z_n,t) = \frac{\Pr(z_n,t|[x],t)\Pr([x],t)}{\Pr(z_n,t)}. \tag{2.117}$$

Actually this is what I believe is the prime motivator for this interpretation. What I don't like about this interpretation is that it is very subjective. If you are doing the measurement the state only exists for you, and other people would ascribe different conditioned states $\rho_n(t)$ depending on the information they have obtained. Also, what is this information about? Fuchs in Ref. [57] has stated that this information represents "the potential consequences of our experimental interventions into nature". I leave to the reader to decide if this is an adequate answer.

Chapter 3

Hidden Variables and Quantum Mechanics

In the previous chapter I presented the orthodox interpretation of quantum mechanics. This interpretation basically says that reality is undefined until we make an observation. Many scientists, including Einstein [51] believe this is because this theory is incomplete. There is something missing, a hidden variable. In this chapter I am going to consider the limitations we must place on a hidden variable interpretation of quantum mechanic in order for it to agree with current experiments and the predictions of the orthodox interpretation. I will show that the hidden variable interpretation has to be both contextual and non-local. I will do this by presenting some simple proofs. These proofs will also explain what I mean by contextual and non-local.

3.1 Contextual nature of quantum mechanics

To do this I am going to consider only simple systems: spin 1/2 systems (e.g. TLA's). A spin-1/2 system is a system which is described by a 2-dimensional Hilbert space. For this system we can define three observables

$$\hat{\sigma}_x = \{(\sigma_{x_+} = +1, \hat{\pi}_+^{(\sigma_x)} = |{\rightarrow}\rangle\langle{\rightarrow}|), (\sigma_{x_-} = -1, \hat{\pi}_-^{(\sigma_x)} = |{\leftarrow}\rangle\langle{\leftarrow}|)\} = \sum_{n=+,-} \sigma_{x_n} \hat{\pi}_n^{(\sigma_x)} \quad (3.1)$$

$$\hat{\sigma}_y = \{(\sigma_{y_+} = +1, \hat{\pi}_+^{(\sigma_y)} = |\otimes\rangle\langle\otimes|), (\sigma_{y_-} = -1, \hat{\pi}_-^{(\sigma_y)} = |\odot\rangle\langle\odot|)\} = \sum_{n=+,-} \sigma_{y_n} \hat{\pi}_n^{(\sigma_y)} \quad (3.2)$$

$$\hat{\sigma}_z = \{(\sigma_{z_+} = +1, \hat{\pi}_+^{(\sigma_z)} = |{\uparrow}\rangle\langle{\uparrow}|), (\sigma_{z_-} = -1, \hat{\pi}_-^{(\sigma_z)} = |{\downarrow}\rangle\langle{\downarrow}|)\} = \sum_{n=+,-} \sigma_{z_n} \hat{\pi}_n^{(\sigma_z)}, \quad (3.3)$$

where $[|{\rightarrow}\rangle,\ |{\leftarrow}\rangle]$, $[|\otimes\rangle,\ |\odot\rangle]$ and $[|{\uparrow}\rangle,\ |{\downarrow}\rangle]$ form 3 different basis sets for the 2-dimensional Hilbert space. Here I would like to note that this is only 3 of the many possible basis sets. These basis sets can be related to each other by

$$|{\uparrow}\rangle = [|{\rightarrow}\rangle + |{\leftarrow}\rangle]/\sqrt{2} = [|\otimes\rangle + |\odot\rangle]/\sqrt{2}\,, \quad (3.4)$$

$$|{\downarrow}\rangle = [|{\rightarrow}\rangle - |{\leftarrow}\rangle]/\sqrt{2} = [-i|\otimes\rangle + i|\odot\rangle]/\sqrt{2}\,. \quad (3.5)$$

Thus we can write any arbitrary state as

$$\begin{aligned}|\psi(t)\rangle &= c_1(t)|\uparrow\rangle + c_2(t)|\downarrow\rangle &(3.6)\\ &= [c_1(t)+c_2(t)]|\rightarrow\rangle/\sqrt{2} + [c_1(t)-c_2(t)]|\leftarrow\rangle/\sqrt{2} &(3.7)\\ &= [c_1(t)-ic_2(t)]|\otimes\rangle/\sqrt{2} + [c_1(t)+ic_2(t)]|\odot\rangle/\sqrt{2}\,. &(3.8)\end{aligned}$$

In the orthodox interpretation this would be interpreted as: If the classical measurement apparatus was designed to measure $\hat{\sigma}_z$ then the state would collapse into $|\uparrow\rangle$ (or $|\downarrow\rangle$) with probability $|c_1(t)|^2$ (or $|c_2(t)|^2$), thereby promoting the value of $\hat{\sigma}_z$, $\upsilon(\hat{\sigma}_z)$, to be $+1$ (or -1) and similarly for the other observables.

In a hidden variable interpretation of quantum mechanics we would like to have a theory which amounts to measurements being secondary. That is, if we measure observable $\hat{\sigma}_z$ then the measurement only reveals the pre-defined value (e.g. $\sigma_{z_-} = -1$). One way to do this is to assign to all observables a definite value, but as we will see this leads to contradictions (due to the contextual nature of quantum mechanics)

The first no-go theorem for hidden variables was outlined by von Neumann [129]. He proposed that since the expectation values of observables can be added then the values of these observables must also obey the additivity equation. This is clearly in contradiction to the predictions of the orthodox interpretation of quantum mechanics. Thus, he concluded that since these predictions are experimentally verified then no hidden variable theory can exists. To illustrate this lets consider the operator $\hat{\sigma}_\theta = [\hat{\sigma}_x + \hat{\sigma}_y]/\sqrt{2}$. From the orthodox theory we know that its expectation value obeys

$$\langle\hat{\sigma}_\theta\rangle = [\langle\hat{\sigma}_x\rangle + \langle\hat{\sigma}_y\rangle]/\sqrt{2}\,. \tag{3.9}$$

and the possible measurement results (eigenvalues) of $\hat{\sigma}_\theta$, $\hat{\sigma}_x$ and $\hat{\sigma}_y$ are all ± 1. Thus we are lead to the contradiction

$$\pm 1 \neq [\pm 1 \pm 1]/\sqrt{2}\,. \tag{3.10}$$

Bell showed in Ref. [10] that the assumption for the additivity of values is invalid, by presenting a counterexample (for a 2-dimensional Hilbert space). He defined a rather complicated assignment procedure (based on a hidden variable λ) which always ensures a ± 1 value for the observable and an average which agrees with the quantum mechanical expectation value. Just because the expectation values obey an additivity equation doesn't mean the values should.

To understand the contextual nature of quantum mechanics we have to define a proposition. A proposition is a question which can only have only a yes (=1) or no (=0) answer. Mathematically it is represented by the value of a projector $\upsilon(\hat{\pi}_n)$ (here $\hat{\pi}_n$ has no time dependence as the questions don't change in time). For example the question "is the value of the observable $\hat{\sigma}_x$ +1?" would be represented by the projector $\hat{\pi}_+^{(\sigma_x)} = |\rightarrow\rangle\langle\rightarrow|$. This has a value $\upsilon(\hat{\pi}_+^{(\sigma_x)}) = 1$ (or $\upsilon(\hat{\pi}_+^{(\sigma_x)}) = 0$) for a yes (or no) answer. A non-contextual (independent of questions asked about other observables) hidden variable theory exists if we can assign to every projector a value 0 or 1 such that the completeness condition (Eq. (2.58)) for projectors in not contradicted. By this, I mean the value of the identity operator ($\hat{1}$) must be 1, and by the completeness condition we get the following constraint

$$\upsilon(\hat{1}) = \sum_n \upsilon(\hat{\pi}_n) = 1. \tag{3.11}$$

Thus for every possible projective measure this constraint must be satisfied (below I will show an example that contradicts this). Note for the 2 dimensional case, quantum mechanics is non-contextual, it is possible to assign a value to every projector so that Eq. (3.11) is satisfied. This is because for every projector there is only one other projector orthogonal to it, and if we set its value to 1 then the orthogonal projector must have value 0. Note, very Recently Cabello [26] has shown that we can get a contradiction if we consider observables represented by POMs, Eq. (2.86). He uses Naimark's theorem to represent the POM elements by projectors in a larger Hilbert space.

Here I am not going to present the original Kochen-Specker [89] non-contextual hidden variable theorem as it requires considering a 3-dimensional Hilbert space and 117 propositions, or Cabello's recent POM version. Instead I am going to review Cabello, Estebaranz, and Alcaine contradiction. They assume a 4-dimensional Hilbert space (two entangled spin 1/2 systems) and show that by defining 9 observables we can write 18 propositions which lead to a contradiction. These are [25]

$$\begin{aligned} Z^{(1)} = \quad & \{(z_1^{(1)}, \hat{\pi}_1^{(1)} = |{\downarrow}\rangle_1\langle{\downarrow}| \otimes |{\downarrow}\rangle_2\langle{\downarrow}| = \hat{P}_1), \\ & (z_2^{(1)}, \hat{\pi}_2^{(1)} = |{\downarrow}\rangle_1\langle{\downarrow}| \otimes |{\uparrow}\rangle_2\langle{\uparrow}| = \hat{P}_2), \\ & (z_3^{(1)}, \hat{\pi}_3^{(1)} = |{\uparrow}\rangle_1\langle{\uparrow}| \otimes |{\rightarrow}\rangle_2\langle{\rightarrow}| = \hat{P}_3), \\ & (z_4^{(1)}, \hat{\pi}_4^{(1)} = |{\uparrow}\rangle_1\langle{\uparrow}| \otimes |{\leftarrow}\rangle_2\langle{\leftarrow}| = \hat{P}_4)\}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} Z^{(2)} = \quad & \{(z_1^{(2)}, \hat{\pi}_1^{(2)} = |{\downarrow}\rangle_1\langle{\downarrow}| \otimes |{\downarrow}\rangle_2\langle{\downarrow}| = \hat{P}_1), \\ & (z_2^{(2)}, \hat{\pi}_2^{(2)} = |{\uparrow}\rangle_1\langle{\uparrow}| \otimes |{\downarrow}\rangle_2\langle{\downarrow}| = \hat{P}_5), \\ & (z_3^{(2)}, \hat{\pi}_3^{(2)} = |{\rightarrow}\rangle_1\langle{\rightarrow}| \otimes |{\uparrow}\rangle_2\langle{\uparrow}| = \hat{P}_6), \\ & (z_4^{(2)}, \hat{\pi}_4^{(2)} = |{\leftarrow}\rangle_1\langle{\leftarrow}| \otimes |{\uparrow}\rangle_2\langle{\uparrow}| = \hat{P}_7)\}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} Z^{(3)} = \quad & \{(z_1^{(3)}, \hat{\pi}_1^{(3)} = |{\rightarrow}\rangle_1\langle{\rightarrow}| \otimes |{\leftarrow}\rangle_2\langle{\leftarrow}| = \hat{P}_8), \\ & (z_2^{(3)}, \hat{\pi}_2^{(3)} = |{\leftarrow}\rangle_1\langle{\leftarrow}| \otimes |{\leftarrow}\rangle_2\langle{\leftarrow}| = \hat{P}_9), \\ & (z_3^{(3)}, \hat{\pi}_3^{(3)} = |{\uparrow}\rangle_1\langle{\uparrow}| \otimes |{\rightarrow}\rangle_2\langle{\rightarrow}| = \hat{P}_3), \\ & (z_4^{(3)}, \hat{\pi}_4^{(3)} = |{\downarrow}\rangle_1\langle{\downarrow}| \otimes |{\rightarrow}\rangle_2\langle{\rightarrow}| = \hat{P}_{10})\}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} Z^{(4)} = \quad & \{(z_1^{(4)}, \hat{\pi}_1^{(4)} = |{\rightarrow}\rangle_1\langle{\rightarrow}| \otimes |{\leftarrow}\rangle_2\langle{\leftarrow}| = \hat{P}_8), \\ & (z_2^{(4)}, \hat{\pi}_2^{(4)} = |{\rightarrow}\rangle_1\langle{\rightarrow}| \otimes |{\rightarrow}\rangle_2\langle{\rightarrow}| = \hat{P}_{11}), \\ & (z_3^{(4)}, \hat{\pi}_3^{(4)} = |{\leftarrow}\rangle_1\langle{\leftarrow}| \otimes |{\uparrow}\rangle_2\langle{\uparrow}| = \hat{P}_7), \\ & (z_4^{(4)}, \hat{\pi}_4^{(4)} = |{\leftarrow}\rangle_1\langle{\leftarrow}| \otimes |{\downarrow}\rangle_2\langle{\downarrow}| = \hat{P}_{12})\}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} Z^{(5)} = \quad & \{(z_1^{(5)}, \hat{\pi}_1^{(5)} = |{\downarrow}\rangle_1\langle{\downarrow}| \otimes |{\uparrow}\rangle_2\langle{\uparrow}| = \hat{P}_2), \\ & (z_2^{(5)}, \hat{\pi}_2^{(5)} = |{\uparrow}\rangle_1\langle{\uparrow}| \otimes |{\downarrow}\rangle_2\langle{\downarrow}| = \hat{P}_5), \\ & (z_3^{(5)}, \hat{\pi}_3^{(5)} = [|{\uparrow}\rangle_1|{\uparrow}\rangle_2 + |{\downarrow}\rangle_1|{\downarrow}\rangle_2][\langle{\uparrow}|_1\langle{\uparrow}|_2 + \langle{\downarrow}|_1\langle{\downarrow}|_2]/2 = \hat{P}_{13}), \\ & (z_4^{(5)}, \hat{\pi}_4^{(5)} = [|{\uparrow}\rangle_1|{\uparrow}\rangle_2 - |{\downarrow}\rangle_1|{\downarrow}\rangle_2][\langle{\uparrow}|_1\langle{\uparrow}|_2 - \langle{\downarrow}|_1\langle{\downarrow}|_2]/2 = \hat{P}_{14})\}, \end{aligned} \tag{3.16}$$

$$\begin{aligned} Z^{(6)} = \quad & \{(z_1^{(6)}, \hat{\pi}_1^{(6)} = |{\leftarrow}\rangle_1\langle{\leftarrow}||{\leftarrow}\rangle_2\langle{\leftarrow}| = \hat{P}_9), \\ & (z_2^{(6)}, \hat{\pi}_2^{(6)} = |{\rightarrow}\rangle_1\langle{\rightarrow}||{\rightarrow}\rangle_2\langle{\rightarrow}| = \hat{P}_{11}), \\ & (z_3^{(6)}, \hat{\pi}_3^{(6)} = [|{\uparrow}\rangle_1|{\uparrow}\rangle_2 - |{\downarrow}\rangle_1|{\downarrow}\rangle_2][\langle{\uparrow}|_1\langle{\uparrow}|_2 - \langle{\downarrow}|_1\langle{\downarrow}|_2]/2 = \hat{P}_{14}), \\ & (z_4^{(6)}, \hat{\pi}_4^{(6)} = [|{\uparrow}\rangle_1|{\downarrow}\rangle_2 - |{\downarrow}\rangle_1|{\uparrow}\rangle_2][\langle{\uparrow}|_1\langle{\downarrow}|_2 - \langle{\downarrow}|_1\langle{\uparrow}|_2]/2 = \hat{P}_{15})\}, \end{aligned} \tag{3.17}$$

$$
\begin{aligned}
Z^{(7)} = \quad & \{(z_1^{(7)}, \hat{\pi}_1^{(7)} = [|\uparrow\rangle_1|\rightarrow\rangle_2 - |\downarrow\rangle_1|\leftarrow\rangle_2][\langle\uparrow|_1\langle\rightarrow|_2 - \langle\downarrow|_1\langle\leftarrow|_2]/2 = \hat{P}_{16}), \\
& (z_2^{(7)}, \hat{\pi}_2^{(7)} = [|\uparrow\rangle_1|\rightarrow\rangle_2 + |\downarrow\rangle_1|\leftarrow\rangle_2][\langle\uparrow|_1\langle\rightarrow|_2 + \langle\downarrow|_1\langle\leftarrow|_2]/2 = \hat{P}_{17}), \\
& (z_3^{(7)}, \hat{\pi}_3^{(7)} = |\uparrow\rangle_1\langle\uparrow||\leftarrow\rangle_2\langle\leftarrow| = \hat{P}_4), \\
& (z_4^{(7)}, \hat{\pi}_4^{(7)} = |\downarrow\rangle_1\langle\downarrow||\rightarrow\rangle_2\langle\rightarrow| = \hat{P}_{10})\}, \qquad (3.18)
\end{aligned}
$$

$$
\begin{aligned}
Z^{(8)} = \quad & \{(z_1^{(8)}, \hat{\pi}_1^{(8)} = [|\uparrow\rangle_1|\rightarrow\rangle_2 - |\downarrow\rangle_1|\leftarrow\rangle_2][\langle\uparrow|_1\langle\rightarrow|_2 - \langle\downarrow|_1\langle\leftarrow|_2]/2 = \hat{P}_{16}), \\
& (z_2^{(8)}, \hat{\pi}_2^{(8)} = [|\downarrow\rangle_1|\rightarrow\rangle_2 - |\uparrow\rangle_1|\leftarrow\rangle_2][\langle\downarrow|_1\langle\rightarrow|_2 - \langle\uparrow|_1\langle\leftarrow|_2]/2 = \hat{P}_{18}), \\
& (z_3^{(8)}, \hat{\pi}_3^{(8)} = |\rightarrow\rangle_1\langle\rightarrow||\uparrow\rangle_2\langle\uparrow| = \hat{P}_6), \\
& (z_4^{(8)}, \hat{\pi}_4^{(8)} = |\leftarrow\rangle_1\langle\leftarrow||\downarrow\rangle_2\langle\downarrow| = \hat{P}_{12})\}, \qquad (3.19)
\end{aligned}
$$

$$
\begin{aligned}
Z^{(9)} = \quad & \{(z_1^{(9)}, \hat{\pi}_1^{(9)} = [|\uparrow\rangle_1|\rightarrow\rangle_2 + |\downarrow\rangle_1|\leftarrow\rangle_2][\langle\uparrow|_1\langle\rightarrow|_2 + \langle\downarrow|_1\langle\leftarrow|_2]/2 = \hat{P}_{17}), \\
& (z_2^{(9)}, \hat{\pi}_2^{(9)} = [|\downarrow\rangle_1|\rightarrow\rangle_2 - |\uparrow\rangle_1|\leftarrow\rangle_2][\langle\downarrow|_1\langle\rightarrow|_2 - \langle\uparrow|_1\langle\leftarrow|_2]/2 = \hat{P}_{18}), \\
& (z_3^{(9)}, \hat{\pi}_3^{(9)} = [|\uparrow\rangle_1|\uparrow\rangle_2 + |\downarrow\rangle_1|\downarrow\rangle_2][\langle\uparrow|_1\langle\uparrow|_2 + \langle\downarrow|_1\langle\downarrow|_2]/2 = \hat{P}_{13}), \\
& (z_4^{(9)}, \hat{\pi}_4^{(9)} = [|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2][\langle\uparrow|_1\langle\downarrow|_2 - \langle\downarrow|_1\langle\uparrow|_2]/2 = \hat{P}_{15})\}. \qquad (3.20)
\end{aligned}
$$

The premise behind the Kochen-Specker argument is to assume that each proposition has a definite answer (0 or 1) and this answer is independent of which observable is being considered (non-contextual). Then we should be able to assign a value (0 or 1) to each of the above 18 projectors, which does not contradict the completeness condition (Eq. (3.11)). Since each projector occurs twice in the nine observables then the total number of values equal to 1 must be even, however, by the nine completeness conditions (nine different observables), it is only possible to assign nine (odd) values equal to one. Therefore contradicting the assumption that we can assign a definite value to all the projectors. Thus we have to conclude that only a subset of the observables (projective measures) can be objectively real (have definite values for there projectors). This subset is the context, by this I mean a hidden variable theory needs to contain both, a method for choosing the context, set of propositions (projectors) which can be given definite values, as well as a way of defining which one of these has a value equal to one. As this determines the actual value of the objectively real observables.

This obviously leads to the question, what subset of projectors can be given definite values? In this thesis, when considering hidden variable theories, I will take the view that definite status can only be given to one complete set of projectors ($\{\hat{\pi}_n\}$) (by complete set I mean a projective measure).

3.2 Non-local nature of quantum mechanics

As illustrated in the above section objective reality can not be given to all observables. Here I am going to consider another requirement of hidden variables, non-locality. In 1964 Bell [9] demonstrated that if a hidden variable theory exists then this theory must be non-local. By non-local I mean if we have (at least) two spacial separated system, which are described by an entangle quantum state (e.g Eq. (2.47)) the values of local observables (local to each system) can be affected instantaneously by operations performed on the distant observables. Bell demonstrated this by deriving a inequality based on statistics from repeated measurements. This was later experimentally verified by Aspect, Dalibard and Roger [2] in 1982.

To see the difference between Bell-type restrictions and the Kochen-Specker theorem, as well as showing the non-local nature of quantum mechanics, I am going to consider *three* entangled spin

1/2 systems (this is effectively an 8 dimensional Hilbert space). This is because in this example the non-local nature of quantum mechanics can be observed directly rather then by a statistical analysis. To do this we start by defining the Kochen-Specker theorem slightly different to that presented in section 3.1 (with respect to observables rather than projectors[94, 100, 103]). Rather then introducing a contradiction in terms of projectors, we consider the six local observables $\hat{\sigma}_{1x}$, $\hat{\sigma}_{2x}$, $\hat{\sigma}_{3x}$, $\hat{\sigma}_{1y}$, $\hat{\sigma}_{2y}$, and $\hat{\sigma}_{3y}$, and the four non-local observables $\hat{\sigma}_{1x}\hat{\sigma}_{2x}\hat{\sigma}_{3x}$, $\hat{\sigma}_{1y}\hat{\sigma}_{2x}\hat{\sigma}_{3y}$, $\hat{\sigma}_{1y}\hat{\sigma}_{2y}\hat{\sigma}_{3x}$, and $\hat{\sigma}_{1x}\hat{\sigma}_{2y}\hat{\sigma}_{3y}$ as well as the five constraints

$$\upsilon(\hat{\sigma}_{1x}\hat{\sigma}_{2y}\hat{\sigma}_{3y}\hat{\sigma}_{1x}\hat{\sigma}_{2y}\hat{\sigma}_{3y}) = \upsilon(\hat{1}) = 1, \tag{3.21}$$

$$\upsilon(\hat{\sigma}_{1y}\hat{\sigma}_{2x}\hat{\sigma}_{3y}\hat{\sigma}_{1y}\hat{\sigma}_{2x}\hat{\sigma}_{3y}) = \upsilon(\hat{1}) = 1, \tag{3.22}$$

$$\upsilon(\hat{\sigma}_{1y}\hat{\sigma}_{2y}\hat{\sigma}_{3x}\hat{\sigma}_{1y}\hat{\sigma}_{2y}\hat{\sigma}_{3x}) = \upsilon(\hat{1}) = 1, \tag{3.23}$$

$$\upsilon(\hat{\sigma}_{1x}\hat{\sigma}_{2x}\hat{\sigma}_{3x}\hat{\sigma}_{1x}\hat{\sigma}_{2x}\hat{\sigma}_{3x}) = \upsilon(\hat{1}) = 1, \tag{3.24}$$

$$\upsilon(\hat{\sigma}_{1x}\hat{\sigma}_{2x}\hat{\sigma}_{3x}\hat{\sigma}_{1y}\hat{\sigma}_{2y}\hat{\sigma}_{3x}\hat{\sigma}_{1y}\hat{\sigma}_{2x}\hat{\sigma}_{3y}\hat{\sigma}_{1x}\hat{\sigma}_{2y}\hat{\sigma}_{3y}) = \upsilon(-\hat{1}) = -1. \tag{3.25}$$

Here we have used the fact that the value of the identify operator is always one (only has one eigenvalue) and

$$\hat{\sigma}_j\hat{\sigma}_k = i\epsilon_{j,k,l}\hat{\sigma}_l + \delta_{j,k}\hat{1}, \tag{3.26}$$

where $\epsilon_{j,k,l}$ is zero if any pair of indices are the same, 1 for cyclic indies, and otherwise equal to -1.

We now make the assumption that for commuting operators, $[\hat{R}, \hat{R}'] = 0$, we can assign a value such that

$$\upsilon(f(\hat{R}, \hat{R}')) = f(\upsilon(\hat{R}), \upsilon(\hat{R}')). \tag{3.27}$$

This directly equates to it being possible to perform simultaneous measurements of two commuting observables in the orthodox interpretation. Doing this we can rewrite the above as

$$\upsilon(\hat{\sigma}_{1x}\hat{\sigma}_{2y}\hat{\sigma}_{3y})\upsilon(\hat{\sigma}_{1x})\upsilon(\hat{\sigma}_{2y})\upsilon(\hat{\sigma}_{3y}) = 1, \tag{3.28}$$

$$\upsilon(\hat{\sigma}_{1y}\hat{\sigma}_{2x}\hat{\sigma}_{3y})\upsilon(\hat{\sigma}_{1y})\upsilon(\hat{\sigma}_{2x})\upsilon(\hat{\sigma}_{3y}) = 1, \tag{3.29}$$

$$\upsilon(\hat{\sigma}_{1y}\hat{\sigma}_{2y}\hat{\sigma}_{3x})\upsilon(\hat{\sigma}_{1y})\upsilon(\hat{\sigma}_{2y})\upsilon(\hat{\sigma}_{3x}) = 1, \tag{3.30}$$

$$\upsilon(\hat{\sigma}_{1x}\hat{\sigma}_{2x}\hat{\sigma}_{3x})\upsilon(\hat{\sigma}_{1x})\upsilon(\hat{\sigma}_{2x})\upsilon(\hat{\sigma}_{3x}) = 1, \tag{3.31}$$

$$\upsilon(\hat{\sigma}_{1x}\hat{\sigma}_{2x}\hat{\sigma}_{3x})\upsilon(\hat{\sigma}_{1y}\hat{\sigma}_{2y}\hat{\sigma}_{3x})\upsilon(\hat{\sigma}_{1y}\hat{\sigma}_{2x}\hat{\sigma}_{3y})\upsilon(\hat{\sigma}_{1x}\hat{\sigma}_{2y}\hat{\sigma}_{3y}) = -1. \tag{3.32}$$

Here we see that on the LHS (left hand side) all the ten observables appear twice in the 5 equations. Since by definition the values (eigenvalues) of all the ten observables are ± 1 the product must be 1. However, for the RHS (right hand side) we get -1 a contradiction. Thus we are left with two possibilities, our assumption about commuting observables, Eq. (3.27) is untrue (this would be devastating for hidden variable theories, as it would contradict the orthodox theory for simultaneous measurements) or we have to conclude that objective reality can not be given to all observables, only to a subset.

To show the non-local nature of quantum mechanics we have to consider a specific quantum state [78, 94, 123]

$$|\Psi\rangle = [|\uparrow\rangle_1|\uparrow\rangle_2|\uparrow\rangle_3 - |\downarrow\rangle_1|\downarrow\rangle_2|\downarrow\rangle_3]/\sqrt{2}\,, \tag{3.33}$$

(this is an entangled state) and the four operators $\hat{\sigma}_{1x}\hat{\sigma}_{2x}\hat{\sigma}_{3x}$, $\hat{\sigma}_{1y}\hat{\sigma}_{2x}\hat{\sigma}_{3y}$, $\hat{\sigma}_{1y}\hat{\sigma}_{2y}\hat{\sigma}_{3x}$, and $\hat{\sigma}_{1x}\hat{\sigma}_{2y}\hat{\sigma}_{3y}$.

It can be shown that with this quantum state the expectation values of these four observables are

$$\langle \hat{\sigma}_{1x}\hat{\sigma}_{2x}\hat{\sigma}_{3x}\rangle = -1, \tag{3.34}$$
$$\langle \hat{\sigma}_{1y}\hat{\sigma}_{2x}\hat{\sigma}_{3y}\rangle = 1, \tag{3.35}$$
$$\langle \hat{\sigma}_{1y}\hat{\sigma}_{2y}\hat{\sigma}_{3x}\rangle = 1, \tag{3.36}$$
$$\langle \hat{\sigma}_{1x}\hat{\sigma}_{2y}\hat{\sigma}_{3y}\rangle = 1. \tag{3.37}$$

Thus we can conclude that since these observables only contain the eigenvalues +1 or −1 then the values of the above four observables must satisfy

$$\upsilon(\hat{\sigma}_{1x}\hat{\sigma}_{2y}\hat{\sigma}_{3y}) = 1, \tag{3.38}$$
$$\upsilon(\hat{\sigma}_{1y}\hat{\sigma}_{2x}\hat{\sigma}_{3y}) = 1, \tag{3.39}$$
$$\upsilon(\hat{\sigma}_{1y}\hat{\sigma}_{2y}\hat{\sigma}_{3x}) = 1, \tag{3.40}$$
$$\upsilon(\hat{\sigma}_{1x}\hat{\sigma}_{2x}\hat{\sigma}_{3x}) = -1. \tag{3.41}$$

If a local hidden variable theory exists then one has to assume that the local observables ($\hat{\sigma}_{1x}$, $\hat{\sigma}_{2x}$, $\hat{\sigma}_{3x}$, $\hat{\sigma}_{1y}$, $\hat{\sigma}_{2y}$ and $\hat{\sigma}_{3y}$) have a definite value independent of each other (this is effectively a weaker non-contextual argument as it is state dependent). Thus we obtain the 4 constraints

$$\upsilon(\hat{\sigma}_{1x})\upsilon(\hat{\sigma}_{2y})\upsilon(\hat{\sigma}_{3y}) = 1, \tag{3.42}$$
$$\upsilon(\hat{\sigma}_{1y})\upsilon(\hat{\sigma}_{2x})\upsilon(\hat{\sigma}_{3y}) = 1, \tag{3.43}$$
$$\upsilon(\hat{\sigma}_{1y})\upsilon(\hat{\sigma}_{2y})\upsilon(\hat{\sigma}_{3x}) = 1, \tag{3.44}$$
$$\upsilon(\hat{\sigma}_{1x})\upsilon(\hat{\sigma}_{2x})\upsilon(\hat{\sigma}_{3x}) = -1. \tag{3.45}$$

which is impossible because the LHS multiplies to 1 (each observable occurs twice) whilst the RHS multiplies to −1. The thing which separates this from a Kochen-Specker argument is we have to specify a state. If we were to consider a separable state the expectation values of the observables would not be the eigenvalues. This means we would not be able to assign a value to the four observables, thus no contradiction for local observables could be reached.

You can argue whether or not this argument supersedes the Kochen-Specker argument, but I believe they are both important. Bell's theorem shows that for entangled quantum states it is impossible to assign a value to all local observables, whereas the observable version of the Kochen-Specker argument shows that it is impossible to assign a value to all local and non-local observables for any quantum state. But to me non-locality is a bit more complicated than shown in this argument. For example we can have observables described by entangled projectors, see observables 5-9 (Eqs. (3.16) – (3.20)) in the Kochen-Specker argument of section 3.1. These obviously represent one type of non-local observables (one in which a non-local measurement is performed). Another type of non-local observables are local projectors with non-local information (see Eqs. (3.12) – (3.15)) which I believe is the non-locality implied by Bell [9]. To be more clear about the later case consider observable 2 (Eq. (3.13)) in the Kochen-Specker argument of section 3.1. This observable represent a situation where the value of system 2 depends (non-locally) on what type of local measurement (either a $\hat{\sigma}_z$ or $\hat{\sigma}_x$ measurement) is done on system 1.

A useful investigation would be to obtain a Kochen-Specker argument showing that it is impossible to assign an objective reality to all observables represented by local projectors (even with above mentioned second type of non-locality). This is not done in section 3.1. If this can be done (which I think is possible) then the Kochen-Specker argument would be more powerful then Bell-type arguments, as it would stress the point that objective reality can only be given to a subset of observables, and this subset is restricted by the projectors. Thereby reinforcing my assumption (or restriction?) at the end of the last section. This being that only one set of projectors ($\{\hat{\pi}_n\}$) can be given definite status. In any case making this restriction does not contradict any of the above limitation on hidden variables. Actually it is a bit too strong, but for the purposes of this thesis this restriction will be satisfactory.

3.3 Summary of chapter

In this chapter I have briefly investigated the restrictions a hidden variable interpretation of quantum mechanics must have in order for it to agree with the prediction of the orthodox interpretation and current experiments (see Ref. [2] for a Bell-type experiment and Ref. [86] for a recent non-contextual experiment). The conclusion which was drawn was a hidden variable interpretation needs to be both non-local and contextual. This means it is impossible (at one time) to assign an objective reality to all observables. Instead only a subset observables can be objectively real, the context. The non-locality nature of quantum mechanics was illustrated when we consider a specific quantum state, namely an entangled state. When considering this state it was observed that the subset of observables can not include all local observables.

In the next chapter I will present a hidden variable interpretation of quantum mechanics which obeys this requirements, namely the Modal interpretation.

Chapter 4

Modal Interpretations of Quantum Mechanics

In the previous chapter I reviewed the proofs that if a hidden variable theory exists then it must be both non-local and contextual. In this chapter I am going to consider a hidden variable theory which satisfies these properties, the modal interpretation of quantum mechanics [4, 11, 23, 39, 124, 125, 62, 80, 90, 114, 126]. Actually I believe it is the only hidden variable interpretation of quantum mechanics. This may sound like a strong statement given the success of Bohmian mechanics [14, 15]. But in this chapter it will be shown that Bohmian mechanics, for an appropriately chosen projective measure, is simply the continuous limit of the modal interpretation (for all natural Hamiltonians).

4.1 General framework

Unlike the orthodox interpretation, the modal interpretation's central goal is to keep the everyday definition of reality intact: things exists even when not observed. However, as shown in Chapter 3, not all observables can be given definite status. That is, not all observables are simultaneously objectively real. To satisfy the contextual requirements of a hidden variable theory, the modal interpretation assumes only one set of projective measures is objectively real. This is referred to as the preferred projective measure. Actually recently in Ref. [62] Wiseman and myself have shown that by using Naimark's theorem this can be extended to include POMs. In this case only one POM is objectively real, the preferred POM. I would like to also point out here that work has been done to extend the property ascription (for projectors) to include more then just one projective measure, but not large enough to get a Kochen-Specker argument (see Ref. [127] and reference within). However, for the purposes of this thesis, this overly strong condition regarding just one set of projectors (or POMs elements) will be assumed as a criteria for the modal interpretation. By this I mean, given the preferred POM $\{\hat{F}_n(t)\}$ [or projective measure $\{\hat{\pi}_n(t)\}$] only observables of the form

$$Z(t) = \{(z_n, \hat{F}_n(t))\} \tag{4.1}$$

(for POMs) or

$$Z(t) = \{(z_n, \hat{\pi}_n(t))\} \tag{4.2}$$

(for projectors) are objectively real. This group of observables will be referred to as a property (or beable after Bell [11]). That is a property refers to an observable that is objectively real. Thus in this interpretation the first problem of the measurement problem is solved, at least for measurements of the preferred observables (property), as a measurement only reveals the pre-defined value. For measurements of other observables see the discussion by Bohm in Ref [15] for their interpretation.

4.1.1 Properties represented by projectors

In this subsection I am going to consider only properties represented by projectors, see Eq. (4.2). This is equivalent to Eq. (2.62) in the orthodox interpretation. To explain why $Z(t)$ has the pre-defined value z_n at time t we have to introduce an extra quantum state, the property state, $|\psi_n(t)\rangle$. That is the system is now described by both $|\psi_n(t)\rangle$ and $|\psi(t)\rangle$.

The property state, $|\psi_n(t)\rangle$, represents the actual state of the system, it determines which value of the possible values $Z(t)$ has. $|\psi(t)\rangle$ (the solution of Eq. (2.28)) in this interpretation simply determines the weights for choosing one property state out of the complete set of property states $\{|\psi_n(t)\rangle\}$ (representing all the other values). Hence the name modal (after modal logic and named by van Fraassen [124]), it contains the necessity (the system *must* be in the state $|\psi_n(t)\rangle$), the possibilities (it *could* be any of the states $\{|\psi_n(t)\rangle\}$) and the probability (determine by $|\psi(t)\rangle$).

The property state is defined as

$$|\psi_n(t)\rangle = \hat{\pi}_n(t)|\psi(t)\rangle/\sqrt{N}, \tag{4.3}$$

where N is the normalization constant and by definition is

$$N = \langle\psi(t)|\hat{\pi}_n(t)|\psi(t)\rangle. \tag{4.4}$$

Here we see that the different property states arises from the different projectors in the complete set $\{\hat{\pi}_n(t)\}$. Thus the property state is equivalent to the conditional state (see Eq. (2.65)) in the orthodox interpretation. However here it represents the actual state of the system, whereas in the orthodox view it represent the state after a measurement with result z_n. The probability that the system will be in this state is given by

$$\Pr(z_n, t) = \langle\psi(t)|\hat{\pi}_n(t)|\psi(t)\rangle, \tag{4.5}$$

the standard Born probability. Thus the average of $Z(t)$ over all possible property states will agree with $\langle Z(t)\rangle$ found by the orthodox theory. However, unlike the orthodox interpretation this probability does not refer to the probability of observing results z_n at time t but to the probability of the system possessing property z_n at time t.

To explicitly show why $Z(t)$ has a definite value (denoted $\upsilon(Z)$) equal to z_n, when the system is the state $|\psi_n(t)\rangle$, we have to consider the higher moments of $Z(t)$ ($Z^m(t)$ etc). If the system is in this state then it can be shown by Eq. (2.63) that the mean value of $Z(t)$ is

$$\langle Z(t)\rangle = \sum_m (z_m)\langle\psi_n(t)|\hat{\pi}_m(t)|\psi_n(t)\rangle = z_n, \tag{4.6}$$

and the variance

$$\langle Z^2(t)\rangle - \langle Z(t)\rangle^2 = \sum_m (z_m)^2\langle\psi_n(t)|\hat{\pi}_m(t)|\psi_n(t)\rangle - z_n = 0. \tag{4.7}$$

Thus we can conclude that $\upsilon(Z(t), t)$ must be certain and equal to z_n at time t. The evolution of this value is effected by both the property states smooth evolution in time (how the projectors change in time) as well the stochastic jumping between different z_n. That is, even for time independent properties, these values may change in time (stochastically). This will be considered in detail in section 4.2.

I would like to note here that because no classical apparatus is needed to measure the system, the system for all intents and purposes can be the complete universe. That is under the modal interpretation quantum mechanics can be extended all the way up the von Neumann chain [129] to include the entire universe. To signify this I will define the property state for the universe as

$$|\Psi_n(t)\rangle = \frac{\hat{\pi}_n(t)|\Psi(t)\rangle}{\sqrt{\Pr(z_n, t)}}, \tag{4.8}$$

where

$$\Pr(z_n, t) = \langle\Psi(t)|\hat{\pi}_n(t)|\Psi(t)\rangle. \tag{4.9}$$

The guiding state, $|\Psi(t)\rangle$, obeys the Schrödinger equation

$$d_t|\Psi(t)\rangle = -\frac{i}{\hbar}\hat{H}_{\text{uni}}(t)|\Psi(t)\rangle \tag{4.10}$$

where $\hat{H}_{\text{uni}}(t)$ is the Hamiltonian of the universe. In terms of all the possible property states the guiding state can be written as

$$|\Psi(t)\rangle = \sum_n \sqrt{\Pr(z_n, t)}\,|\Psi_n(t)\rangle. \tag{4.11}$$

Thus in this interpretation, not only does it solve problem one of the measurement problem (reality is well defined and measurement only reveals this pre-defined value), there is no Heisenberg cut. However the problem of choice (problem 2) still remains as we can choose a different projective measure as preferred (different sets of $\hat{\pi}_n(t)$). This will give a different group of observables property status. This problem, I believe, in the modal interpretations has not been resolved in a wholly satisfactory way and may never be. Presently there is many variants which try to answer the problem of choice or accept the arbitrariness of choice.

Beable (or Bell) variant

The first possibility is that we accept choice as being essential. If we do this then we arrive at the beable variant [11, 120, 128, 121]. In fact in Bell's original version [11] he chooses the projectors to describe the fermion number configuration of the world, $n = \{n_1, n_2, ...\}$. The form of this projector would be

$$\hat{\pi}_n = \hat{\pi}_{n_1} \otimes \hat{\pi}_{n_2} ... \otimes \hat{1}_{\text{other}} \tag{4.12}$$

where $\{\hat{\pi}_{n_1}\}$, $\{\hat{\pi}_{n_2}\}$, and $\{...\}$ label the projective measure for each fermion mode (each subsystem of the universe) and $\hat{1}_{\text{other}}$ labels any other subsystems of the total universe not required when considering only the fermion number configuration property.

The above example illustrate that actually choice enters into this variant in two ways. Given a Hilbert space $\mathcal{H}_{\text{uni}}$ we can choose both how to factorize it (what subsystems exist), as well as within this factorization what properties (of the subsystems) will be objectively real? For example, we could chose to factorize the total universe as $\mathcal{H}_{\text{uni}} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ (or any other of the infinite factorizations) and in this factorization we could choose to describe the three properties:

$$Z^{(1)}(t) = \{(z_n^{(1)}, \hat{\pi}_n^{(1)}(t) = \hat{\pi}_{n_1}(t) \otimes \hat{1}_2 \otimes \hat{1}_3)\}, \tag{4.13}$$

$$Z^{(2)}(t) = \{(z_n^{(2)}, \hat{\pi}_n^{(2)}(t) = \hat{1}_1 \otimes \hat{\pi}_{n_2}(t) \otimes \hat{1}_3)\}, \tag{4.14}$$

$$Z^{(3)}(t) = \{(z_n^{(3)}, \hat{\pi}_n^{(3)}(t) = \hat{1}_1 \otimes \hat{1}_2 \otimes \hat{\pi}_{n_3}(t))\}, \tag{4.15}$$

each referring only to one of the three subsystems. Thus if our theory needed to describe one and only one of the above three properties then we would chose $\{\hat{\pi}_n(t)\}$ equal to either $\{\hat{\pi}_n^{(1)}(t)\}$, $\{\hat{\pi}_n^{(2)}(t)\}$, or $\{\hat{\pi}_n^{(3)}(t)\}$ depending on which property we were intending on describing. If however we wanted to describe a property dependent on all subsystems as well as the above three properties then

$$Z^{(4)}(t) = \{(z_n^{(4)}, \hat{\pi}_n^{(4)}(t))\} \tag{4.16}$$

where $\hat{\pi}_n^{(4)}(t) = \hat{\pi}_{n_1}(t) \otimes \hat{\pi}_{n_2}(t) \otimes \hat{\pi}_{n_3}(t)$. This combination of projectors is known as the principle of composition [4]. This principle states that if (at least) two subsystems, 1 and 2, posses the properties described by the set of projectors $\{\hat{\pi}_{n_1}(t) \otimes \hat{1}_{\text{rest}}\}$ and $\{\hat{1}_{\text{rest}} \otimes \hat{\pi}_{n_2}(t)\}$, respectively, then the property describing the system composed of both subsystems 1 and 2 must be described by the set of projectors $\{\hat{\pi}_{n_1}(t) \otimes \hat{\pi}_{n_2}(t) \otimes \hat{1}_{\text{rest}}\}$. Here $\hat{1}_{\text{rest}}$ refers to the identity operator for the rest of the universe not included in the space described by the projectors.

An argument against this variant is that if reality is not defined until we specify what set (or group since it satisfies all the constraint of a group) of the properties are going to be considered (Eq. (4.2)), then with respect to the orthodox view we have not really made any improvement, so why bother? I personally don't agree with this statement as I feel the orthodox view tries to force the idea that reality can't be defined until a measurement is made, and this simple variant shows otherwise (which I feel is merit enough to consider it as a counter example to the orthodox interpretation). Also if we are prepared to accept that we live outside of the quantum world (which is required in the orthodox interpretation) then it is not hard to envisage that the arrangement of the experiment is what determines which properties are going to be given definite status. However by doing this we again develop the problem of the Heisenberg cut.

Bub's variant

If for now we assume that there is a classical world outside of the quantum world, which decides the arrangement of the experimental apparatus we get Bub's version of the Modal interpretation [21, 22, 23]. He assumes that certain observables are always objectively real. These observables include the pointer readings in an apparatus. When the apparatus comes in contact with a system (via an interaction) then the composite quantum state ($|\Psi(t)\rangle$) for the system-apparatus will be an entangled state. He suggests that when (and only when) this composite state forms a Schmidt decomposition, with one orthogonal set being the eigenset of the apparatus observable, then the system can be given definite status. To explain this let's assume that the apparatus has definite values for properties containing the projector $\hat{\pi}_n^{(\text{app})}(t) = \hat{1} \otimes |\phi_n(t)\rangle_2\langle\phi_n(t)|$. Then provided the guiding state can be written as

$$|\Psi(t)\rangle = \sum_n^D c_n(t)|\phi_n'(t)\rangle_1|\phi_n(t)\rangle_2, \tag{4.17}$$

where $\{|\phi_n'(t)\rangle_1\}$ forms an orthonormal set in $\mathcal{H}_{\text{sys}}$, we can assign definite status to system properties with the projector set $\{\hat{\pi}_n^{(\text{sys})}(t) = |\phi_n'(t)\rangle_1\langle\phi_n(t)| \otimes \hat{1}\}$. Thus in this variant for both the system and apparatus to be objectively real the property state will be of the form $|\Psi_n(t)\rangle = |\phi_n'(t)\rangle_1|\phi_n(t)\rangle_2$, perfectly correlated.

Thus in this variant reality is kept intact and as the problem of choice is partially answered. By partially I mean we have to conjure something outside of the system (or system and apparatus) which determines what properties are going to be given definite status (choice of $|\phi_n(t)\rangle_2$). This can be something as simple as how we arrange the experiment. Arguments against this variant are obviously where is the Heisenberg cut? and as pointed out by Elby [52], if the measurement is non-ideal (no Schmidt decomposition exists with the set $\{|\phi_n(t)\rangle_2\}$ forming the apparatus basis) then we can't assign definite status to any properties for the system. I personally don't feel the latter is a problem as it simply means (in this cases) the system just acts as a guiding system (or phantom system) which influences the properties of the apparatus, the thing which we actually measure.

Healey's and Dieks' variant

In these variants Healey [80, 81] and Dieks [39, 40, 41, 42, 43, 126] assume that the preferred projective measure is determined by the guiding state for the universe. Let's assume that the universe can be factorized into 2 subsystems (the system and the rest of the world). If this is the case then Healey and Dieks assume that the set of projectors that occur in the unique Schmidt decomposition, see Eq. (4.17), (or more specially the spectral resolution of the reduced state of each subsystem) determines the preferred projective measure. That is, if the guiding state of universe is represented by Eq. (4.17) then we can define the two reduced states

$$\rho^1_{\text{red}}(t) = \sum_n c_n(t)|\phi'_n(t)\rangle_1\langle\phi'_n(t)|, \tag{4.18}$$

$$\rho^2_{\text{red}}(t) = \sum_n c_n(t)|\phi_n(t)\rangle_2\langle\phi_n(t)|. \tag{4.19}$$

This in turn defines $\{\hat{\pi}^{(1)}_n(t) = |\phi'_n(t)\rangle_1\langle\phi'_n(t)| \otimes \hat{1}\}$ and $\{\hat{\pi}^{(2)}_n(t) = \hat{1} \otimes |\phi_n(t)\rangle_2\langle\phi_n(t)|\}$, as the preferred projective measures, each local to one subsystem. In the case of degeneracy (where there is not a unique Schmidt decomposition) Dieks proposes [41, 126] that we just group the degenerate projectors into one projector, (for example $\hat{\pi}_n(t) = \sum_{i\in I_n} |\phi_i(t)\rangle_2\langle\phi_i(t)|$, where I_n is the set of indies with identical $c_i(t)$) and use this to define the preferred projective measure. To describe properties related to both systems we have to invoke the principle of property composition. That is all non-local properties (assuming no degeneracies) must be described by the preferred projective measure $\{\hat{\pi}^{(3)}_n(t) = |\phi'_{n_1}(t)\rangle_1\langle\phi'_{n_1}(t)| \otimes |\phi_{n_2}(t)\rangle_2\langle\phi_{n_2}(t)|\}$ where n here refers to (n_1, n_2).

To extend this to more than 2 subsystems they proposes that it is the reduced state spectral resolution for each subsystem, which determines the preferred projective measure for each subsystem, and any larger system are determine by the principle of compositions. That is if the universe has a m^{th}-order factorization $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_m$ then the spectral resolution for each subsystems $\rho^\alpha_{\text{red}}(t) = \sum_{n_\alpha} w_{n_\alpha}\hat{\pi}_{n_\alpha}(t)$ determines the overall preferred projective measure for the universe by the principle of composition. Thus the preferred projective measure for a property of the universe is

$$\{\hat{\pi}^{(\text{uni})}_n(t) = \hat{\pi}_{n_1}(t)\hat{\pi}_{n_2}(t)...\hat{\pi}_{n_m}(t)\}, \tag{4.20}$$

where $n = (n_1, n_2, ..., n_m)$. This may seem like it has rectified the problem of choice, but choice still remains here in the choice of the preferred factorization. Actually in Dieks' earlier proposal he assumed that factorization was arbitrary and we should be able to give definite status to all factorizations (thus choice would completely disappear), but in 1995 Bacciagaluppi [3] showed that if this is the case then a Kochen-Specker type contradiction exists. Because of this Dieks has changed his stance and in Ref. [43] he assumes a preferred factorization, which he now calls the atomic factorization. In this factorization each subsystems represents the atomic components of the universe and all properties for composite systems must obey the principle of composition. This is motivated by the analogy; in a classical systems all physical properties of a composite system (eg a molecule) depend on the properties of their elementary constituents (atoms).

I would also like to note that in this variant (for non degenerate projectors) the property state takes the form

$$|\Psi_n(t)\rangle = |\phi_{n_1}\rangle_1 \otimes |\phi_{n_2}\rangle_2 \otimes ... \otimes |\phi_{n_m}\rangle_m \tag{4.21}$$

and represent the actual state of the universe. Here we see that, unlike the Bub variant, for a 2^{nd} order factorization we can have property states (that have definite values for properties for both the system and the rest of the world) with non-perfect correlations.

The entropy (Spekkens-Sipe) variant

This variant arises from the work of Spekkens and Sipe [113, 114]. As in the above variant we still have to assume a preferred factorization of the universe (thus presently there is no way to get rid of choice completely). However instead of proposing a preferred projective measuere, they assume that the decomposition of the guiding state into property states is of product form. That is

$$|\Psi(t)\rangle = \sum_n c_n(t)|\Psi_n(t)\rangle \tag{4.22}$$

where the property state is

$$|\Psi_n(t)\rangle = |\phi_n\rangle_1 \otimes |\phi_n\rangle_2 \otimes ... \otimes |\phi_n\rangle_m \tag{4.23}$$

They propose that given a preferred factorization and a product decomposition we can define a measure for entropy as

$$S_{|\Psi(t)\rangle} = -\sum_n |c_n(t)|^2 \log(|c_n(t)|^2). \tag{4.24}$$

They then assume that the preferred decomposition is the one that minimizes this entropy. This is primarily motivated by the fact that by minimizing the entropy we are actually determining which product decomposition has the least possibilities for the property state. Spekkens and Sipe show that if the preferred factorization has only two factor spaces, then the decomposition that minimizes the entropy is the bi-orthogonal decomposition. However, if the preferred factorization has N factor spaces, then if there is a bi-orthogonal decomposition with respect to some bi-factorization that is a coarse-graining of the N-partite factorization, and this decomposition happens to be a product decomposition with respect to the N-partite factorization, then this decomposition minimizes the entropy.

4.1.2 Properties represented by POMs

In a recent paper [62] Wiseman and myself have shown that there is actually *more* choices to be made than previously realized. Specifically, it is not necessary to restrict the properties to be of the form displayed in Eq. (4.2). We can also consider properties described by POM (see Eq. (4.1)). This extra choice is motivated by the fact that in the orthodox interpretation of quantum mechanics we can extend the theory of measurement from that using projective measures to that using positive operator measures (POMs) (see chapter 2).

A typical example (in the orthodox interpretation) of an observable made from a POM is

$$Z = \{(z_n, \frac{1}{B}|z_n\rangle\langle z_n|)\}, \tag{4.25}$$

where $\{|z_n\rangle\}$ forms an overcomplete basis in $\mathcal{H}_{\text{uni}}$, with $\sum_n |z_n\rangle\langle z_n| = \hat{1}B$. For argument sake lets assume that we can represent the universe by the two states $(|\Psi_n(t)\rangle, |\Psi(t)\rangle)$, where

$$|\Psi_n(t)\rangle = \frac{\frac{1}{\sqrt{B}}|z_n\rangle\langle z_n|\Psi(t)\rangle}{\sqrt{\Pr(z_n,t)}}, \tag{4.26}$$

with

$$\Pr(z_n,t) = \frac{1}{B}\langle\Psi(t)|z_n\rangle\langle z_n|\Psi(t)\rangle, \tag{4.27}$$

and the guiding state, $|\Psi(t)\rangle$, still obeys the Schrödinger equation. That is, the property state is effectively the conditional composite state for POM observables in the orthodox interpretation (see section 2.2.2). In doing this we find that if the universe is in this property state then a definite value can't be attributed to Z. This is due to the fact that $\{|\Psi_n(t)\rangle\}$, for POM observables, are not generally orthogonal. That is, with a finite probability when the universe is described by this property state, in the orthodox theory, a measurement of Z is not certain. This clearly disagrees with the above description of modal dynamics. Thus we cannot treat observables represented by non-orthogonal states in the same manner as orthogonal states.

Also the above actually does not represent all POM-type observables (POMs are not always generated from non-orthogonal states). Given any set of projectors $\{\hat{\Pi}_n(t)\}$ in a larger Hilbert space $\mathcal{K} = \mathcal{H}_{\text{uni}} \otimes \mathcal{H}_{\text{aux}}$ where $\mathcal{H}_{\text{aux}}$ is some auxiliary Hilbert space, it is well know that POM can always be found by [84]

$$\hat{F}_n(t) = \text{Tr}_{\text{aux}}[(\hat{1}_{\text{uni}} \otimes \rho_{\text{aux}})\hat{\Pi}_n(t)] \tag{4.28}$$

where ρ_{aux} is a state in $\mathcal{H}_{\text{aux}}$. For simplicity we define it as $\rho_{\text{aux}} = |\phi\rangle\langle\phi|$ (see section 2.3). Thus we have the dilemma that not all observables can be described by the current projective type modal theories.

To extend modal theory to include POM observables we invoke Naimark's theorem [24, 84]. This basically says that given a POM $\{F_n(t)\}$ it is always possible to generate a projective measure $\{\hat{\Pi}_n(t)\}$ in a larger Hilbert space $\mathcal{H}_{\text{uni}} \otimes \mathcal{H}_{\text{aux}}$, where $\dim[\mathcal{H}_{\text{aux}}]$ is not necessarily equal to $\sum_n 1$. Mathematical this reads as, we can define a projector $\hat{\Pi}_n(t)$ such that

$$\text{Tr}_{\mathcal{H}_{\text{uni}}}[|\Psi(t)\rangle\langle\Psi(t)|\hat{F}_n(t)] = \text{Tr}_{\mathcal{K}}[|\Psi(t)\rangle\langle\Psi(t)| \otimes |\phi\rangle\langle\phi|\hat{\Pi}_n(t)], \tag{4.29}$$

or in another way

$$\hat{F}_n(t) = \langle\phi|\hat{\Pi}_n(t)|\phi\rangle, \tag{4.30}$$

for all $|\Psi(t)\rangle \in \mathcal{H}_{\text{uni}}$ and for all possible $\hat{F}_n(t)$ $(n = 1, ..., N)$. $|\phi\rangle\langle\phi|$ is called the Naimark projection of $\mathcal{K}$ onto $\mathcal{H}_{\text{uni}}$. To work out the set $\{\hat{\Pi}_n(t)\}$ it is necessary to introduce another projector $\hat{\Pi}_{N+1}(t)$, such that

$$\sum_n^{N+1} \hat{\Pi}_n(t) = \hat{1}_{\text{uni+aux}}, \tag{4.31}$$

and

$$\hat{\Pi}_n(t)\hat{\Pi}_m(t) = \hat{\Pi}_n(t)\delta_{nm}, \tag{4.32}$$

is satisfied for n, $m = 1, ..., N+1$. The set of projectors in this enlarged Hilbert space is labeled the Naimark extension of $\hat{F}_n(t)$ [84]. A worked example of this is shown later in this section.

We now propose that to calculate modal dynamics for POMs, the observables defined by Eqs. (4.1) and (4.25) become properties in $\mathcal{K}$. The set of properties which are objectively real is now defined by

$$Z(t) = \{(z_n, \hat{\Pi}_n(t))\}, \tag{4.33}$$

and if z_n is a real (or complex) number this is represented by the Hermitian (or normal) operator

$$\hat{Z}(t) = \sum_n z_n \hat{\Pi}_n(t). \tag{4.34}$$

I would like to note here that the value of z_{N+1} is arbitrary. The guiding state becomes

$$|\Phi(t)\rangle = |\Psi(t)\rangle \otimes |\phi\rangle, \tag{4.35}$$

where $|\Psi(t)\rangle$ is still the solution to the Schrödinger equation (Eq. (4.10)). With this guiding state and the enlarged-Hilbert space projector $\hat{\Pi}_n(t)$ we can write the probability for $Z(t)$ having the definite value z_n as

$$\Pr(z_n, t) \quad = \quad \langle\Phi(t)|\hat{\Pi}_n(t)|\Phi(t)\rangle, \tag{4.36}$$

where $\Pr(z_{N+1}, t) = 0$ for all time as the projector $\hat{\Pi}_{N+1}(t)$ by definition projects into the null space of $|\Phi(t)\rangle$. The property states are defined in $\mathcal{K}$ as

$$|\Phi_n(t)\rangle = \frac{\hat{\Pi}_n(t)|\Phi(t)\rangle}{\sqrt{\Pr(z_n, t)}} \tag{4.37}$$

which as in the projective case now forms an orthogonal (distinguishable) set. Thus the guiding state can be written as

$$|\Phi(t)\rangle = \sum_n \sqrt{\Pr(z_n, t)}|\Phi_n(t)\rangle. \tag{4.38}$$

Here we see that to explain properties represented by POM elements, we have to enlarge the Hilbert space of the universe and in this larger Hilbert space we can define both a guiding wave and the property state. That is, the universe is now described by the two states ($|\Phi_n(t)\rangle$, $|\Phi(t)\rangle$). Thus the standard analysis of modal dynamics now applies. However, because the property state is generally in an entangled state (between the universe and the auxiliary system), we can no longer really call it the state of the universe. This may raise interpretational difficulties, but I will leave this question for later work and just treat the above as a purely mathematical procedure to dealing with properties representing POM.

To illustrate Naimark's theorem I will consider a simple example, a universe consisting of only a spin-1/2 system, and consider the observable Z defined by

$$Z = \{(z_n, \hat{F}_n = \frac{2}{3}|z_n\rangle\langle z_n|)\}, \tag{4.39}$$

with n being 1, 2, and 3 and $z_n = \exp(i2\pi n/3)$ [84]. The states $|z_n\rangle$ are defined by

$$|z_n\rangle = \frac{1}{\sqrt{2}}\Big(z_n|\downarrow\rangle + z_n^*|\uparrow\rangle\Big), \tag{4.40}$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of σ_z (the Pauli spin matrix). In the Bloch sphere these states all lay in the $x - y$ plane with an angular separation of $2\pi/3$.

Using Naimark's theorem we extend this 2-dimensional Hilbert space to a 4-dimensional Hilbert space, where it can be shown (using Eqs. (4.31) and (4.32)) that the four projectors are [84]

$$\begin{aligned}
\hat{\Pi}_1 &= \hat{F}_1 \otimes |\phi\rangle\langle\phi| - \sqrt{2}/3|z_1\rangle\langle z_2| \otimes |\phi\rangle\langle\phi'| - \sqrt{2}/3|z_2\rangle\langle z_1| \otimes |\phi'\rangle\langle\phi| + 1/3|z_2\rangle\langle z_2| \otimes |\phi'\rangle\langle\phi'|, \\
\hat{\Pi}_2 &= \hat{F}_2 \otimes |\phi\rangle\langle\phi| + \sqrt{2}/3|z_2\rangle\langle z_2| \otimes |\phi\rangle\langle\phi'| + \sqrt{2}/3|z_2\rangle\langle z_2| \otimes |\phi'\rangle\langle\phi| + 1/3|z_2\rangle\langle z_2| \otimes |\phi'\rangle\langle\phi'|, \\
\hat{\Pi}_3 &= \hat{F}_3 \otimes |\phi\rangle\langle\phi| + \sqrt{2}/3|z_3\rangle\langle z_2| \otimes |\phi\rangle\langle\phi'| + \sqrt{2}/3|z_2\rangle\langle z_3| \otimes |\phi'\rangle\langle\phi| + 1/3|z_2\rangle\langle z_2| \otimes |\phi'\rangle\langle\phi'|, \\
\hat{\Pi}_4 &= \hat{1} \otimes |\phi'\rangle\langle\phi'| - |z_2\rangle\langle z_2| \otimes |\phi'\rangle\langle\phi'|. \qquad (4.41)
\end{aligned}$$

The two states $|\phi\rangle$ and $|\phi'\rangle$ form a set of orthogonal basis states of the auxiliary Hilbert space and $\hat{\Pi}_4$ is an added projector needed to complete the set of projectors.

The modal dynamics for these states is formulated as follows. By Eq. (4.35) and Eq. (4.37) the three possible property states are $\{|\Phi_n(t)\rangle = \hat{\Pi}_n|\phi\rangle|\Psi(t)\rangle/\sqrt{\Pr(z_n,t)}\}$, where $n = 1, 2, 3$. The fourth projector is not included as for all possible states it projects into the null space of $|\Phi(t)\rangle$. The observable $\hat{Z}$ becomes the property $\hat{Z} = \sum_n z_n \hat{\Pi}_n$, and for this example the possible values are; $z_1 = e^{i2\pi/3}$, $z_2 = e^{i4\pi/3}$, $z_3 = 1$, and depending on which property state represents the extended universe determines which one of these three values Z has.

4.2 Dynamics in modal interpretations

In the above section I have presented the general outline of the modal interpretation of quantum mechanics. We saw that by introducing an extra quantum state, the property state, the collapse postulate in the orthodox interpretation is no longer needed. However, I have said nothing about how this state evolves in time. In this section I will briefly outline how this states evolution can be calculated for both the projective case (which is basically a reproduction of the results presented in [11, 120, 128] and generalized in [37, 4, 121] to include time-dependent projectors) and the POM case [62]. We will find it contains both a smooth evolution, due to the time evolution on the projectors (or POM elements), and a stochastic evolution, jumps between different values of n.

4.2.1 The projective case

Before reproducing the standard results for the projector case, lets assume we have a system (or universe) consisting of N distinct states, representing some property $Z(t)$, with possible values $\{z_n\}$. We will also assume that there is a probability distribution $\Pr(z_n, t)$ for this system. That is $\Pr(z_n, t)$ is the probability that the property will have the value z_n at time t. We will now show that we can associate with this probability distribution a stochastic trajectory, jumping between the different states z_n, for the value of this property $(v(Z(t), t))$.

To do this we start by defining a parameter T_{nm} as

$$T_{nm}(t) = \lim_{dt\to 0}[\Pr(z_n, t + dt|z_m, t) - \Pr(z_n, t|z_m, t)]/dt. \qquad (4.42)$$

Here $\Pr(z_n, t + dt|z_m, t)$ is a conditional probability and is read as the probability of the system being in state z_n at time $t + dt$ given it was in state z_m at time t. From this definition it follows that

$$\sum_n T_{nm} = 0. \qquad (4.43)$$

For distinct states the conditional probability $\Pr(z_n, t|z_m, t)$ must be 0 for $n \neq m$. This allows us to interpret $T_{nm}(t)dt = \Pr(z_n, t + dt|z_m, t)$, for $n \neq m$, as a transitional probability, so we call $T_{nm}(t)$

the transition rate, it is a measure of the rate at which z_n gains probability at the expense of z_m. For $n = m$, $\Pr(z_n, t + dt|z_n, t) = 1$ and T_{nn} (which is negative) is a measure of the rate at which state z_n loses probability. Once these transitions rates are known we can easily imagine that given an initial value for $Z(t_0)$ at time t_0, $\upsilon(Z(t_0), t_0) = z_n$ we can calculate a stochastic trajectory for $\upsilon(Z(t), t)$ which when averaged over all trajectories agrees with the probability distribution.

To determine these transition rates we use

$$\Pr(z_n, t + dt) = \sum_m \Pr(z_n, t + dt|z_m, t)\Pr(z_m, t) \tag{4.44}$$

and

$$\Pr(z_n, t) = \sum_m \Pr(z_m, t + dt|z_n, t)\Pr(z_n, t), \tag{4.45}$$

to find a evolution equation for $\Pr(z_n, t)$. Taking the difference of these two equations gives

$$\Pr(z_n, t + dt) - \Pr(z_n, t) = \sum_m \Pr(z_n, t + dt|z_m, t)\Pr(z_m, t) - \Pr(z_m, t + dt|z_n, t)\Pr(z_n, t). \tag{4.46}$$

Dividing by dt and taking the continuous limit, assuming that this limit exists (T_{nm} is well defined), we find that $\Pr(z_n, t)$ is differentiable and obeys the master equation

$$d_t\Pr(z_n, t) = \sum_m [T_{nm}(t)\Pr(z_m, t) - T_{mn}\Pr(z_n, t)]. \tag{4.47}$$

Defining a probability current $J_{nm}(t)$ as

$$J_{nm}(t) = T_{nm}(t)\Pr(z_m, t) - T_{mn}\Pr(z_n, t), \tag{4.48}$$

results in $J_{nm}(t) = -J_{mn}(t)$ and allows us to rewrite the probability master equation as

$$d_t Pr(z_n, t) = \sum_m J_{nm}(t). \tag{4.49}$$

Given $J_{nm}(t)$ and $Pr(z_n, t)$ for all n it is possible to calculate the transition rates. Actually there is an infinite range of solutions. One solution, chosen by Bell [11] is as follows: For $J_{nm}(t) < 0$,

$$T_{nm}(t) = 0, \tag{4.50}$$

$$T_{mn}(t) = -J_{nm}(t)/\Pr(z_n, t), \tag{4.51}$$

and for $J_{nm}(t) > 0$

$$T_{nm}(t) = J_{nm}(t)/\Pr(z_m, t), \tag{4.52}$$

$$T_{mn}(t) = 0. \tag{4.53}$$

The other solutions are found by adding an extra term $T^0_{nm}(t)$ to $T_{nm}(t)$ [4, 37, 128], where $T^0_{nm}(t)$ is constrained only by

$$T^0_{nm}(t)\Pr(z_m, t) - T^0_{mn}(t)\Pr(z_n, t) = 0. \tag{4.54}$$

To make the link with quantum mechanics we say that the N distinct states are the property states $\{|\Psi_n(t)\rangle\}$, and the property or properties which are objectively real form the group $\mathcal{G}$, defined by elements

$$\hat{Z}(t) = \sum_n z_n \hat{\pi}_n(t), \tag{4.55}$$

with the value of property $\hat{Z}(t)$ being the corresponding z_n. That is, for the property state $|\Psi_n(t)\rangle$, $\upsilon(Z(t),t) = z_n$. The evolution of $|\Psi_n(t)\rangle$ (the stochastic trajectory for $\upsilon(Z(t),t)$, jumping between different z_n) is determined by the rates $T_{nm}(t)$, which themselves depend upon $J_{nm}(t)$.

To work out $J_{nm}(t)$ we take the time derivative of Eq. (4.5). Doing this we obtain the differential equation

$$d_t\text{Pr}(z_n,t) = 2\text{Im}[\langle\Psi(t)|\hat{\pi}_n(t)\hat{H}_{\text{uni}}(t)|\Psi(t)\rangle]/\hbar + \langle\Psi(t)|d_t[\hat{\pi}_n(t)]|\Psi(t)\rangle, \tag{4.56}$$

where we have used Eq. (4.10). This can be simplified by defining the Hermitian operator, $\hat{R}(t)$, for which

$$d_t\hat{\pi}_n(t) = -\frac{i}{\hbar}[\hat{R}(t),\hat{\pi}_n(t)]. \tag{4.57}$$

This allows us to rewrite Eq. (4.56) as

$$d_t\text{Pr}(z_n,t) = 2\text{Im}\{\langle\Psi(t)|\hat{\pi}_n(t)[\hat{H}_{\text{uni}}(t) - \hat{R}(t)]|\Psi(t)\rangle\}/\hbar. \tag{4.58}$$

Comparing this with Eq. (4.49) and using the fact that $\sum_m \hat{\pi}_m(t) = \hat{1}$, one possible probability current (that was chosen by Bell [11]) is

$$J_{nm}(t) = 2\text{Im}\{\langle\Psi(t)|\hat{\pi}_n(t)[\hat{H}_{\text{uni}}(t) - \hat{R}(t)]\hat{\pi}_m(t)|\Psi(t)\rangle\}/\hbar. \tag{4.59}$$

Note that this is only one of infinitely many possible currents, as we can add any current $J^0_{nm}(t)$ to $J_{nm}(t)$ which satisfies $\sum_m J^0_{nm} = 0$, to give a valid probability current. For the purposes of this thesis I will only consider the simple solutions (not containing the extra $T^0_{nm}(t)$ and $J^0_{nm}(t)$ terms). For a discussion about these solution see Refs. [128] and [37].

Once we have $J_{nm}(t)$ we can work out the stochastic trajectories for the values of the property Z, $\upsilon(Z(t),t)$. This in turn allows us to work out a trajectory for the property states, this I will denote by $|\Psi_{\upsilon(Z(t),t)}(t)\rangle$. It represents the actual evolution of the universe, and in the second part of this thesis we will see that the evolution equation $d_t|\Psi_{\upsilon(Z(t),t)}(t)\rangle$ is non-linear and is equivalent to a stochastic Schrödinger equation for an appropriately chosen preferred projective measure.

4.2.2 The POM case

To work out the transitional rates for properties described by POM elements, as in the projector case we need to find the probability current. Taking the time derivative of Eq. (4.36) and using the Schrödinger equation (Eq. (4.10)) we get

$$d_t\text{Pr}(z_n,t) = 2\text{Im}\{\langle\Phi(t)|\hat{\Pi}_n(t)[\hat{H}_{\text{uni}}(t)\otimes\hat{1}_{\text{aux}} - \hat{R}'(t)]|\Phi(t)\rangle\}/\hbar. \tag{4.60}$$

Comparing this with Eq. (4.49) and using the fact that $\sum_m \hat{\Pi}_m(t) = \hat{1}$, Bell's probability current is

$$J_{nm}(t) = 2\text{Im}\{\langle\Phi(t)|\hat{\Pi}_n(t)[\hat{H}_{\text{uni}}(t)\otimes\hat{1}_{\text{aux}} - \hat{R}'(t)]\hat{\Pi}_m(t)|\Phi(t)\rangle\}/\hbar, \tag{4.61}$$

where $\hat{R}'(t)$ is Hermitian and defines the evolution of the projectors $\hat{\Pi}_n(t)$, by

$$d_t\hat{\Pi}_n = -\frac{i}{\hbar}[\hat{R}'(t),\hat{\Pi}_n]. \tag{4.62}$$

In this case an actual trajectory is represented by $|\Phi_{\upsilon(Z(t),t)}(t)\rangle$. In the second part of the thesis we will see that the evolution equation for this property state, for an appropriately chosen POM, gives the Diósi, Gisin, and Strunz stochastic Schrödinger equation [48], in the non-Markovian limit and Gisin and Percival stochastic Schrödinger equation in the Markovian limit [76, 77]. I would also like to point out here that this equation corresponds to the heterodyne quantum trajectory in the orthodox interpretation [137].

4.3 Bohmian Mechanics

In Bohmian mechanics [14, 15, 35] the preferred projective measure is the position projective measure. That is the (dimensionless) position $\{x_j\}$ (vector notation) of the system is assumed to be objectively real and the wavefunction $\Psi(\{x_j\},t) = \langle\{x_j\}|\Psi(t)\rangle$ is then interpreted as an objective real field which guides the position in a non-classical way. Here $\langle\{x_j\}|$ is the dual vector of $|\{x_j\}\rangle = |x_1\rangle \otimes |x_2\rangle \otimes .. \otimes |x_m\rangle$, a position eigenstate in a m^{th}-order factorization of the universe. In Bohm's original papers he showed that this non-classical behavior could be represented by an extra potential in the Hamiltonian-Jacobi equation, the quantum potential which depends on $\Psi(\{x_j\},t)$.

In this thesis I will not introduce the quantum potential, but instead describe Bohmian trajectories with reference to a continuous probability current $J_k(\{x_j\},t)$. I will also consider $|\Psi(t)\rangle$ to belong to a tensor product $\mathcal{H}_x \otimes \mathcal{H}'$, where $\mathcal{H}_x$ is the Hilbert space containing the position eigenstates $|\{x_j\}\rangle$. Then the wavefunction becomes a vector given by

$$|\tilde{\psi}(\{x_j\},t)\rangle = \langle\{x_j\}|\Psi(t)\rangle \in \mathcal{H}'. \tag{4.63}$$

This allows us to take into account a Hilbert space for the universe which is larger than that for the positions of the system (for example one that includes spin as well). Here we see that $|\{x_j\}\rangle|\tilde{\psi}(\{x_j\},t)\rangle$ is the continuous equivalent of our unnormalized property state, and the set of properties with definite status, are the (dimensionless) position operators $\{\hat{X}^{(j)}\}$. That is, in my notation, the j^{th} property is

$$Z \equiv \{X_j\} = \Big\{ \Big(\{x_j\}, \hat{\pi}(\{x_j\})d\{x_j\} = |\{x_j\}\rangle\langle\{x_j\}|d\{x_j\} \otimes \hat{1}_{\text{rest}} \Big) \Big\}, \tag{4.64}$$

where the results here are not a number but a string of continuous numbers. Thus the possible values for the property Z at time t [denoted $\upsilon(Z,t) = \{\upsilon(X_j,t)\}$] are $\{x_j\}$.

With $\hat{\pi}(\{x_j\})$ we can define the probability density as

$$P(\{x_j\},t) = \langle\Psi(t)|\hat{\pi}(\{x_j\})|\Psi(t)\rangle \tag{4.65}$$

which is a conserved quantity that obeys the continuity equation

$$\partial_t P(\{x_j\},t) \quad = \quad -\sum_k \frac{\partial}{\partial x_k} J_k(\{x_j\},t). \tag{4.66}$$

The functional form of $J_k(\{x_j\},t)$ depends on the form of $H_{\text{uni}}(t)$. As in the discrete (modal) case there is not a unique solution to $J_k(\{x_j\},t)$. For example in three dimensions we can add any vector field $\nabla \times \mathbf{A}(\mathbf{x},t)$ to $\mathbf{J}(\mathbf{x},t)$ as $\nabla \cdot \nabla \times \mathbf{A}(\mathbf{x},t) \equiv 0$.

To work out the actually trajectory $\{\upsilon(X_k,t)\}$ we use the fact that the continuous probability current $J_k(\{x_j\},t)$, by definition, is related to the velocity field $\mathrm{v}_k(\{x_j\},t)$ by

$$J_k(\{x_j\},t) = \mathrm{v}_k(\{x_j\},t)P(\{x_j\},t). \tag{4.67}$$

This then allows us to determine the Bohmian trajectories for the set of values $\{\upsilon(X_k,t)\}$ by

$$d_t\upsilon(X_k,t) = \mathrm{v}_k(\{x_j\},t)|_{\{x_j=\upsilon(X_j,t)\}}. \tag{4.68}$$

Because this is a deterministic equation, probability enters only through the initial conditions, $\{\upsilon(X_k,0)\}$. Eq. (4.68) is only one of the possible continuous trajectories which satisfy Eq. (4.65), other possibilities include stochastic approaches; see Ref. [128] and references within.

It should be noted that so far nothing has specified that the properties $\{X_j\}$ must be positions, and in fact Brown and Hiley in Ref. [19] develop a formalism where $\{X_j\}$ can be either position or momentum. As examples, they consider a simple universe (a single 1-D particle) and derive $d_t\upsilon(X,t)$ and $d_t\upsilon(P,t)$ for a linear, quadratic and cubic potential.

4.4 Continuous modal dynamics

In this section I investigate the continuum limit of modal dynamics. This has been previously done by Sudbery and Vink in Refs. [120] and [128] respectively, where it was shown that Bohmian mechanics can be obtained by taking the appropriate decomposition (property states). In this section I will briefly outline their work, then extend it to show that the Brown and Hiley's generalization of Bohmian mechanics to include the momentum representations [19] is not the continuum limit of Bell's modal dynamics. I also present an alternative (I believe easier) method for finding the continuous trajectories, that works when the modal dynamics has a continuum limit [62].

4.4.1 When Bohmian-type theories are the continuous limit of modal dynamics

To demonstrate that the modal dynamics does give Bohmian mechanics as its continuum limit, let's first consider the Hamiltonian

$$\hat{H}_{\mathrm{uni}}(t) = \Big[\hat{A}(t) + \sum_k \hat{B}_k(t)\hat{Y}_k + \sum_k \hat{Y}_k[\hat{B}_k(t)]^\dagger + \sum_k \hat{C}_k(t)\hat{Y}_k\hat{Y}_k + \sum_k \hat{Y}_k\hat{Y}_k\hat{C}_k^\dagger(t)\Big], \tag{4.69}$$

where $\hat{A}(t)$, $\hat{B}_k(t)$ and $\hat{C}_k(t)$ are arbitrary functions of the operators $\{\hat{X}_k\}$ and the rest of the universe, and $\{\hat{Y}_k\}$ are the conjugate operators to $\{\hat{X}_k\}$. That is, $[\hat{Y}_j, \hat{X}_k] = -i\delta_{jk}$. The set $\{\hat{Y}_k\}$ can be view as dimensionless momentum operators.

To calculate $d_t\upsilon(X_k,t)$ in Eq. (4.68) we need to calculate the velocity field $\mathrm{v}_k(\{x_j\},t)$, which in turn requires calculation of $J_k(\{x_j\},t)$. Taking the derivative of Eq. (4.65) and using the Schrödinger equation (Eq. (4.10)) we can write

$$\begin{aligned} d_tP(\{x_j\},t) &= 2\mathrm{Im}[\langle\Psi(t)|\{x_j\}\rangle\langle\{x_j\}|\hat{H}_{\mathrm{uni}}(t)|\Psi(t)\rangle]/\hbar \\ &= -2\sum_k \mathrm{Re}\Big[\langle\tilde{\psi}(\{x_j\},t)|\{2\mathrm{Re}[\hat{B}_k(\{x_j\},t)]\partial_{x_k} \\ &\quad +\partial_{x_k}\mathrm{Re}[\hat{B}_k(\{x_j\},t)]\}|\tilde{\psi}(\{x_j\},t)\rangle\Big]/\hbar \\ &\quad -2\sum_k \mathrm{Im}\Big[\langle\tilde{\psi}(\{x_j\},t)|\{2\mathrm{Re}[\hat{C}_k(\{x_j\},t)]\partial^2_{x_k} + 2\partial_{x_k}\hat{C}_k^\dagger(\{x_j\},t)\partial_{x_k} \\ &\quad +\partial^2_{x_k}\hat{C}_k^\dagger(\{x_j\},t)\}|\tilde{\psi}(\{x_j\},t)\rangle\Big]/\hbar, \end{aligned} \tag{4.70}$$

which can be simplified to

$$\begin{aligned} d_tP(\{x_j\},t) &= -\sum_k \partial_{x_k}\Big\{\mathrm{Re}\Big[\langle\tilde{\psi}(\{x_j\},t)|\{2\mathrm{Re}[\hat{B}_k(\{x_j\},t)] - 4i\mathrm{Re}[\hat{C}_k(\{x_j\},t)]\partial_{x_k} \\ &\quad -2i\partial_{x_k}[\hat{C}_k^\dagger(\{x_j\},t)]\}|\tilde{\psi}(\{x_j\},t)\rangle\Big]\Big\}/\hbar. \end{aligned} \tag{4.71}$$

Comparing this with Eq. (4.66) gives one solution for $J_k(\{x_j\},t)$,

$$\begin{aligned} J_k(\{x_j\},t) &= \mathrm{Re}\Big[\langle\tilde{\psi}(\{x_j\},t)|\{2\mathrm{Re}[\hat{B}_k(\{x_j\},t)] - 4i\mathrm{Re}[\hat{C}_k(\{x_j\},t)]\partial_{x_k} \\ &\quad -2i\partial_{x_k}[\hat{C}_k^\dagger(\{x_j\},t)]\}|\tilde{\psi}(\{x_j\},t)\rangle\Big]/\hbar. \end{aligned} \tag{4.72}$$

For illustration we consider the 1-dimensional case and assume $\hat{A}(t) = \hat{V}(\hat{X})$, $\hat{B}(t) = 0$, and $\hat{C}(t) = \hat{C}^\dagger(t) = \hbar^2 \hat{1}/4M$, describing for example an electron in a 1-D potential, with the operator nature of $\hat{1}$ and $\hat{V}$ signifying operation on the Hilbert space for the internal structure of the electron. That is

$$\hat{H}_{\rm uni}(t) = \frac{\hbar^2 \hat{Y}^2}{2M} + \hat{V}(\hat{X}), \tag{4.73}$$

For this example $J(x,t)$ becomes

$$J(x,t) = \frac{\hbar}{M} {\rm Im}[\langle\tilde{\psi}(x,t)|\partial_x|\tilde{\psi}(x,t)\rangle], \tag{4.74}$$

and thus

$$d_t \upsilon(X,t) = \frac{\hbar}{M} {\rm Im}\Big[\frac{\langle\tilde{\psi}(x,t)|\partial_{x_k}|\tilde{\psi}(x,t)\rangle}{\langle\tilde{\psi}(x,t)|\tilde{\psi}(x,t)\rangle}\Big]\Big|_{x=\upsilon(X,t)}. \tag{4.75}$$

To simplify this we can rewrite $|\tilde{\psi}(x,t)\rangle$ as

$$|\tilde{\psi}(x,t)\rangle = \sum_j R_j(x,t)\exp[iS_j(x,t)]|j\rangle, \tag{4.76}$$

where $R_j(x,t)$ and $S_j(x,t)$ are real functions and $\{|j\rangle\}$ is an orthonormal basis set, which for example spans the Hilbert space of the internal structure of the electron. Then Eq. (4.75) simplifies to

$$d_t \upsilon(X,t) = \hbar \frac{\sum_j R_j^2(x,t)\partial_x[S_j(x,t)]}{M\sum_j R_j^2(x,t)}\Big|_{x=\upsilon(X,t)}. \tag{4.77}$$

To compare this to the modal dynamics defined in Sec. II we have to discretize x. In Refs. [128] and [120] this is done by defining a lattice of size N and lattice separation ϵ. Thus x becomes $x_n = \epsilon n$, and the projectors $\hat{\pi}_n = |\epsilon n\rangle\langle\epsilon n| \otimes \hat{1}$. With these projectors the property states Eq. (4.3) become $|\tilde{\Psi}_n(t)\rangle = |\epsilon n\rangle\langle\epsilon n|\Psi(t)\rangle = |\epsilon n\rangle|\tilde{\psi}_n(t)\rangle$ where $|\tilde{\psi}_n(t)\rangle$ is an unnormalized state representing the internal structure of the electron. Using the results of Sec. 4.2.1 the probability current is

$$J_{nm}(t) = 2{\rm Im}[\langle\tilde{\psi}_n(t)|\langle\epsilon n|\hat{H}_{\rm uni}|\epsilon m\rangle|\tilde{\psi}_m(t)\rangle]/\hbar, \tag{4.78}$$

and the discretized version of the Hamiltonian is

$$\langle\epsilon n|\hat{H}_{\rm uni}|\epsilon m\rangle = [-\hbar^2(\delta_{n,m+1} + \delta_{n,m-1} - 2\delta_{n,m})/2M\epsilon^2 + \hat{V}(\epsilon n)\delta_{n,m}]. \tag{4.79}$$

This gives

$$J_{nm}(t) = -\frac{\hbar}{M\epsilon^2}{\rm Im}\Big[\langle\tilde{\psi}_n(t)|\tilde{\psi}_{n-1}(t)\rangle\delta_{n,m+1} + \langle\tilde{\psi}_n(t)|\tilde{\psi}_{n+1}(t)\rangle\delta_{n,m-1}\Big]. \tag{4.80}$$

Taylor expanding $|\tilde{\psi}_{n-1}(t)\rangle$ and $|\tilde{\psi}_{n-1}(t)\rangle$ gives

$$J_{nm}(t) = \frac{\hbar}{M\epsilon}{\rm Im}\{\langle\tilde{\psi}_n(t)|\Delta_\epsilon[|\tilde{\psi}_n(t)\rangle]\delta_{n,m+1} - \langle\psi_n(t)|\Delta_\epsilon[|\tilde{\psi}_n(t)\rangle]\delta_{n,m-1} + O(\epsilon)\}, \tag{4.81}$$

where Δ_ϵ is the discretized version of a derivative. That is

$$\Delta_\epsilon|\tilde{\psi}_n(t)\rangle = \frac{|\tilde{\psi}_{n+1}(t)\rangle - |\tilde{\psi}_n(t)\rangle}{\epsilon}. \tag{4.82}$$

As in the continuous case we write $|\tilde{\psi}_n(t)\rangle$ in terms of the real functions $S_j(\epsilon n,t)$ and $R_j(\epsilon n,t)$ which results in $J_{nm}(t)$ becoming

$$J_{nm}(t) = \frac{\hbar}{M\epsilon}\{\sum_j R_j^2(\epsilon n)\Delta_\epsilon[S_j(\epsilon n)]\delta_{n,m+1} - \sum_j R_j^2(\epsilon n)\Delta_\epsilon[S_j(\epsilon n)]\delta_{n,m-1} + O(\epsilon)\}. \tag{4.83}$$

In the $\epsilon \to 0$ limit (continuum limit) we can neglect the higher order terms in the above expression for $J_{nm}(t)$. That is the only terms which will contribute are the transitions from m to $m-1$ (or $m+1$) state. Because of this we can write

$$J_{n(n+1)}(t) = -\frac{\hbar}{M\epsilon}\sum_j R_j^2(\epsilon n)\Delta_\epsilon[S_j(\epsilon n)] \tag{4.84}$$

$$J_{n(n-1)}(t) = \frac{\hbar}{M\epsilon}\sum_j R_j^2(\epsilon n)\Delta_\epsilon[S_j(\epsilon n)]. \tag{4.85}$$

If $\sum_j R_j^2\Delta_\epsilon[S_j(\epsilon n)] > 0$ then $J_{n(n+1)}(t) < 0$ and $J_{n(n-1)}(t) > 0$ then by Eqs. (4.50) – (4.53),

$$T_{(n+1)n}(t) = \hbar\frac{\sum_j R_j^2(\epsilon n)\Delta_\epsilon[S_j(\epsilon n)]}{M\epsilon\sum_j R_j^2(\epsilon n)} \tag{4.86}$$

$$T_{(n-1)n}(t) = 0. \tag{4.87}$$

If $\sum_j R_j^2\Delta_\epsilon[S_j(\epsilon n)] < 0$ then $J_{n(n+1)}(t) > 0$ and $J_{n(n-1)}(t) < 0$ then by Eqs. (4.50) – (4.53)

$$T_{(n+1)n}(t) = 0 \tag{4.88}$$

$$T_{(n-1)n}(t) = -\hbar\frac{\sum_j R_j^2(\epsilon n)\Delta_\epsilon[S_j(\epsilon n)]}{M\epsilon\sum_j R_j^2(\epsilon n)}. \tag{4.89}$$

These transition rates imply that in an interval dt the average displacement dx will be

$$\begin{aligned} M[dx] &= \epsilon T_{(n+1)n}dt - \epsilon T_{(n-1)n}dt \\ &= \hbar\frac{\sum_j R_j^2(\epsilon n)\Delta_\epsilon[S_j(\epsilon n)]}{M\sum_j R_j^2(\epsilon n)}dt + O(\epsilon)dt. \end{aligned} \tag{4.90}$$

Provided $S_j(\epsilon n)$ and $R_j(\epsilon n)$ are continuous, the average $M[dx]$ reduces to Eq. (4.77) as $\epsilon \to 0$. However, to show that the trajectories are smooth and deterministic from the initial $v(X, t_0)$ we require also that the dispersion $M[dx^2]$ goes to zero in the continuum limit. That is

$$M[dx^2] = \epsilon^2 T_{(n+1)n} + \epsilon^2 dt T_{(n-1)n}dt = O(\epsilon)dt \tag{4.91}$$

which goes to zero as $\epsilon \to 0$. This can be extended to the general case, Eq. (4.69).

4.4.2 When Bohmian-type theories are not the continuous limit of modal dynamics

The above demonstrates that in the continuum limit Bell's modal dynamics becomes Bohmian Mechanics. However, if we consider the Hamiltonian

$$\hat{H}_{\text{uni}} = [\kappa\hat{Y}^3 + V(\hat{X})], \tag{4.92}$$

We will show that this is not the case. This Hamiltonian is unphysical if $\hat{X}$ is position as this says we have a cubic dependence on momentum, which is not present in any natural Hamiltonians. However, if $\hat{X}$ corresponds to momentum (and $-\hat{Y}$ to position), as in Brown and Hiley's [19] extension to include the momentum representation, then this Hamiltonian is possible.

As before if we discretize x to $x_n = \epsilon n$, the probability current again will be given by Eq. (4.78). However, the discretized version of the Hamiltonian in this case is

$$\langle\epsilon n|\hat{H}_{\text{uni}}|\epsilon m\rangle = \frac{\kappa}{\epsilon^3}[i\delta_{n,m+3} - i\delta_{n,m-3} - 3i\delta_{n,m+1} + 3i\delta_{n,m-1}] + V(\epsilon n)\delta_{n,m}, \tag{4.93}$$

which results in

$$J_{nm}(t) = \frac{\kappa}{\epsilon^3}\mathrm{Im}\Big[i\delta_{n,m+3}\langle\tilde{\psi}_n(t)|\tilde{\psi}_{n-3}(t)\rangle - i\delta_{n,m-3}\langle\tilde{\psi}_n(t)|\tilde{\psi}_{n+3}(t)\rangle - 3i\delta_{n,m+1}\langle\tilde{\psi}_n(t)|\tilde{\psi}_{n-1}(t)\rangle + 3i\delta_{n,m-1}\langle\tilde{\psi}_n(t)|\tilde{\psi}_{n+1}(t)\rangle\Big]/\hbar. \quad (4.94)$$

Taylor expanding this gives a rather large expression, but since $\langle\tilde{\psi}_n(t)|\tilde{\psi}_n(t)\rangle$ is a real number and $1/\epsilon^3 >> 1/\epsilon^2$ in the $\epsilon \to 0$ limit we can ignore all orders of the Taylor expansion. This allows us to write

$$J_{nm}(t) = \frac{\kappa\langle\tilde{\psi}_n(t)|\tilde{\psi}_n(t)\rangle}{\epsilon^3}[\delta_{n,m+3} - \delta_{n,m-3} - 3\delta_{n,m+1} + 3\delta_{n,m-1}]/\hbar. \quad (4.95)$$

Since $\langle\tilde{\psi}_n(t)|\tilde{\psi}_n(t)\rangle$ is always positive, for $\kappa > 0$ the transitions rates defined in Eqs. (4.50) – (4.53) become

$$T_{(n-3)n}(t) = 0 \quad (4.96)$$
$$T_{(n+3)n}(t) = \frac{\kappa}{\hbar\epsilon^3} \quad (4.97)$$
$$T_{(n-1)n}(t) = \frac{3\kappa}{\hbar\epsilon^3} \quad (4.98)$$
$$T_{(n+1)n}(t) = 0. \quad (4.99)$$

These transition rates imply that in an interval dt the average displacement dx will be

$$M[dx] = 3\epsilon T_{(n+3)n}dt - \epsilon T_{(n-1)n}dt = 0 \quad (4.100)$$

and the dispersion will be

$$M[dx^2] = 9\epsilon^2 T_{(n+1)n} + \epsilon^2 dt T_{(n-1)n}dt = \frac{12\kappa dt}{\hbar\epsilon}, \quad (4.101)$$

which diverges as $\epsilon \to 0$. Thus the continuum limit does not exists. This implies that Brown and Hiley's [19] extension of Bohmian mechanics to include momentum (where Hamiltonians of this form can exist) is not the continuum limit of Bell's modal dynamics. It is possible that a different choice for $J_{nm}(t)$ or $T_{nm}(t)$ would allow their equations to be derived, but that is beyond the scope of this thesis. However I would like to speculate that I believe this will most likely be a different choice of $J_{nm}(t)$ as different $T_{nm}(t)$ in the $\hat{Y}^2$ case results only in the actual trajectories becoming stochastic rather than the deterministic ones given by Eq. (4.68), see Ref. [128].

4.4.3 The velocity operator technique

In the above section we have demonstrated that when using the Bell solution for the transition rates, modal dynamics for some continuous property only reduces to a deterministic theory (apart from the random initial conditions) if the Hamiltonian is at most quadratic in the conjugate variable to the property. If the property is position then this deterministic limit is Bohmian Mechanics and the trajectories are then found using Eq. (4.68), which requires calculation of the probability current density, $J_k(\{x_j\}, t)$. Here we present an alternative to this, a method to calculate $\mathrm{v}_k(\{x_j\}, t)$ directly:

$$\mathrm{v}_k(\{x_j\}, t) = \frac{\mathrm{Re}[\langle\Psi(t)|\{x_j\}\rangle\langle\{x_j\}|\hat{\mathrm{v}}_k(t)|\Psi(t)\rangle]}{\langle\Psi(t)|\{x_j\}\rangle\langle\{x_j\}|\Psi(t)\rangle}. \quad (4.102)$$

Here $\hat{\mathrm{v}}_k(t)$ is the k^{th} component of the velocity operator. This operator is defined as

$$\hat{\mathrm{v}}_k(t) = -\frac{i}{\hbar}[\hat{X}_k, \hat{H}_{\mathrm{uni}}(t)]. \quad (4.103)$$

To show that this does give the same trajectories as Bohmian mechanics, we note that the top line of Eq. (4.102) should be $J_k(\{x_j\},t)$ by definition. Now using the Hamiltonian defined in Eq. (4.69), the velocity operator is

$$\hat{\mathrm{v}}_k(t) \quad = \quad \{\hat{B}_k(t) + [\hat{B}_k(t)]^\dagger + 2\hat{C}_k(t)\hat{Y}_k + 2\hat{Y}_k\hat{C}_k^\dagger(t)\}/\hbar. \tag{4.104}$$

This results in the following velocity field

$$\begin{aligned}\mathrm{v}_k(\{x_j\},t) \quad &= \quad \frac{1}{\hbar\langle\tilde{\psi}(\{x_j\},t)|\tilde{\psi}(\{x_j\},t)\rangle} \times \mathrm{Re}\Big[\langle\tilde{\psi}(\{x_j\},t)|\{2\mathrm{Re}[\hat{B}_k(\{x_j\},t)] \\ &\quad -4i\mathrm{Re}[\hat{C}_k(\{x_j\},t)]\partial_{x_k} - 2i\partial_{x_k}[\hat{C}_k^\dagger(\{x_j\},t)]\}|\tilde{\psi}(\{x_j\},t)\rangle\Big].\end{aligned} \tag{4.105}$$

Comparing this with Eq. (4.72) we see that the top line is indeed $J_k(\{x_j\},t)$. This completes our proof that our velocity method does generate the same trajectories as Bohmian mechanics for Hamiltonians of the form displayed in Eq. (4.69). However, by extending this argument to higher orders it can shown that our velocity method does not agree with Bohmian mechanics. That is, this is another example of a method that only works for Hamiltonians that do not contain terms of order $\hat{Y}_k^3$ and higher.

4.5 Simple example: Harmonic oscillator

4.5.1 Husimi POM

To illustrate modal dynamics for properties representing POMs, I will investigate a simple model; a universe consisting of a one dimensional harmonic oscillator of frequency ω. This system of course has the Hamiltonian

$$\hat{H}_{\mathrm{uni}}(t) = \frac{1}{2}m\omega^2\hat{Q}^2 + \frac{\hat{P}^2}{2m}, \tag{4.106}$$

where m is the mass of the oscillator and $\hat{Q}$ and $\hat{P}$ are positions and momentum operators respectively. This can be written in terms of the dimensionless position ($\hat{X}$) and momentum operators ($\hat{Y}$) as

$$\hat{H}_{\mathrm{uni}}(t) = \frac{\hbar\omega}{2}(\hat{X}^2 + \hat{Y}^2), \tag{4.107}$$

where

$$\hat{X} = \sqrt{\frac{m\omega}{\hbar}}\,\hat{Q} \tag{4.108}$$

and

$$\hat{Y} = \sqrt{\frac{1}{m\omega\hbar}}\,\hat{P}. \tag{4.109}$$

If we now define an annihilation operator $\hat{a}$ as

$$\hat{a} = \frac{(\hat{X} + i\hat{Y})}{\sqrt{2}}, \tag{4.110}$$

we can write the Hamiltonian as

$$\hat{H}_{\mathrm{uni}}(t) = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}). \tag{4.111}$$

It should be noted that the commutation relation for the annihilation and creation operator is

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{4.112}$$

The preferred POM we consider is the Husimi POM [87], which has infinitesimal POM elements (or effects) given by

$$\hat{F}(a) = \frac{1}{\pi}|a\rangle\langle a|d^2a, \tag{4.113}$$

where $|a\rangle$ is defined by the eigenvalue equation $\hat{a}|a\rangle = a|a\rangle$. The observable (one of the many) we associate with this POM is

$$A = \{(a, \frac{1}{\pi}|a\rangle\langle a|d^2a)\}, \tag{4.114}$$

with complex results a. This result can be represented as a point in some complex space (phase space), which we denote by (x^+, y^-). That is $a = x^+ + iy^-$ in complex notation. In the orthodox theory this POM corresponds to a measure of both position and momentum with minimal additional uncertainty [84].

Before analyzing the modal dynamics that corresponds to this POM, we need to define a few operators that act in the enlarged Hilbert space $\mathcal{K}$. We define

$$\hat{x}^+ = [\hat{a} + \hat{a}^\dagger + \hat{b} + \hat{b}^\dagger]/2, \tag{4.115}$$
$$\hat{x}^- = [\hat{a} + \hat{a}^\dagger - \hat{b} - \hat{b}^\dagger]/2, \tag{4.116}$$
$$\hat{y}^+ = [-i\hat{a} + i\hat{a}^\dagger - i\hat{b} + i\hat{b}^\dagger]/2, \tag{4.117}$$
$$\hat{y}^- = [-i\hat{a} + i\hat{a}^\dagger + i\hat{b} - i\hat{b}^\dagger]/2, \tag{4.118}$$

where $\hat{b}$ and $\hat{b}^\dagger$ are annihilation and creation operators which act in $\mathcal{H}_{\mathrm{aux}}$. These four operators obey the commutator relations

$$[\hat{x}^+, \hat{y}^+] = [\hat{x}^-, \hat{y}^-] = i, \tag{4.119}$$
$$[\hat{x}^+, \hat{y}^-] = [\hat{x}^-, \hat{y}^+] = 0, \tag{4.120}$$
$$[\hat{x}^+, \hat{x}^-] = [\hat{y}^+, \hat{y}^-] = 0, \tag{4.121}$$

thus $\hat{x}^+$ and $\hat{y}^-$ have joint eigenstates, which we denote as $|x^+, y^-\rangle$. Theses are given by

$$|x^+, y^-\rangle = \int \frac{dx'}{\sqrt{2\pi}} \left|\frac{x^+ - x'}{\sqrt{2}}\right\rangle_{\mathrm{aux}} \left|\frac{x^+ + x'}{\sqrt{2}}\right\rangle_{\mathrm{uni}} e^{iy^- x'}, \tag{4.122}$$

where $|(x^+ + x')/\sqrt{2}\rangle_{\mathrm{uni}}$ is an x-state (an eigenstate of $\hat{X} = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$) and $|(x^+ - x')/\sqrt{2}\rangle_{\mathrm{aux}}$ is a x-state for the auxiliary system.

Because $\hat{x}^+$ and $\hat{y}^-$ can be well-defined simultaneously, we interpreted them as being suitable modal properties to represent simultaneously the position and momentum of the harmonic oscillator. This can be justified on the grounds that for $|\phi\rangle = |0\rangle$ (a vacuum state)

$$\langle\Phi(t)|\hat{x}^+|\Phi(t)\rangle = \langle\Psi(t)|\hat{X}|\Psi(t)\rangle/\sqrt{2} = \sqrt{\frac{m\omega}{2\hbar}}\,\langle\Psi(t)|\hat{Q}|\Psi(t)\rangle, \tag{4.123}$$
$$\langle\Phi(t)|\hat{y}^-|\Phi(t)\rangle = \langle\Psi(t)|\hat{Y}|\Psi(t)\rangle/\sqrt{2} = \sqrt{\frac{1}{2m\omega\hbar}}\,\langle\Psi(t)|\hat{P}|\Psi(t)\rangle, \tag{4.124}$$

where $|\Phi(t)\rangle = |\Psi(t)\rangle|0\rangle$ is the guiding wave for $\mathcal{K}$ and $|\Psi(t)\rangle$ is the solution of the Schrödinger equation. Thus we see that their average value is equal to the average value of the dimensionless momentum and position (up to a factor of $\sqrt{2}$). Considering the higher moments we find that

$$\langle\Phi(t)|\hat{x}^+\hat{x}^+|\Phi(t)\rangle = \frac{\langle\Psi(t)|\hat{X}^2|\Psi(t)\rangle}{2} + \frac{\langle 0|\hat{b}\hat{b}^\dagger|0\rangle}{4} = \frac{m\omega}{2\hbar}\langle\Psi(t)|\hat{Q}^2|\Psi(t)\rangle + \frac{1}{4}, \tag{4.125}$$
$$\langle\Phi(t)|\hat{y}^-\hat{y}^-|\Phi(t)\rangle = \frac{\langle\Psi(t)|\hat{Y}^2|\Psi(t)\rangle}{2} + \frac{\langle 0|\hat{b}\hat{b}^\dagger|0\rangle}{4} = \frac{1}{2m\omega\hbar}\langle\Psi(t)|\hat{P}^2|\Psi(t)\rangle + \frac{1}{4}. \tag{4.126}$$

In the classical limit ($\hbar \to 0$) these operators have essentially the same statistics as a dimensionless position and momentum operators (as the 1/4 term becomes negligible compared to the first terms in each equation).

In terms of this larger Hilbert space, $\mathcal{K}$, we can rewrite Eq. (4.107) as

$$\hat{H}_{\rm uni} \otimes \hat{1}_{\rm aux} = \frac{\hbar\omega[\hat{x}^+\hat{x}^+ + \hat{x}^-\hat{x}^- + 2\hat{x}^+\hat{x}^- + \hat{y}^+\hat{y}^+ + \hat{y}^-\hat{y}^- + 2\hat{y}^+\hat{y}^-]}{4}, \tag{4.127}$$

where we have used $\hat{X} \otimes \hat{1}_{\rm aux} = [\hat{x}^+ + \hat{x}^-]/\sqrt{2}$ and $\hat{Y} \otimes \hat{1}_{\rm aux} = [\hat{y}^+ + \hat{y}^-]/\sqrt{2}$

To define the modal dynamics in $\mathcal{K}$ we use Naimark's theorem, with a Naimark projector $|\phi\rangle = |0\rangle$, to extend the POM elements defined by Eq. (4.113) to the projector

$$\hat{\Pi}(a)d^2a = |x^+, y^-\rangle\langle x^+, y^-|dx^+dy^-. \tag{4.128}$$

This implies

$$\frac{1}{\pi}|a\rangle\langle a| = \langle 0|x^+, y^-\rangle\langle x^+, y^-|0\rangle \tag{4.129}$$

To show this we consider only the term $\langle 0|x^+, y^-\rangle = |a\rangle/\sqrt{\pi}$. Using Eq. (4.122) and the standard definition of a x-state

$$|x\rangle = \frac{1}{\pi^{1/4}}\exp(-x^2/2)\sum_n \frac{H_n(x)}{\sqrt{2^n n!}}|n\rangle, \tag{4.130}$$

where $H_n(x)$ is a $n^{\rm th}$ order Hermite polynomial. This term can be written as

$$\langle 0|x^+, y^-\rangle = \int \frac{dx'}{\sqrt{2\pi}}\frac{1}{\pi^{1/4}}\exp\left[\frac{-(x^+ - x')^2}{4}\right]\Big|\frac{x^+ + x'}{\sqrt{2}}\Big\rangle_{\rm uni} e^{iy^- x'}. \tag{4.131}$$

Defining $\aleph = (x^+ + x')/\sqrt{2}$ allows us to rewrite this as

$$\langle 0|x^+, y^-\rangle = \int \frac{d\aleph}{\sqrt{\pi}}\frac{1}{\pi^{1/4}}\exp\left[\frac{-(2x^+ - \sqrt{2}\aleph)^2}{4}\right]|\aleph\rangle\sqrt{2}_{\rm uni} e^{iy^-(\sqrt{2}\aleph - x^+)}, \tag{4.132}$$

which with definition Eq. (4.130) can be expanded to

$$\langle 0|x^+, y^-\rangle = \sum_n \int \frac{d\aleph}{\pi}\exp(-x^+x^+ + \sqrt{2}\aleph x^+ - \aleph^2 + iy^-\sqrt{2}\aleph - iy^-x^+)\frac{H_n(\aleph)}{\sqrt{2^n n!}}|n\rangle_{\rm uni} \tag{4.133}$$

and simplified by using $a = x^+ + iy^-$ to

$$\langle 0|x^+, y^-\rangle = \exp(-|a|^2/2)\sum_n \int \frac{d\aleph}{\pi}\exp(-\aleph^2)\exp(2a\aleph/\sqrt{2} - a^2/2)\frac{H_n(\aleph)}{\sqrt{2^n n!}}|n\rangle_{\rm uni}. \tag{4.134}$$

Then using the identity $\sum_m t^m H_m(\aleph)/m! = \exp(2t\aleph - t^2)$ [8] this can be written as

$$\langle 0|x^+, y^-\rangle = \frac{\exp(-|a|^2/2)}{\pi}\sum_{n,m}\frac{a^m}{\sqrt{2^m m!}\sqrt{2^n n!}}|n\rangle_{\rm uni}\int d\aleph\exp(-\aleph^2)H_m(\aleph)H_n(\aleph) \tag{4.135}$$

and since $\int d\aleph\exp(-\aleph^2)H_m(\aleph)H_n(\aleph) = 2^n n!\sqrt{\pi}\delta_{nm}$ [8] this becomes

$$\langle 0|x^+, y^-\rangle = \frac{\exp(-|a|^2/2)}{\sqrt{\pi}}\sum_n \frac{a^n}{\sqrt{n!}}|n\rangle_{\rm uni} = \frac{1}{\sqrt{\pi}}|a\rangle_{\rm uni}, \tag{4.136}$$

QED.

With this enlarged projector the observable A becomes the property,

$$A = \{(a = x^+ + iy^-, \hat{\Pi}(a)d^2a = |x^+, y^-\rangle\langle x^+, y^-|dx^+dy^-)\} \tag{4.137}$$

or in operator notation

$$\hat{A} = \int\int (x^+ + iy^-)|x^+, y^-\rangle\langle x^+, y^-|dx^+dy^-. \tag{4.138}$$

Since $\{|x^+, y^-\rangle\langle x^+, y^-|\}$ forms a complete orthogonal basis and Eq. (4.127) contains no cubic or higher order terms involving $\hat{x}^-$ or $\hat{y}^+$, the results of section 4.4.3 are applicable (the velocity operator method). That is, a deterministic differential equation for the $\upsilon(A, t) = \upsilon(x^+, t) + i\upsilon(y^-, t)$ can be determined.

Using Eq. (4.103) with the Hamiltonian displayed in Eq. (4.127) gives the following two velocity operators

$$\hat{\mathrm{v}}_+(t) = \frac{\omega}{2}\hat{y}^+ + \frac{\omega}{2}\hat{y}^-, \tag{4.139}$$

$$\hat{\mathrm{v}}_-(t) = -\frac{\omega}{2}\hat{x}^+ - \frac{\omega}{2}\hat{x}^-. \tag{4.140}$$

Substituting these into Eq. (4.102) (with $|\Psi(t)\rangle \to |\Phi(t)\rangle$) gives

$$\mathrm{v}_+(x^+, y^-, t) = \frac{\omega \mathrm{Re}[-i\langle\Phi(t)|x^+, y^-\rangle\partial_{x^+}\langle x^+, y^-|\Phi(t)\rangle]}{2\langle\Phi(t)|x^+, y^-\rangle\langle x^+, y^-|\Phi(t)\rangle} + \frac{\omega}{2}y^-, \tag{4.141}$$

$$\mathrm{v}_-(x^+, y^-, t) = -\frac{\omega \mathrm{Re}[i\langle\Phi(t)|x^+, y^-\rangle\partial_{y^-}\langle x^+, y^-|\Phi(t)\rangle]}{2\langle\Phi(t)|x^+, y^-\rangle\langle x^+, y^-|\Phi(t)\rangle} - \frac{\omega}{2}x^+. \tag{4.142}$$

Thus the differential equations determining the trajectories for the actual values are

$$\begin{aligned} d_t\upsilon(x^+, t) &= \frac{\omega \mathrm{Re}[-i\langle\Phi(t)|x^+, y^-\rangle\partial_{x^+}\langle x^+, y^-|\Phi(t)\rangle]}{2\langle\Phi(t)|x^+, y^-\rangle\langle x^+, y^-|\Phi(t)\rangle}\Big|_{x^+=\upsilon(x^+,t), y^-=\upsilon(y^-,t)} \\ &\quad + \frac{\omega}{2}\upsilon(y^-, t), \end{aligned} \tag{4.143}$$

$$\begin{aligned} d_t\upsilon(y^-, t) &= -\frac{\omega \mathrm{Re}[i\langle\Phi(t)|x^+, y^-\rangle\partial_{y^-}\langle x^+, y^-|\Phi(t)\rangle]}{2\langle\Phi(t)|x^+, y^-\rangle\langle x^+, y^-|\Phi(t)\rangle}\Big|_{x^+=\upsilon(x^+,t), y^-=\upsilon(y^-,t)} \\ &\quad - \frac{\omega}{2}\upsilon(x^+, t). \end{aligned} \tag{4.144}$$

Using the fact that

$$\langle x^+, y^-|\Phi(t)\rangle = \langle x^+, y^-|0\rangle|\Psi(t)\rangle = \frac{\exp[-({x^+}^2 + {y^-}^2)/2]}{\sqrt{\pi}}\sum_m \frac{(x^+ - iy^-)^m}{\sqrt{m!}}\langle m|\Psi(t)\rangle \tag{4.145}$$

the partial derivatives can be written as

$$\begin{aligned} \partial_{x^+}\langle x^+, y^-|\Phi(t)\rangle &= \frac{\exp[-({x^+}^2 + {y^-}^2)/2]}{\sqrt{\pi}}\sum_m m\frac{(x^+ - iy^-)^{m-1}}{\sqrt{m!}}\langle m|\Psi(t)\rangle \\ &\quad - x^+\langle x^+, y^-|\Phi(t)\rangle, \end{aligned} \tag{4.146}$$

$$\begin{aligned} \partial_{y^-}\langle x^+, y^-|\Phi(t)\rangle &= -i\frac{\exp[-({x^+}^2 + {y^-}^2)/2]}{\sqrt{\pi}}\sum_m m\frac{(x^+ - iy^-)^{m-1}}{\sqrt{m!}}\langle m|\Psi(t)\rangle \\ &\quad - y^-\langle x^+, y^-|\Phi(t)\rangle. \end{aligned} \tag{4.147}$$

which allows us to write

$$d_t\upsilon(x^+, t) = +\frac{\omega}{2}\upsilon(y^-, t) + \frac{\omega}{2}\mathrm{Im}[\chi_\Psi(\upsilon(x^+, t), \upsilon(y^-, t))], \tag{4.148}$$

$$d_t\upsilon(y^-, t) = -\frac{\omega}{2}\upsilon(x^+, t)) - \frac{\omega}{2}\mathrm{Re}[\chi_\Psi(\upsilon(x^+, t), \upsilon(y^-, t))], \tag{4.149}$$

where

$$\chi_\Psi(\upsilon(x^+,t),\upsilon(y^-,t)) = \frac{\sum_m m(x^+-iy^-)^{m-1}\langle m|\Psi(t)\rangle/\sqrt{m!}}{\sum_m (x^+-iy^-)^m\langle m|\Psi(t)\rangle/\sqrt{m!}}\Big|_{x^+=\upsilon(x^+,t),y^-=\upsilon(y^-,t)}. \tag{4.150}$$

Thus the differential equation for the value of property A is

$$d_t\upsilon(A,t) = -\frac{i\omega}{2}\upsilon(A,t) - \frac{i\omega}{2}\chi_\Psi(\upsilon(x^+,t),\upsilon(y^-,t)). \tag{4.151}$$

When $|\Psi(t)\rangle$ is a number state

Let's first of all consider a number state $|n\rangle$ as the initial condition for $|\Psi(t_0)\rangle$. Then with the Schrödinger equation,

$$d_t|\Psi(t)\rangle = -i\omega\hat{a}^\dagger\hat{a}|\Psi(t)\rangle, \tag{4.152}$$

the guiding wave will be $|\Psi(t)\rangle = e^{-i\omega n(t-t_0)}|n\rangle$. Substituting this into Eq. (4.150), gives

$$\chi_\Psi(\upsilon(x^+,t),\upsilon(y^-,t)) \quad = \quad \frac{n[\upsilon(x^+,t)-i\upsilon(y^-,t)]^{n-1}}{[\upsilon(x^+,t)-i\upsilon(y^-,t)]^n} = \frac{n}{\upsilon(A,t)^*} = \frac{n\upsilon(A,t)}{|\upsilon(A,t)|^2}. \tag{4.153}$$

Thus

$$d_t\upsilon(A,t) = -\frac{i\omega}{2}\Big(1+\frac{n}{|\upsilon(A,t)|^2}\Big)\upsilon(A,t). \tag{4.154}$$

This has the solution

$$\upsilon(A,t) = \upsilon(A,t_0)e^{-i\omega'(t-t_0)}, \tag{4.155}$$

where $\omega' = \omega\Big(1+n/|\upsilon(A,t_0)|^2\Big)/2$ and $\upsilon(A,t_0)$ is the initial value of A (which is stochastic in nature, as it is chosen from the initial distribution).

When $|\Psi(t)\rangle$ is a coherent state

If we assume that initially the system in is a coherent state $|\Psi(t_0)\rangle = |\beta\rangle$, then by Eq. (4.152),

$$|\Psi(t)\rangle = \exp(|\beta|^2/2)\sum_n \frac{\beta^n e^{-i\omega n(t-t_0)}}{\sqrt{n!}}|n\rangle. \tag{4.156}$$

Substituting this into Eq. (4.150), gives

$$\chi_\psi(\upsilon(x^+,t),\upsilon(y^-,t)) = \beta e^{-i\omega(t-t_0)}\frac{\sum_m m(\upsilon(A,t)^*\beta e^{-i\omega(t-t_0)})^{m-1}/m!}{\sum_m(\upsilon(A,t)^*\beta e^{-i\omega t})^m/m!} = \beta e^{-i\omega(t-t_0)}. \tag{4.157}$$

Thus

$$d_t\upsilon(A,t) = -\frac{i\omega}{2}\upsilon(A,t) - \frac{i\omega}{2}\beta e^{-i\omega t}. \tag{4.158}$$

This has the solution

$$\upsilon(A,t) = [\upsilon(A,t_0)-\beta]e^{-i\omega(t-t_0)/2} + \beta e^{-i\omega(t-t_0)}. \tag{4.159}$$

4.5.2 Bohmian mechanics, the position projector

In this section we consider the modal dynamics for the case when the preferred projective measure is the position projective measure. That is, the preferred projective measure is $\{\hat{\pi}(x)dx = |x\rangle\langle x|dx\}$. Since this is already a projective measure, there is no need to enlarge the universe, and as shown above (and in Refs. [120] and [128]) the modal dynamics for this case is just Bohmian mechanics.

Using the Hamiltonian depicted in Eq. (4.107) and the velocity operator technique (as there are no terms of order $\hat{Y}^3$ and higher) it can easily be shown that the velocity field is

$$\mathrm{v}(x,t) = \frac{\omega \mathrm{Re}[-i\langle\Psi(t)|x\rangle\partial_x\langle x|\Psi(t)\rangle]}{\langle\Psi(t)|x\rangle\langle x|\Psi(t)\rangle}, \tag{4.160}$$

as $\hat{\mathrm{v}} = \omega\hat{Y}$. Using Eq. (4.68) this gives

$$d_t\upsilon(X,t) = \frac{\omega \mathrm{Im}[\langle\Psi(t)|x\rangle\partial_x\langle x|\Psi(t)\rangle]}{\langle\Psi(t)|x\rangle\langle x|\Psi(t)\rangle}\Big|_{x=\upsilon(X,t)}. \tag{4.161}$$

Since $\hat{Y}$ does not commute with $\hat{X}$, we can not give both $\hat{X}$ and $\hat{Y}$ definite status (to give $\hat{Y}$ property status we would have to chose the preferred projective measure to be the momentum projective measure). However, as in Bohmian mechanics, we can define a momentum field, $y(x,t)$, by

$$y(x,t) \quad = \quad \frac{\mathrm{Re}[\langle\Psi(t)|x\rangle\langle x|\hat{Y}|\Psi(t)\rangle]}{\langle\Psi(t)|x\rangle\langle x|\Psi(t)\rangle} = \frac{\mathrm{Im}[\langle\Psi(t)|x\rangle\partial_x\langle x|\Psi(t)\rangle]}{\langle\Psi(t)|x\rangle\langle x|\Psi(t)\rangle}. \tag{4.162}$$

We interpret this momentum field as, if the system has the position $\upsilon(X,t)$ then its momentum is $y(x,t)|_{x=\upsilon(X,t)}$. Thus with this momentum and position we can define a point in phase space $(\upsilon(X,t),\, y(x,t)|_{x=\upsilon(X,t)})$, which in complex notation is written as

$$\alpha(t) = \frac{\upsilon(X,t) + iy(x,t)|_{x=\upsilon(X,t)}}{\sqrt{2}}. \tag{4.163}$$

The factor $1/\sqrt{2}$ is to scale this to agree with the preceding section (value of observable A).

When $|\Psi(t)\rangle$ is a number state

As before when we assume a number state initial condition the guiding wave at time t is $|\Psi(t)\rangle = e^{-i\omega n(t-t_0)}|n\rangle$. Substituting this into Eq. (4.161) and using Eq. (4.130) we get

$$d_t\upsilon(X,t) \quad = \quad \frac{\omega \mathrm{Im}\{\partial_x[\exp(-x^2/2)H_n(x)]/\sqrt{2^n n!}\}}{\exp(-x^2/2)H_n(x)/\sqrt{2^n n!}}\Big|_{x=\upsilon(X,t)} = 0. \tag{4.164}$$

Using a similar argument and Eq. (4.162) we get $y(x,t)_{x=\upsilon(X,t)} = 0$. That is, once the initial value $\upsilon(X,t_0)$ is picked from the quantum mechanical distribution is stays there for all time. In terms of the complex notation $\alpha(t)$ we get

$$\alpha(t) = \upsilon(X,t_0)/\sqrt{2}. \tag{4.165}$$

When $|\Psi(t)\rangle$ is a coherent state

If we assume that initially the system in is a coherent state $|\Psi(t_0)\rangle = |\beta\rangle$, then Eq. (4.156) in the position representation is

$$|\Psi(t)\rangle \quad = \quad \frac{\exp(|\beta|^2/2)}{\pi^{1/4}}\int dx' \exp(-x'^2/2)\exp(\sqrt{2}\beta e^{-i\omega(t-t_0)}x - \beta^2 e^{-i2\omega(t-t_0)}/2)|x'\rangle. \tag{4.166}$$

Substituting this into Eq. (4.161) gives

$$d_t\upsilon(X,t) = \frac{\omega \mathrm{Im}\{\partial_x[\exp(-x'^2/2)\exp(\sqrt{2}\beta e^{-i\omega(t-t_0)}x)]\}}{\exp(-x'^2/2)\exp(\sqrt{2}\beta e^{-i\omega(t-t_0)}x)}\Big|_{x=\upsilon(X,t)} = \omega\sqrt{2}\mathrm{Im}[\beta e^{-i\omega(t-t_0)}], \tag{4.167}$$

and using Eq. (4.162) we get $y(x,t)_{x=\upsilon(X,t)} = \sqrt{2}\mathrm{Im}[\beta e^{-i\omega(t-t_0)}]$. Taking the derivative of this gives

$$\dot{y}(x,t)_{x=\upsilon(X,t)} = -\omega\sqrt{2}\mathrm{Re}[\beta e^{-i\omega(t-t_0)}]. \tag{4.168}$$

Thus

$$\dot{\alpha}(t) = \omega\{\mathrm{Im}[\beta e^{-i\omega(t-t_0)}] - i\mathrm{Re}[\beta e^{-i\omega(t-t_0)}]\} = -i\omega\beta e^{-i\omega(t-t_0)}. \tag{4.169}$$

This has the solution

$$\alpha(t) = \beta e^{-i\omega(t-t_0)} + (\alpha(t_0) - \beta) \tag{4.170}$$

where $\alpha(t_0)$ is defined by

$$\alpha(t_0) = \frac{\upsilon(X,t_0) + iy(x,t_0)|_{x=\upsilon(X,t_0)}}{\sqrt{2}}. \tag{4.171}$$

4.5.3 Classical limit

In classical mechanics the value of all properties can be defined simultaneously. Hence at the same time we can assign a value to position, X, and momentum, Y. Let these be denoted $\upsilon(X,t)$ and $\upsilon(Y,t)$. With these values we can determine the trajectories via Hamiltonian's equations. In terms of the dimensionless position and momentum these are

$$d_t\upsilon(X,t) = \frac{1}{\hbar}\frac{\partial H_{\mathrm{uni}}(x,y,t)}{\partial y}\Big|_{(x=\upsilon(X,t),y=\upsilon(Y,t))}, \tag{4.172}$$

$$d_t\upsilon(Y,t) = -\frac{1}{\hbar}\frac{\partial H_{\mathrm{uni}}(x,y,t)}{\partial x}\Big|_{(x=\upsilon(X,t),y=\upsilon(Y,t))}. \tag{4.173}$$

Here I have introduced the Hamiltonian as a function of the generalized coordinates (x, y) for position and momentum. To work out the classical trajectories for a harmonic oscillator we substitute the Hamiltonian (Eq. (4.107)) with the operators changed to the corresponding generalized coordinates into these equations. Doing this we get

$$d_t\upsilon(X,t) = \omega\upsilon(Y,t), \tag{4.174}$$

$$d_t\upsilon(Y,t) = -\omega\upsilon(X,t), \tag{4.175}$$

which in terms of a property, $A = (X+iY)/\sqrt{2}$ (a complex representation of X and Y in phase space), we can write a differential equation for the value of A as

$$d_t\upsilon(A,t) = -i\omega\upsilon(A,t). \tag{4.176}$$

This has the well known solution

$$\upsilon(A,t) = \upsilon(A,t_0)e^{-i\omega(t-t_0)}. \tag{4.177}$$

Now lets compare the above classical result to the classical limit of the modal dynamics presented in Secs. 4.5.1 (the preferred measure is the Husimi POM) and 4.5.2 (the preferred measure is the position projective measure). In Sec. 4.5.1 we saw that when the preferred measure is the Husimi POM, in the enlarged Hilbert space, the property is $\hat{A} = \hat{x}^+ + i\hat{y}^-$. In the classical limit $\hat{x}^+$ and $\hat{y}^-$ have the same statistics as the dimensionless position $(X(t)/\sqrt{2})$ and momentum $(Y(t)/\sqrt{2})$ properties. I now ask the question what is the classical limit for the trajectory of $\upsilon(A,t)$? First I will consider a number state initial condition, Eq. (4.155). From the probability formula ($\Pr(a,t) = \langle\Phi(t)|\hat{\Pi}(a)|\Phi(t)\rangle d^2a$) it can be shown that for the number state initial condition,

$$\Pr(a,t_0) = \frac{1}{\pi}\exp(-|a|^2)\frac{(|a|^2)^n}{n!}\,d^2a, \tag{4.178}$$

which in turn implies

$$\langle |A|^2 \rangle = n + 1, \tag{4.179}$$

and

$$\Delta |A|^2 = \sqrt{\langle [|A|^2]^2 \rangle - \langle |A|^2 \rangle^2} = \sqrt{n+2}\,. \tag{4.180}$$

Thus the value of property $|A|^2$, as $n \to \infty$, will be approximately n. Now $\upsilon(|A|^2, t_0) = |\upsilon(A, t_0)|^2$ (since properties $|A|^2$ and A commute, see Eq. (3.27)), so in the large n limit, $\omega' \to \omega$ in Eq. (4.155), thus reproducing the classical dynamics.

When considering the case with a initial coherent state we can similarly argue that $\beta \approx \upsilon(A, t_0)$ with high probability. Then in the limit $|\beta| \to \infty$, the difference between the classical formula and Eq. (4.159) is negligible.

By contrast, in the position case (Bohmian mechanics), for the number state, $\alpha(t) = \upsilon(X, t_0)/\sqrt{2}$ for all n, thus no classical limit exists. However, it can be argued that when we consider the second case (the coherent state initial condition), again the difference between the classical formula and Eq. (4.170) is negligible. This is not surprising, as the coherent state is a classical-like state. What is surprising is that it is possible to obtain classical modal dynamics even for a non-classical state, by using POMs.

4.6 Summary of chapter

In this section I have presented the modal interpretation of quantum mechanics. This interpretation, unlike the orthodox interpretation, provides an answer to problem one in the measurement problem. That is things really do have a value prior to measurement; measurement only reveals the pre-existing value. Because of this I introduce the notation that an observables, $Z(t)$ should be called a property, and the value of this property at time t is denoted $\upsilon(Z(t), t)$. I also showed the standard modal dynamics [11, 37, 4, 120, 121, 128] for calculating how this value changes in time can be extended to include observables represented by a POM [62].

Although this interpretation provides an answer to problem one of the measurement problem, because of Kochen-Specker type arguments not all observables can be given property status. This means the problem of choice still remains, what chooses which observables are objectively real? This question I feel may never be answered (especially with the extension of modal dynamics by Wiseman and myself), but for the purposes of this thesis I will not consider this a problem. I will sometimes take the view that we have to live with choice, it is essential (similar to Beable variant) and when trying to explain measurements I will take the view that we live outside the quantum world and the choice is made by the way we arrange the experiment (the whole apparatus is important). However, this introduces the problem of the Heisenberg cut.

In this chapter I also showed that Bohmian mechanics [14, 15], when we chose a position projector, is simply the continuous limit of the modal interpretation when the Hamiltonian is at most quadratic in momentum. This by definition is true for all natural Hamiltonian. However Brown and Hiley's [19] extension of Bohmian mechanics to include the momentum representation is not the continuous limit of modal dynamics (at least for Bell-type dynamics) for all natural Hamiltonian, as it is possible to get potentials of cubic order and higher in position.

To illustrate the modal dynamics I considered a simple example: a universe considering of a single Harmonic oscillator. I then chose two preferred measures, the first one being the Husimi POM (made from coherent state POM elements) and the second the position projective measure. It is observed that for the two initial state conditions $|\Psi(t_0)\rangle = |n\rangle$ (a number state) and $|\Psi(t_0)\rangle = |\beta\rangle$ (a coherent state) the trajectories for the value of the Husimi POM have a classical limit which agrees with our classical theories. When comparing to the modal dynamics for the position projective measure (Bohmian mechanics) we find that only position is defined, and for $|\Psi(t)\rangle = |n\rangle$ the dynamics are highly non-classical. Only for a classical-like state $|\Psi(t_0)\rangle = |\beta\rangle$, and by defining a momentum field, can a classical limit be obtained.

Chapter 5

Other Interpretations of Quantum Mechanics

Although not important for this thesis I would like to point out that the orthodox and modal interpretations are not the only interpretations of quantum mechanics. To give credit to the second part of my title "interpretations of quantum mechanics" I feel that mentioning (even if very briefly) the relative state (or many worlds) interpretation and the dynamical reduction interpretation is necessary. Thus in this Chapter I will very briefly outline these two interpretations.

5.1 Many worlds interpretation

The many world interpretation arises form the earlier work of Everett [54, 130], and later expanded in Ref. [55]. However here it was referred to as the relative state interpretation. It was later named the many worlds (or universes) interpretation by De Witt [36]. In this interpretation instead of introducing an extra dynamical equation or supplementing the theory with an extra quantum state (or hidden variable) we take the theory as being complete and assume that the universe as we know it is a lot more complicated than we currently are aware (experience it).

To explain this interpretation and how it is different from the orthodox and modal interpretations, I will consider the quantum state $|\Psi(t)\rangle$ and the unique bi-orthogonal (or Schmidt) decomposition (if there are no degeneracies)

$$|\Psi(t)\rangle = \sum_{n}^{D} c_n(t)|\phi'_n(t)\rangle_1|\phi_n(t)\rangle_2, \tag{5.1}$$

where $\{|\phi'_n(t)\rangle_1\}$ and $\{|\phi_n(t)\rangle_2\}$ form a orthogonal basis set in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. In the many worlds (or relative state) interpretation, like the modal interpretation, we say that the universe is described by $|\Psi(t)\rangle$. Thus there is no classical world outside of the quantum world. Instead of introducing an extra quantum state to describe measurement we conclude that the universe is not the standard universe we are all used to, it is a multi-universe. For the above example $|\Psi(t)\rangle$ would correspond to a D dimensional multi-unverse, with the n^{th} universe being described by the state $|\phi'_n(t)\rangle_1|\phi_n(t)\rangle_2$. Here $|\phi'_n(t)\rangle_1$ describes the state of the system and $|\phi_n(t)\rangle_2$ describes the rest of the universe, including us. Thus the unique bi-orthogonal decomposition has chosen that we are measuring an observable described by

$$Z = \{(z_n, |\phi'_n(t)\rangle_1\langle\phi'_n(t)|)\}. \tag{5.2}$$

That is all possible results of Z occur in the multi-universe, each result z_n occur in the corresponding universe. The measurement problem is never encountered as we are part of the multi-universe and relative to each universe (where we have the state $|\phi_n(t)\rangle_2$) we will only see one of the results of the measurement (the one corresponding to projector $|\phi'_n(t)\rangle_1\langle\phi'_n(t)|$). Thus, because all results occur reality can be assigned before a measurement.

The problem of choice is partially answered by the uniqueness of the Schmidt decompositions [36]. The reason why only "partially" is if there is degeneracy then there is more than one possible Schmidt decomposition for $|\Psi(t)\rangle$. This is called the preferred decomposition problem. Bub and Elby [53] have partially answered this problem by showing that if the multi-universe can represented by a tri-orthogonal decomposition for $|\Psi(t)\rangle$,

$$|\Psi(t)\rangle = \sum_n^D c_n(t)|\phi'_n(t)\rangle_1|\phi''_n(t)\rangle_2|\phi_n(t)\rangle_3, \tag{5.3}$$

then this decomposition is unique even if the coefficients, $c_n(t)$, are degenerate. Note not all states have a tri-orthogonal decomposition [53, 29, 102].

Personally I feel that this view is similar to the modal (except the range of possibilities are replaced by the range of universes) and because of this the introduction of a multi-universe is not really necessary. Also in the orthodox interpretation (as well as the modal interpretation) it has been shown that projective type measurements are only a subset of all possible measurements. We can measure observables described by POM elements. However for this to occur in this theory we would have to enlarge the universe. This is similar to the work Wiseman and myself have done in the modal interpretation [62] (also see chapter 4), but in this view since $|\Phi\rangle$ represents the state of the universe this enlargement does not make sense. This argument to some extent can be applied to the modal interpretation, but because $|\Psi(t)\rangle$ in this interpretation is just a guiding state for the hidden variables it is easier to accept this enlargement.

5.2 Dynamical reduction interpretation

Another interpretation which I feel is necessary to mention, as it provides another view for both non-Markovian SSEs [5, 6, 7] and Markovian SSEs [7, 44, 45, 70, 71, 73, 74, 75, 76, 77, 99] is the dynamical reduction interpretation. This theory also covers the non-linear stochastic evolution equation of Ghirardi, Rimini, and Weber [72, 7]. In this interpretation the wavefunction is considered to be the basic element of reality and to describe the measurement problem the linear schrodinger equation has to be replaced by a modified stochastic evolution equation. Currently there are three versions of this interpretation: the spontaneous localization model, the continuous spontaneous localization model and the dynamical reduction with gaussian noises model. Here I will briefly explain each of these models.

5.2.1 Spontaneous localization models

The spontaneous localization model was first introduced by Ghirardi, Rimini, and Weber in 1986 [72]. In this model they assume that the there is some spontaneous localization process which occurs with a mean rate, λ. This localization is described by the equation

$$|\psi(t)\rangle \rightarrow \frac{|\tilde{\psi}_x(t)\rangle}{\sqrt{\langle\tilde{\psi}_x(t)|\tilde{\psi}_x(t)\rangle}} \tag{5.4}$$

where $|\tilde{\psi}_x(t)\rangle = \hat{L}_x|\psi(t)\rangle$ and x represents a point in real space. $\hat{L}_x$ is positive, self-adjoint, linear operator that represents the localization of a quantum state $|\psi(t)\rangle$. The probability density for the occurrence of a localization at point x is assumed to be

$$P(x,t) = \langle\tilde{\psi}_x(t)|\tilde{\psi}_x(t)\rangle. \tag{5.5}$$

Thus

$$\int dx\hat{L}_x^2 = \hat{1}. \tag{5.6}$$

The form of $\hat{L}_x$ is assumed to be

$$\hat{L}_x = \left(\frac{\alpha}{\pi}\right)^{1/4}\exp[-\alpha(\hat{Q}-x)^2/2], \tag{5.7}$$

where $\hat{Q}$ is the position operator and α is a free parameter.

With the above assumptions we can derive the non-linear stochastic master equation by

$$\rho_{\text{GRW}}(t+dt) = (1+\lambda dt)\{\rho_{\text{GRW}}(t) - \frac{i}{\hbar}[\hat{H}_{\text{uni}}(t), \rho_{\text{GRW}}(t)]dt\} + \lambda dt\mathcal{T}[\hat{L}_x]\rho_{\text{GRW}}(t), \tag{5.8}$$

where $\hat{\mathcal{T}}$ is a superoperator and represents the operation

$$\hat{\mathcal{T}}\rho(t) = \int dx\hat{L}_x\rho(t)\hat{L}_x \tag{5.9}$$

and $(1-\lambda dt)$ is the probability for no localization (thus λdt is the probability for a localization). Rearranging this, and ignoring dt^2 terms, gives

$$d_t\rho_{\text{GRW}}(t) = -\frac{i}{\hbar}[\hat{H}_{\text{uni}}(t), \rho_{\text{GRW}}(t)] - \lambda(\rho(t) - \hat{\mathcal{T}}\rho_{\text{GRW}}(t)), \tag{5.10}$$

the Ghirardi, Rimini, and Weber evolution equation [72, 7]. The free parameter α, as well as λ are chosen such that when describing a macroscopic system, superpositions quickly disappear (thus there is no Schrödinger cat paradox [109]) and when describing micro systems, like a free particle, the average dynamics agree with the orthodox theory. I would also like to note that when $\lambda = 0$ this reduces to the standard Schrödinger equation evolution. With this equation is can be shown that the average of the $\hat{Q}$ and $\hat{P}$, for a free particle, obey

$$\begin{aligned} \langle\hat{Q}\rangle &= \text{Tr}[\hat{Q}\rho_{\text{GRW}}(t)] = \text{Tr}[\hat{Q}\rho(t)] &(5.11)\\ \langle\hat{P}\rangle &= \text{Tr}[\hat{P}\rho_{\text{GRW}}(t)] = \text{Tr}[\hat{P}\rho(t)], &(5.12)\end{aligned}$$

where $\rho(t)$ is found by the standard Schrödinger evolution. That is the average values for the position and momentum agree with the orthodox interpretation.

I would like to point out that while this equation does provide an alternative view to quantum mechanics, 5.10 can also be derived under the orthodox theory. In the orthodox interpretation $\hat{L}_x$ is effectively a measurement operator, corresponding to the continuous results x (see section 2.2.2). Thus under this interpretation Ghirardi, Rimini, and Weber evolution equation simply represents the average evolution of a continuous trajectory for the following situation. We have a classical measuring apparatus that measure observables described by the complete set of POM elements $\hat{L}_x^2$. This apparatus is designed such that in every interval dt it flips a weighted coin (or any other classical device which generates binary outcomes with a weighting factor for success equal to λ) and depending on the result it make a measurement or not.

5.2.2 Continuous Spontaneous Localization (CSL) models

Continuous spontaneous localization models are simply the continuous limit of the spontaneous localization models. The continuous limit is found by letting $\lambda \to \infty$, $\alpha \to 0$ and $\alpha\lambda = 2\gamma$. Doing this the GRW equation reduces to

$$d_t\rho_{CSL}(t) = -\frac{i}{\hbar}[\hat{H}_{\text{uni}}(t), \rho_{\text{CSL}}(t)] - \gamma[\hat{L}_x\rho_{\text{CSL}}(t)\hat{L}_x - \hat{L}_x^2\rho_{\text{CSL}}(t)/2 - \rho_{\text{CSL}}(t)\hat{L}_x^2/2], \tag{5.13}$$

which is in Lindblad form [92]. Assuming the more general Lindblad form ($\hat{L}$ is not necessary defined by Eq. (5.7)) we can rewrite the above master equation as

$$d_t\rho_{CSL}(t) = -\frac{i}{\hbar}[\hat{H}_{\text{uni}}(t), \rho_{\text{CSL}}(t)] - \gamma[\hat{L}\rho_{\text{CSL}}(t)\hat{L}^\dagger - \hat{L}^\dagger\hat{L}\rho_{\text{CSL}}(t)/2 - \rho_{\text{CSL}}(t)\hat{L}^\dagger\hat{L}/2]. \tag{5.14}$$

With this equation believers of CSL looked for a modified Schrödinger equation (stochastic in nature) which when the outer project is averaged gives $\rho_{CSL}(t)$. That is

$$\rho_{\text{CSL}}(t) = E_P[|\psi(t)\rangle\langle\psi(t)|], \tag{5.15}$$

where P is the probability distribution for the stochastic process. Diósi fist published a non-linear modified Schrödinger in 1988 [44] and in the same year applied it to Brownian motion [45]. A short time later Pearle [99] (also with Ghirardi and Rimini [71]) along with Gisin [73] (also with Gatarek [70]) proposed one for arbitrary $\hat{L}$.

The standard method for deriving these equations is to assume the simplest stochastic differential equation (SDE) for $|\bar{\psi}(t)\rangle$. Here a bar has been used because the simplest case does not persevere the norm. However the average norm must be 1 (otherwise Eq. (5.15) would not be satisfied). Doing this we start by proposing

$$d_t|\bar{\psi}(t)\rangle = [-\frac{i}{\hbar}\hat{H}_{\text{uni}}(t) + \hat{A} + \sqrt{\gamma}\,\hat{L}\xi(t)]|\bar{\psi}(t)\rangle, \tag{5.16}$$

where $\hat{A}$ is an arbitrary operators and $\xi(t)$ is a white noise function satisfying

$$E[\xi(t)] = 0, \tag{5.17}$$
$$E[\xi(t)\xi(t')] = \delta(t-t'). \tag{5.18}$$

Integrating this equation gives

$$|\bar{\psi}(t)\rangle - |\psi(\bar{t}_0)\rangle = \int_{t_0}^{t} ds[-\frac{i}{\hbar}\hat{H}_{\text{uni}}(s) + \hat{A}]|\bar{\psi}(s)\rangle + \sqrt{\gamma}\int_{t_0}^{t}\hat{L}|\bar{\psi}(s)\rangle\xi(s)ds, \tag{5.19}$$

which in the infinitesimal limit is

$$d|\bar{\psi}(t_0)\rangle = [-\frac{i}{\hbar}\hat{H}_{\text{uni}}(t_0)dt + \hat{A}dt]|\bar{\psi}(t_0)\rangle + \sqrt{\gamma}\int_{t_0}^{t_0+dt}\hat{L}|\bar{\psi}(s)\rangle\xi(s)ds. \tag{5.20}$$

The last term in this equation due to the white noise correlations can not be treated like a standard integral.

To work with this integral we have to consider a Wiener process. This process is described by the probability density [64]

$$\Lambda(w,t) = \frac{1}{\sqrt{2\pi(t-t_0)}}\exp[-(w-w_0)^2/2(t-t_0)]. \tag{5.21}$$

With this distribution is can be shown that

$$
\begin{aligned}
E_\Lambda[r(W,t)] &= w_0, && (5.22)\\
E_\Lambda[r(W,t)r(W,t)] &= t - t_0 + w_0^2, && (5.23)\\
E_\Lambda[r(W,t_0)r(W,t_0)] &= w_0^2. && (5.24)
\end{aligned}
$$

where $r(W,t)$ is my notation for the random variable associated with the Wiener process at time t. We can redefine this process by defining the variable $\Delta w = w - w_0$ this allows us to rewrite Eq. (5.22) as

$$\Lambda(\Delta w, t) = \frac{1}{\sqrt{2\pi(t-t_0)}} \exp[-(\Delta w)^2/2(t-t_0)], \tag{5.25}$$

thus

$$
\begin{aligned}
E_\Lambda[r(\Delta W,t)] &= 0, && (5.26)\\
E_\Lambda[r(\Delta W,t)r(\Delta W,t)] &= t - t_0. && (5.27)
\end{aligned}
$$

To make the connection with the white noise we define

$$r(\Delta W, t) = \int_{t_0}^{t} \xi(s) ds. \tag{5.28}$$

This can be shown to be correct by substituting it into Eq. (5.26) (and Eq. (5.22)) and using the white noise properties defined in Eq. (5.17) and Eq. (5.18). In the infinitesimal limit $t - t_0 = dt$, $r(\Delta W, t) = \xi(t_0)dt$ and for short hand it is customary to use $dw(t_0) = r(\Delta W, t) = \xi(t_0)dt$, $dw(t)$ is called the Wiener increment and satisfies the infinitesimal limit of the correlations displayed in Eqs. (5.26) and (5.27). That is it is a Gaussian random variable of mean 0 and variance $\sqrt{dt}$. With this Wiener increment we can rewrite any integral containing $\xi(t)$ as

$$\int_{t_0}^{t} b(\psi(s), \psi^*(s))\xi(s) ds = \int_{t_0}^{t} b(\psi(s), \psi^*(s)) dw(s) \tag{5.29}$$

which can be evaluated by using either the Itô method or Stratonovich method [64]. These methods involve defining the integral as a Riemann-Stieltjes integral, namely we divide the interval $[t_0, t]$ into N subintervals such that $t_0 \leq t_1 \leq t_2 ... \leq t_N$, and define intermediate points τ_i such that $t_{i-1} \leq \tau_i \leq t_i$. That is, we define this integral as,

$$\int_{t_0}^{t} b(\psi(s), \psi^*(s)) dw(s) = \sum_{i=1}^{N} b(\psi(\tau_i), \psi^*(\tau_i)) dw(t_{i-1}). \tag{5.30}$$

There are many ways we can define the midpoints. The Itô method chooses them such that $\tau_i = t_{i-1}$, and this allows us to write the integral as,

$$\mathcal{I} \int_{t_0}^{t} b(\psi(s), \psi^*(s)) dw(s) = \sum_{i=1}^{N} b[\psi(t_{i-1}), \psi^*(t_{i-1})] dw(t_{i-1}). \tag{5.31}$$

By contrast the Stratonovich (mid point) method is defined by

$$\mathcal{S} \int_{t_0}^{t} b(\psi(s), \vec{\psi}^*(s)) dw(s) = \sum_{i=1}^{N} b[\psi(t_{i-1}) + \tfrac{1}{2} d\psi(t_{i-1}), \psi^*(t_{i-1}) + \tfrac{1}{2} d\psi^*(t_{i-1})] dw(t_{i-1}). \tag{5.32}$$

Here I will use the Itô method as Itô calculus is easier to work with. Doing this Eq. (5.20) can be written as

$$d|\bar{\psi}(t_0)\rangle = [-\frac{i}{\hbar}\hat{H}_{\rm uni}(t_0)dt + \hat{A}dt + \sqrt{\gamma}\,\hat{L}dw(t_0)]|\bar{\psi}(t_0)\rangle. \tag{5.33}$$

To determine the form of the operator $\hat{A}$ we use the fact that the average of the norm of $|\bar{\psi}(t)\rangle$ must be 1 ($E_\Lambda[\langle\bar{\psi}(t)|\bar{\psi}(t)\rangle] = 1$). Using this constraint we find that $\hat{A} = -\frac{1}{2}\gamma\hat{L}^\dagger\hat{L}$, thus

$$d|\bar{\psi}(t)\rangle = [-\frac{i}{\hbar}\hat{H}_{\rm uni}(t)dt + -\tfrac{1}{2}\gamma\hat{L}^\dagger\hat{L}dt + \sqrt{\gamma}\,\hat{L}dw(t)]|\bar{\psi}(t)\rangle. \tag{5.34}$$

This equation may average to the correct master equation, but to have an evolution equation we can associate a meaning to we need it to be norm conserving. To find the normalization modified Schrödinger equation we simply use Itô calculus [64]. Doing this we obtain

$$d|\psi(t)\rangle = \frac{d|\bar{\psi}(t)\rangle}{\sqrt{\langle\bar{\psi}(t)|\bar{\psi}(t)\rangle}} + |\bar{\psi}(t)\rangle d\frac{1}{\sqrt{\langle\bar{\psi}(t)|\bar{\psi}(t)\rangle}} + d|\bar{\psi}(t)\rangle\ d\frac{1}{\sqrt{\langle\bar{\psi}(t)|\bar{\psi}(t)\rangle}}, \tag{5.35}$$

where

$$d\frac{1}{\sqrt{\langle\bar{\psi}(t)|\bar{\psi}(t)\rangle}} = -\frac{d[\langle\bar{\psi}(t)|\bar{\psi}(t)\rangle]}{2[\langle\bar{\psi}(t)|\bar{\psi}(t)\rangle]^{3/2}} + \frac{3d[\langle\bar{\psi}(t)|\bar{\psi}(t)\rangle]^2}{8[\langle\bar{\psi}(t)|\bar{\psi}(t)\rangle]^{5/2}}. \tag{5.36}$$

Using the above it can be shown that

$$d[\langle\bar{\psi}(t)|\bar{\psi}(t)\rangle] = \sqrt{\gamma}\,\langle\bar{\psi}(t)|\hat{L}+\hat{L}^\dagger|\bar{\psi}(t)\rangle dw(t) \tag{5.37}$$

and thus

$$d\frac{1}{\sqrt{\langle\bar{\psi}(t)|\bar{\psi}(t)\rangle}} = \frac{\sqrt{\gamma}}{\sqrt{\langle\bar{\psi}(t)|\bar{\psi}(t)\rangle}}\Big[-\frac{\langle\hat{L}+\hat{L}^\dagger\rangle_t dw(t)}{2} + \sqrt{\gamma}\,\frac{3\langle\hat{L}+\hat{L}^\dagger\rangle_t^2 dt}{8}\Big], \tag{5.38}$$

where $\langle\hat{A}\rangle_t$ is shorthand notation for $\langle\psi(t)|\hat{A}|\psi(t)\rangle$. Substituting this into Eq. (5.35) gives

$$d|\psi(t)\rangle = \Big[\Big(-\tfrac{i}{\hbar}\hat{H}_{\rm uni}(t) - \tfrac{1}{2}\gamma\hat{L}^\dagger\hat{L} + \tfrac{3}{8}\gamma\langle\hat{L}+\hat{L}^\dagger\rangle_t^2 - \tfrac{1}{2}\gamma\hat{L}\langle\hat{L}+\hat{L}^\dagger\rangle_t\Big)dt + \sqrt{\gamma}\,\Big(\hat{L} - \tfrac{1}{2}\langle\hat{L}+\hat{L}^\dagger\rangle_t\Big)dw(t)\Big]|\psi(t)\rangle, \tag{5.39}$$

a stochastic modified Schrödinger equation that preserves the norm. However using the Wiener increment defined by the distribution $\Lambda(dw, t+dt) = 2\pi dt^{-1/2}\exp[-(dw)^2/2dt]$, this equation does not average to $\rho_{\rm CSL}(t)$. This can be seen by

$$\rho_{\rm CSL}(t) = E_\Lambda[|\bar{\psi}(t)\rangle\langle\bar{\psi}(t)|] = E_\Lambda[\langle\bar{\psi}(t)|\bar{\psi}(t)\rangle|\psi(t)\rangle\langle\psi(t)|] = E_P[|\psi(t)\rangle\langle\psi(t)|] \tag{5.40}$$

where

$$P(dw, t) = \langle\bar{\psi}(t)|\bar{\psi}(t)\rangle\Lambda(dw, t). \tag{5.41}$$

This I will call the Girsanov transformation after [70]; others refer to it "cooking the probability" [71]. To work out the new set of increments, $dw(t)$ we use the fact the Wiener increments are Markovian in nature. This allows us to rewrite Girsanov transformation as

$$P(dw, t_0+dt) = (1 + d\langle\bar{\psi}(t_0)|\bar{\psi}(t_0)\rangle)\Lambda(dw, t_0+dt) = (1 + \sqrt{\gamma}\,\langle\hat{L}+\hat{L}^\dagger\rangle_{t_0}dw)\Lambda(dw, t_0+dt). \tag{5.42}$$

Since $dw^2 = dt$ this can be written as,

$$P(dw, t_0+dt) \quad = \quad \frac{\exp[-(dw)^2/2dt]}{\sqrt{2\pi dt}}[1 + \sqrt{\gamma}\,\langle\hat{L}+\hat{L}^\dagger\rangle_{t_0}dw + \gamma\langle\hat{L}+\hat{L}^\dagger\rangle_{t_0}^2(dw^2 - dt)/2]. \tag{5.43}$$

defining $\chi = \sqrt{\gamma}\,\langle \hat{L}+\hat{L}^\dagger\rangle_{t_0} dw - \gamma\langle \hat{L}+\hat{L}^\dagger\rangle_{t_0}^2 dt/2$ we can write the above as,

$$\begin{aligned}
P(dw, t_0+dt) &= \frac{\exp[-dw^2/2dt]}{\sqrt{2\pi dt}}[1+\chi+\chi^2/2] && (5.44)\\
&= \frac{\exp[-dw^2/2dt]}{\sqrt{2\pi dt}}\exp(\sqrt{\gamma}\,\langle \hat{L}+\hat{L}^\dagger\rangle_{t_0} dw - \gamma\langle \hat{L}+\hat{L}^\dagger\rangle_{t_0}^2 dt/2) && (5.45)\\
&= \frac{\exp[-(dw^2 - 2\sqrt{\gamma}\,\langle \hat{L}+\hat{L}^\dagger\rangle_{t_0} dwdt + \gamma\langle \hat{L}+\hat{L}^\dagger\rangle_{t_0}^2 dt^2)/2dt]}{\sqrt{2\pi dt}} && (5.46)\\
&= \frac{\exp[-(dw - \sqrt{\gamma}\,\langle \hat{L}+\hat{L}^\dagger\rangle_{t_0} dt)^2/2dt]}{\sqrt{2\pi dt}} && (5.47)
\end{aligned}$$

Thus the new increment $dw(t)$ (which is not necessary a Wiener increment) must satisfy

$$\begin{aligned}
E_\Lambda[dw(t)] &= \sqrt{\gamma}\,\langle \hat{L}+\hat{L}^\dagger\rangle_t dt, && (5.48)\\
E_\Lambda[dw(t)^2] &= dt. && (5.49)
\end{aligned}$$

Since the above modified Schrödinger equation was developed, Gisin and Percival have consider cases when the noise is complex [74, 75, 76, 77]. In this situation (Ref. [76, 77]) the normalized modified Schrödinger equation is

$$\begin{aligned}
d|\psi(t)\rangle = \Big[\Big(&-\tfrac{i}{\hbar}\hat{H}_{\rm uni}(t) - \tfrac{1}{2}\gamma\hat{L}^\dagger\hat{L} + \tfrac{3}{4}\gamma\langle\hat{L}\rangle_t\langle\hat{L}^\dagger\rangle_t - \tfrac{1}{2}\gamma\hat{L}\langle\hat{L}^\dagger\rangle_t\Big)dt + \sqrt{\gamma}\,\Big(\hat{L} - \tfrac{1}{2}\langle\hat{L}\rangle_t\Big)dw^*(t)\\
&-\sqrt{\gamma}\,\tfrac{1}{2}\langle\hat{L}^\dagger\rangle_t dw(t)\Big]|\psi(t)\rangle, && (5.50)
\end{aligned}$$

where

$$\begin{aligned}
E_P[dw(t)] &= \sqrt{\gamma}\,\langle\hat{L}\rangle_t dt, && (5.51)\\
E_P[dw(t)dw^*(t)] &= dt, && (5.52)\\
E_P[dw(t)dw(t)] &= 0. && (5.53)
\end{aligned}$$

For those who are familiar with the form of Gisin's and Percival's CSL model [76, 77] (which they call quantum state diffusion) will note that my equation appears different. I would like to point out that they are only different by a time-dependent phase factor and since $|\psi(t)\rangle$ and $\exp(-i\varphi(t))|\psi(t)\rangle$ are actually the same state then in fact they are the same equation [135].

My view on the CSL model is although it does propose a new interpretation of quantum mechanics, as we will see in the second part of this thesis (chapters 7 and 9) these equations can actually be derive within the orthodox and modal interpretations, by introducing a Markovian bath. That is the two Eqs. (5.39) and (5.50) are actually just a member of the class of evolution equations known as Markovian stochastic Schrödinger equations.

5.2.3 Dynamical reduction with general Gaussian noises

The last dynamical reduction model I am going to consider is dynamical reduction with general Gaussian noises. This was very recently proposed by Bassi and Ghirardi [5, 6, 7]. In this model we extend the ideas of the CSL model to include noises which are not necessarily white. That is, to start with we assume a modified Schrödinger equation of the form

$$d_t|\bar{\psi}(t)\rangle = [-\frac{i}{\hbar}\hat{H}_{\rm uni}(t) + \hat{A} + w(t)\hat{L}]|\bar{\psi}(t)\rangle, \qquad (5.54)$$

here $w(t)$ is a complex Gaussian stochastic process (not white noise) whose first two moments are

$$E[w(t)] = 0, \tag{5.55}$$

$$E[w(t)w(s)] = \gamma D(t-s) \tag{5.56}$$

and $\hat{L} = \hat{L}^\dagger$. As in the CSL case we can work out the $\hat{A}$ by using the requirement that the average of the norm must be 1. Bassi and Ghirardi show that by using the Furutsu-Novikov formula,

$$E[F[w(t)]w(t)] = \gamma \int_0^t D(t-s)E\Big[\frac{\delta}{\delta w(s)}F[w(t)]\Big] \tag{5.57}$$

where $F[w(t)]$ is a functional of the path $w(t)$ and $\delta_{w(s)}$ (which is short hand for $\delta/\delta w(s)$) is a functional derivative, the correct modified Schrödinger equation is

$$d_t|\bar{\psi}(t)\rangle = [-\frac{i}{\hbar}\hat{H}_{\rm uni}(t) + \hat{L}w(t) + 2\gamma\hat{L}\int_0^t D(t,s)\frac{\delta}{\delta w(s)}]|\bar{\psi}(t)\rangle. \tag{5.58}$$

This is actually equivalent to the linear form of a non-Markovian SSE derived by Wiseman and myself in [59] (with $\hat{L} = \hat{L}^\dagger$ see chapter 9) and by considering complex noise we can derived the linear non-Markovian SSE first derive by Diósi, Gisin and Strunz [48] using this method. Thus dynamical reduction with gaussian noise is one interpretation of non-Markovian SSE. Bassi and Ghirardi take the view that, Eq. (5.58) when normalized, represents an objective description of the world when the dynamical reduction mechanism is controlled by general Gaussian noises. However as shown by Wiseman and myself, by introducing a bath, non-Markovian SSE have an interpretation in both the orthodox and modal interpretation of quantum mechanics. This will be shown in chapter 9.

5.2.4 Summary of dynamical reduction interpretation

To summarize the dynamical reduction interpretation, I believe that since it is possible (although presently not done for the random spontaneous localization model) to explain these equations under either the orthodox or modal interpretation by introducing an underlying bath (this is what causes the noise conditioning) then this is only a different way of looking at quantum mechanics. Furthermore in my opinion the motivation behind this method is limited in comparison to the modal and orthodox interpretation. By this I mean dynamical reduction models cannot provide reasons for preferring one stochastic equation over the others, whereas in the orthodox and modal interpretation all these stochastic equation can be explained under the one theory. However, I can see the usefulness in proceeding down these paths as they provide alternative mathematical procedures for developing very complicated non-linear SSEs. I would also like to say that the non-Markovian SSE presented by Bassi and Ghirradi [5] was developed simultaneously by Wiseman and myself [59] in 2002 .

Part II

STOCHASTIC SCHRÖDINGER EQUATIONS

Chapter 6

Open Quantum Systems

In nature it is very unlikely to find a system existing in isolation; usually it is immersed in an environment (or bath). In quantum mechanics we label this type of system an open quantum system. Due to the many degrees of freedom of the bath it is impractical (in general impossible) to solve the Schrödinger equation for this system. Instead it is best to introduced a reduced state $\rho_{\rm red}(t)$ for the system, as

$$\rho_{\rm red}(t) = {\rm Tr}_{\rm env}[|\Psi(t)\rangle\langle\Psi(t)|], \tag{6.1}$$

where $|\Psi(t)\rangle$ is the composite state of the system and bath (which is effectively the universe). In this chapter I will show that we can derive a closed evolution equation for $\rho_{\rm red}(t)$, known as the master equation. I will present this equation for both a Markovian and non-Markovian bath and in doing this I will point out explicitly what the difference is between a Markovian and non-Markovian bath. I will also present the numerical interpretation of SSEs.

6.1 Underlying dynamics

For the purposes of this thesis when considering an open quantum system I will always be referring to the following underlying dynamics. The total Hamiltonian of the universe (system and bath) is

$$\hat{H}_{\rm uni} = \hat{H}_{\rm sys} \otimes \hat{1} + \hat{1} \otimes \hat{H}_{\rm env} + \hat{V}. \tag{6.2}$$

The system Hamiltonian is split into two parts, defined by $\hat{H}_{\rm sys} = \hat{H}_{\omega_{\rm sys}} + \hat{H}$. The bath is modeled by a collection of κ harmonic oscillators, so the Hamiltonian for the environment is

$$\hat{H}_{\rm env} = \hbar \sum_k^{\kappa} \omega_k \hat{a}_k^\dagger \hat{a}_k, \tag{6.3}$$

where ω_k is the angular frequency and $\hat{a}_k$ ($\hat{a}_k^\dagger$) is the annihilation (creation) operators for the $k^{\rm th}$ mode respectively. The annihilation and creation operators are defined such that they satisfy the commutation relation $[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{k,k'}$. This is the standard model for the electromagnetic field. I will assume the interaction Hamiltonian, $\hat{V}$, is linear in the bath amplitude and of the form

$$\hat{V} = i\hbar \sum_k^{\kappa} (g_k^* \hat{L} \hat{a}_k^\dagger - g_k \hat{L}^\dagger \hat{a}_k), \tag{6.4}$$

where g_k is the coupling amplitude of the $k^{\rm th}$ mode to the system. This is motivated by fact that if the harmonic oscillators correspond to the electromagnetic field, then the interaction of an atom (or a single-mode cavity) with the field can be modeled by the dipole approximation [1, 110]. Here we assume that the wavelength of light is much greater than the dimension of the atom (this implies the atom see an uniform field). If this is the case then

$$\hat{V} = -e\hat{\mathbf{r}} \cdot \hat{\mathbf{E}}(\mathbf{r_0}) \tag{6.5}$$

where $\hat{\mathbf{r}}$ is the position vector of the electron and $\hat{\mathbf{E}}(\mathbf{r_0})$ is the electromagnetic field calculated at the center of mass position of the atom ($\mathbf{r_0}$). The form of $\hat{\mathbf{E}}(\mathbf{r_0})$ (neglecting polarization) is

$$\hat{\mathbf{E}}(\mathbf{r_0}) = -i\sum_k^{\kappa} \epsilon_{\mathbf{k}} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\,[\hat{a}_k^\dagger \exp(-i\mathbf{k}\cdot\mathbf{r_0}) - \hat{a}_k \exp(i\mathbf{k}\cdot\mathbf{r_0})], \tag{6.6}$$

where ϵ_0 is the permittivity of free space and V is the quantization volume. The system part of $\hat{V}$ can be written as

$$e\hat{\mathbf{r}} = \sum_{j,j'} e|\phi_j\rangle\langle\phi_j|\hat{\mathbf{r}}|\phi_{j'}\rangle\langle\phi_{j'}| = \sum_{j,j'} \mathbf{d}_{j,j'}|\phi_j\rangle\langle\phi_{j'}| \tag{6.7}$$

where $\mathbf{d}_{j,j'} = \mathbf{d}l_{j.j'}$ is the electron-dipole transition matrix. Using this we can rewrite Eq. (6.5) as

$$\hat{V} = i\hbar\sum_k^{\kappa} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\left[\mathbf{d}\sum_{j,j'} l_{j,j'}|\phi_j\rangle\langle\phi_{j'}| + \mathbf{d}\sum_{j,j'} l_{j,j'}|\phi_j\rangle\langle\phi_{j'}|\right]\cdot\epsilon_{\mathbf{k}}[\hat{a}_k^\dagger \exp(-i\mathbf{k}\cdot\mathbf{r_0}) - \hat{a}_k \exp(i\mathbf{k}\cdot\mathbf{r_0})]. \tag{6.8}$$

This can be simplified by invoking the rotating wave approximation (RWA) (neglecting non-conserving energy terms) to

$$\hat{V} = i\hbar\sum_k^{\kappa}[g_k^*\hat{a}_k^\dagger \sum_{j<j'} l_{j,j'}|\phi_j\rangle\langle\phi_{j'}| - g_k\hat{a}_k\sum_{j<j'} l_{j',j}^*|\phi_{j'}\rangle\langle\phi_j|], \tag{6.9}$$

where

$$g_k = \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\,\frac{\mathbf{d}\cdot\epsilon_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r_0})}{\hbar}. \tag{6.10}$$

Thus if $\hat{L} = \sum_{j<j'} l_{j,j'}|\phi_j\rangle\langle\phi_{j'}|$ this is equivalent to Eq. (6.4).

For calculational purposes we define dynamics for open quantum systems in the interaction picture (see section 2.1.3). This allows us to move the fast dynamics placed on the state by the Hamiltonians $\hat{H}_{\omega_{\rm sys}}$ and $\hat{H}_{\rm env}$ to the operators. The unitary evolution operator for this transformations is

$$\hat{U}_0(t,t_0) = \exp[-\frac{i}{\hbar}(\hat{H}_{\omega_{\rm sys}}\otimes\hat{1} + \hat{1}\otimes\hat{H}_{\rm env})(t-t_0)]. \tag{6.11}$$

This allows us to write the Schrödinger equation (in the interaction picture as) as

$$d_t|\Psi(t)\rangle = -\frac{i}{\hbar}[\hat{H}_{\rm int}(t) + \hat{V}_{\rm int}(t)]|\Psi(t)\rangle, \tag{6.12}$$

where the Hamiltonians are

$$\hat{H}_{\rm int}(t) \quad = \quad U_0^\dagger(t,t_0)\hat{H}U_0(t,t_0), \tag{6.13}$$

which is still just a system operator and

$$\hat{V}_{\rm int}(t) \quad = \quad i\hbar\sum_k^{\kappa}[g_k^*\hat{L}\hat{a}_k^\dagger e^{i\Omega_k(t-t_0)} - g_k\hat{L}^\dagger\hat{a}_k e^{-i\Omega_k(t-t_0)}] \tag{6.14}$$

where $\Omega_k = \omega_k - \omega_{\rm sys}$. Here we have finally restricted the form of $\hat{H}_{\omega_{\rm sys}}$ to be such that $\hat{L}$ in the interaction picture simply rotates in the complex plane at frequency $\omega_{\rm sys}$. That is $\hat{L}_{\rm int}(t) = \hat{L}e^{-i\omega_{\rm sys}(t-t_0)}$.

6.2 Markovian master equation

To derive the master equation we start by using the Schrödinger equation for the composite state matrix $W(t)$ ($\rho_{\text{red}} = \text{Tr}_{\text{env}}[W(t)]$), in the interaction picture this is

$$d_t W(t) = -\frac{i}{\hbar}[\hat{H}_{\text{int}}(t) + \hat{V}_{\text{int}}(t), W(t)]. \tag{6.15}$$

Integrating this equation gives

$$W(t) = W(t_0) - \frac{i}{\hbar}\int_{t_0}^{t} dt'[\hat{H}_{\text{int}}(t') + \hat{V}_{\text{int}}(t), W(t')]. \tag{6.16}$$

Tracing Eq. (6.15) over the environment gives

$$d_t \rho_{\text{red}}(t) = -\frac{i}{\hbar}[\hat{H}_{\text{int}}(t), \rho_{\text{red}}(t)] - \frac{i}{\hbar}\text{Tr}_{\text{env}}[\hat{V}_{\text{int}}(t), W(t)]. \tag{6.17}$$

Substituting Eq. (6.16) into this gives

$$\begin{aligned} d_t \rho_{\text{red}}(t) &= -\frac{i}{\hbar}[\hat{H}_{\text{int}}(t), \rho_{\text{red}}(t)] - \frac{i}{\hbar}\text{Tr}_{\text{env}}[\hat{V}_{\text{int}}(t), W(t_0)] \\ &\quad -\frac{1}{\hbar^2}\int_{t_0}^{t} dt'\text{Tr}_{\text{env}}[\hat{V}_{\text{int}}(t), [\hat{H}_{\text{int}}(t') + \hat{V}_{\text{int}}(t'), W(t')]]. \end{aligned} \tag{6.18}$$

This equation is exact. I have simply cast Eq. (6.15) into a convenient form to make reasonable approximations.

I will firstly assume that at time t_0 the bath and the system are separable ($W(t_0) = \rho_{\text{red}}(t_0)\rho_{\text{env}}(t_0)$, where $\rho_{\text{red}}(t_0)$ is the initial state of the system and $\rho_{\text{env}}(t_0)$ is the initial state of the environment). That is, at the initial time no correlations exists (this assumption will be assumed for all open quantum systems dealt with in this thesis). With this assumption we can eliminate the term $-i\text{Tr}_{\text{env}}[\hat{V}_{\text{int}}(t), W(t_0)]/\hbar$ as $\text{Tr}_{\text{env}}[\hat{V}_{\text{int}}(t), \rho_{\text{env}}(t_0)] = 0$ for most initial environment states and if not we can always rearrange the system Hamiltonian to include this term.

The first approximation we make is the Born (weak coupling, g_k is very small) approximation [18, 28]. This consists in approximating $W(t')$ for the last term in Eq. (6.18) by

$$W(t') = \rho_{\text{red}}(t')\rho_{\text{env}}(t_0) + O(\hat{V}_{\text{int}}(t')). \tag{6.19}$$

That is, we assume the environment is so large that to second order in the interaction Hamiltonian (second order in g_k) the state matrix is separable. Making this approximation to Eq. (6.18) gives

$$d_t \rho_{\text{red}}(t) = -\frac{i}{\hbar}[\hat{H}_{\text{int}}(t), \rho_{\text{red}}(t)] - \frac{1}{\hbar^2}\int_{t_0}^{t} dt'\text{Tr}_{\text{env}}[\hat{V}_{\text{int}}(t), [\hat{H}_{\text{int}}(t') + \hat{V}_{\text{int}}(t'), \rho_{\text{red}}(t')\rho_{\text{env}}(t_0)]]. \tag{6.20}$$

Note do not confuse this to mean that the composite state matrix $W(t)$ is always approximated by Eq. (6.19), if this was the case then we could substitute this approximation into Eq. (6.15). It is only valid when made to second order interaction Hamiltonian terms.

Eq. (6.20) can be further simplified by making the Markovian approximation [18, 28]. This approximation is based on the assumption that the correlation time of the environment, $\tau_{\text{env}}^{\text{corr}}$, (time over which the relevant bath operators decorrelate) is (very) small compared to the time scale $\tau_{\text{sys}}^{\text{coh}}$ (time over which a typical pure system state decoheres). Thus we can replace $\rho_{\text{red}}(t')$ by $\rho_{\text{red}}(t)$ in Eq. (6.20). This yields a closed differential equation of motion for the reduced state [contains only

$\rho_{\rm red}(t)]$, namely

$$d_t\rho_{\rm red}(t) = -\frac{i}{\hbar}[\hat{H}_{\rm int}(t),\rho_{\rm red}(t)] - \frac{1}{\hbar^2}\int_{t_0}^{t} dt'{\rm Tr}_{\rm env}[\hat{V}_{\rm int}(t),[\hat{H}_{\rm int}(t')+\hat{V}_{\rm int}(t'),\rho_{\rm red}(t)\rho_{\rm env}(t_0)]] \tag{6.21}$$

$$= -\frac{i}{\hbar}[\hat{H}_{\rm int}(t),\rho_{\rm red}(t)] + \hat{\mathcal{D}}(t)\rho_{\rm red}(t), \tag{6.22}$$

where $\hat{\mathcal{D}}$ is a the superoperator representing the effect of the Markovian environment on the reduced state evolution. A superoperator is defined such that it takes $\rho(t) \to \rho'(t)$. Defining $\mathcal{B}$ as the space containing all possible state matrices then we can mathematically define an arbitrary superoperator by the map

$$\hat{\mathcal{A}} : \mathcal{B} \to \mathcal{B}. \tag{6.23}$$

For this thesis, when refereing to Markovian master equations, I am only going to consider the initial environment state $\rho_{\rm env}(t_0) = |\{0_k\}\rangle_{\rm env}\langle\{0_k\}|$ (vacuum initial conditions). Using this initial environment state and the underlying dynamics presented above $\hat{\mathcal{D}}(t)\rho_{\rm red}(t)$ becomes

$$\hat{\mathcal{D}}(t)\rho_{\rm red}(t) = \int_{t_0}^{t} dt'{\rm Tr}_{\rm env}\Big[\sum_{k}^{\kappa}[g_k^*\hat{L}\hat{a}_k^\dagger e^{i\Omega_k(t-t_0)} - g_k\hat{L}^\dagger\hat{a}_k e^{-i\Omega_k(t-t_0)}],\Big[\sum_{k'}^{\kappa}[gk'^*\hat{L}\hat{a}_{k'}^\dagger \times e^{i\Omega_{k'}(t'-t_0)} - g_{k'}\hat{L}^\dagger\hat{a}_{k'}e^{-i\Omega_{k'}(t'-t_0)}],\rho_{\rm red}(t)|\{0_k\}\rangle_{\rm env}\langle\{0_k\}|\Big]\Big]. \tag{6.24}$$

This expands to give 16 terms however, for the vacuum bath initial condition, only 4 are non-zero. Thus

$$\hat{\mathcal{D}}(t)\rho_{\rm red}(t) = \int_{t_0}^{t} dt'\Big[\alpha^*(t-t')\hat{L}\rho_{\rm red}(t)\hat{L}^\dagger - \alpha^*(t-t')\rho_{\rm red}(t)\hat{L}^\dagger\hat{L} - \alpha(t-t')\hat{L}^\dagger\hat{L}\rho_{\rm red}(t) + \alpha(t-t')\hat{L}\rho_{\rm red}(t)\hat{L}^\dagger\Big], \tag{6.25}$$

where

$$\alpha(t-t') = \sum_{k}^{\kappa}|g_k|^2 e^{-i\Omega_k(t-t')}. \tag{6.26}$$

The above, although appears correct, is not consistent with the Markovian approximation. In the Markovian limit there is an infinite number of modes, each of which is infinite in volume, so $|g_k|^2$ is infinitesimal [see for example Eq. (6.10)]. Thus we can replace the sum in Eq. (6.26) by and integral and introduce $p(\omega)$ (the density of field modes). Doing this Eq. (6.26) becomes

$$\alpha(t-t') = \int_0^\infty d\omega p(\omega)|g(\omega)|^2 e^{-i(\omega-\omega_{\rm sys})(t-t')}. \tag{6.27}$$

Note $|g(\omega)|^2$ may be infinitesimal but the product $p(\omega)|g(\omega)|^2$ is finite. In Eq. (6.25) $\alpha(t-t')$ is integrated over time, so it is this integral which we are interested in. Performing this integration gives

$$\lim_{t\to\infty}\int_0^{t-t_0} d\tau\alpha(\tau) = \int_0^{t-t_0} d\tau\int_0^\infty d\omega p(\omega)|g(\omega)|^2 e^{-i(\omega-\omega_{\rm sys})\tau}$$

$$= \int_0^\infty d\omega p(\omega)|g(\omega)|^2\int_0^{t-t_0} d\tau\Big[\cos[(\omega-\omega_{\rm sys})\tau] + i\sin[(\omega-\omega_{\rm sys})\tau]\Big]$$

$$= \int_0^\infty d\omega p(\omega)|g(\omega)|^2\Big[\pi\delta(\omega-\omega_{\rm sys}) + i\frac{P}{\omega-\omega_{\rm sys}}\Big], \tag{6.28}$$

where P indicates the Cauchy principal value. Note I have change the integration variable from dt' to $d\tau$ where $\tau = t - t'$ and assumed that the integration limit can be extended to infinity. This can be simplified to

$$\lim_{t\to\infty} \int_0^{t-t_0} d\tau \alpha(\tau) = \frac{\gamma}{2} + i\Delta \tag{6.29}$$

where

$$\gamma = 2\pi p(\omega_{\text{sys}})|g(\omega_{\text{sys}})|^2 \tag{6.30}$$

$$\Delta = P \int_0^{\infty} d\omega \frac{p(\omega)|g(\omega)|^2}{\omega - \omega_{\text{sys}}}. \tag{6.31}$$

Here γ is radiative decay rate and Δ has the effect of adding a rotating term to the master equation.

Substituting Eq. (6.29) into Eq. (6.25) gives

$$\hat{\mathcal{D}}(t)\rho_{\text{red}}(t) = \frac{\gamma}{2}\Big[2\hat{L}\rho_{\text{red}}(t)\hat{L}^\dagger - \rho_{\text{red}}(t)\hat{L}^\dagger\hat{L} - \hat{L}^\dagger\hat{L}\rho_{\text{red}}(t)\Big] - i\Delta[\hat{L}^\dagger\hat{L}, \rho_{\text{red}}(t)]. \tag{6.32}$$

In this thesis, from now on I will assume that $\Delta = 0$. Thus

$$\hat{\mathcal{D}}(t)\rho_{\text{red}}(t) = \frac{\gamma}{2}\Big[2\hat{L}\rho_{\text{red}}(t)\hat{L}^\dagger - \rho_{\text{red}}(t)\hat{L}^\dagger\hat{L} - \hat{L}^\dagger\hat{L}\rho_{\text{red}}(t)\Big]. \tag{6.33}$$

With this superoperator the general Markovian master equation for an initial quantum state of the form $W(t_0) = \rho_{\text{red}}(t_0) \otimes |\{0_k\}\rangle_{\text{env}}\langle\{0_k\}|$ is

$$d_t\rho_{\text{red}}(t) = -\frac{i}{\hbar}[\hat{H}_{\text{int}}(t), \rho_{\text{red}}(t)] + \gamma\Big[\hat{L}\rho_{\text{red}}(t)\hat{L}^\dagger - \rho_{\text{red}}(t)\hat{L}^\dagger\hat{L}/2 - \hat{L}^\dagger\hat{L}\rho_{\text{red}}(t)/2\Big], \tag{6.34}$$

which is equivalent to the CSL master equation (Eq. (5.14)). This is why Gisin and Percival in Ref. [76] quote that their Markovian SSE represents "the evolution of a quantum system in interaction with its environment".

6.3 General non-Markovian master equation

In this section I am going to use the Nakajima-Zwanzig projection (super-) operator technique to define the general master equation [96, 141]. This technique is based on a partition of the state of the system into a relevant and irrelevant part by defining a projection superoperator, $\hat{\mathcal{P}}$, which projects the state matrix of the composite system $W(t)$ into a relevant part, and the projector $\hat{\mathcal{Q}} = \hat{1} - \hat{\mathcal{P}}$ which projects into the irrelevant part. For the above system-environment model the projector $\hat{\mathcal{P}}$ is defined as

$$\hat{\mathcal{P}}W(t) = \text{Tr}_{\text{env}}[W(t)] \otimes \rho_{\text{env}}. \tag{6.35}$$

When $W(t)$ corresponds to the solution of

$$d_tW(t) = -\frac{i}{\hbar}[\hat{H}_{\text{int}}(t) + \hat{V}_{\text{int}}(t), W(t)] = \hat{\mathcal{L}}(t)W(t), \tag{6.36}$$

we have

$$\hat{\mathcal{P}}W(t) = \rho_{\text{red}}(t)\rho_{\text{env}}. \tag{6.37}$$

Here ρ_{env} is currently an arbitrary state for the environment. Since the commutators $[\hat{\mathcal{P}}, d_t]W(t) = 0$ and $[\hat{\mathcal{Q}}, d_t]W(t) = 0$, the equation of motion for the two components $\hat{\mathcal{P}}W(t)$ and $\hat{\mathcal{Q}}W(t)$ can be obtained directly from Eq. (6.36):

$$d_t[\hat{\mathcal{P}}W(t)] = \hat{\mathcal{P}}\hat{\mathcal{L}}(t)\hat{\mathcal{P}}W(t) + \hat{\mathcal{P}}\hat{\mathcal{L}}(t)\hat{\mathcal{Q}}W(t), \tag{6.38}$$

$$d_t[\hat{\mathcal{Q}}W(t)] = \hat{\mathcal{Q}}\hat{\mathcal{L}}(t)\hat{\mathcal{P}}W(t) + \hat{\mathcal{Q}}\hat{\mathcal{L}}(t)\hat{\mathcal{Q}}W(t). \tag{6.39}$$

The second of these equations can be solved formally for $\hat{\mathcal{Q}}W(t)$ in terms of $\hat{\mathcal{Q}}W(t_0)$ and $\hat{\mathcal{P}}W(t)$. To do this we start by noting that

$$\begin{aligned}\partial_t\Big\{T_{\leftarrow}\exp\Big[-\int_{t_0}^{t}\hat{\mathcal{Q}}\hat{\mathcal{L}}(s)ds\Big]\hat{\mathcal{Q}}W(t)\Big\} &= -T_{\leftarrow}\exp\Big[-\int_{t_0}^{t}\hat{\mathcal{Q}}\hat{\mathcal{L}}(s)ds\Big]\hat{\mathcal{Q}}\hat{\mathcal{L}}(t)\hat{\mathcal{Q}}W(t)\\ &\quad +T_{\leftarrow}\exp\Big[-\int_{t_0}^{t}\hat{\mathcal{Q}}\hat{\mathcal{L}}(s)ds\Big]\partial_t[\hat{\mathcal{Q}}W(t)], \end{aligned}\tag{6.40}$$

where $T_{\leftarrow}$ indicates the chronological time ordering. Comparing this with Eq. (6.39) gives

$$\partial_t\Big\{T_{\leftarrow}\exp\Big[-\int_{t_0}^{t}\hat{\mathcal{Q}}\hat{\mathcal{L}}(s)ds\Big]\hat{\mathcal{Q}}W(t)\Big\} = T_{\leftarrow}\exp\Big[-\int_{t_0}^{t}\hat{\mathcal{Q}}\hat{\mathcal{L}}(s)ds\Big]\hat{\mathcal{Q}}\hat{\mathcal{L}}(t)\hat{\mathcal{P}}W(t). \tag{6.41}$$

Integrating this equation (from t_0 to t) gives

$$T_{\leftarrow}\exp\Big[-\int_{t_0}^{t}\hat{\mathcal{Q}}\hat{\mathcal{L}}(s)ds\Big]\hat{\mathcal{Q}}W(t) = \hat{\mathcal{Q}}W(t_0) + \int_{t_0}^{t}T_{\leftarrow}\exp\Big[-\int_{t_0}^{t'}\hat{\mathcal{Q}}\hat{\mathcal{L}}(s)ds\Big]\hat{\mathcal{Q}}\hat{\mathcal{L}}(t')\hat{\mathcal{P}}W(t')dt', \tag{6.42}$$

which can be rearranged to

$$\hat{\mathcal{Q}}W(t) = T_{\leftarrow}\exp\Big[+\int_{t_0}^{t}\hat{\mathcal{Q}}\hat{\mathcal{L}}(s)ds\Big]\hat{\mathcal{Q}}W(t_0) + \int_{t_0}^{t}T_{\leftarrow}\exp\Big[\int_{t'}^{t}\hat{\mathcal{Q}}\hat{\mathcal{L}}(s)ds\Big]\hat{\mathcal{Q}}\hat{\mathcal{L}}(t')\hat{\mathcal{P}}W(t')dt'. \tag{6.43}$$

If we now make our first approximation, we assume that initially there is no correlations, that is $W(t_0) = \rho_{\mathrm{red}}(t_0)\rho_{\mathrm{env}}(t_0)$. We can write $\hat{\mathcal{Q}}W(t_0)$ as

$$\hat{\mathcal{Q}}W(t_0) = (1-\hat{\mathcal{P}})W(t_0) = W(t_0) - \rho_{\mathrm{red}}(t_0)\rho_{\mathrm{env}} \tag{6.44}$$

if $\rho_{\mathrm{env}}(t_0) = \rho_{\mathrm{env}}$ then $\hat{\mathcal{Q}}W(t_0) = 0$, and since ρ_{env} is an arbitrary state we can take this as being true. Thus we can simplify Eq. (6.40) to

$$\hat{\mathcal{Q}}W(t) = \int_{t_0}^{t}T_{\leftarrow}\exp\Big[\int_{t'}^{t}\hat{\mathcal{Q}}\hat{\mathcal{L}}(s)ds\Big]\hat{\mathcal{Q}}\hat{\mathcal{L}}(t')\hat{\mathcal{P}}W(t')dt'. \tag{6.45}$$

Substituting this into the equation of motion for the relevant part of $W(t)$ (Eq. (6.38)) we obtain

$$d_t[\hat{\mathcal{P}}W(t)] = \hat{\mathcal{P}}\hat{\mathcal{L}}(t)\hat{\mathcal{P}}W(t) + \int_{t_0}^{t}\hat{\mathcal{P}}\hat{\mathcal{L}}(t)T_{\leftarrow}\exp\Big[\int_{t'}^{t}\hat{\mathcal{Q}}\hat{\mathcal{L}}(s)ds\Big]\hat{\mathcal{Q}}\hat{\mathcal{L}}(t')\hat{\mathcal{P}}W(t')dt', \tag{6.46}$$

which by definition (6.35) becomes

$$\begin{aligned}d_t[\rho_{\mathrm{red}}(t)]\rho_{\mathrm{env}}(t_0) &= \mathrm{Tr}_{\mathrm{env}}[\hat{\mathcal{L}}(t)\rho_{\mathrm{red}}(t)\rho_{\mathrm{env}}(t_0)]\rho_{\mathrm{env}}(t_0) + \int_{t_0}^{t}\mathrm{Tr}_{\mathrm{env}}\Big[\hat{\mathcal{L}}(t)T_{\leftarrow}\exp\Big[\int_{t'}^{t}\hat{\mathcal{Q}}\hat{\mathcal{L}}(s)ds\Big]\\ &\quad \times\hat{\mathcal{Q}}\hat{\mathcal{L}}(t')\rho_{\mathrm{red}}(t')\rho_{\mathrm{env}}(t_0)\Big]\rho_{\mathrm{env}}(t_0)dt'. \end{aligned}\tag{6.47}$$

We can simplify the first term in this equation by considering the underlying dynamics, doing this we find that

$$\mathrm{Tr}_{\mathrm{env}}[\hat{\mathcal{L}}(t)\rho_{\mathrm{red}}(t)\rho_{\mathrm{env}}(t_0)] = -\frac{i}{\hbar}[\hat{H}_{\mathrm{int}}(t),\rho_{\mathrm{red}}(t)] - \frac{i}{\hbar}\mathrm{Tr}_{\mathrm{env}}[\hat{V}_{\mathrm{int}}(t),\rho_{\mathrm{red}}(t)\rho_{\mathrm{env}}(t_0)] \tag{6.48}$$

and for all linear coupling interactions, and using the same arguments in the Markovian case, we can set $\mathrm{Tr}_{\mathrm{env}}[\hat{V}_{\mathrm{int}}(t),\rho_{\mathrm{red}}(t)\rho_{\mathrm{env}}(t_0)] = 0$ [this is guaranteed for $\rho_{\mathrm{env}}(t_0) = |\{0_k\}\rangle_{\mathrm{env}}\langle\{0_k\}|$, a vacuum initial bath state, and for $\hat{V}_{\mathrm{int}}(t)$ given by Eq. (6.14)]. The second term is where the difficulty lies. In general, even by considering the underlying dynamics, this can not be simplified. Instead we define the system two-time superoperator $\hat{\mathcal{K}}(t,t')$ as

$$\hat{\mathcal{K}}(t,t')\rho_{\text{red}}(t') = \text{Tr}_{\text{env}}\Big[\hat{\mathcal{L}}(t)T_{\leftarrow}\exp\Big[\int_{t'}^{t}\hat{\mathcal{Q}}\hat{\mathcal{L}}(s)ds\Big]\hat{\mathcal{Q}}\hat{\mathcal{L}}(t')\rho_{\text{env}}(t_0)\Big]\rho_{\text{red}}(t'). \tag{6.49}$$

This allows us to rewrite Eq. (6.47) as

$$d_t\rho_{\text{red}}(t) \quad = \quad -\frac{i}{\hbar}[\hat{H}_{\text{int}}(t),\rho_{\text{red}}(t)] + \int_{t_0}^{t}dt'\hat{\mathcal{K}}(t,t')\rho_{\text{red}}(t'). \tag{6.50}$$

We interpret $\hat{\mathcal{K}}(t,t')$ as the memory-time superoperator, it represents the effect of the environment on the average evolution of the system (for a Markovian environment $\hat{\mathcal{K}}(t,t') = 2\hat{\mathcal{D}}(t)\delta(t-t')$). I would like to note that this equation is exact (provided the initial state is separable), but due to complexity of $\hat{\mathcal{K}}(t,t')$ [Eq. (6.49)] in general this can not be explicitly evaluated. However, perturbative techniques do exit [18, 31, 68, 69, 105, 106, 112], but in general these techniques do not preserve the positivity requirements of the master equation at small times. That is, the reduced state is not always a positive operator (with all eigenvalues non-negative).

6.4 Stochastic Schrödinger equations: Numerical interpretation

In the above section I illustrated how to derive the master equation for both a Markovian and non-Markovian environment. For some systems (Hilbert space dimension D) it is possible that the size of $\rho_{\text{red}}(t)$ might be too large to store on a computer (D^2). In this case we need to reformulate the problem in terms of pure states. To do this we define a pure state, $|\psi_z(t)\rangle$, such that

$$\rho_{\text{red}}(t) = E[|\psi_z(t)\rangle\langle\psi_z(t)|] \tag{6.51}$$

where $E[...]$ denotes an ensemble average over the parameter $z(t,t)$. The different functional forms of $z(t,t)$ define different classes of $|\psi_z(t)\rangle$. These different classes are referred to as unravelings of the master equation. With this pure state we can evaluate specific averages by

$$\text{Tr}[\rho_{\text{red}}(t)\hat{A}_{\text{sys}}] = E[\langle\psi_z(t)|\hat{A}_{\text{sys}}|\psi_z(t)\rangle]. \tag{6.52}$$

The evolution equation of this pure state is a SSE. Diósi, Gisin and Strunz [46, 47, 48, 116, 117] were the first to propose a diffusive non-Markovian SSE which satisfies this constraint.

In the Markovian limit along with diffusive unravelings (satisfying Eq. (6.51)) we can also define jump-like SSEs, these satisfy

$$\rho_{\text{red}}(t) = E[|\psi_{dN}(t)\rangle\langle\psi_{dN}(t)|] \tag{6.53}$$

where $E[...]$ denotes an ensemble average over the noise function $dN(t)$, which is either 0 or 1 $[dN(t) = dN(t)^2(t)]$. In terms of a numerical application these were first proposed by Dalibard, Castin, and Mølmer [34, 95] and later by Dum, Parkins, Zoller, and Gardiner [50, 66], around the same time as when diffusive Markovian SSE were been developed under the CSL model of quantum mechanics (see section 5.2.2 and references within).

6.5 Summary of chapter

In the chapter I have defined what an open quantum system is and presented its underlying dynamics. I have also shown that by introducing a Markovian bath the master equation for the reduced state evolution is equivalent to the CSL master equation. For the non-Markovian bath I presented the Nakajima-Zwanzig method for deriving the non-Markovian master equation. Lastly I defined the numerical interpretation of all (Markovian and non-Markovian) SSEs.

Chapter 7

Markovian SSEs: Quantum Trajectories

In this chapter I am going to present quantum trajectory theory [28, 136, 137, 134]. This is an application of quantum measurement theory to continuous-in-time monitoring. I will show that the jump-like and diffusive Markovian SSEs can be interpreted as continuous evolution equations (quantum trajectories) for the system state conditioned on continuous monitoring of the bath. Thus under this interpretation all Markovian SSE can be given an interpretation. Performing different bath measurements defines which class of Markovian SSE the system state will obey (the unraveling). In this chapter I will consider 5 measurement schemes: direct, heterodyne, homodyne-x, homodyne-y, and an adaptive detection. But before I do this I want to consider the general framework

7.1 General detection

In chapter 2 (namely sections 2.3 and 2.4) I introduced the concept of continuous measurement theory. There it was observed that we can represent all bath measurements as POM measurements on the system. To derive Markovian SSEs we assume that the bath measurement is described by the observable

$$Z(t) \equiv \{Z_k(t)\} = \Big\{\Big(\{z_{n_k}\}, \hat{F}_{\{z_{n_k}\}}(t) = \frac{1}{N}|\{z_{n_k}(t)\}\rangle_{\rm env}\langle\{z_{n_k}(t)\}| \otimes \hat{1}_{\rm sys}\Big)\Big\} \tag{7.1}$$

which outputs a string of bath results $\{z_{n_k}\}$. However upon measurement we can couple all the results together by defining a noise function $z(t,s) = f(\{r(Z_k(t),t)\},s)$. Here I am again using the notation $r(Z_k(t),t)$, corresponding to the result of the measurement of the observable $Z_k(t)$ [that is it is the random variable associated with probability distribution for the measurement of observable $Z_k(t)$]. Equally valid, we could define an observable for the entire bath as

$$I(t) = \{(I, \hat{F}_I(t) = \frac{1}{N}|I(t)\rangle_{\rm env}\langle I(t)| \otimes \hat{1}_{\rm sys})\}, \tag{7.2}$$

where $|I(t)\rangle_{\rm env}$ labels the state the bath is projected into. The results of this measurement are $r(I(t),t) \propto z(t,t)$ (usually referred to as the current observable). That is, the entire bath is treated as a single system. For this type of bath observable one decomposition into measurement operators is

$$\hat{M}_I(t)_{\rm env} = \frac{1}{\sqrt{N}}|0\rangle_{\rm env}\langle I(t)|. \tag{7.3}$$

Thus $|\chi\rangle_{\rm env}$ in Eq. (2.93) is $|0\rangle \equiv |\{0_k\}\rangle$. This is a reasonable assumption as in general Markovian SSEs are applied to systems immersed in a electromagnetic fields, and measurement is performed by a photodetector; to detect a photon the photodetector absorbs it.

With this definition for a bath measurement operator the system measurement operators [denoted $\hat{M}_I(t,t_0)_{\rm sys}$] are found via Eq. (2.105) [except $\hat{U}(t,t_0)$ must be replaced by $\hat{U}_{\rm int}(t,t_0)$ as we are in the interaction picture]. That is

$$\hat{M}_I(t,t_0)_{\rm sys} = \frac{1}{\sqrt{N}}\langle I(t)|\hat{U}_{\rm int}(t,t_0)|0\rangle. \tag{7.4}$$

To find $\hat{U}_{\rm int}(t,t_0)$ we note that $|\Psi(t)\rangle = \hat{U}_{\rm int}(t,t_0)|\Psi(t_0)\rangle$, which can also be calculated by integrating Eq. (6.12). Doing this we get

$$|\Psi(t)\rangle - |\Psi(t_0)\rangle = \int_{t_0}^{t} ds\Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(s) + \sum_k^{\kappa}[g_k^*\hat{L}\hat{a}_k^\dagger e^{i\Omega_k s} - g_k\hat{L}^\dagger\hat{a}_k e^{-i\Omega_k s}]\Big\}|\Psi(s)\rangle. \tag{7.5}$$

However, since g_k is infinitesimal and real, the above can be simplified by replacing the discrete frequency modes by continuous frequency modes and $\sum_k^\infty \to \int_0^\infty p(\omega)d\omega$, where $p(\omega)$ is the density of field modes. Doing this gives

$$\begin{aligned}|\Psi(t)\rangle - |\Psi(t_0)\rangle \;\;=\;\; & \int_{t_0}^{t} ds\Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(s) + \int_0^\infty d\omega\sqrt{p(\omega)}\,g(\omega)[\hat{L}\hat{a}^\dagger(\omega)e^{i(\omega-\omega_{\rm sys})s} \\ & -\hat{L}^\dagger\hat{a}(\omega)e^{-i(\omega-\omega_{\rm sys})s}]\Big\}|\Psi(s)\rangle.\end{aligned} \tag{7.6}$$

Note the continuous frequency modes are related to discrete frequency modes by $\hat{a}_k = \hat{a}(\omega)/\sqrt{P(\omega)}$ and have the commutator relation $[\hat{a}(\omega),\hat{a}^\dagger(\omega')] = \delta(\omega-\omega')$. This can be further simplified by changing the integration variable ω to $\tilde{\omega} = \omega - \omega_{\rm sys}$ and noting that for a Markovian system we can assume that $\sqrt{p(\omega)}\,g(\omega)$ is approximately constant and equal to $\sqrt{\gamma/2\pi}$ [see Eq. (6.30)]. Doing this gives

$$|\Psi(t)\rangle - |\Psi(t_0)\rangle = \int_{t_0}^{t} ds\Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(s) + \sqrt{\frac{\gamma}{2\pi}}\int_{-\omega_{\rm sys}}^{\infty-\omega_{\rm sys}} d\tilde{\omega}[\hat{L}\hat{a}^\dagger(\tilde{\omega})e^{i\tilde{\omega}s} - \hat{L}^\dagger\hat{a}(\tilde{\omega})e^{-i\tilde{\omega}s}]\Big\}|\Psi(s)\rangle. \tag{7.7}$$

To make the next simplification we write this in terms temporal modes.

In general a temporal mode is best defined by considering the electromagnetic field operator $\hat{\mathbf{E}}(\mathbf{r})$. The standard definition of this (neglecting polarization) is [122]

$$\hat{\mathbf{E}}(\mathbf{r}) = -i\sum_k^{\kappa}\hat{a}_k^\dagger\mathbf{u}_k^*(\mathbf{r}) + i\sum_k^{\kappa}\hat{a}_k\mathbf{u}_k(\mathbf{r}) \tag{7.8}$$

where $\mathbf{u}_k(\mathbf{r})$ is the mode function for the $k^{\rm th}$ frequency (spectral) mode. This can be rewritten as

$$\hat{\mathbf{E}}(\mathbf{r}) = -i\sum_\tau^{\kappa}\hat{b}_\tau^\dagger\mathbf{v}_\tau^*(\mathbf{r}) + i\sum_\tau^{\kappa}\hat{b}_\tau\mathbf{v}_\tau(\mathbf{r}), \tag{7.9}$$

where $\mathbf{v}_\tau(\mathbf{r})$ labels a new type of mode with annihilation and creation operators $\hat{b}_\tau$ and $\hat{b}_\tau^\dagger$. These new modes can be related to the frequency modes by

$$\hat{b}_\tau = \sum_k^{\kappa}\hat{a}_k\gamma_{\tau,k}^*, \tag{7.10}$$

where $\gamma_{\tau,k}$ are the elements of a unitary matrix ($\sum_\tau \gamma^*_{\tau,k}\gamma_{\tau,k'} = \delta_{k,k'}$). Choosing $\gamma_{\tau,k}$ to give a discrete fourier transform,

$$\hat{b}_\tau = \frac{1}{\sqrt{\kappa}}\sum_k^\kappa \hat{a}_k \exp(-i2\pi\tau k/\kappa), \tag{7.11}$$

$$\hat{a}_k = \frac{1}{\sqrt{\kappa}}\sum_\tau^\kappa \hat{b}_\tau \exp(i2\pi\tau k/\kappa), \tag{7.12}$$

results in $\mathbf{v}_\tau(\mathbf{r})$ having the functional form of a temporal mode. For the continuous case the temporal modes, denoted by $\hat{b}(\tau)$, become continuous in time and are related to $\hat{a}(\tilde{\omega})$ by the fourier transform

$$\hat{b}(\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{a}(\tilde{\omega})\exp(-i\tilde{\omega}\tau)d\tilde{\omega}., \tag{7.13}$$

$$\hat{a}(\tilde{\omega}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{b}(\tau)\exp(i\tilde{\omega}\tau)d\tau \tag{7.14}$$

Thus we can rewrite Eq. (7.7) as

$$|\Psi(t)\rangle - |\Psi(t_0)\rangle = \int_{t_0}^t ds\Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(s) + \sqrt{\gamma}\int_{-\infty}^{\infty} d\tau[\hat{L}\hat{b}^\dagger(\tau)\delta(\tau-s) - \hat{L}^\dagger\hat{b}(\tau)\delta(s-\tau)]\Big\}|\Psi(s)\rangle. \tag{7.15}$$

Here we have used the fact that in general $\omega_{\rm sys}$ is large and replaced the lower limit of the $\tilde{\omega}$ integral in Eq. (7.7) by $-\infty$.

For continuous monitoring [the time interval $(t, t_0) \to (t_0 + dt, t_0)$] Eq. (7.15) becomes

$$d|\Psi(t_0+dt)\rangle = -\frac{i}{\hbar}\hat{H}_{\rm int}(t_0)dt|\Psi(t_0)\rangle + \sqrt{\gamma}\int_{t_0}^{t_0+dt} ds[\hat{L}\hat{\xi}^\dagger(s) - \hat{L}^\dagger\hat{\xi}(s)]|\Psi(s)\rangle, \tag{7.16}$$

where the operator $\hat{\xi}(t)$ is

$$\hat{\xi}(t) = \int_{-\infty}^{\infty} d\tau\delta(t-\tau)\hat{b}(\tau), \tag{7.17}$$

and obeys

$$[\hat{\xi}(t), \hat{\xi}^\dagger(t')] = \delta(t, t'). \tag{7.18}$$

This delta-function singularity stops us from directly integrating Eq. (7.16). This is the quantum equivalent of white noise, and to treat it we need to define the quantum equivalent of the classical Wiener process ΔW (see section 5.2.2). Gardiner, Parkins and Zoller [66] define this be the operator $\Delta\hat{B}(t)$. Following there method for a vacuum state initial condition we obtain the following correlations

$$\langle 0|\Delta\hat{B}(t)|0\rangle = \langle 0|\Delta\hat{B}^\dagger(t)|0\rangle = 0, \tag{7.19}$$

$$\langle 0|\Delta\hat{B}^\dagger(t)\Delta\hat{B}(t_0)|0\rangle = \langle 0|\Delta\hat{B}^\dagger(t)\Delta\hat{B}^\dagger(t_0)|0\rangle = \langle 0|\Delta\hat{B}(t)\Delta\hat{B}(t_0)|0\rangle = 0, \tag{7.20}$$

$$\langle 0|\Delta\hat{B}(t)\Delta\hat{B}^\dagger(t_0)|0\rangle = t - t_0. \tag{7.21}$$

To make the connection with the white noise operators $\hat{\xi}(t)$ we note that

$$\Delta\hat{B}(t) = \int_{t_0}^t \hat{\xi}(s)ds, \tag{7.22}$$

$$\Delta\hat{B}^\dagger(t) = \int_{t_0}^t \hat{\xi}^\dagger(s)ds. \tag{7.23}$$

As with the Wiener process this can be proven by substituting these into Eqs. (7.19) – (7.21) and using Eq. (7.18). Thus, in the infinitesimal limit ($t = t_0 + dt$) $\Delta\hat{B}(t) = \hat{\xi}(t_0)dt$ and $\Delta\hat{B}^\dagger(t) = \hat{\xi}^\dagger(t_0)dt$ and for short hand it is customary to use the $d\hat{B}(t_0)$ and $d\hat{B}^\dagger(t_0)$. $d\hat{B}(t_0)$ and $d\hat{B}^\dagger(t_0)$ are quantum non-commutative analogues of the Wiener increment. With these quantum Wiener increment we can write any integral containing $\hat{\xi}(t)$ as

$$\int_{t_0}^{t} ds\hat{\xi}(s)|\Psi(s)\rangle = \int_{t_0}^{t} d\hat{B}(s)|\Psi(s)\rangle, \tag{7.24}$$

$$\int_{t_0}^{t} ds\hat{\xi}^\dagger(s)|\Psi(s)\rangle = \int_{t_0}^{t} d\hat{B}^\dagger(s)|\Psi(s)\rangle., \tag{7.25}$$

which can be evaluated by the quantum Stratonovich or quantum Itô method (or any other method between). These methods involve defining the integral as a (operator) Riemann-Stieltjes integral. That is, we divide the interval $[t_0, t]$ into N subintervals such that $t_0 \leq t_1 \leq t_2 ... \leq t_N$, and define intermediate points τ_i such that $t_{i-1} \leq \tau_i \leq t_i$ and define this integral as

$$\int_{t_0}^{t} d\hat{B}(s)|\Psi(s)\rangle = \sum_{i=1}^{N} d\hat{B}(t_{i-1})|\Psi(\tau_i)\rangle, \tag{7.26}$$

$$\int_{t_0}^{t} d\hat{B}(s)|\Psi(s)\rangle = \sum_{i=1}^{N} d\hat{B}(t_{i-1})|\Psi(\tau_i)\rangle. \tag{7.27}$$

There are many ways we can define the midpoints. The Itô method chooses them such that $\tau_i = t_{i-1}$ allowing us to write the integral as,

$$\mathcal{I}\int_{t_0}^{t} d\hat{B}(s)|\Psi(s)\rangle = \sum_{i=1}^{N} d\hat{B}(t_{i-1})|\Psi(t_{i-1})\rangle, \tag{7.28}$$

$$\mathcal{I}\int_{t_0}^{t} d\hat{B}(s)|\Psi(s)\rangle = \sum_{i=1}^{N} d\hat{B}(t_{i-1})|\Psi(t_{i-1})\rangle. \tag{7.29}$$

By contrast the Stratonovich (mid point) method chooses them such that

$$\mathcal{S}\int_{t_0}^{t} d\hat{B}(s)|\Psi(s)\rangle = \sum_{i=1}^{N} d\hat{B}(t_{i-1})[|\Psi(t_i)\rangle + |\Psi(t_{i-1})\rangle]/2 = \sum_{i=1}^{N} d\hat{B}(t_{i-1})[|\Psi(t_{i-1})\rangle + \tfrac{1}{2}d|\Psi(t_{i-1})\rangle], \tag{7.30}$$

$$\mathcal{S}\int_{t_0}^{t} d\hat{B}(s)|\Psi(s)\rangle = \sum_{i=1}^{N} d\hat{B}(t_{i-1})[|\Psi(t_i)\rangle + |\Psi(t_{i-1})\rangle]/2 = \sum_{i=1}^{N} d\hat{B}(t_{i-1})[|\Psi(t_{i-1})\rangle + \tfrac{1}{2}d|\Psi(t_{i-1})\rangle]. \tag{7.31}$$

With this Stratonovich integral (as this is the more natural for the limit of a delta function) we can rewrite Eq. (7.16) as

$$d|\Psi(t_0 + dt)\rangle = -\frac{i}{\hbar}\hat{H}_{\text{int}}(t_0)dt|\Psi(t_0)\rangle + \sqrt{\gamma}\int_{t_0}^{t_0+dt} ds[\hat{L}d\hat{B}^\dagger(s) - \hat{L}^\dagger d\hat{B}(s)]|\Psi(s)\rangle. \tag{7.32}$$

In Ref. [66, 67] Gardiner, Parkins and Zoller have shown explicitly how to work out the Itô correction terms needed to convert this to a Itô integral. For the above equation, with vacuum-bath initial conditions, they show that the conversion is

$$\mathcal{S}\int_{t_0}^{t}[\hat{L}d\hat{B}^\dagger(s) - \hat{L}^\dagger d\hat{B}(s)]|\Psi(s)\rangle = \mathcal{I}\int_{t_0}^{t}[\hat{L}d\hat{B}^\dagger(s) - \hat{L}^\dagger d\hat{B}(s)]|\Psi(s)\rangle + \frac{\sqrt{\gamma}}{2}\int_{t_0}^{t} ds\hat{L}^\dagger\hat{L}|\Psi(s)\rangle, \tag{7.33}$$

which in the continuous monitoring limit reduces to

$$\mathcal{S}\int_{t_0}^{t_0+dt}[\hat{L}d\hat{B}^\dagger(s)-\hat{L}^\dagger d\hat{B}(s)]|\Psi(s)\rangle=[\hat{L}d\hat{B}^\dagger(t_0))-\hat{L}^\dagger d\hat{B}(t_0)-\sqrt{\gamma}\,\hat{L}^\dagger\hat{L}/2]|\Psi(t_0)\rangle dt. \tag{7.34}$$

Thus Eq. (7.16) becomes

$$d|\Psi(t_0+dt)\rangle=[-dt\frac{i}{\hbar}\hat{H}_{\rm int}(t_0)+\sqrt{\gamma}\,\hat{L}d\hat{B}^\dagger(t_0))-\sqrt{\gamma}\,\hat{L}^\dagger d\hat{B}(t_0)-dt\gamma\hat{L}^\dagger\hat{L}/2]|\Psi(t_0)\rangle, \tag{7.35}$$

which in turn implies

$$U_{\rm int}(t_0+dt,t_0) = 1+[-dt\frac{i}{\hbar}\hat{H}_{\rm int}(t_0)+\sqrt{\gamma}\,\hat{L}d\hat{B}^\dagger(t_0)-\sqrt{\gamma}\,\hat{L}^\dagger d\hat{B}(t_0)-dt\gamma\hat{L}^\dagger\hat{L}/2]. \tag{7.36}$$

Note the above shows that for each interval dt (due to the delta correlations) each $U_{\rm int}(t_0+dt,t_0)$ exists in it own Hilbert space. In some situation it is easier if we introduce a harmonic oscillator to model the entire bath in the interval dt, with annihilation and creation operators $\hat{a}_{dt}$ and $\hat{a}_{dt}^\dagger$ defined by

$$\hat{a}_{dt} = \sqrt{dt}\,\hat{\xi}(t_0)=d\hat{B}(t_0)/\sqrt{dt}\,, \tag{7.37}$$

$$\hat{a}_{dt}^\dagger = \sqrt{dt}\,\hat{\xi}^\dagger(t_0)=d\hat{B}^\dagger(t_0)/\sqrt{dt}\,. \tag{7.38}$$

This is valid because

$$[\hat{a}_{dt},\hat{a}_{dt}^\dagger]=\lim_{dt\to 0}[\hat{\xi}(t_0+dt),\hat{\xi}^\dagger(t_0)]=\lim_{dt\to 0}\delta(t_0+dt-t_0)dt=1, \tag{7.39}$$

the standard requirement for a harmonic oscillator. With this model Eq. (7.36) becomes

$$U_{\rm int}(t_0+dt,t_0) = 1+[-dt\frac{i}{\hbar}\hat{H}_{\rm int}(t_0)+\sqrt{\gamma dt}\,\hat{a}_{dt}^\dagger\hat{L}-\sqrt{\gamma dt}\,\hat{a}_{dt}\hat{L}^\dagger-dt\gamma\hat{L}^\dagger\hat{L}/2]. \tag{7.40}$$

This is useful as with respect to this model we can define a complete set of bath basis states $\{|n\rangle\}$ for ecah dt interval, which are the eigenstate of the operator $\hat{a}_{dt}^\dagger\hat{a}_{dt}$. Thus the infinitesimal measurement operator for the system is

$$\hat{M}_I(t_0+dt,t_0)_{\rm sys}=\frac{1}{\sqrt{N}}\langle I(t_0+dt)|0\rangle\{1-[dt\frac{i}{\hbar}\hat{H}_{\rm int}(t_0)+dt\gamma\hat{L}^\dagger\hat{L}/2]\}+\sqrt{\gamma dt}\,\langle I(t_0+dt)|\hat{a}_{dt}^\dagger|0\rangle\hat{L} \tag{7.41}$$

and the state conditioned on continuous monitoring of the bath (with the bath after measurement being projected into the vacuum state) is given by

$$|\psi_{\mathbf{I}}(t)\rangle=\frac{\hat{M}_{I_k}(t,t-dt)\hat{M}_{I_{k-1}}(t-dt,t-2dt)\ldots\hat{M}_{I_1}(t_0+dt,t_0)|\psi(t_0)\rangle}{N}, \tag{7.42}$$

with $\mathbf{I}_{[t_0,t)}=[r_1(I(t_0+dt),t_0+dt),...,r_k(I(t),t)]$, the string of results for measurement of the observable I and N is the normalization constant for the measurement operator.

It can be shown that the ensemble average of these condition states over all possible currents $\mathbf{I}_{[t_0,t)}$ is

$$\rho_{red}(t)=E[|\psi_{\mathbf{I}}(t)\rangle\langle\psi_{\mathbf{I}}(t)|]. \tag{7.43}$$

Thus $|\psi_{\mathbf{I}}(t)\rangle$ by definition is the solution of a SSE.

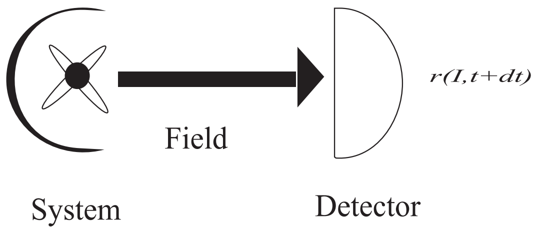

Figure 7.1: A illustration depicting direct detection of a system. The system is placed at the focus of a parabolic mirror so that all the fluorescence emitted by the system is detector by the photodetector.

7.2 Direct detection

7.2.1 General theory

For direct detection all the light emitted by the atom (the entire Markovian bath) is detected by a photodetector [28, 137]. Figure 7.1 illustrates this type of measurement. To understand direct detection we need to consider how this measurement is performed in an optical bath. For an electromagnetic field [Eq. (6.6)] the photocurrent is given by [28]

$$\hat{I} = 2\epsilon_0 c \hat{E}^{(-)}(\mathbf{r})\hat{E}^{(+)}(\mathbf{r}), \tag{7.44}$$

where

$$\hat{E}^{(+)}(\mathbf{r}) = \epsilon \cdot \hat{\mathbf{E}}^{(+)}(\mathbf{r}) = i\sum_k^{\kappa} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\,\hat{a}_k \exp(i\mathbf{k}\cdot\mathbf{r}), \tag{7.45}$$

$$\hat{E}^{(-)}(\mathbf{r}) = \epsilon \cdot \hat{\mathbf{E}}^{(-)}(\mathbf{r}) = -i\sum_k^{\kappa} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\,\hat{a}_k^\dagger \exp(-i\mathbf{k}\cdot\mathbf{r}), \tag{7.46}$$

are the positive and negative components of the electric field operator evaluated at the location of the detector $\mathbf{r}$, and the factor $2\epsilon_0 c$ is needed to give the correct units for photocurrent. Substituting Eqs. (7.45) and (7.46) into Eq. (7.44) gives

$$\hat{I} = \frac{\hbar c}{V}\sum_k^{\kappa}\sqrt{\omega_k}\,\exp(-i\mathbf{k}\cdot\mathbf{r})\hat{a}_k^\dagger \sum_{k'}^{\kappa}\sqrt{\omega_{k'}}\,\exp(i\mathbf{k}'\cdot\mathbf{r})\hat{a}_{k'}. \tag{7.47}$$

Moving to the interaction picture this becomes

$$\hat{I}_{\rm int}(t-t_0) = \frac{\hbar c}{V}\sum_k^{\kappa}\sqrt{\omega_k}\,\exp(-i\mathbf{k}\cdot\mathbf{r})\hat{a}_k^\dagger \exp[i\omega_k(t-t_0)]\sum_{k'}\sqrt{\omega_{k'}}\,\exp(i\mathbf{k}'\cdot\mathbf{r})\hat{a}_{k'}\exp[-i\omega_k'(t-t_0)]. \tag{7.48}$$

Using Eq. (7.13) provided we assume that $\sqrt{\omega_k}$ is approximately constant, it can be shown that

$$\hat{I}_{\rm int}(t-t_0) \propto \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \delta(\tau - t + t_0)\delta(t - t_0 - \tau')\hat{b}^\dagger(\tau)\hat{b}(\tau'). \tag{7.49}$$

Thus for this thesis I will define the observable for the direct detection photocurrent as

$$\hat{I}_{\rm int}(t) = \hat{b}^\dagger(t)\hat{b}(t). \tag{7.50}$$

That is, the direct detection corresponds to detection of the temporal modes.

For continuous-in-time monitoring of this photocurrent (repeated measurements every dt) in each interval the intensity of the field (bath) will be collapsed into the basis set $\{|n\rangle_{\rm env}\}$ defined as the eigenset of $\hat{a}_{dt}^\dagger \hat{a}_{dt}$. That is, the observable (which is time independent) for this measurement scheme is

$$I = \Big((I=0, |0\rangle_{\rm env}\langle 0|), (I=1/dt, |1\rangle_{\rm env}\langle 1|), ...\Big). \tag{7.51}$$

Using Eq. (7.41) the system measurement operators are

$$\hat{M}_{I=0}(t_0+dt,t_0)_{\rm sys} = 1-[dt\frac{i}{\hbar}\hat{H}_{\rm int}(t_0)+dt\gamma\hat{L}^\dagger\hat{L}/2], \tag{7.52}$$

$$\hat{M}_{I=1/dt}(t_0+dt,t_0)_{\rm sys} = \sqrt{\gamma dt}\,\hat{L}. \tag{7.53}$$

Thus only results 0 and $1/dt$ are allowed. Thus $\mathbf{I}_{[t_0,t)}$ will be a string of 0's and 1/dt's, with the ones occurring will probability

$$\begin{aligned}\Pr(I=1/dt, t_0+dt) &= \langle\psi(t_0)|\hat{M}_{I=1/dt}^\dagger(t_0+dt,t_0)_{\rm sys}\hat{M}_{I=1/dt}(t_0+dt,t_0)_{\rm sys}|\psi(t_0)\rangle \\ &= dt\gamma\langle\psi(t_0)|\hat{L}^\dagger\hat{L}|\psi(t_0)\rangle.\end{aligned} \tag{7.54}$$

Using these measurement operators and Eq. (7.42), the quantum trajectory for direct detection can be written as

$$\begin{aligned}d|\psi_{\mathbf{I}}(t)\rangle &= r(I,t+dt)dt\Big(\frac{\hat{L}}{\sqrt{\langle\psi_{\mathbf{I}}(t)|\hat{L}^\dagger\hat{L}|\psi_{\mathbf{I}}(t)\rangle}}-1\Big)|\psi_{\mathbf{I}}(t)\rangle - dt\Big(\frac{i}{\hbar}\hat{H}_{\rm int}(t)+\frac{\gamma}{2}\hat{L}^\dagger\hat{L} \\ &\quad -\frac{\gamma}{2}\langle\psi_{\mathbf{I}}(t)|\hat{L}^\dagger\hat{L}|\psi_{\mathbf{I}}(t)\rangle\Big)|\psi_{\mathbf{I}}(t)\rangle,\end{aligned} \tag{7.55}$$

where $r(I,t+dt) = dN(t)/dt$. $dN(t)$ is a stochastic random variable that equals one if there is a detection in the interval dt and equals zero otherwise. Formally, $dN(t)$ is defined by

$$dN(t)^2 = dN(t), \tag{7.56}$$

$$E[dN(t)] = \Pr(I=1/dt, t+dt). \tag{7.57}$$

By averaging over this stochastic increment and by using Eqs. (7.56) and (7.57) it is easily seen that

$$\rho_{\rm red}(t+dt) = E[d|\psi_{\mathbf{I}}(t)\rangle\langle\psi_{\mathbf{I}}(t)| + |\psi_{\mathbf{I}}(t)\rangle d\langle\psi_{\mathbf{I}}(t)| + d|\psi_{\mathbf{I}}(t)\rangle d\langle\psi_{\mathbf{I}}(t)|] \tag{7.58}$$

equals Eq. (6.34).

7.2.2 Application: Driven TLA

The system used to illustrate quantum trajectories is a classically driven two level atom, immersed in a electromagnetic field. With no monitoring of the field, the average state evolution when all the dynamical parameters are known is given by the master equation (6.34), with $\hat{L} = \hat{\sigma} = |{\downarrow}\rangle\langle{\uparrow}|$. Here $|{\downarrow}\rangle$ labels the ground state and $|{\uparrow}\rangle$ the excited state of the TLA. In the interaction picture, $\hat{H}_{\rm int}(t)$ for a driven TLA is found using the dipole interaction Hamiltonian, Eq. (6.5), except here the electromagnetic field is assumed to be classical. That is

$$\hat{H}(t) = -e\hat{\mathbf{r}}\cdot\mathbf{E}(\mathbf{r_0},t) \tag{7.59}$$

where

$$\mathbf{E}(\mathbf{r_0}) = \mathbf{E}_0(\mathbf{r_0})\{\exp[-i\omega_{\rm c}(t-t_0)]+\exp[i\omega_{\rm c}(t-t)]\}. \tag{7.60}$$

Defining $\mathbf{d} = \mathbf{d}^* = e\langle\downarrow|\hat{\mathbf{r}}|\uparrow\rangle$ we can rewrite Eq. (7.59) under the RWA [1, 110] as

$$\hat{H}(t) = \frac{\hbar\Omega_{\rm dri}}{2}\{\hat{\sigma}^\dagger \exp[-i\omega_{\rm c}(t-t_0)] + \hat{\sigma}\exp[i\omega_{\rm c}(t-t_0)]\}, \tag{7.61}$$

where $\Omega_{\rm dri} = -2\mathbf{d}\cdot\mathbf{E}_0(\mathbf{r_0})/\hbar$ is the Rabi frequency. Moving to the interaction picture and choosing $\omega_{\rm sys} = \omega_{\rm c}$ this becomes

$$\hat{H}_{\rm int} = \frac{\hbar\Omega_{\rm dri}}{2}[\hat{\sigma}^\dagger + \hat{\sigma}] = \frac{\hbar\Omega_{\rm dri}}{2}\hat{\sigma}_x. \tag{7.62}$$

Thus the master equation is

$$\dot{\rho}_{\rm red}(t) = -\frac{i\Omega_{\rm dri}}{2}[\hat{\sigma}_x, \rho_{\rm red}(t)] + \gamma[\hat{\sigma}\rho_{\rm red}(t)\hat{\sigma}^\dagger - \tfrac{1}{2}\hat{\sigma}^\dagger\hat{\sigma}\rho_{\rm red}(t) - \tfrac{1}{2}\rho_{\rm red}(t)\hat{\sigma}^\dagger\hat{\sigma}]. \tag{7.63}$$

The solution of this equation can be described by the Bloch vectors $x(t) = \langle\hat{\sigma}_x\rangle$, $y(t) = \langle\hat{\sigma}_y\rangle$, $z(t) = \langle\hat{\sigma}_z\rangle$), with $\rho(t)$ written as

$$\rho = \tfrac{1}{2}[1 + x(t)\hat{\sigma}_x + y(t)\hat{\sigma}_y + z(t)\hat{\sigma}_z]. \tag{7.64}$$

The purity p is equal to

$$p(t) = \tfrac{1}{2}[1 + x^2(t) + y^2(t) + z^2(t)]. \tag{7.65}$$

Using this representation, Eq. (7.63) generates the three coupled differential equations

$$\begin{aligned} d_t x(t) &= -\frac{\gamma}{2}x(t), && (7.66)\\ d_t y(t) &= -\Omega_{\rm dri}z(t) - \frac{\gamma}{2}y(t), && (7.67)\\ d_t z(t) &= \Omega_{\rm dri}y(t) - \gamma - z(t)\gamma. && (7.68)\end{aligned}$$

Using the Matrix method for coupled differential equation (given the set $d_t\vec{x}(t) = A\vec{x}(t) + \vec{b}$ the solution will be $\vec{x}(t) = \sum_\lambda c_\lambda \vec{u}_\lambda e^{\lambda t} - A^{-1}\vec{b}$, where λ and $\vec{u}_\lambda$ refer to the eigenvalues and eigenvectors of A and c_λ are constants determined by the initial conditions) the above has the solutions

$$\begin{aligned} x(t) &= x(t_0)\exp[-\gamma(t-t_0)/2], && (7.69)\\ y(t) &= c_+\exp[\lambda_+(t-t_0)] + c_-\exp[\lambda_-(t-t_0)] + y_{ss}, && (7.70)\\ z(t) &= c_+\frac{\gamma - 4i\sqrt{(\Omega_{\rm dri}^2 - (\gamma/4)^2)}}{4\Omega_{\rm dri}}\exp[\lambda_+(t-t_0)] + c_-\frac{\gamma + 4i\sqrt{(\Omega_{\rm dri}^2 - (\gamma/4)^2)}}{4\Omega_{\rm dri}} \\ &\quad \times\exp[\lambda_- t - t_0)] + z_{ss}. && (7.71)\end{aligned}$$

where

$$\begin{aligned} \lambda_\pm &= -\frac{3\gamma}{4} \pm i\sqrt{(\Omega_{\rm dri}^2 - (\gamma/4)^2)}, && (7.72)\\ c_\pm &= \frac{1}{8i\sqrt{(\Omega_{\rm dri}^2 - (\gamma/4)^2)}}\{\mp 4\Omega_{\rm dri}(z(t_0) - z_{ss}) \pm [\gamma \pm 4i\sqrt{(\Omega_{\rm dri}^2 - (\gamma/4)^2)}\,] \\ &\quad \times(y(t_0) - y_{ss})\} && (7.73)\end{aligned}$$

and x_{ss}, y_{ss}, and z_{ss} are the steady state solutions of the master equation. That is the solution of Eq. (4.49) is a state that rotates about the x-axis at frequency $\Omega_{\rm dri}$, with damping in all variables towards the steady state value of

$$x_{\rm ss} = 0, \qquad y_{\rm ss} = \frac{2\Omega_{\rm dri}\gamma}{2\Omega_{\rm dri}^2 + \gamma^2}, \qquad z_{\rm ss} = \frac{-\gamma^2}{2\Omega_{\rm dri}^2 + \gamma^2}. \tag{7.74}$$

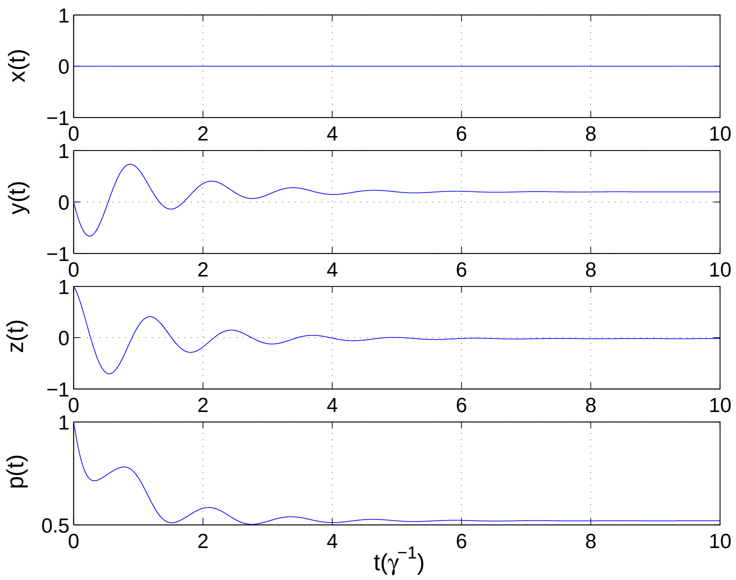

Figure 7.2: Solution of the driven TLA Markovian master equation for a driving $\Omega_{\rm dri} = 5\gamma$, $dt = 0.001$, and excited state initial conditions.

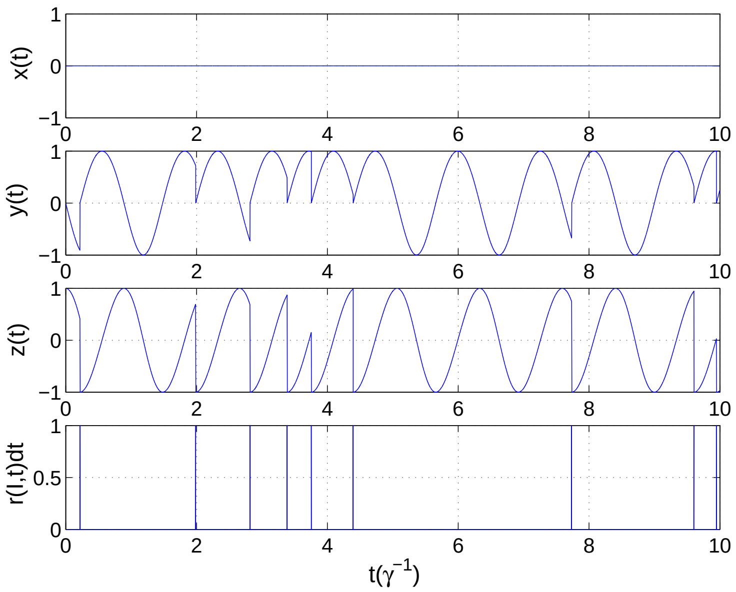

Figure 7.3: A quantum trajectory for direct detection of a driven TLA. Also shown is the photocurrent (times dt). Other details are as in figure 7.2.

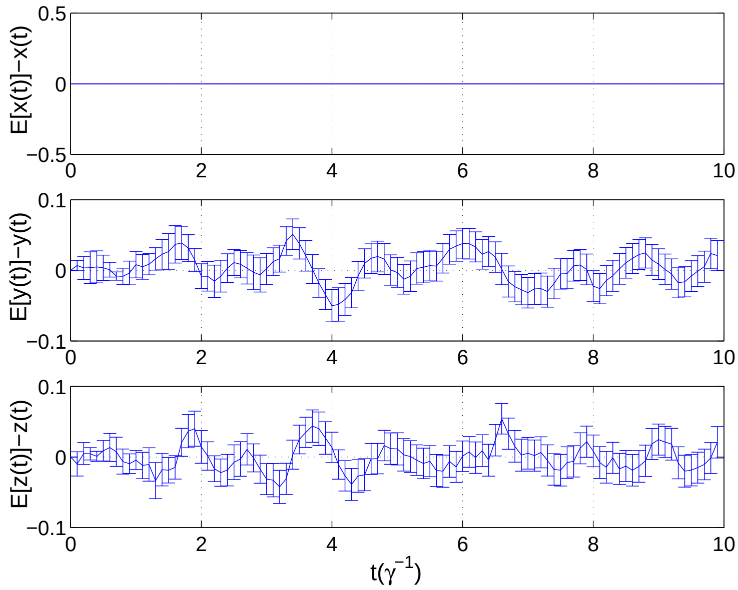

Figure 7.4: The difference between the ensemble average of 1000 direct detection quantum trajectories and the master equation for a driven TLA. Other details are as in figure 7.2.

Figure 7.2 illustrates these solutions for an excited state initial condition.

Applying this system to direct detection we simply replace $\hat{L}$ and $\hat{H}_{\rm int}(t)$ in Eq. (7.55) with $\hat{\sigma}$ and $\hbar\Omega_{\rm dri}\hat{\sigma}_x$. Evaluating this equation gives the direct detection quantum trajectory for a driven TLA. An example trajectory is shown in figure 7.3. This figure illustrates the unique nature of direct detection. It shows that while no detection is occurring the system state rotates around the x-axis of the Bloch sphere, and upon detection the system jumps into the ground state. Also shown in this figure is the current (times by dt), which as expected represents a sequence of 0's and 1's which is predominantly zero. To show that the ensemble average of direct detection does reproduce the master equation the ensemble average of 1000 trajectories was calculated. These results are displayed in figure 7.4 where it is observed that the difference between the ensemble average and the master equation can be taken to be zero within one standard deviation in the mean (this is how the error bars in this figure are defined).

7.3 Heterodyne detection

7.3.1 General theory

To perform a heterodyne measurement on the bath (for optical situations) one can use one of two possible arrangements, one using a local oscillator with a time varying phase [137] or the apparatus depicted in fig. 7.5. Here I consider only the later case, it involves 3 beam splitters; one to split the field from the atom (source) into two outputs and the other two perform two simultaneous measurements of the operators $\hat{I}_1$ and $\hat{I}_2$ respectively [137, 135]. We define the overall operator (to be measured) as $\hat{I} = (\hat{I}_1 + i\hat{I}_2)/\sqrt{2}\,.$

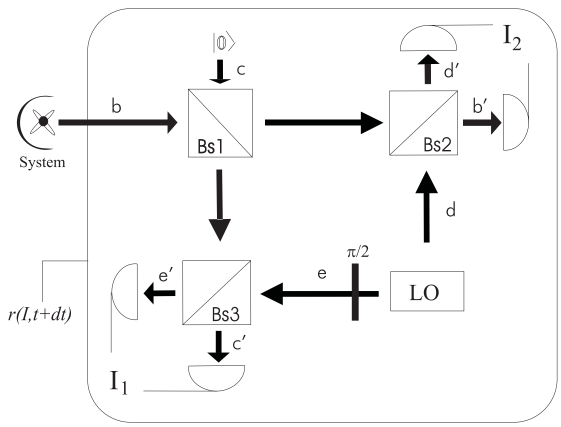

Figure 7.5: A illustration depicting the heterodyne detection scheme. It involve three beam splitters which coherently mix the fluorescence from the system with two local oscillators (LO) with a phase difference of $\pi/2$.

To mathematically model the measurement device, the insides of the box in fig. 7.5, we label the three modes (each equivalent to a temporal mode) $\hat{b}(t)$ for the source (field), $\hat{c}(t)$ an extra vacuum mode, $\hat{d}(t)$ and $\hat{e}(t)$ represent a coherent local oscillator (LO) tuned to the frequency of the system $\omega_{\rm sys}$ (or more correctly the frequently of the interaction picture) and amplitude $|\beta|$. There is a phase difference between mode $\hat{d}(t)$ and $\hat{e}(t)$ of $\pi/2$. If we were to work out the total state (in the Schrödinger picture) prior to all detection events it would be given by,

$$|\Psi_{\rm sch}(t)\rangle = \hat{U}_{\rm Bs3}\hat{U}_{\rm Bs2}\hat{U}_{\rm Bs1}\hat{U}_0(t,t_0)|\Psi(t)\rangle|0\rangle|\beta\rangle, \tag{7.75}$$

which is a large and involved entangled state, where as in the interaction picture with free dynamics, and the three beam splitters removed the total state is $|\Psi(t)\rangle|0\rangle|\beta\rangle$. In this picture the annihilation operators will contain all the unitary transformations, for example the operator $\hat{b}(t)$ will be,

$$\hat{b}'(t) = \hat{U}^\dagger_{\rm Bs2}\hat{U}^\dagger_{\rm Bs1}\ \hat{b}(t)\ \hat{U}_{\rm Bs1}\hat{U}_{\rm Bs2} \tag{7.76}$$

where $\hat{U}_{\rm Bs1}$ for example performs the transformation

$$\hat{b}(t) \to \frac{1}{\sqrt{2}}(\hat{b}(t) + i\hat{c}(t)) \tag{7.77}$$

$$\hat{c}(t) \to \frac{1}{\sqrt{2}}(i\hat{b}(t) + \hat{c}(t)). \tag{7.78}$$

Extending this to all the modes we get,

$$\hat{b}'(t) = \frac{1}{2}(\hat{b}(t) + i\hat{c}(t) + i\sqrt{2}\,\hat{d}(t)) \tag{7.79}$$

$$\hat{c}'(t) = \frac{1}{2}(i\hat{b}(t) - \hat{c}(t) + \sqrt{2}\,\hat{e}(t)) \tag{7.80}$$

$$\hat{d}'(t) = \frac{1}{2}(i\hat{b}(t) - \hat{c}(t) + \sqrt{2}\,\hat{d}(t)) \tag{7.81}$$

$$\hat{e}'(t) = \frac{1}{2}(\hat{b}(t) + i\hat{c}(t) + i\sqrt{2}\,\hat{e}(t)). \tag{7.82}$$

The current $\hat{I}_1(t)$ is measured by taking the difference between the photocurrent at the exits of beam splitter 3 and normalizing by $|\beta|$, whereas $\hat{I}_2(t)$ is found using beam splitter 2. Thus in the interaction picture

$$\hat{I}_{1\ \mathrm{int}}(t) = \frac{\hat{c}'^\dagger(t)\hat{c}'(t) - \hat{e}'^\dagger(t)\hat{e}'(t)}{|\beta|} = \frac{i\hat{e}^\dagger(t)\hat{b}(t) - i\hat{b}^\dagger(t)\hat{e}(t) - \hat{e}^\dagger(t)\hat{c}(t) - \hat{c}^\dagger(t)\hat{e}(t)}{\sqrt{2}\,|\beta|}, \tag{7.83}$$

$$\hat{I}_{2\ \mathrm{int}}(t) = \frac{\hat{b}'^\dagger(t)\hat{b}'(t) - \hat{d}'^\dagger(t)\hat{d}'(t)}{|\beta|} = \frac{i\hat{b}^\dagger(t)\hat{d}(t) - i\hat{d}^\dagger(t)\hat{b}(t) + \hat{c}^\dagger(t)\hat{d}(t) + \hat{d}^\dagger(t)\hat{c}(t)}{\sqrt{2}\,|\beta|}. \tag{7.84}$$

Since we know that modes $\hat{d}(t)$ and $\hat{e}(t) = i\hat{d}(t)$ correspond to a coherent state, then $\hat{d}(t)|\beta\rangle = \beta|\beta\rangle$. The current operator can be average over these input modes to give,

$$\hat{I}_{1\ \mathrm{int}}(t) = \frac{\hat{b}(t) + \hat{b}^\dagger(t) - i\hat{c}(t) + i\hat{c}^\dagger(t)}{\sqrt{2}} \tag{7.85}$$

$$\hat{I}_{2\ \mathrm{int}}(t) = \frac{-i\hat{b}(t) + i\hat{b}^\dagger(t) + \hat{c}(t) + \hat{c}^\dagger(t)}{\sqrt{2}} \tag{7.86}$$

and thus

$$\hat{I}_{\mathrm{int}}(t) = \hat{b}(t) + i\hat{c}^\dagger(t). \tag{7.87}$$

Since initially mode $\hat{c}$ is in a vacuum state, for the infinitesimal interval dt the heterodyne observable corresponds to projection into the overcomplete basis of the operator $\hat{a}_{dt}$. That is $|I\rangle_{\mathrm{env}}$ (in terms of $|n\rangle_{\mathrm{env}}$) is

$$|I\rangle_{\mathrm{env}} = \sqrt{dt}\,\exp(-|I|^2 dt/2)\sum_n \frac{(I\sqrt{dt}\,)^n}{\sqrt{n!}}|n\rangle_{\mathrm{env}}. \tag{7.88}$$

With this state and Eq. (7.41) the system measurement operator is

$$\hat{M}_I(t_0+dt,t_0)_{\mathrm{sys}} = \sqrt{\frac{dt}{\pi}}\,\exp(-|I|^2dt/2)\{1-[dt\frac{i}{\hbar}\hat{H}_{\mathrm{int}}(t_0) - I^*dt\sqrt{\gamma}\,\hat{L} + dt\gamma\hat{L}^\dagger\hat{L}/2]\}, \tag{7.89}$$

Using these measurement operators and Eq. (7.42), the quantum trajectory for heterodyne detection is the solution of

$$\begin{aligned} d|\psi_{\mathbf{I}}(t)\rangle &= \Big(-dt\frac{i}{\hbar}\hat{H}_{\mathrm{int}}(t) + \sqrt{\gamma}\,[\hat{L} - \langle\hat{L}\rangle_t/2]r(I^*,t+dt)dt - \sqrt{\gamma}\,\langle\hat{L}^\dagger\rangle_t r(I,t+dt)dt/2 \\ &\quad -\gamma dt[\hat{L}^\dagger\hat{L} + \hat{L}\langle\hat{L}^\dagger\rangle_t - \frac{3}{2}\langle\hat{L}\rangle_t\langle\hat{L}^\dagger\rangle_t]/2\Big)|\psi_{\mathbf{I}}(t)\rangle, \end{aligned} \tag{7.90}$$

where $\langle\hat{L}\rangle_t = \langle\psi_{\mathbf{I}}(t)|\hat{L}|\psi_{\mathbf{I}}(t)\rangle$, and $r(I,t+dt)$ is a stochastic random variable that corresponds to the distribution

$$\Pr([I],t_0+dt) = \langle\Psi(t_0+dt)|I\rangle\langle I|\Psi(t_0+dt)\rangle d^2I. \tag{7.91}$$

Using Eq. (7.35) we can write this as,

$$\Pr([I],t_0+dt) = (1+\sqrt{\gamma}\,I^*dt\langle\hat{L}\rangle_t + \sqrt{\gamma}\,Idt\langle\hat{L}^\dagger\rangle_t)dt\exp(-|I|^2dt)d^2I. \tag{7.92}$$

To order dt this is,

$$\begin{aligned} \Pr([I],t_0+dt) &= \Big(1+\sqrt{\gamma}\,I^*dt\langle\hat{L}\rangle_t + \sqrt{\gamma}\,Idt\langle\hat{L}^\dagger\rangle_t + \gamma\langle\hat{L}^\dagger\rangle_t\langle\hat{L}\rangle_t(|I|^2dt^2 - dt)\Big) \\ &\quad \times dt\exp(-|I|^2dt)d^2I, \end{aligned} \tag{7.93}$$

as $|I|^2dt^2 = dt + O(dt^2)$. Defining

$$\chi = I^*dt\sqrt{\gamma}\,\langle\hat{L}\rangle_t + Idt\sqrt{\gamma}\,\langle\hat{L}^\dagger\rangle_t - \langle\hat{L}^\dagger\rangle_t\langle\hat{L}\rangle_t\gamma dt \tag{7.94}$$

we can write the above as,

$$\begin{aligned}\Pr([I], t_0 + dt) &= \frac{dt}{\pi}\exp(-dt|I|^2)(1 + \chi + \chi^2/2)d^2I \\ &= \frac{dt}{\pi}\exp[-dt|I - \sqrt{\gamma}\,\langle\hat{L}\rangle_t|^2]d^2I. \end{aligned} \tag{7.95}$$

Therefore $r(I, t + dt)$ is a complex Gaussian random variable of mean $\sqrt{\gamma}\,\langle\hat{L}\rangle_t$ and variance $1/dt$. We can write this in terms of the complex Wiener increment as,

$$r(I, t_0 + dt)dt = dw(t_0) + \sqrt{\gamma}\,\langle\hat{L}\rangle_t dt. \tag{7.96}$$

Thus

$$E[r(I, t_0 + dt)] = \sqrt{\gamma}\,\langle\hat{L}\rangle_t, \tag{7.97}$$

$$E[r(I, t_0 + dt)r(I^*, t_0 + dt)] = 1/dt. \tag{7.98}$$

That is the heterodyne quantum trajectory is equivalent to Gisin, and Percival's CSL model (see section 5.2.2).

7.3.2 Application: Driven TLA

Under heterodyne detection the driven TLA system state will obey a diffusive Markovian SSE. Applying the driven TLA system to heterodyne detection, as in direct detection, we simply replace $\hat{L}$ and $\hat{H}_{\rm int}(t)$ in Eq. (7.90) with $\hat{\sigma}$ and $\hbar\Omega_{\rm dri}\hat{\sigma}_x$. Evaluating this equation gives a heterodyne quantum trajectory. An example trajectory is shown in figure 7.6. This figure illustrates the unique nature of heterodyne detection. It shows that this quantum trajectory unlike direct detection is diffusive in nature, the state stochasticly moves around the block sphere. Also shown in this figure is the current, here we see that it contains both a real (solid line) and a complex (dotted line) part. To show that the ensemble average reproduces the master equation the difference between the ensemble average of 1000 trajectories and the master equation is shown in figure 7.7. Here we see that within the error bars this difference is zero.

7.4 Homodyne-x and homodyne-y detection

7.4.1 General theory

To perform a homodyne measurement of the bath one uses the apparatus depicted in fig. 7.8. It uses a beam splitter to couple a local oscillator, tuned to the frequency of the system but out of phase by an amount ϑ [28, 136, 137, 134, 135] to the system field. The photocurrent from the two outputs is then measured to give a current which is represent by the operator $\hat{I}(t)$.

As in the heterodyne case to model the measurement apparatus it is easier to work with temporal modes. We label the two modes; $\hat{b}(t)$ for the source, $\hat{c}(t)$ represent the local oscillator which is a coherent state of frequency $\omega_{\rm sys}$, phase ϑ, and amplitude $|\beta|$. In the interaction picture with the free dynamics and the beam splitters removed the operators are,

$$\hat{b}'(t) = \frac{1}{\sqrt{2}}(\hat{b}(t) + i\hat{c}(t)) \tag{7.99}$$

$$\hat{c}'(t) = \frac{1}{\sqrt{2}}(i\hat{b}(t) + \hat{c}(t)). \tag{7.100}$$

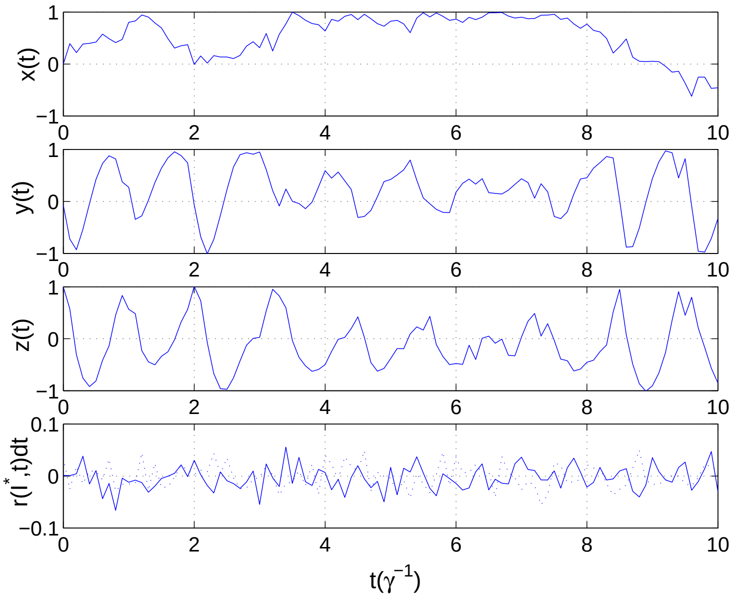

Figure 7.6: A quantum trajectory in Bloch representation for heterodyne detection of a driven TLA. Also shown is the complex photocurrent with the solid line representing the real part (current at beam splitter 1) and the dotted line representing the imaginary part (current at beam splitter 2). Other details are as in figure 7.2.

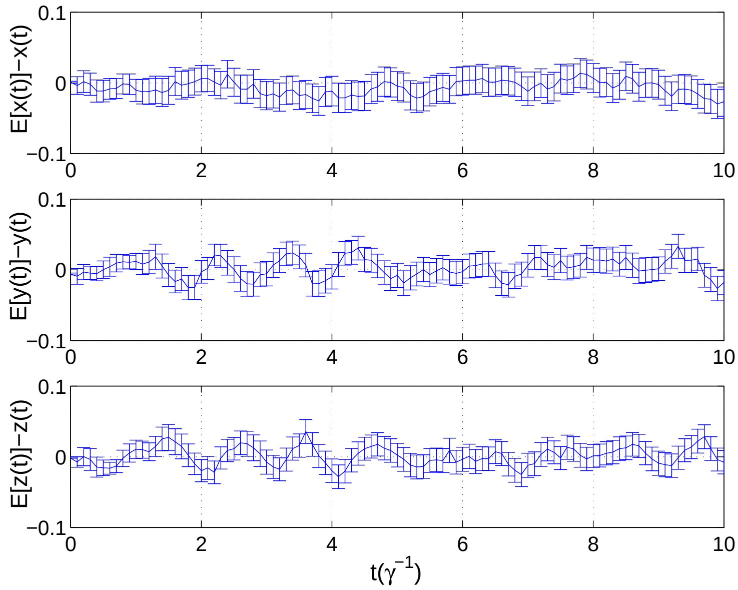

Figure 7.7: The difference between the ensemble average of 1000 heterodyne quantum trajectories and the master equation for a driven TLA. Other details are as in figure 7.2.

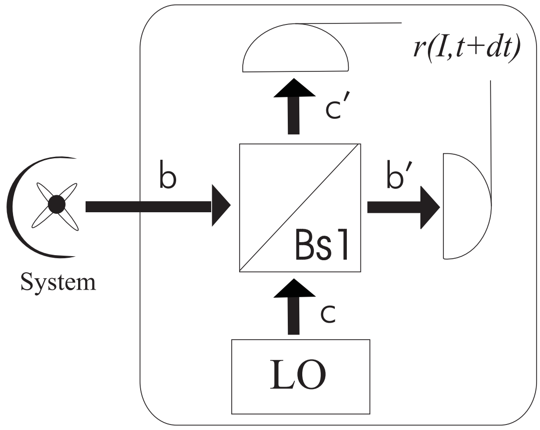

Figure 7.8: An illustration depicting the homodyne detection scheme. The fluorescence from the system is mixed coherently with a local oscillators (LO) via a beam splitter.

The current $\hat{I}(t)$ is measured by taking the difference between the photon number at the exits of the beam splitter and normalizing by $|\beta|$. That is in the interaction picture

$$\hat{I}_{\rm int}(t) = \frac{\hat{b}'^\dagger(t)\hat{b}'(t) - \hat{c}'^\dagger(t)\hat{c}'(t)}{|\beta|} = \frac{i\hat{b}^\dagger(t)\hat{c}(t) - i\hat{c}(t)^\dagger\hat{b}(t)}{2|\beta|}. \tag{7.101}$$

Since we know that mode $\hat{c}$ corresponds to the coherent state, $\hat{c}|\beta\rangle = \beta|\beta\rangle$, then the current operator average over the coherent state input is,

$$\hat{I}(t) = \frac{-i\hat{b}(t)e^{-i\vartheta} + i\hat{b}^\dagger(t)e^{i\vartheta}}{2} \tag{7.102}$$

Defining $\vartheta = \phi + \pi/2$ the above becomes,

$$\hat{I}(t) = \frac{\hat{b}(t)e^{-i\phi} + \hat{b}^\dagger(t)e^{i\phi}}{2}. \tag{7.103}$$

Taking $\phi = 0$ gives homodyne-x detection, and $\phi = \pi/2$ gives homodyne-y detection.

In the infinitesimal interval dt, homodyne detection corresponds to projection into the basis state of the operator $\sqrt{dt}\,a_{dt} + \sqrt{dt}\,a_{dt}^\dagger$ (here I have assumed $\phi = 0$ for simplicity). That is $|I\rangle_{\rm env}$

$$|I\rangle_{\rm env} = \left(\frac{dt}{2\pi}\right)^{1/4} \exp(-I^2dt/4)\sum_n \frac{H_n(I\sqrt{dt/2}\,)}{\sqrt{2^n n!}}|n\rangle_{\rm env}. \tag{7.104}$$

Since $\hat{a}|0\rangle = 0$ the system measurement operator (from Eq. (7.41)) is

$$\hat{M}_I(t_0+dt,t_0)_{\rm sys} = \left(\frac{dt}{2\pi}\right)^{1/4} \exp(-I^2dt/4)\{1 - [dt\frac{i}{\hbar}\hat{H}_{\rm int}(t_0) - Idt\sqrt{\gamma}\,\hat{L} + dt\gamma\hat{L}^\dagger\hat{L}/2]\} \tag{7.105}$$

(for homodyne-y $\hat{L} \to \hat{L}e^{-i\pi/2}$). Using these measurement operators and Eq. (7.42), the quantum trajectory for homodyne detection can be written as

$$\begin{aligned} d|\psi_{\mathbf{I}}(t)\rangle &= \Big(- dt\frac{i}{\hbar}\hat{H}_{\rm int}(t) + \sqrt{\gamma}\,[\hat{L} - \langle\hat{L}+\hat{L}^\dagger\rangle_t/2]r(I,t+dt)dt \\ &\quad -\gamma dt[\hat{L}^\dagger\hat{L} + \hat{L}\langle\hat{L}+\hat{L}^\dagger\rangle_t - \frac{3}{4}\langle\hat{L}+\hat{L}^\dagger\rangle_t^2]/2\Big)|\psi_{\mathbf{I}}(t)\rangle, \end{aligned} \tag{7.106}$$

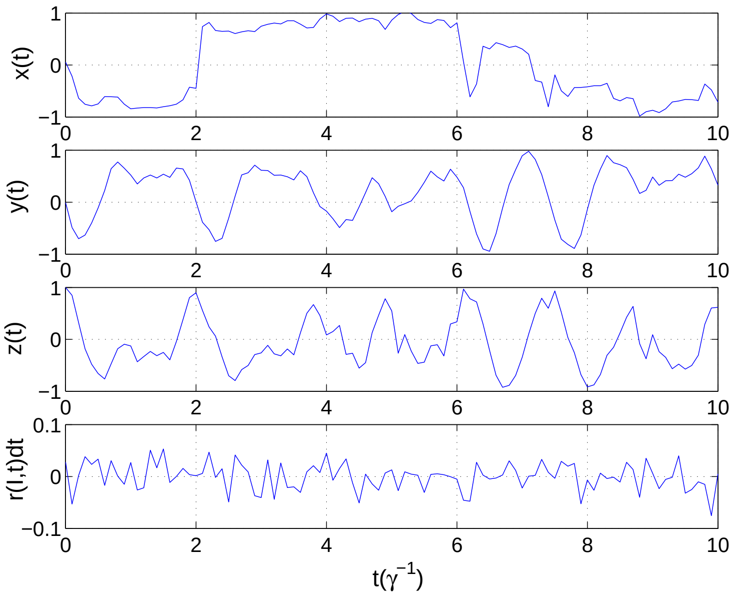

Figure 7.9: A quantum trajectory for homodyne-x detection of a driven TLA. Other details are as in figure 7.2.

where $r(I, t + dt)$ is a stochastic random variable that corresponds to the distribution

$$\Pr([I], t_0 + dt) = \left(\frac{dt}{2\pi}\right)^{1/2} \exp[-(I - \sqrt{\gamma}\,\langle \hat{L} + \hat{L}^\dagger \rangle_t)^2 dt/2] dI. \tag{7.107}$$

Therefore $r(I, t+dt)$ is a Gaussian random variable of mean $\sqrt{\gamma}\,\langle \hat{L} + \hat{L} \rangle_t$ and variance $1/dt$, we can write this in terms of the Wiener increment as,

$$r(I, t_0 + dt)dt = dw(t_0) + \sqrt{\gamma}\,\langle \hat{L} + \hat{L}^\dagger \rangle_t dt. \tag{7.108}$$

Thus

$$E[r(I, t_0 + dt)] = \sqrt{\gamma}\,\langle \hat{L} + \hat{L}^\dagger \rangle_t, \tag{7.109}$$

$$E[r(I, t_0 + dt) r(I, t_0 + dt)] = 1/dt. \tag{7.110}$$

That is the homodyne quantum trajectory is equivalent to Ghirardi, Pearle and Rimini CSL model (see section 5.2.2).

7.4.2 Application: Driven TLA

As in heterodyne detection the homodyne quantum trajectory will be a diffusive Markovian SSE. Applying the driven TLA system to this detection scheme simply means we replace $\hat{L}$ and $\hat{H}_{\text{int}}(t)$ in Eq. (7.106) with $\hat{\sigma} \exp(-i\phi)$ and $\hbar\Omega_{\text{dri}}\hat{\sigma}_x$. Evaluating this equation for $\phi = 0$ gives the homodyne-x quantum trajectory and for $\phi = \pi/2$ we obtain the homodyne-y quantum trajectory. These two trajectories are shown in figure 7.9 and figure 7.11 respectively.

Figure 7.9 shows that under Homodyne-x detection the state is pushed stochastically towards $\hat{\sigma}_x$ eigenstates, with relatively small oscillations in $y(t)$ and $z(t)$ directions. However for homodyne-y

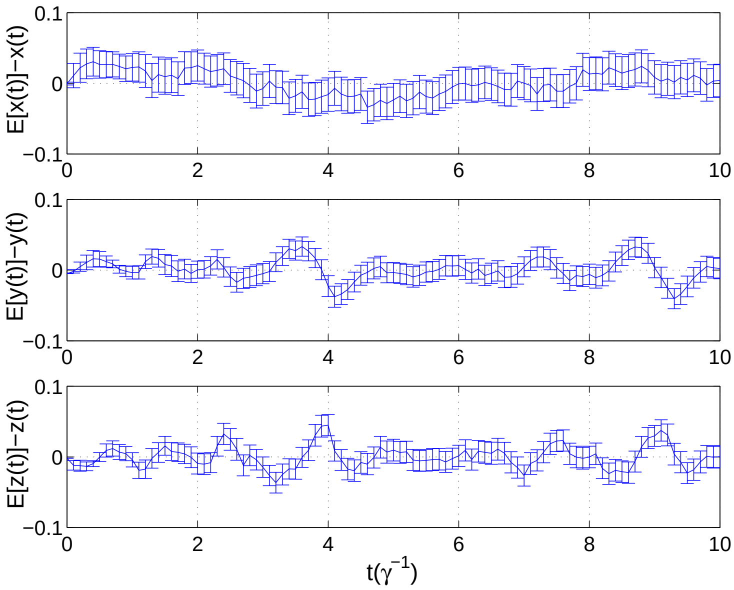

Figure 7.10: The difference between the ensemble average of 1000 homodyne x quantum trajectories and the master equation. Other details are as in figure 7.2.

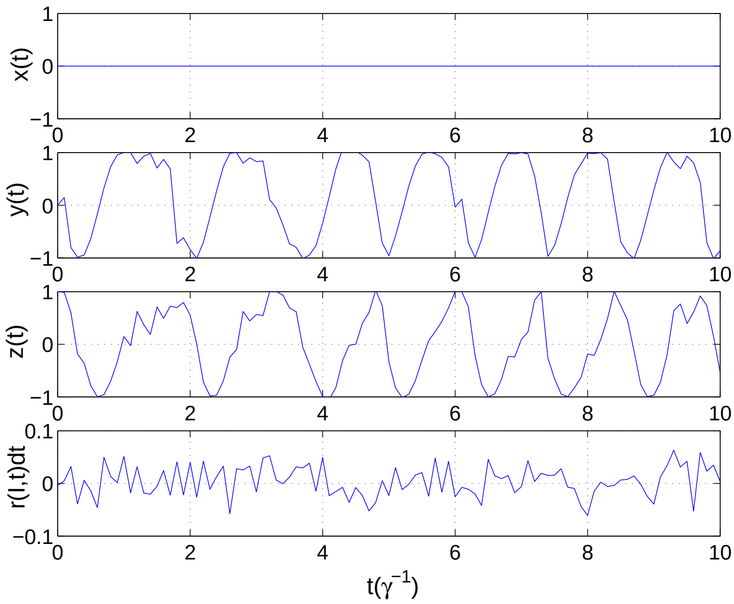

Figure 7.11: A quantum trajectory for homodyne-y detection of a driven TLA. Other details are as in figure 7.2.

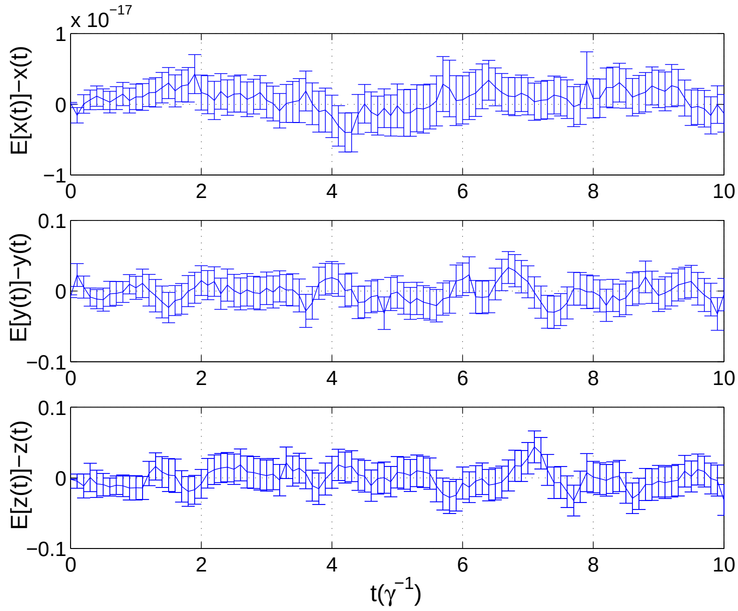

Figure 7.12: The difference between the ensemble average of 1000 homodyne y quantum trajectories and the master equation for a driven TLA. Other details are as in figure 7.2.

the state of the system oscillates (stochastically) around the x-axis at frequency $\Omega_{\rm dri}$ (figure 7.11). Thus by simply changing the phase of the local oscillator we can drastically change the evolution of the state. The convergence, under the ensemble average, of homodyne detection to the master equation is shown in figures 7.10 and 7.12. Here it is observed that the ensemble average of 1000 trajectories for both homodyne x and y is the solution of the master equation (figure 7.63).

7.5 An adaptive detection technique for a driven TLA

The last measurement scheme I am going to consider is adaptive detection. This scheme is designed to keep the system (TLA) jumping between two fixed states. For large classical driving, $\Omega_{\rm dri}$, these fixed states turn out to be close to $\hat{\sigma}_x$ eigenstates [138]. This two-state jumping is achieved by coherently mixing the fluorescence emitted from the atom with a weak local oscillator (LO) via a low-reflectance beam splitter (see Figure 7.13). The amplitude μ of the local oscillator is switched between $\pm\frac{1}{2}\sqrt{\gamma}$ each time a detection is registered by the photodetector.

For this detection scheme the measurement operators are [138]

$$M_{I=1/dt}(t_0+dt,t_0)_{\rm sys} = \sqrt{dt\gamma}\,(\hat{\sigma}+\mu), \qquad (7.111)$$

$$M_{I=0}(t_0+dt,t_0)_{\rm sys} = 1-(i\frac{\Omega_{\rm dri}}{2}\hat{\sigma}_x+\frac{\gamma}{2}\hat{\sigma}^\dagger\hat{\sigma}+\mu\gamma\hat{\sigma}+\frac{\gamma\mu^2}{2})dt. \qquad (7.112)$$

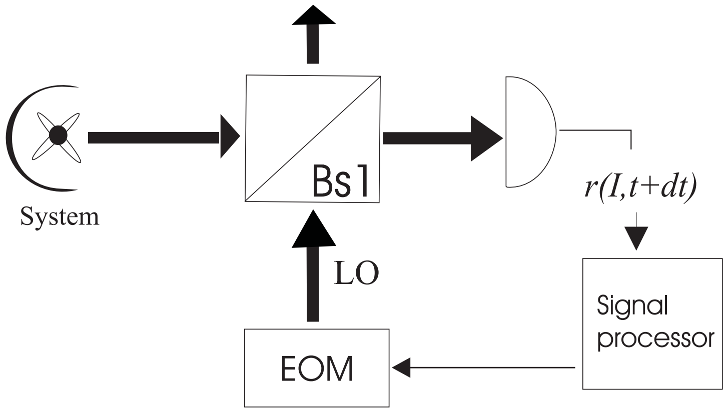

Figure 7.13: A schematic for adaptive detection. The fluorescence emitted by the atom is coherently mixed with a weak local oscillator (LO) via a low reflectivity beam splitter. The electro-optic modulator (EOM) reverses the amplitude of the LO every time the photodetector fires.

These measurement operators result in the following quantum trajectory

$$\begin{aligned} d|\psi_{\mathbf{I}}(t)\rangle &= r(I,t+dt)dt\Big(\frac{(\hat{\sigma}+\mu)}{\sqrt{\langle\psi_{\mathbf{I}}(t)|(\hat{\sigma}^\dagger+\mu)(\hat{\sigma}+\mu)|\psi_{\mathbf{I}}(t)\rangle}}-1\Big)|\psi_{\mathbf{I}}(t)\rangle+dt\Big(-i\frac{\Omega_{\rm dri}}{2}\hat{\sigma}_x \\ &\quad -\frac{\gamma}{2}\hat{\sigma}^\dagger\hat{\sigma}-\mu\gamma\hat{\sigma}+\frac{\gamma}{2}\langle\hat{\sigma}^\dagger\hat{\sigma}\rangle_t+\frac{\gamma\mu}{2}\langle\hat{\sigma}^\dagger+\hat{\sigma}\rangle\Big)|\psi_{\mathbf{I}}(t)\rangle \end{aligned} \tag{7.113}$$

where $r(I,t+dt) = dN(t)/dt$. As in direct detection $dN(t)$ is a stochastic random variable that equals one if there is a detection in the interval dt and equals zero otherwise. Formally, $dN(t)$ is defined by

$$dN(t)^2 = dN(t), \tag{7.114}$$

$$E[dN(t)] = \Pr(I=1/dt,t+dt) = dt\gamma\langle\psi_{\mathbf{I}}(t)|(\hat{\sigma}^\dagger+\mu)(\hat{\sigma}+\mu)|\psi_{\mathbf{I}}(t)\rangle. \tag{7.115}$$

Figure 7.14 shows the quantum trajectory for this adaptive technique. It is observed that when the initial transients have passed this state jumps between two fixed states. In Ref. [138] it is shown that these two states in the Bloch representation are

$$x = \frac{\mp 2\Omega_{\rm dri}^2}{2\Omega_{\rm dri}^2+\gamma^2}, \qquad y = \frac{2\Omega_{\rm dri}\gamma}{2\Omega_{\rm dri}^2+\gamma^2}, \qquad z = \frac{-\gamma^2}{2\Omega_{\rm dri}^2+\gamma^2}, \tag{7.116}$$

Figure 7.15 shows that this detection scheme, like all others, averages to the correct master equation.

7.6 Summary of chapter

In this chapter I have shown that Markovian SSE can be derived and interpreted under the orthodox interpretation of quantum mechanics by introducing a Markovian bath. They represent evolution equations for the system state conditioned on continuous measurement of the bath. The different unravelings correspond to different detector arrangements. Here I consider the 5 schemes, direct, heterodyne, homodyne-x, homodyne-y and an adaptive technique. It is observed that by changing

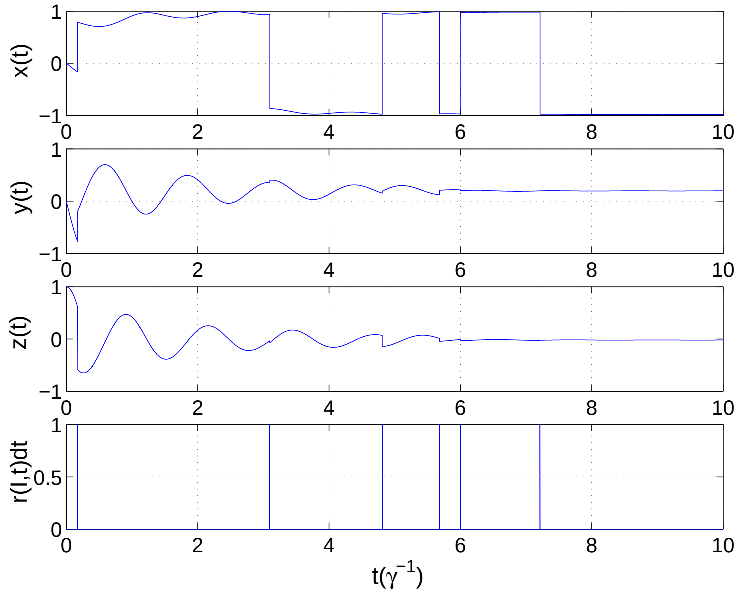

Figure 7.14: A quantum trajectory for an adaptive detection procedure for a driven TLA. Other details are as in figure 7.2.

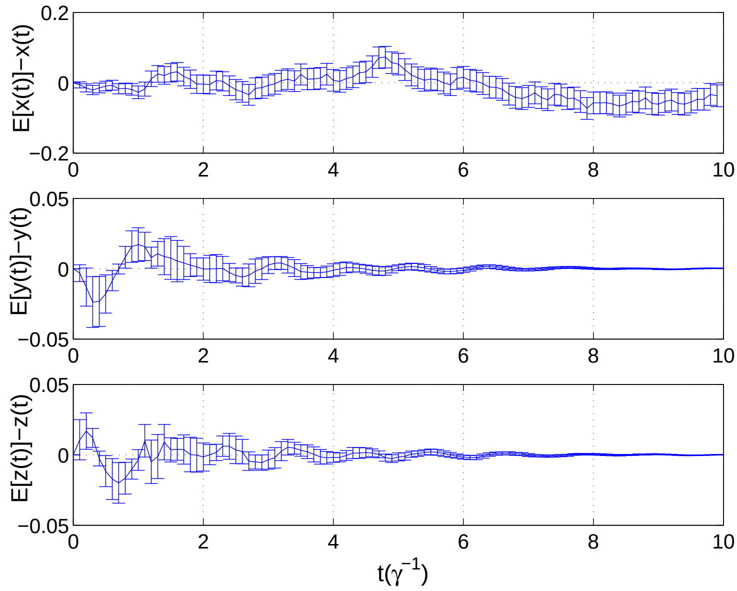

Figure 7.15: The difference between the ensemble average of 1000 adaptive quantum trajectories and the master equation for a driven TLA. Other details are as in figure 7.2.

the way we detect the bath drastically affects the system-state evolution (can be diffusive or jump-like). This highlights the orthodox position, "no phenomenon is a phenomenon until it is an observed phenomenon" [12] as well as the epistemic nature of quantum states (states of knowledge). To proceed down this line of thought, in the next chapter I will present a simple problem which allows us to couple epistemic quantum states with classical epistemic probabilities.

Chapter 8

State and Parameter Estimation

Following the evolution of an open quantum system requires full knowledge of its dynamics. In this chapter I consider an open quantum system for which the Hamiltonian is "uncertain" (which was also considered by Wiseman and myself in Ref. [58]). In particular, I treat in detail a simple system: a radiatively damped atom driven by an unknown Rabi frequency $\Omega_{\rm dri}$ (as would occur for an atom at an unknown point in a standing light wave). By measuring the environment of the system, knowledge about the system state, and about the uncertain dynamical parameter, can be acquired. I will show that these two sorts of knowledge acquisition (quantified by the posterior distribution for $\Omega_{\rm dri}$, and the conditional purity of the system, respectively) are quite distinct processes, which are not strongly correlated. Also, the quality and quantity of knowledge gain depend strongly on the type of monitoring scheme. I will demonstrate this by considering the five different detection schemes (direct, adaptive, homodyne-x, homodyne-y, and heterodyne) using four different measures of the knowledge gain (Shannon information about $\Omega_{\rm dri}$, variance in $\Omega_{\rm dri}$, long-time system purity, and short-time system purity).

8.1 General formalism

8.1.1 Quantum state and parameter estimation

Quantum parameter estimation is a well-established area [84, 85], which is usually formulated as follows. A known quantum state enters an apparatus that performs an operation on the state. The operation, which is usually unitary but need not be [104, 32], is parameterized by one or more unknown parameters. The goal is to estimate these parameters by making a measurement on the (unknown) output state. Except in special cases, it is not possible precisely to find out the unknown parameters from a measurement on a single system. Rather, the operation and measurement must be performed repeatedly, on a sequence of identically prepared quantum systems.

There is a trivial sense in which it is possible to obtain complete information about the unknown parameters from a single system. That is by taking the output state after the measurement, and using it as the next input state, having perhaps transformed it first. If the transformation required is as difficult as preparing a new system from scratch, then there is nothing to be gained by reusing the same system. However, this scenario of repeated measurements on a single system is useful pedagogically to make the transition to continuously monitored systems with unknown dynamical parameters. This transition is made by considering the limit where the unknown transformation is infinitesimally different from the identity, and the repeat time is infinitesimal.

To the best of my knowledge, a theoretical treatment of estimating an unknown dynamical parameter by continuous observation of a system was first done by Mabuchi [93]. His system was a two-level atom driven by a classically electromagnetic field mode in a cavity. The unknown parameter was the position of the atom. This is a dynamical parameter because it determines the strength of the coupling between the atom and field (the Rabi frequency). The continuous monitoring considered was counting the photons that escape through one of the cavity mirrors. Mabuchi used Bayesian statistics to determine the posterior probability distribution for the Rabi frequency. This represents the knowledge the experimenter would have about the Rabi frequency given a particular (typical) measurement record. The measurement is continuous in time (monitoring) because in any instant of time a photon may or may not be detected.

Here I am concerned with the same question, namely how would an experimenter gain knowledge of an unknown dynamical parameter from the measurement record resulting from monitoring the system. We even choose a similar (but even simpler) quantum system to that of Ref. [93], namely an atom driven by a classical field of unknown Rabi frequency. However, the analysis goes beyond, and has additional aims to, that of Ref. [93] (although we should note that extensions similar to the first three outlined below were suggested in a footnote of that work.)

First, I will consider the entire ensemble of possible measurement records and parameter values, rather than just one (typical) measurement record from one parameter value.

Second, I will quantitatively characterize this ensemble by calculating the average information gained (in bits) by the measurement, as a function of time.

Third, I will consider different ensembles resulting from different measurement schemes on the system. I would like to emphasize that the choice of measurement scheme does not affect the evolution of the system on average. That is, for all measurement schemes, averaging over the possible results and the possible values of the Rabi frequency yields the same equation of motion for the system state. Physically, this is because the average behaviour of the system is determined by its immediate environment, whereas the different measurement schemes are effected by detecting the light emitted by the system in different ways. However, the different measurement schemes give very different typical posterior distributions, and very different rates of information gain.

Fourth, and perhaps most distinctively, I will consider not just the estimation of the unknown parameter, but also the estimation of the state of the system conditioned on the measurement results. I do this using the same Bayesian method as for the parameter estimation. In this respect, this work could be seen as an extension of quantum trajectory theory (see last chapter) to systems with unknown dynamical parameters.

If the dynamical parameters for an open quantum system are known then conditioning the system on efficient detection of its emissions is guaranteed to monotonically increase its average purity in time, as information is gained about the system. But if dynamical parameters are not known then the average purity may decrease, as the different possible evolutions are summed incoherently. On the other hand, the measurement record also contains information about these parameters, so that these parameters become better defined over time. Hence one might expect that the system will eventually become pure anyway. It is one of the main results of this work that this expectation is not met.

8.1.2 Quantum trajectories with an unknown parameter

In the last chapter I presented quantum trajectory theory. To extend this theory to include unknown parameters, we have to consider repeated measurements on mixed states rather than pure states. Using mixed states the quantum trajectory for the record $\mathbf{I}_{[t_0,t)}$ is given by

$$\rho_{\mathbf{I}}(t) = \frac{\tilde{\rho}_{\mathbf{I}}(t)}{\mathrm{Tr}[\tilde{\rho}_{\mathbf{I}}(t)]}, \tag{8.1}$$

where $\tilde{\rho}_{\mathbf{I}}(t)$ is defined by

$$\tilde{\rho}_{\mathbf{I}}(t) = \hat{M}_{I_k}(t, t-dt)_{\mathrm{sys}} \ldots \hat{M}_{I_1}(t+dt, t_0)_{\mathrm{sys}} \rho(t_0) M^{\dagger}_{I_1}(t+dt, t_0)_{\mathrm{sys}} \ldots \hat{M}^{\dagger}_{I_k}(t, t-dt)_{\mathrm{sys}} \tag{8.2}$$

where $\rho(t_0)$ is the initial system state.

If we now consider the situation where there is an unknown dynamical parameter λ in the measurement operators $\hat{M}^{\dagger}_{I}(t+dt, t)_{\mathrm{sys}}$. This is done by simply noting that for each λ there will be a conditioned state. This gives a doubly conditioned state of the form

$$\rho_{\mathbf{I},\lambda}(t) = \frac{\tilde{\rho}_{\mathbf{I},\lambda}(t)}{\mathrm{Pr}(\mathbf{I}_{[t_0,t)}|\lambda)}, \tag{8.3}$$

where $\mathrm{Pr}(\mathbf{I}_{[t_0,t)}|\lambda)$ is the probability of getting $\mathbf{I}_{[t_0,t)}$ *given* λ. It is obtained by

$$\mathrm{Pr}(\mathbf{I}_{[t_0,t)}|\lambda) = \mathrm{Tr}[\tilde{\rho}_{\mathbf{I},\lambda}(t)]. \tag{8.4}$$

The posterior probability distribution $P(\lambda|\mathbf{I}_{[t_0,t)})$ is found by using a Bayesian inference formula [17]. This is

$$P(\lambda|\mathbf{I}_{[t_0,t)}) = \frac{\mathrm{Pr}(\mathbf{I}_{[t_0,t)}|\lambda) P_0(\lambda)}{\int \mathrm{Pr}(\mathbf{I}_{[t_0,t)}|\lambda) P_0(\lambda) d\lambda}, \tag{8.5}$$

where $P_0(\lambda)$ is the prior distribution for λ. For a "good measurement" of λ, as time increases, we would expect this prior distribution to converge to a δ-distribution.

Theoretically, Eq. (8.5) is complete for determining $P(\lambda|\mathbf{I}_{[t_0,t)})$. However, in general $\mathrm{Pr}(\mathbf{I}_{[t_0,t)}|\lambda)$ is very small and in numerical simulations it will incur large computer roundoff errors. The small magnitude of $\mathrm{Pr}(\mathbf{I}_{[t_0,t)}|\lambda)$ is due to the many possible trajectories the system could follow.

To overcome this problem, linear quantum trajectories [79] were used (see section 2.2.3). Linear quantum trajectories arise if we assume an ostensible distribution for the current I, $\Lambda\mathrm{r}(I)$ [134]. These $\Lambda\mathrm{r}(I)$ are independent of λ and the only condition they must satisfy is that they sum (integrate) to one. With these ostensible probabilities, the linear stochastic master equation (LSME) is derived from [134]

$$\bar{\rho}_{\mathbf{I},\lambda}(t) = \frac{\tilde{\rho}_{\mathbf{I},\lambda}(t)}{\Lambda\mathrm{r}(\mathbf{I}_{[t_0,t)})}, \tag{8.6}$$

where the ostensible probability for getting $\mathbf{I}_{[t_0,t)}$ is

$$\Lambda\mathrm{r}(\mathbf{I}_{[t_0,t)}) = \Lambda\mathrm{r}(I_k)\Lambda\mathrm{r}(I_{k-1}) \ldots \Lambda\mathrm{r}(I_1). \tag{8.7}$$

The actual probability of getting $\mathbf{I}_{[t_0,t)}$ is related to the ostensible distribution by

$$\mathrm{Pr}(\mathbf{I}_{[t_0,t)}|\lambda) = \Lambda\mathrm{r}(\mathbf{I}_{[t_0,t)}) \mathrm{Tr}[\bar{\rho}_{\mathbf{I},\lambda}(t)], \tag{8.8}$$

the Girsanov transformation. Substituting $\mathrm{Pr}(\mathbf{I}_{[t_0,t)}|\lambda)$ into Eq. (8.5) we obtain

$$P(\lambda|\mathbf{I}_{[t_0,t)}) = \frac{\mathrm{Tr}[\bar{\rho}_{\mathbf{I},\lambda}(t)] P_0(\lambda)}{\int \mathrm{Tr}[\bar{\rho}_{\mathbf{I},\lambda}(t)] P_0(\lambda) d\lambda}. \tag{8.9}$$

From Eq. (8.9) we see that to calculate $\Pr(\mathbf{I}_{[t_0,t)}|\lambda)$, the norm of the linear conditioned state [Eq. (8.6)] is needed. The order of magnitude of this norm is dependent on the ostensible probability we chose. By Eq. (8.8), if $\Lambda\mathrm{r}(\mathbf{I}_{[t_0,t)})$ is chosen to be of the same order as the true probability, this norm will be of order unity. This avoids the problem of large computer roundoff error.

8.1.3 Quantifying the information gained

One of the main aims of this chapter is to classify the information gained about the unknown parameter. The posterior probability calculated by Eq. (8.9) contains all the information about λ for a particular record. However the question remains, how can this information be quantified? Two measures are investigated. The first is the variance:

$$V_{\mathbf{I}}(t) = \int P(\lambda|\mathbf{I}_{[t_0,t)})\lambda^2 d\lambda - \Big(\int P(\lambda|\mathbf{I}_{[t_0,t)})\lambda d\lambda\Big)^2. \tag{8.10}$$

The second is the information gain, $B_{\mathbf{I}}(t)$ defined as

$$B_{\mathbf{I}}(t) \quad = \quad \int P(\lambda|\mathbf{I}_{[t_0,t)}) \log_2 P(\lambda|\mathbf{I}_{[t_0,t)}) d\lambda - \int P_0(\lambda) \log_2 P_0(\lambda) d\lambda. \tag{8.11}$$

This measures the number of bits of information gained by the observer about the parameter λ. It can be thought of as the negative change in entropy of λ. The greatest information gain corresponds to the transition from a flat (most disordered) distribution to a peaked (most ordered) distribution.

These parameters give an indication of the quality of knowledge gained by an observer, for a particular run of the experiment. To characterize a particular measurement scheme, it is necessary to calculate the ensemble averages of $V_{\mathbf{I}}(t)$ and $B_{\mathbf{I}}(t)$, which we denote as $V(t)$ and $B(t)$. The ensemble average of a parameter $A_{\mathbf{I}}$ is defined as

$$A = \mathrm{E}[A_{\mathbf{I}}] = \sum_{\mathbf{I}_{[t_0,t)}} A_{\mathbf{I}} \Pr(\mathbf{I}_{[t_0,t)}) = \sum_{\mathbf{I}_{[t_0,t)}} \int A_{\mathbf{I}} \Pr(\mathbf{I}_{[t_0,t)}|\lambda) P_0(\lambda) d\lambda. \tag{8.12}$$

Numerically, this is done by picking a true λ, λ_{true}, randomly from $P_0(\lambda)$, and then simulating a quantum trajectory for this λ_{true}, yielding $\mathbf{I}_{[t_0,t)}$. This gives a typical record as would be obtained experimentally. This $\mathbf{I}_{[t_0,t)}$ is then used to calculate $\mathrm{Tr}[\bar{\rho}_{\mathbf{I},\lambda}(t)]$ for all λ's in the range of P_0. This allows the calculation of $P(\lambda|\mathbf{I}_{[t_0,t)})$, with this probability the parameter of interest $A_{\mathbf{I}}$ can be calculated. By storing this value and repeating the above procedure $n \gg 1$ times, the ensemble average A of $A_{\mathbf{I}}$ is obtained.

8.1.4 Best estimate of conditioned state

Another aim of this chapter is to determine the best estimate of the state given the knowledge we have obtained from a measurement. In Eq. (8.3) we defined the doubly conditioned state that arose when the state was conditioned on both $\mathbf{I}_{[t_0,t)}$ and λ. From Eq. (8.3) there are two best estimate states that can be calculated. They are ρ_λ and $\rho_{\mathbf{I}}$ and can be interpreted as the best estimate state, when λ or $\mathbf{I}_{[t_0,t)}$ is known respectively. They are defined as follows

$$\rho_\lambda(t) \quad = \quad \sum_{\mathbf{I}_{[t_0,t)}} \tilde{\rho}_{\mathbf{I},\lambda}(t), \tag{8.13}$$

$$\rho_{\mathbf{I}}(t) \quad = \quad \frac{\int \tilde{\rho}_{\mathbf{I},\lambda}(t) P_0(\lambda) d\lambda}{\int \Pr(\mathbf{I}_{[t_0,t)}|\lambda) P_0(\lambda) d\lambda}. \tag{8.14}$$

It should be noted that the average of each of these states will give the same average state $\rho(t)$.

Eq. (8.13) describes the best estimate state that arises when the dynamical parameter is known and the record is not (i.e. a non-monitored system). This obeys the master equation displayed in Eq. (6.34). Of more interest to us is the best estimate state described by Eq. (8.14), which is the state conditioned on some observed record $\mathbf{I}_{[t_0,t)}$, when the true value of λ is unknown.

In calculating $\rho_{\mathbf{I}}$, if we use Eq. (8.14), we again run into the problem that the magnitude of $\tilde{\rho}_{\mathbf{I},\lambda}(t+dt)$ will typically be very small. Again this is overcome by using linear quantum trajectories, replacing Eq. (8.14) by

$$\rho_{\mathbf{I}}(t) = \frac{\int \bar{\rho}_{\mathbf{I},\lambda}(t) P_0(\lambda) d\lambda}{\int \mathrm{Tr}[\bar{\rho}_{\mathbf{I},\lambda}(t)] P_0(\lambda) d\lambda}. \tag{8.15}$$

To quantify the information gained about the state, the purity for a particular record ($p_{\mathbf{I}}$) can be determined,

$$p_{\mathbf{I}}(t) = \mathrm{Tr}[\rho_{\mathbf{I}}(t)^2]. \tag{8.16}$$

The ensemble average purity ($\mathrm{E}[p_{\mathbf{I}}]$) will give us an indication of how well the measurement scheme is at producing pure states. One might expect that a high $p(t)$ would correspond to a high $B(t)$. However it will be seen that this is not true.

8.2 Application: A driven TLA with unknown driving

The system we are considering is a classically driven two level atom, immersed in the vacuum. With no monitoring of the vacuum field, the average state evolution when all the dynamical parameters (γ and Ω_{dri}) are known is given by the master equation. The Lindblad form [92] of the master equation for the TLA, in the interaction picture (with respect to the free evolution of the atom) is defined by Eq. (7.63). The most obvious choice for the unknown dynamical parameter is Ω_{dri} (that is $\lambda = \Omega_{\mathrm{dri}}$). This can be physically motivated as follows: if we placed a laser-cooled atom (with no center-of-mass motion) in a classical standing field, then the Ω_{dri} it would experience is

$$\Omega_{\mathrm{dri}} = \Omega_{\mathrm{max}} \sin(kr), \tag{8.17}$$

where k is the wavevector for the classical field and r is the position of center of mass of the atom. We assume that the placement of the atom in the field is not biased in any way. That is, in one wavelength (λ) of the field the atom position distribution is given by $P_0(r) = 1/\lambda$. Using Eq. (8.17), $P_0(r)$ can be transformed into a probability distribution in Ω_{dri} space,

$$P_0(\Omega_{\mathrm{dri}}) = \frac{1}{\pi\sqrt{\Omega_{\mathrm{max}}^2 - \Omega_{\mathrm{dri}}^2}}. \tag{8.18}$$

This is the prior distribution for Ω_{dri}, that will be used in the rest of this paper, with $\Omega_{\mathrm{max}} = 10\gamma$. Along with this prior distribution the initial condition that we will use for our simulations, unless otherwise stated, is the steady state of the general master equation, Eq. (7.63), average over all possible Ω_{true}'s.

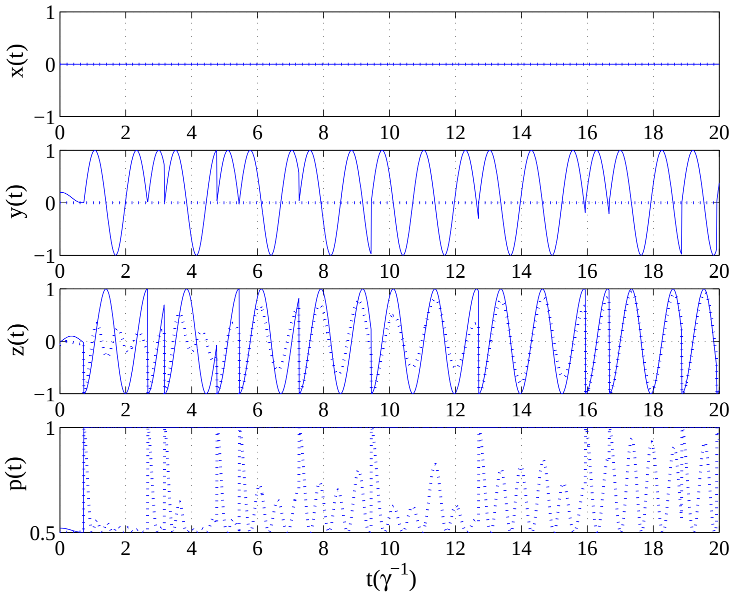

Figure 8.1: The best estimate states when Ω is known (Solid line) and unknown (dotted line) for $\Omega_{\rm true} = 5\gamma$ when direct detection is used. Time is measured in units of γ^{-1}. The initial states are the steady state for the known case and the average steady state for the unknown case.

8.2.1 Direct detection

The first measurement scheme investigated was direct detection (see section 7.2). Here the system measurement operators are

$$\hat{M}_{I=1/dt}(t+dt,t)_{\rm sys} = \sqrt{dt\gamma}\,\hat{\sigma}, \tag{8.19}$$

$$\hat{M}_{I=0}(t+dt,t)_{\rm sys} = 1 - \Big(i\frac{\Omega_{\rm dri}}{2}\hat{\sigma}_x + \frac{\gamma}{2}\hat{\sigma}^\dagger\hat{\sigma}\Big)dt. \tag{8.20}$$

Using these measurement operators and Eq. (8.1), a SME for direct detection can be written as

$$d\rho_{\mathbf{I},\Omega}(t) = r(I,t+dt)dt\hat{\mathcal{G}}[\sqrt{dt\gamma}\,\hat{\sigma}]\rho_{\mathbf{I},\Omega}(t) - dt\hat{\mathcal{H}}[i\frac{\Omega_{\rm dri}}{2}\hat{\sigma}_x + \frac{\gamma}{2}\hat{\sigma}^\dagger\hat{\sigma}]\rho_{\mathbf{I},\Omega}(t), \tag{8.21}$$

where $\hat{\mathcal{G}}$ and $\hat{\mathcal{H}}$ are the nonlinear superoperators defined for arbitrary $\hat{L}$ and ρ by

$$\hat{\mathcal{G}}[\hat{L}]\rho = \frac{\hat{L}\rho\hat{L}^\dagger}{{\rm Tr}[\hat{L}\rho\hat{L}^\dagger]} - \rho, \tag{8.22}$$

$$\hat{\mathcal{H}}[\hat{L}]\rho = \hat{L}\rho + \rho\hat{L}^\dagger - {\rm Tr}[\hat{L}\rho + \rho\hat{L}^\dagger]\rho \tag{8.23}$$

and $r(I,t+dt) = dN(t)/dt$. $dN(t)$ is defined by Eqs. (7.56) and (7.57).

To consider the case when $\Omega_{\rm dri}$ is unknown, a linear SME had to be developed. Using the direct detection measurement operators and Eq. (8.6), with $\Lambda{\rm r}(I)$ defined by

$$\Lambda{\rm r}(1/dt) = \epsilon dt = 1 - \Lambda{\rm r}(0), \tag{8.24}$$

where ϵ is an arbitrary parameter, the linear SME is

$$d\bar{\rho}_{\mathbf{I},\Omega}(t) = r(I,t+dt)dt\bar{\mathcal{G}}[\sqrt{dt\gamma}\,\hat{\sigma}]\bar{\rho}_{\mathbf{I},\Omega}(t) - dt\bar{\mathcal{H}}[i\frac{\Omega_{\rm dri}}{2}\sigma_x + \frac{\gamma}{2}\hat{\sigma}^\dagger\hat{\sigma} - \frac{\epsilon}{2}]\bar{\rho}_{\mathbf{I},\Omega}(t). \tag{8.25}$$

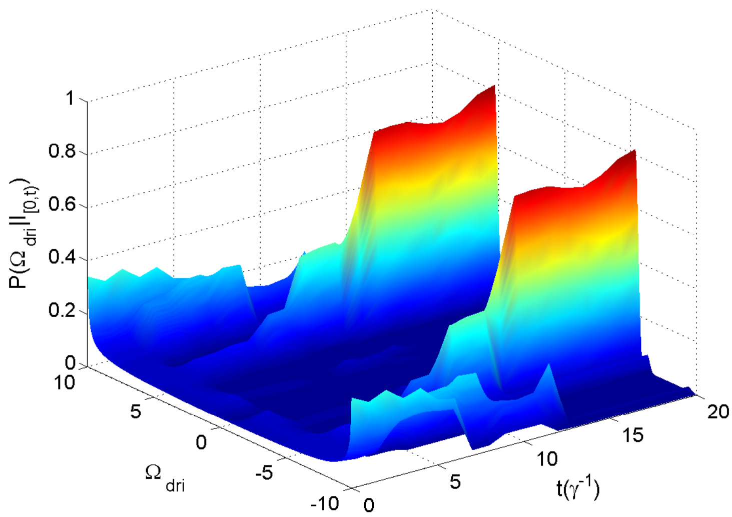

Figure 8.2: A plot of a typical $P(\Omega_{\rm dri}|\mathbf{I}_{[t_0,t)})$ when $\Omega_{\rm true}=5\gamma$ for the direct detection scheme. $\Omega_{\rm dri}$ is measured in units of γ and time is measured in units of γ^{-1}.

The $\bar{\mathcal{G}}$ and $\bar{\mathcal{H}}$ linear superoperators are defined as

$$\bar{\mathcal{G}}[\hat{L}]\bar{\rho} = \frac{\hat{L}\bar{\rho}\hat{L}^\dagger}{\epsilon dt} - \bar{\rho}, \tag{8.26}$$

$$\bar{\mathcal{H}}[\hat{L}]\bar{\rho} = \hat{L}\bar{\rho} + \bar{\rho}\hat{L}^\dagger. \tag{8.27}$$

To obtain the general master equation from Eq. (8.25), $E[dN(t)] = \Lambda {\rm r}(1)$ has to be used, for numerical speed [$\Lambda{\rm r}(1/dt)$ close to the actual distribution] it is best to use

$$\epsilon dt = {\rm Tr}[\rho_{ss}\hat{M}^\dagger_{I=1/dt}(t+dt,t)_{\rm sys}\hat{M}_{I=1/dt}(t+dt,t)_{\rm sys}] = \frac{\gamma\Omega^2_{\rm dri}}{2\Omega^2_{\rm dri}+\gamma^2}dt. \tag{8.28}$$

To determine the parameters of interest to us, namely $\rho_{\mathbf{I}}(t)$ and $P(\Omega_{\rm dri}|\mathbf{I}_{[t_0,t)})$, Eq. (8.25) is numerically simulated for all possible $\Omega_{\rm dri}$ in $P_0(\Omega_{\rm dri})$ with $r(I,t+dt)$ specified by $\mathbf{I}_{[t_0,t)}$. This record would ideally be obtained experimentally but for the purpose of this paper it is calculated by numerically evaluating Eq. (8.21) for a known $\Omega_{\rm dri}$, which we will refer to as $\Omega_{\rm true}$. This $\mathbf{I}_{[t_0,t)}$ is then used in Eq. (8.25) to generate ${\rm Tr}[\bar{\rho}_{\mathbf{I},\Omega_{\rm dri}}]$ for all the $\Omega_{\rm dri}$'s between $-\Omega_{\rm max}$ and $\Omega_{\rm max}$. Then with Eq. (8.15) and Eq. (8.9) one can obtain both $\rho_{\mathbf{I}}(t)$ and $P(\Omega_{\rm dri}|\mathbf{I}_{[t_0,t)})$.

For a $\mathbf{I}_{[t_0,t)}$ based on $\Omega_{\rm true}=5\gamma$ the best estimate state and the posterior distribution were calculated and are shown in figure 8.1 and 8.2 respectively. It is observed that, in contrast to the known $\Omega_{\rm dri}$ case, the best estimate of $y(t)$ is identically zero. This is because positive and negative $\Omega_{\rm dri}$ are initially equally likely, so that $y_{\rm ss}$ in Eq. (7.74) averages to zero. Moreover, the sign of $\Omega_{\rm dri}$ is not determinable by this measurement scheme, because the rate of detections depends only on $z(t)$, which is independent of the sign of $\Omega_{\rm dri}$.

Another difference apparent with the unknown $\Omega_{\rm dri}$ case is that $z(t)$ oscillates with a different frequency to the known $\Omega_{\rm dri}$ case, in this case a faster frequency [since $P_0(\Omega)$ is peaked at the end points $|\Omega| = \Omega_{\rm max} = 10\gamma$]. However as time increases its frequency tends to that of the known case.

This is due to the fact that for direct detection the rate of detections is dependent on the magnitude of $\Omega_{\rm dri}$, so as time goes on one would expect to gain more information about the magnitude of $\Omega_{\rm dri}$.

These interpretations of the conditioned dynamics are confirmed in figure 8.2. With increasing time, the posterior distribution localizes at $\pm\Omega_{\rm true}$. The mean is always zero and thus is not an unbiased estimator of $\Omega_{\rm dri}$. The reason that the magnitude is determinable and the sign is not, can be formulated as follows. In the Bloch representation of Eq. (8.25), with the transformation $y \to -y$, $\Omega_{\rm dri} \to -\Omega_{\rm dri}$ the equations stay invariant. Since this transformation changes the direction of rotation around the x-axis, I will call it the rotation transformation.

With an indeterminable direction of rotation and this measurement scheme, it can be seen that the best estimate state will never become more pure than a state that is a mixture of two states that rotate is opposite directions around the $x = 0$ great circle of the Bloch sphere. Thus the best estimate state oscillates up and down the z-axis of the Bloch sphere.

We turn now to quantifying the measurement scheme's ability to gain knowledge, by numerically determining the ensemble average purity, $V(t)$ and $B(t)$. These ensemble averages were calculated for $\Omega_{\rm true}$'s weighted on the prior distribution, Eq. (8.18). These numerical simulations are depicted in figure 8.3 for two initial states; one is the steady state (solid line) and the other is the ground state (dotted line). It is observed that in both cases the average purity of the state never attains one, with the purity in the second case initially decreasing from one. The long time purity ($\simeq 0.75$) is due to the best estimate being a mixture of two states as explained above. This figure can be obtained analytically, if we make the follow two assumptions. The first is that $\Omega_{\rm true} \gg \gamma$. This is valid as $P_0(\Omega)$ from which $\Omega_{\rm true}$ is drawn is peaked at $\pm\Omega_{\rm max}$, and in our calculations $\Omega_{\rm max} \gg \gamma$. The second assumption is that in the long time limit the posterior distribution localizes on $\pm\Omega_{\rm true}$, which is what is seen in figure 8.2. With these two assumption the long-time best estimate state in Bloch representation will be

$$x(t) = 0, \qquad y(t) = 0, \qquad z(t) \simeq -\cos\Omega_{\rm true}(t - t_{\rm last}), \tag{8.29}$$

where $t_{\rm last}$ is the time of the last jump, which is typically more than one Rabi cycle before t. With this state the average purity (for the long time limit) can be estimated as

$$p(t) = \frac{\Omega}{2\pi}\int_0^{2\pi/\Omega} \frac{1 + z(s)^2}{2} ds = \frac{3}{4}. \tag{8.30}$$

From figure 8.3, it is also observed that the simulated ensemble average variance $V(t)$ is approximately constant for all time. In fact, given that no information about the sign of Ω is determinable, and that the initial distribution $P_0(\Omega_{\rm dri})$ is symmetric, it is easy to prove that $V(t)$ is exactly constant.

For the third parameter $B(t)$, it is observed that, on average, direct detection yields information about $\Omega_{\rm dri}$ as time increases, for both initial states. It is observed that the initial slope of $B(t)$ is zero for the ground state, while it is non-zero for the initial steady state case. The initial flatness in the first case is due to the fact that if the system starts in the ground, the rate of detections (proportional to the excited state component) scales as $(\Omega_{\rm true}t)^2$, and with out any detections it would not be possible to gain any information. By contrast, for the steady state case there will be some excited state fraction (depending on $\Omega_{\rm dri}$) and thus a finite detection rate even at $t = 0$. Figure 8.3 also show that, after the initial flatness, the $B(t)$ in the first case rapidly overtakes that in the second case. This jump in $B(t)$ occurs at roughly $t = 1/\Omega_{\rm max}$, which is when one would expect a significant excited state fraction to have developed (Recall that $P_0(\Omega_{\rm dri})$ is sharply peaked at $\Omega = \pm\Omega_{\rm max}$).

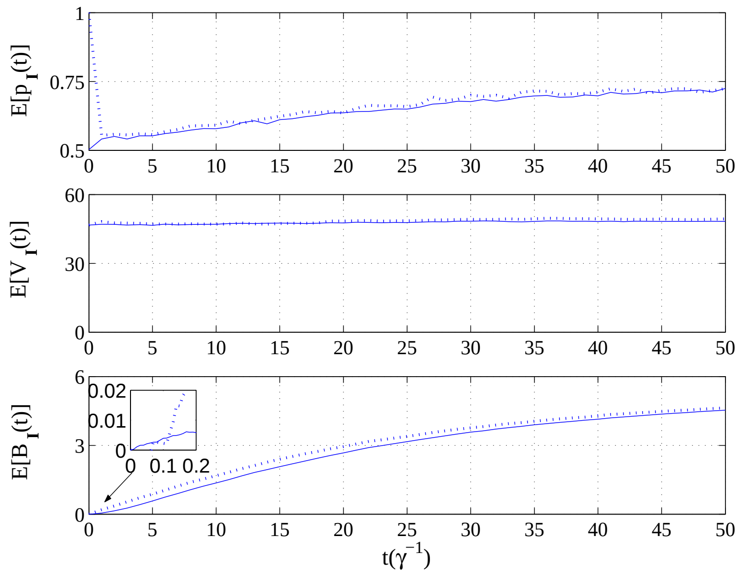

Figure 8.3: The ensemble average ($n = 1000$) of the purity, variance and information gain when direct detection is used, for two initial states, the steady state (solid) and ground state (dotted). Time is measured in units of γ^{-1}.

8.2.2 Adaptive detection

The second measurement scheme investigated was the adaptive scheme of Wiseman and Toombes [138] (see section 7.5).

For this detection scheme the measurement operators are

$$M_{I=1/dt}(t_0+dt,t_0)_{\rm sys} = \sqrt{dt\gamma}\,(\hat{\sigma}+\mu), \quad (8.31)$$

$$M_{I=0}(t_0+dt,t_0)_{\rm sys} = 1-(i\frac{\Omega_{\rm dri}}{2}\hat{\sigma}_x+\frac{\gamma}{2}\hat{\sigma}^\dagger\hat{\sigma}+\mu\gamma\hat{\sigma}+\frac{\gamma\mu^2}{2})dt. \quad (8.32)$$

These measurement operators result in a SME of the form

$$d\rho_{\mathbf{I},\Omega_{\rm dri}}(t) = r(I,t+dt)dt\hat{\mathcal{G}}[\sqrt{dt\gamma}\,(\hat{\sigma}+\mu)]\rho_{\mathbf{I},\Omega_{\rm dri}}(t)-dt\hat{\mathcal{H}}[i\frac{\Omega_{\rm dri}}{2}\hat{\sigma}_x+\frac{\gamma}{2}\hat{\sigma}^\dagger\hat{\sigma} +\mu\gamma\hat{\sigma}+\frac{\gamma\mu^2}{2}]\rho_{\mathbf{I},\Omega_{\rm dri}}(t), \quad (8.33)$$

where $r(I,t+dt) = dN(t)/dt$ with $dN(t)$ satisfying Eqs. (7.114) and (7.114). Using the same ostensible distribution $\Lambda\mathrm{r}(I)$ as in direct detection, the linear SME is

$$d\bar{\rho}_{\mathbf{I},\Omega_{\rm dri}}(t) = r(I,t+dt)dt\bar{\mathcal{G}}[\sqrt{dt\gamma}\,(\hat{\sigma}+\mu)]\bar{\rho}_{\mathbf{I},\Omega_{\rm dri}}(t)-dt\bar{\mathcal{H}}[i\frac{\Omega_{\rm dri}}{2}\hat{\sigma}_x+\frac{\gamma}{2}\hat{\sigma}^\dagger\hat{\sigma}+\mu\gamma\hat{\sigma} +\frac{\gamma\mu^2}{2}-\frac{\epsilon}{2}]\bar{\rho}_{\mathbf{I},\Omega_{\rm dri}}(t), \quad (8.34)$$

Figure 8.4 shows the best estimate state for a known (solid) and unknown $\Omega_{\rm dri}$ (dotted), with $\Omega_{\rm true} = 5\gamma$. It is observed that with the known $\Omega_{\rm dri}$ case after the initial transients, the state jumps between the two fixed states given by Eq. (7.116). For the unknown $\Omega_{\rm dri}$ case the y component

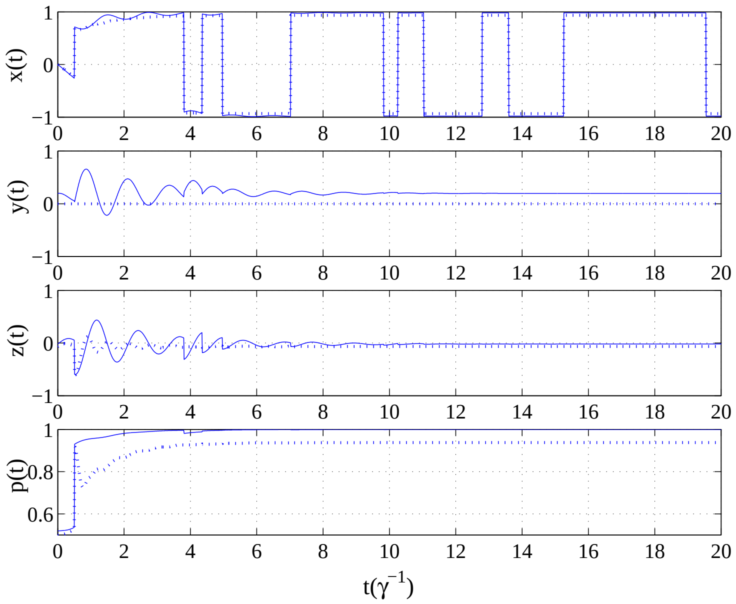

Figure 8.4: The best estimate states when adaptive detection is used. Details are as in Figure 8.1

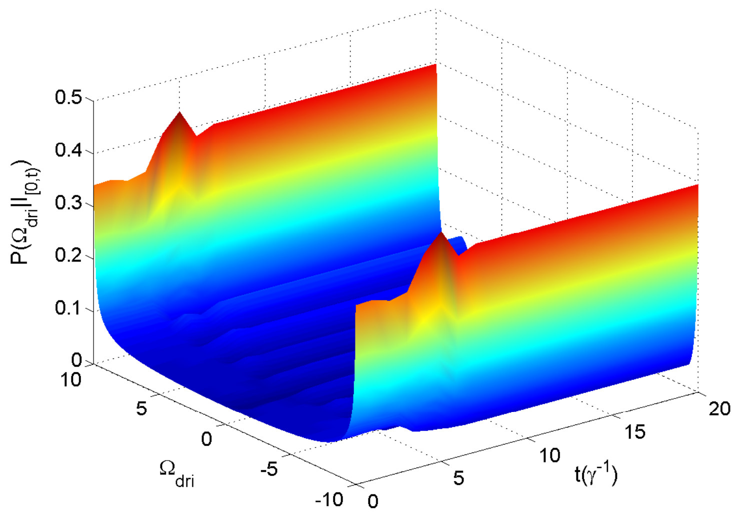

Figure 8.5: A plot of $P(\Omega|\mathbf{I}_{[t_0,t)})$ for the adaptive scheme. Details are as in Figure 8.2.

averages to zero, and the x and z components both appear to be slightly different to the known $\Omega_{\rm dri}$ case.

Similarly to the direct detection case, a better understanding of this state can be obtained by considering $P(\Omega_{\rm dri}|\mathbf{I}_{[t_0,t)})$. This is shown in Figure 8.5 and it can be seen that as time increases under this adaptive measurement, the typical posterior probability distribution $P(\Omega_{\rm dri}|\mathbf{I}_{[t_0,t)})$ scarcely changes from $P_0(\Omega_{\rm dri})$. This is not unexpected, as for this detection scheme it can be shown that at steady state the jumps are Poissonian, with rate $\gamma/4$ (this was also used as ϵ in the ostensible distribution). That is, the jumps are independent of $\Omega_{\rm dri}$ and hence yield no information about it. Since $P(\Omega|\mathbf{I}_{[t_0,t)}) \approx P_0(\Omega)$, we can use this approximation to obtain analytically an indication of the best estimate state by solving Eq. (8.14). For this detection scheme this is simply the mean of Eq. (7.116) under the distribution $P_0(\Omega)$. This gives

$$x = \mp(1-\frac{\gamma^2}{\sqrt{2\Omega_{\rm max}^2+\gamma^2}}), \quad y=0, \quad z=\frac{-\gamma}{\sqrt{2\Omega_{\rm max}^2+\gamma^2}}. \tag{8.35}$$

Comparing this with the numerical simulation it is observed that they agree very well.

To quantify this detection scheme, the ensemble average of the variance, purity and B were numerically calculated and are shown in Figure 8.6. The purity rapidly becomes, and remains, relatively high. This is because the best estimate state of Eq. (8.35) is the same no matter what $\Omega_{\rm true}$ is chosen. For $\Omega_{\rm max} = 10\gamma$ the numerical value of the stationary purity is 0.934 and by using Eq. (8.35) an analytical value of the purity can be obtained,

$$p = 1+\frac{\gamma^2}{\gamma^2+2\,{\Omega_{\rm max}}^2} - \frac{\gamma}{\sqrt{\gamma^2+2\,{\Omega_{\rm max}}^2}}. \tag{8.36}$$

For $\Omega_{\rm max} = 10\gamma$ this gives a value of 0.934, which is equal to the numerical value.

Since this state has a high purity one might expect that the unknown parameter must also be well defined. However this is not true as already discussed. This lack of knowledge about $\Omega_{\rm dri}$ is seen in Figure 8.6. Like direct detection, the sign of $\Omega_{\rm dri}$ cannot be determined so the average variance remains precisely constant. However unlike direct detection, the information gain is bounded, with a maximum ΔI of less than 0.06 bits.

An interesting point to note is this scheme would be well suited to estimating γ (if there were some uncertainty in that parameter) even if $\Omega_{\rm dri}$ was also uncertain. That is because the detection rate is proportional to γ, almost independent of $\Omega_{\rm dri}$. Of course this would require the local oscillator amplitude to be adjusted from an initial guess according to the best estimate of γ.

8.2.3 Homodyne x detection

The next detection scheme I am going to consider is Homodyne x detection. As illustrated in section 7.4 this detection scheme results in a diffusive SSE, with the current record $\mathbf{I}_{[t_0,t)}$ containing a string of continuous currents $r(I, t+dt)$ every dt interval. Since I is a continuous variable we can define a measurement operator, $\hat{M}_I(t+dt,t)_{\rm sys}$, a continuous function of I, to represent this measurement scheme,

$$\hat{M}_I(t+dt,t)_{\rm sys} = \sqrt{\Upsilon_I}[1-(i\frac{\Omega_{\rm dri}}{2}\hat\sigma_x+\frac{\gamma}{2}\hat\sigma^\dagger\hat\sigma-\sqrt{\gamma}\,\hat\sigma e^{-i\phi}I)dt]. \tag{8.37}$$

Here ϕ is the phase of the local oscillator and

$$\Upsilon_I dI = \frac{1}{\sqrt{2\pi/dt}}e^{-\frac{1}{2}I^2dt}dI, \tag{8.38}$$

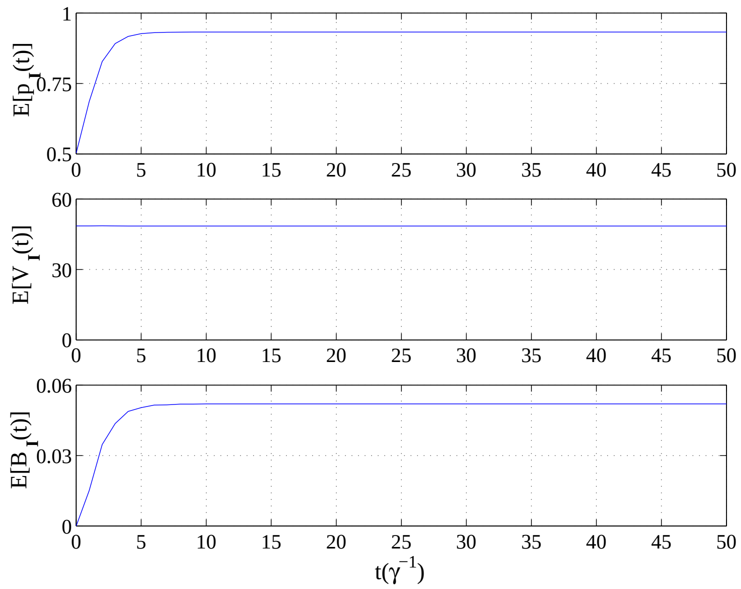

Figure 8.6: The ensemble average ($n = 1000$) of the purity, variance and information gain when the adaptive detection technique was used. Note for $B(t)$ the scale has been change when compared to figure 8.3

is a Gaussian probability measure.

With this continuous measurement operator the SME in Itô form is

$$\begin{aligned} d\rho_{\mathbf{I},\Omega_{\rm dri}}(t) &= -\frac{i\Omega_{\rm dri}}{2}[\hat{\sigma}_x,\rho_{\mathbf{I},\Omega_{\rm dri}}(t)]dt+\hat{\mathcal{D}}\rho_{\mathbf{I},\Omega_{\rm dri}}(t)+\sqrt{\gamma}\,\hat{\mathcal{H}}[\hat{\sigma}e^{-i\phi}]\rho_{\mathbf{I},\Omega_{\rm dri}}(t)(r(I,t+dt)dt \\ &\quad -\sqrt{\gamma}\,\mathrm{Tr}[\hat{\sigma}e^{-i\phi}\rho_{\mathbf{I},\Omega_{\rm dri}}(t)+\rho_{\mathbf{I},\Omega_{\rm dri}}(t)\hat{\sigma}^{\dagger}e^{i\phi}]dt), \end{aligned} \tag{8.39}$$

where $r(I, t + dt)$ is the current element for the interval dt and is equal to

$$r(I,t+dt)=\sqrt{\gamma}\,\mathrm{Tr}[\sigma e^{-i\phi}\rho_{\mathbf{I},\mathbf{\Omega}_{\rm dri}}(t)+\rho_{\mathbf{I},\mathbf{\Omega}_{\rm dri}}(t)\sigma^{\dagger}e^{i\phi}]+\xi(t), \tag{8.40}$$

where $\xi(t)$ represents Gaussian white noise.

For the linear SME we take the ostensible probability for the current to be equal to that which would arise from the LO alone. This results in $\Lambda\mathrm{r}([I]) = \Upsilon_I dI$ so that $r(I, t + dt)$ is ostensibly a GRV with mean zero and variance dt^{-1}, like $\xi(t)$. The linear SME in Itô form is

$$d\bar{\rho}_{\mathbf{I},\Omega_{\rm dri}}(t) = -\frac{i\Omega_{\rm dri}}{2}[\hat{\sigma}_x,\bar{\rho}_{\mathbf{I},\Omega_{\rm dri}}(t)]dt+\hat{\mathcal{D}}\bar{\rho}_{\mathbf{I},\Omega}(t)+\sqrt{\gamma}\,\bar{\mathcal{H}}[\hat{\sigma}e^{-i\phi}]\bar{\rho}_{\mathbf{I},\Omega}(t)Idt. \tag{8.41}$$

It can be seen that both the linear SME and the SME reduce to Eq. (4.49) when the ensemble average is taken. Similarly to the previous schemes, to determine an unknown $\Omega_{\rm dri}$, $\mathbf{I}_{[t_0,t)}$ is generated by the SME for a preset $\Omega_{\rm dri}$, $\Omega_{\rm true}$, which may then be "forgotten". The linear SME is then used to generate both $\rho_{\mathbf{I}}(t)$ and $P(\Omega|\mathbf{I}_{[t_0,t)})$ for the predetermined record $\mathbf{I}_{[t_0,t)}$.

For homodyne x quadrature measurement, the ϕ of the LO is set to zero (as $x = \langle\sigma + \sigma^{\dagger}\rangle$). With this phase and $\Omega_{\rm true} = 5\gamma$, the best estimate state for a known and unknown $\Omega_{\rm dri}$ are shown in Figure 8.7. It is observed that for the known $\Omega_{\rm dri}$ case, the state seems to localize itself relatively fast into pure states that have a large x contribution, and small oscillations in the y and z directions.

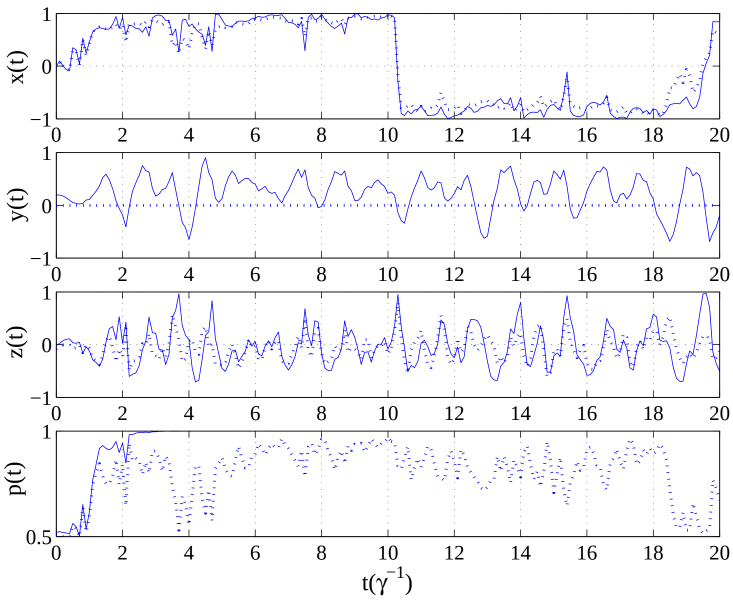

Figure 8.7: The best estimate states for the homodyne x scheme. Details are as in Figure 8.1.

By contrast, when $\Omega_{\rm dri}$ is unknown, the best estimate state still contains a large x contribution, but the y is strictly zero and the amplitude of the z oscillations is reduced. As in the previous cases, this zero y component can be understood by considering $P(\Omega_{\rm dri}|\mathbf{I}_{[t_0,t)})$, shown in Figure 8.8. It is seen that, like direct detection, this measurement scheme has an even posterior distribution that localizes at $\pm\Omega_{\rm true}$. This is again due to the stochastic Bloch equations being invariant under the previously considered rotational transformation. However, the rate at which this localization occurs is much slower than under direct detection.

The slower rate of information gain is confirmed with the calculation of the ensemble average of $B(t)$, shown in Figure 8.9. It is seen that within $50\gamma^{-1}$ units of time, $B(t=50\gamma^{-1})$ for homodyne x is about half that of direct detection. Physically this comes about because, for the system we are investigating, the underlying dynamics cause the states to rotate around the x-axis with frequency $\Omega_{\rm true}$. The measurement scheme tends to produce states oriented mainly in the $\pm x$ directions. This can be understood from the measurement effect $\hat{F}_I$, which, using Eq. (2.2.2), can be shown to be

$$\hat{F}_I(t+dt,t)_{\rm sys}dI = \frac{1}{\sqrt{2\pi/dt}}e^{-\frac{1}{2}(I-\hat{\sigma}_x)^2dt}dI. \tag{8.42}$$

This effect is a Gaussian with a mean equal to the $\hat{\sigma}_x$ quadrature operator and variance dt^{-1}. Thus, it is an unsharp measurement of $\hat{\sigma}_x$. Thus, for a measurement scheme that makes the conditioned state mainly oriented in the $\pm x$ directions, one would expect that this state would be less affected by an unknown $\Omega_{\rm dri}$ than a state on the $x=0$ plane as produced by direct detection. Thus less information about $\Omega_{\rm dri}$ comes out of the measurement record. In Figure 8.8 it is observed that the ensemble average of the purity of this state increases quickly to about 0.75, then increases only slowly afterwards. This quick increase is also a result of the state becoming predominately $\pm x$ oriented. (similarly to the adaptive detection scheme) and the slow increase is due to the slow increase in

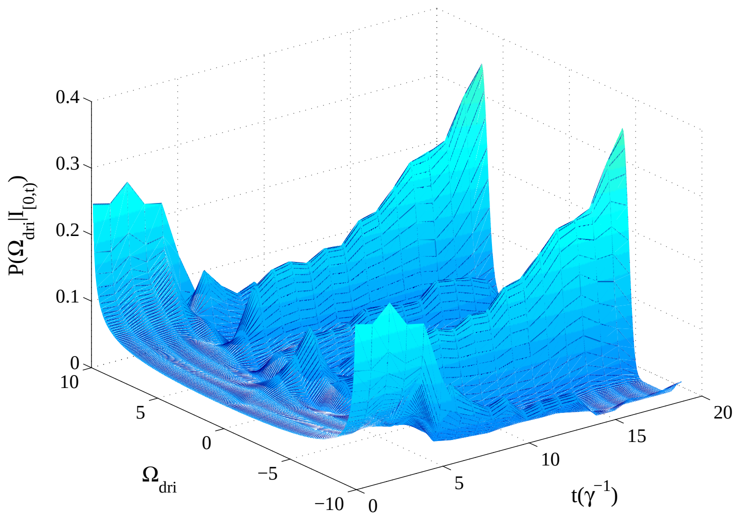

Figure 8.8: A plot of $\mathrm{P}(\Omega|\mathbf{I}_{[\mathrm{t}_0,\mathrm{t})})$ for the homodyne x scheme. Details are as in Figure 8.2.

the knowledge of $\Omega_{\rm dri}$ (similarly to direct detection). As with direct detection, the system state will never become fully pure. This is due to the double peaks in $P(\Omega_{\rm dri}|\mathbf{I}_{[t_0,t)})$, which insures the y component of the state always averages to zero.

8.2.4 Homodyne y detection

Setting the ϕ of the local oscillator to $\pi/2$ allows measurement of the y quadrature (as $y = \langle -i\hat{\sigma} + i\hat{\sigma}^\dagger\rangle$). The best estimate states for the known and unknown $\Omega_{\rm dri}$ are shown in Figure 8.10, for $\Omega_{\rm true} = 5\gamma$. It is seen that when $\Omega_{\rm dri}$ is known (solid) this measurement scheme makes the state coarsely rotate around the Bloch sphere with a purity of one. When $\Omega_{\rm dri}$ is unknown (dotted line), Figure 8.10 shows that, unlike the previous schemes, the y component does not average to zero. As time increases the oscillations in the y and z components for the unknown $\Omega_{\rm dri}$ case gradually converge to those for the known $\Omega_{\rm dri}$ case. This suggests that this scheme can determine $\Omega_{\rm true}$. This is confirmed by the calculation of $P(\Omega_{\rm dri}|\mathbf{I}_{[t_0,t)})$ shown in Figure 8.11. The ability of this scheme to distinguish the sign of $\Omega_{\rm dri}$ can be physically understood by considering the Bloch representation of Eq. (8.41) for $\phi = \pi/2$. These stochastic equations are *not* invariant under the rotation transformation.

To understand how this scheme reduces the uncertainty in $\Omega_{\rm dri}$, consider the effect for this measurement scheme

$$\hat{F}_I(t+dt,t)_{\rm sys}dI = \frac{1}{\sqrt{2\pi/dt}}e^{-\frac{1}{2}(I-\hat{\sigma}_y)^2dt}dI. \tag{8.43}$$

That is, $\hat{F}_I(t+dt,t)_{\rm sys}$ is an unsharp measurement of y. Now y is a variable that is directly affected by $\Omega_{\rm true}$, and indeed the sign of y reverses if the sign of $\Omega_{\rm dri}$ reverses. Even though in each interval dt, y is measured unsharply, over time this detection scheme will result in a narrowing of our knowledge of $\Omega_{\rm dri}$, until infinite time where it would be fully known. This is further confirmed by the calculation

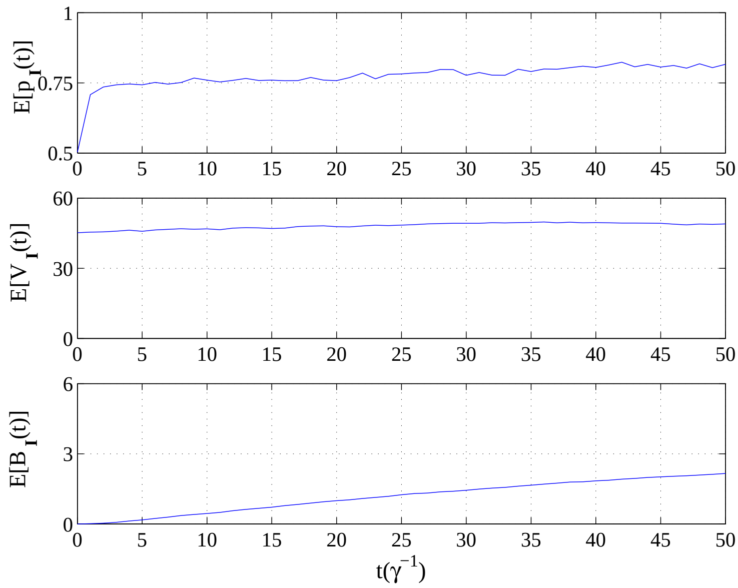

Figure 8.9: The ensemble average ($n = 500$) of the purity, variance and information gain when homodyne x was used. Time is measured in units of γ^{-1}.

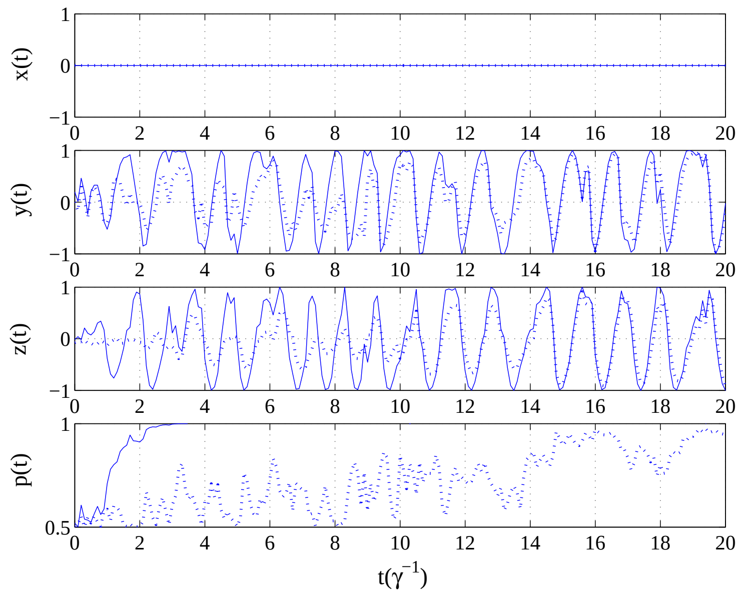

Figure 8.10: The best estimate states for homodyne y measurement. Details are as in Figure 8.1.

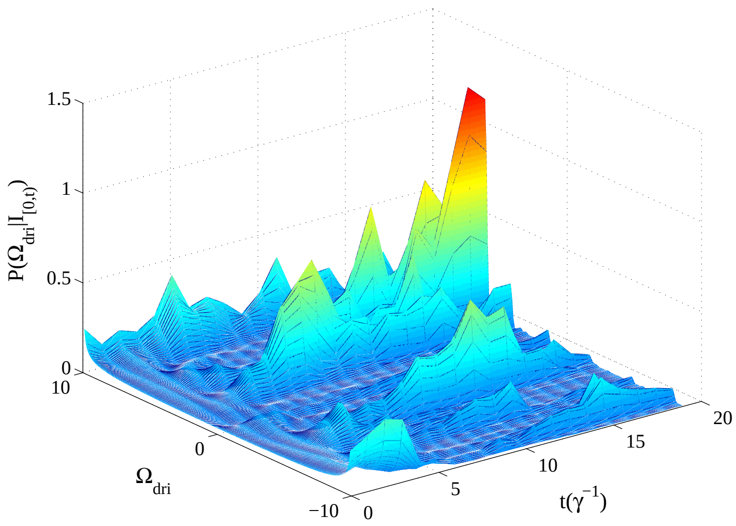

Figure 8.11: A plot of $P(\Omega_{\rm dri}|\mathbf{I}_{[t_0,t)})$ for homodyne y measurement. Details are as in Figure 8.2.

of the ensemble averages of the three parameters, purity, $V(t)$ and $B(t)$ (Figure 8.12). It is observed that the purity of this state increases up to one, the V in Ω reduces substantially in the $50\gamma^{-1}$ units of time and $B(t)$ increases to a value larger than that for all other schemes.

8.2.5 Heterodyne detection

The last detection scheme considered uses the heterodyne technique (see section 7.3). In this detection scheme, as in homodyne detection, $\mathbf{I}_{[t_0,t)}$ will comprises of a string of continuous numbers I, but they will be complex. The continuous set of measurement operators are

$$\hat{M}_I(t+dt,t)_{\rm sys} = \sqrt{\Upsilon_I}[1-(i\frac{\Omega_{\rm dri}}{2}\hat{\sigma}_x + \frac{\gamma}{2}\hat{\sigma}^\dagger\hat{\sigma} - \sqrt{\gamma}\,\hat{\sigma}I^*)dt], \tag{8.44}$$

where

$$\Upsilon_I d^2I = \frac{dt}{\pi}e^{-|I|^2dt}d^2I. \tag{8.45}$$

Using the above measurement operators, the heterodyne SME in Itô form is

$$\begin{aligned} d\rho_{\mathbf{I},\Omega_{\rm dri}}(t) \;\; = \;\; & -\frac{i\Omega_{\rm dri}}{2}[\hat{\sigma}_x,\rho_{\mathbf{I},\Omega_{\rm dri}}(t)]dt + \hat{\mathcal{D}}\rho_{\mathbf{I},\Omega_{\rm dri}}(t) \\ & +\sqrt{\gamma}\,(\hat{\sigma}\rho_{\mathbf{I},\Omega_{\rm dri}}(t) - \langle\hat{\sigma}\rangle_t\rho_{\mathbf{I},\Omega_{\rm dri}}(t))(r(I^*,t+dt)dt - \sqrt{\gamma}\,\langle\hat{\sigma}^\dagger\rangle_t dt) \\ & +\sqrt{\gamma}\,(\rho_{\mathbf{I},\Omega_{\rm dri}}(t)\hat{\sigma}^\dagger - \langle\hat{\sigma}^\dagger\rangle_t\rho_{\mathbf{I},\Omega_{\rm dri}}(t))(r(I,t+dt)dt - \sqrt{\gamma}\,\langle\hat{\sigma}\rangle_t dt), \end{aligned} \tag{8.46}$$

where $r(I,t+dt)$ is the current elements for the interval dt and is equal to

$$r(I,t+dt) = [\sqrt{\gamma}\,\langle\hat{\sigma}\rangle_t + \xi(t)], \tag{8.47}$$

where $\xi(t)$ is complex Gaussian white noise.

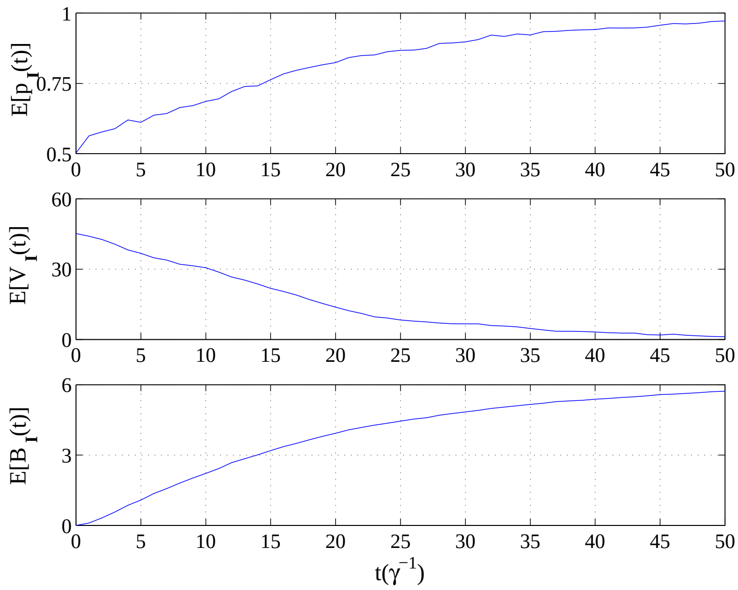

Figure 8.12: The ensemble average ($n = 500$) of the purity, variance and information gain when homodyne detection of the y quadrature was used.

For the linear SME we again assume that the ostensible probability is that due just to the LO, which results in a heterodyne current $r(I, t+dt)$ with the same statistics as $\xi(t)$. With this complex current the ostensible probability $\Lambda\mathrm{r}([I])$ is equal to $\Upsilon_I d^2 I$. This gives a linear SME in Itô form of

$$\begin{aligned} d\bar{\rho}_{\mathbf{I},\Omega_{\mathrm{dri}}}(t) &= -\frac{i\Omega_{\mathrm{dri}}}{2}[\hat{\sigma}_x, \bar{\rho}_{\mathbf{I},\Omega_{\mathrm{dri}}}(t)]dt + \hat{\mathcal{D}}\bar{\rho}_{\mathbf{I},\Omega_{\mathrm{dri}}}(t) + \sqrt{\gamma}\,\hat{\sigma}\bar{\rho}_{\mathbf{I},\Omega_{\mathrm{dri}}}(t) r(I^*, t+dt)dt \\ &\quad + \sqrt{\gamma}\,\bar{\rho}_{\mathbf{I},\Omega_{\mathrm{dri}}}(t)\hat{\sigma}^\dagger r(I, t+dt)dt. \end{aligned} \tag{8.48}$$

Using $\Omega_{\mathrm{true}} = 5\gamma$, the best estimate state for known and unknown Ω are shown in Figure 8.13. It is observed that for a known Ω_{dri}, the state contains attributes of both the homodyne x and y measurement schemes. By this we mean that the state tends to have a distinct x components, whilst keeping the coarse rotations of the homodyne y scheme. This is not unexpected as heterodyne is equivalent to simultaneous homodyne x and y measurements, each of 50% efficiency. In the unknown Ω_{dri} case it is observed that the y component does not average to zero, suggesting that $P(\Omega_{\mathrm{dri}}|\mathbf{I}_{[t_0,t)})$ localizes to Ω_{true}, which is confirmed by figure 8.14. However, the rate at which $P(\Omega_{\mathrm{dri}}|\mathbf{I}_{[t_0,t)})$ converges to $\delta(\Omega_{\mathrm{dri}} - \Omega_{\mathrm{true}})$ is much slower than that of the homodyne y measurement. This is also illustrated in figure 8.15 as the ensemble average $B(t)$ is not as high. Figure 8.15 also shows the ensemble average of the purity and from this figure it is seen that it contains similar properties of both the homodyne x and y schemes. In particular, it has an initial sharp increase, which is due the state obtaining a large x component (similar to the homodyne x scheme) and as time goes on the purity increases to one due to the localization of $P(\Omega_{\mathrm{dri}}|\mathbf{I}_{[t_0,t)})$ (similar to homodyne y).

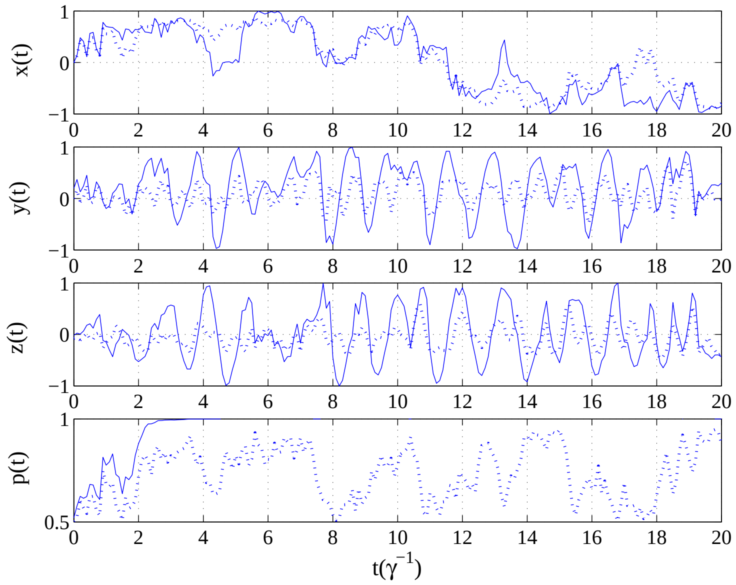

Figure 8.13: The best estimate states, when heterodyne is used. Details are as in figure 8.1.

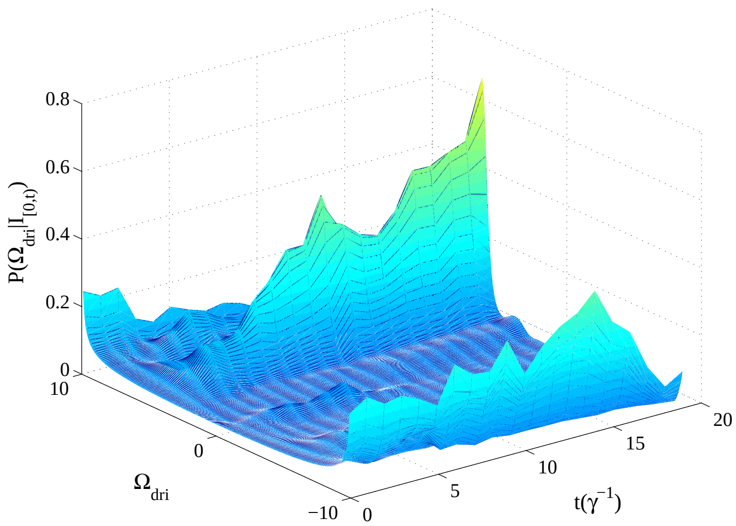

Figure 8.14: A plot of $\mathrm{P}(\Omega|\mathbf{I}_{[t_0,t)})$ for heterodyne detection. Details are the same as figure 8.2.

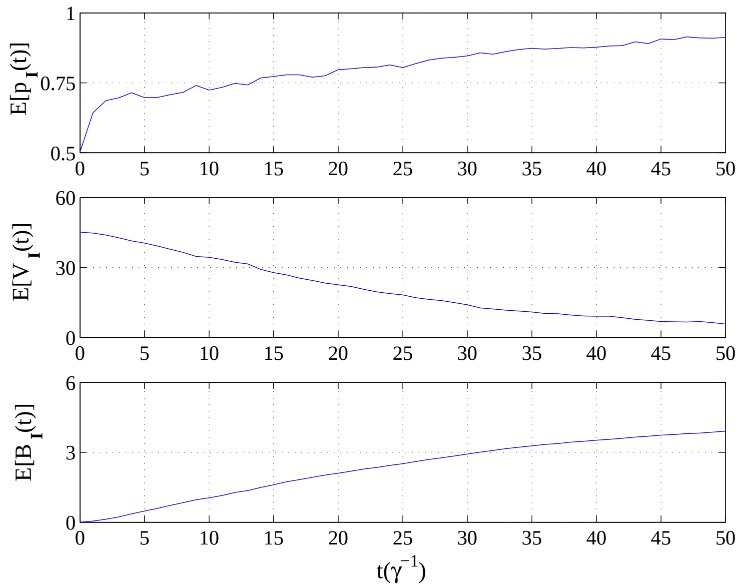

Figure 8.15: The ensemble average ($n = 250$) of the purity, variance and information gain for heterodyne detection. Time is measured in units of γ^{-1}.

8.3 Summary of chapter

The results of this chapter demonstrate that quantum parameter and state estimation for a continuously monitored open system is greatly affected by the measuring scheme. It was observed that as the measurement time increased, some detection schemes had the ability of both reducing our uncertainty in the unknown dynamical parameter, and producing a conditioned state of high purity, whereas other schemes could only do one of these, or none (depending on how the uncertainty in the unknown parameter is quantified). I would like to re-emphasize that all of the measurement schemes arise from the same coupling of the system to the environment; all that is different is how the environment is measured.

The system which I considered was a two-level atom with Hamiltonian $\Omega_{\rm dri}\hat{\sigma}_x/2$, with spontaneous decay rate γ. The unknown dynamical parameter is $\Omega_{\rm dri}$, the Rabi frequency. I began with the atom in its stationary mixed state (depending on $\Omega_{\rm dri}$) and the prior distribution of $\Omega_{\rm dri}$ was that appropriate to an atom at a random point in a standing wave with a maximum Rabi frequency $\Omega_{\rm max} = 10\gamma$. I analyzed five different measurement schemes, direct detection, a particular adaptive scheme [138], homodyne detection of the x quadrature, homodyne of the y, and heterodyne. The results of the chapter can be summarized using four different measures of the effectiveness of the measurement. The first two relate to the knowledge obtained about $\Omega_{\rm dri}$. One is B_l, the long-time ($t >> \gamma^{-1}$) increase in the average information about the parameter $\Omega_{\rm dri}$. The other is V_l, the long-time average variance in $\Omega_{\rm dri}$. The next two relate to the knowledge obtained about the system. One is p_l, the long-time purity. This measures how much is known about the system, given the long-time knowledge about the unknown parameter $\Omega_{\rm dri}$. The other is p_s, the short-time ($t =$ a few γ^{-1}) purity. This time is long enough that, if $\Omega_{\rm dri}$ were known, the system would have been more-or-less completely purified, but short enough that the actual amount of information obtained about $\Omega_{\rm dri}$ is small. That is, it measures how well the measurement can purify the state despite the large initial uncertainty in the dynamics.

The results of this work is summarized in the table below, using the four measures of effectiveness for the five different detection schemes. Rather than quote figures for these four measures, I use a rating system ($\star$ to $\star\star\star\star$), the details of which are explained in the caption. This allows the results to be taken in at a glance.

	Detection Schemes				
Measure	Direct	Adaptive	Homodyne x	Homodyne y	Heterodyne
B_l	$\star\star\star$	$\star$	$\star\star$	$\star\star\star\star$	$\star\star\star$
V_l	$\star$	$\star$	$\star$	$\star\star\star\star$	$\star\star\star$
p_l	$\star$	$\star\star$	$\star$	$\star\star\star\star$	$\star\star\star$
p_s	$\star$	$\star\star\star\star$	$\star\star\star$	$\star$	$\star\star$

Table 8.1: Ratings for the five different detection schemes, for four different measures. Four $\star$s is the best rating and one $\star$ the worst. For B_l, any rating above $\star$ indicates that the information about $\Omega_{\rm dri}$ continues to increase with time, with the lower cut-offs for $\star\star\star$ and $\star\star\star\star$ being $B_l = 2.5$ and 5 bits respectively at $t = 50\gamma^{-1}$. For V_l, any rating above $\star$ indicates a variance in Ω that decreases, with the lower cut-offs for $\star\star\star$ and $\star\star\star\star$ being $V_l = 10\gamma^2$ and γ^2 respectively at $t = 50\gamma^{-1}$. For p_l, a rating above $\star\star$ indicates a purity that continues to increase with time. For schemes where the purity saturates, the lower cut-off for $\star\star$ is $p_l = 0.9$. For schemes where the purity continues to increase, the lower cut-off for $\star\star\star\star$ is $p_l = 0.95$ at $t = 50\gamma^{-1}$. Finally, for p_s, the lower cut-offs for $\star\star$, $\star\star\star$, and $\star\star\star\star$ are, respectively, $p_s = 0.65, 0.75, 0.85$ at $t = 3\gamma^{-1}$. In all cases $\Omega_{\rm max} = 10\gamma$.

From the table it is observed that homodyne y (Sec. 8.2.4) was the best detection scheme by all measures except for the short-time purification, for which it was the worst. Both of these aspects are explained by the fact that this scheme measures σ_y, the dynamics of which depend strongly on $\Omega_{\rm dri}$. Hence the measurement record contains a lot of information about $\Omega_{\rm dri}$, including its sign (because rotations over the top of the Bloch sphere are different from rotations under the bottom). This also enables the purity to approach unity as time increases. However, for short times, when little information about $\Omega_{\rm dri}$ has been obtained, a y measurement is actually very poor for purifying the state. That is because the measurement tends to produce states with well-defined values of y, and these are states that are very sensitive to the rotation around the x-axis at rate $\Omega_{\rm dri}$. For a poorly known $\Omega_{\rm dri}$, this tends to make the system state more mixed, so that the purity grows only as the information about $\Omega_{\rm dri}$ increases.

After homodyne y detection, the method that provided most information about $\Omega_{\rm dri}$ was direct detection (Sec. 7.2). Under direct detection, the count rate is proportional to $\sigma_z + 1$, and (like σ_y), the dynamics of σ_z depend strongly upon $\Omega_{\rm dri}$, due to the Rabi rotations around the x-axis. However, in terms of σ_z, rotations around the $+x$-axis from the ground state are indistinguishable from rotations around the $-x$-axis. Hence the measurement cannot distinguish the sign of $\Omega_{\rm dri}$ and there is no change in the ensemble averaged variance as time increases. As a consequence, the purity saturates at a low value. The short time purification is poor also, for a similar reason to that for homodyne y detection.

The adaptive detection is almost complementary in its qualities to homodyne y detection. As explained in Sec. 8.2.2, it yields almost no information about $\Omega_{\rm dri}$, because the rate of detections in steady state is independent of $\Omega_{\rm dri}$. In particular, it yields no information about the sign of $\Omega_{\rm dri}$, so the variance is constant. As a consequence, the purity does not approach unity. Nevertheless, it does approach a quite high value, of over $1-\gamma/(\sqrt{2}\,\Omega_{\max})$, which is 0.93 for $\Omega_{\max}=10\gamma$. This is because the conditioned states are, for large $\Omega_{\rm dri}$, asymptotically independent of $\Omega_{\rm dri}$, as they approach $\hat{\sigma}_x$ eigenstates. This explains why the adaptive scheme gives the best results for short-time purification: the conditioned states are almost unaffected by the uncertainty in $\Omega_{\rm dri}$.

Homodyne x detection (Sec. 8.2.3) is in many ways similar to the adaptive scheme, and this is readily understandable since it would be expected to produce conditioned states tending towards $\hat{\sigma}_x$ eigenstates. Like adaptive (and direct) detection, the sign of $\Omega_{\rm dri}$ is indeterminable so the variance is constant. Hence the final purity does not approach unity. Although its asymptotic value is not as high as that for adaptive detection, it is higher than that for direct detection. This is as expected, since the conditioned states, being imperfectly localized towards the x-eigenstates, are still affected by $\Omega_{\rm dri}$. This also explains why the initial purification is not quite as good as for adaptive detection, and why information continues to be gained (albeit slowly) as time increases.

The final scheme, heterodyne detection (Sec. 8.2.5), is most easily understood by viewing it as an equal mixture of homodyne x and homodyne y detection, which is in fact a completely rigorous viewpoint. All of the ratings for heterodyne detection are intermediate between those for the two homodyne schemes.

In conclusion, I have shown that gaining knowledge about an unknown dynamical parameter by monitoring the system is a quite different phenomenon from gaining knowledge about the system itself. We have also distinguished different sorts of knowledge acquisition with distinct characteristics: for the unknown parameter, information gain (in bits) versus reducing the variance; and for the system, short-time purity gain versus long-time purity gain. The ability to acquire knowledge in these various ways is extremely sensitive to the choice of monitoring scheme (which does not affect the average evolution of the system). For the system we investigated, explaining the particulars of this sensitivity depends upon a detailed understanding of the conditional dynamics of the system. The discoveries of this work may have important implications for the suitability of different quantum feedback-control techniques [49, 133] in experimental systems with unknown dynamical parameters.

Chapter 9

Non-Markovian SSEs

In this chapter I will present non-Markovian stochastic Schrödinger equations. In particular I will consider only diffusive non-Markovian SSE. I will present three different unravelings; these are the coherent-state, quadrature and position-state unraveling. I will also discuss their interpretation under both the orthodox and modal interpretation of quantum mechanics. I will show that only under the modal theory non-Markovian SSE can be given a physical interpretation. That is, in the non-Markovian limit SSEs are not quantum trajectories. I will apply the theory to two simple systems: a TLA coupled linearly to a single and two mode bath respectively.

9.1 Non-Markovian SSE under the orthodox interpretation

9.1.1 General derivation

In the non-Markovian regime, because the master equation, in general, can not be evaluated (namely $\hat{\mathcal{K}}(t,t')$) non-Markovian SSEs as seen in section 6.4 provide a method for determining the reduced state, which always insures the positivity requirements of $\rho_{\rm red}(t)$. Here, however, I asked the question do non-Markovian stochastic unravelings and their family of SSEs have a physical interpretation under the orthodox view of quantum mechanics. I will investigate this question by deriving the Diósi, Gisin, and Strunz (coherent-state) non-Markovian SSE [46, 47, 48, 116, 117], and two new unravelings: the quadrature [59] and position-state [61] non-Markovian SSEs. That is I will derive these three unraveling using quantum measurement theory (QMT) [91]. In sections 2.2.1 and 2.2.2 we observed that when a rank one measurement is performed on the bath the system is projected into a pure state, the conditional system state. To derive non-Markovian SSEs we assume that the measurement is described by the observable

$$Z(t) \equiv \{Z_k(t)\} = \Big\{\Big(\{z_k\}, \hat{F}_{\{z_k\}}(t) = \frac{1}{N}|\{z_k(t)\}\rangle\langle\{z_k(t)\}| \otimes \hat{1}_{\rm sys}\Big)\Big\} \tag{9.1}$$

which outputs a string of bath results $\{z_k\}$. However upon measurement we can couple all the results together by defining a noise function

$$z(t,t) = f(\{r(Z_k(t),t)\},t), \tag{9.2}$$

where $\{r(Z_k(t),t)\}$ corresponds to the results of the measurement of the observables $\{Z_k(t)\}$ at time t (measurement time). The two time notation will become clearer when we consider how to derive stochastic Schrödinger equations. With this noise function we can associate a noise operator

$$\hat{z}(t) = f(\{\hat{Z}_k(t)\}, t), \tag{9.3}$$

such that $\{|\{z_k(t)\}\rangle\}$ corresponds to an eigenstate of the noise operator with an eigenvalue equal to the noise function.

With this general bath measurement we can define the conditional system state as (see section 2.2.2, namely Eq. (2.99))

$$|\psi_{\{z_k\}}(t)\rangle = \frac{\langle\{z_k\}(t)|\Psi(t)\rangle}{\sqrt{N\ P(\{z_k\}, t)}}, \tag{9.4}$$

where

$$P(\{z_k\}, t) = \langle\Psi(t)|\hat{F}_{\{z_k\}}(t)|\Psi(t)\rangle. \tag{9.5}$$

This conditional system state has the property that

$$\rho_{\rm red}(t) = \int P(\{z_k\}, t)|\psi_{\{z_k\}}(t)\rangle\langle\psi_{\{z_k\}}(t)|d\{z_k\}. \tag{9.6}$$

Here the set $\{z_k\}$ is really just configuration coordinates. If we now let this set be the set of random variables $\{r(Z_k(t), t)\}$ chosen from the distribution $P(\{z_k\}, t)$ then we can rewrite the reduced state as

$$\rho_{\rm red}(t) = E[|\psi_{\{r(Z_k(t),t)\}}(t)\rangle\langle\psi_{\{r(Z_k(t),t)\}}(t)|], \tag{9.7}$$

where E denotes an ensemble average over the probability density $P(\{z_k\}, t)$. This is the requirement outlined in Sec. 6.4 for a solution to a SSE. This suggests that the time derivative of $|\psi_{\{r(Z_k(t),t)\}}(t)\rangle$ will be a non-Markovian SSE. To calculate this we need to be able to generate a self consistent differential equation for $|\psi_{\{r(Z_k(t),t)\}}(t)\rangle$. To do this we use the fact that an infinitesimal change in $|\psi_{\{z_k\}}(t)\rangle$ is given by

$$d|\psi_{\{z_k\}}(t)\rangle = \frac{\partial}{\partial t}|\psi_{\{z_k\}}(t)\rangle dt + \sum_k^{\kappa} \frac{\partial}{\partial z_k}|\psi_{\{z_k\}}(t)\rangle dz_k, \tag{9.8}$$

which evaluated along the path $\{z_k = r(Z_k(t), t)\}$ gives

$$\frac{d}{dt}|\psi_{\{r(Z_k(t),t)\}}(t)\rangle = \frac{\partial}{\partial t}|\psi_{\{r(Z_k(t),t)\}}(t)\rangle + \sum_k d_t r(Z_k(t), t)\frac{\partial}{\partial r(Z_k(t), t)}|\psi_{\{r(Z_k(t),t)\}}(t)\rangle. \tag{9.9}$$

This is complicated as it is highly non-linear and non-Markovian. To simplify this procedure we introduce linear quantum measurement theory (see section 2.2.3).

Linear QMT uses the same principles as QMT except we use an ostensible distribution (density), $\Lambda(\{z_k\})$, in place of the actual distribution. Using this distribution the linear conditional state is

$$|\bar{\psi}_{\{z_k\}}(t)\rangle = \frac{\langle\{z_k(t)\}|\Psi(t)\rangle}{\sqrt{N\ \Lambda(\{z_k\})}}. \tag{9.10}$$

The bar above this linear state signifies that the state is not normalized to one. As before the reduced state can be written as

$$\rho_{\rm red}(t) = \int \Lambda(\{z_k\})|\bar{\psi}_{\{z_k\}}(t)\rangle\langle\bar{\psi}_{\{z_k\}}(t)|d\{z_k\}. \tag{9.11}$$

Thus $\rho_{\rm red}(t) = \bar{E}[|\bar{\psi}_{\{\bar{r}(Z_k(t))\}}(t)\rangle\langle\bar{\psi}_{\{\bar{r}(Z_k(t))\}}(t)|]$ where the bar over E and $\{r(Z_k(t), t)\}$ denotes that the random variable is chosen from the ostensible distribution $\Lambda(\{z_k\})$. Because $\Lambda(\{z_k\})$ is time independent $\{r(Z_k(t),t)\} = \{\bar{r}(Z_k(t))\}$ for all t. We also assume that the observable is time independent (this implies $Z_k(t) = Z_k$ and $|\{z_k(t)\}\rangle = |\{z_k(t)\}\rangle$) then differentiating Eq. (9.10) with respect to t gives

$$\frac{\partial}{\partial t}|\bar{\psi}_{\{z_k\}}(t)\rangle = \frac{\langle\{z_k\}|d_t|\Psi(t)\rangle}{\sqrt{\Lambda(\{z_k\})}}, \tag{9.12}$$

where $d_t|\Psi(t)\rangle$ is determined by the Schrödinger equation. In terms of the random variable this implies

$$\frac{d}{dt}|\bar{\psi}_{\{\bar{r}(Z_k)\}}(t)\rangle = \frac{\partial}{\partial t}|\bar{\psi}_{\{\bar{r}(Z_k)\}}(t)\rangle, \tag{9.13}$$

as $d_t\bar{r}(Z_k) = 0$ for all k. This means provided we know the Hamiltonian for the universe (system and bath) the linear SSE simply corresponds to writing the Schrödinger equation in the bath basis $|\{z_k\}\rangle$.

To calculate the actual non-Markovian SSE we use the fact that

$$|\psi_{\{z_k\}}(t)\rangle = \frac{|\bar{\psi}_{\{z_k\}}(t)\rangle}{\sqrt{\langle\bar{\psi}_{\{z_k\}}(t)|\bar{\psi}_{\{z_k\}}(t)\rangle}}. \tag{9.14}$$

To show this simply rearrange and substitute Eq. (9.10) into Eq. (9.4). Thus with Eq. (9.14) the infinitesimal change in $|\psi_{\{z_k\}}(t)\rangle$ is

$$\begin{aligned} d|\psi_{\{z_k\}}(t)\rangle &= \frac{1}{|\bar{\psi}_{\{z_k\}}(t)|}\frac{\partial}{\partial t}|\bar{\psi}_{\{z_k\}}(t)\rangle dt + |\bar{\psi}_{\{z_k\}}(t)\rangle\frac{\partial}{\partial t}\frac{1}{|\bar{\psi}_{\{z_k\}}(t)|}dt \\ &+ \sum_k\Big(\frac{1}{|\bar{\psi}_{\{z_k\}}(t)|}\frac{\partial}{\partial z_k}|\bar{\psi}_{\{z_k\}}(t)\rangle dz_k + |\bar{\psi}_{\{z_k\}}(t)\rangle\frac{\partial}{\partial z_k}\frac{1}{|\bar{\psi}_{\{z_k\}}(t)|}dz_k\Big), \end{aligned} \tag{9.15}$$

where

$$|\bar{\psi}_{\{z_k\}}(t)| = \sqrt{\langle\bar{\psi}_{\{z_k\}}(t)|\bar{\psi}_{\{z_k\}}(t)\rangle}\,. \tag{9.16}$$

Here I have assumed z_k is real; for complex variables we will have two more terms to evaluate. Moving on, we can evaluate the above along the path $\{z_k = r(Z_k,t)\}$ to get

$$\begin{aligned} d_t|\psi_{\{r(Z_k,t)\}}(t)\rangle &= \frac{1}{|\bar{\psi}_{\{r(Z_k,t)\}}(t)|}\frac{\partial}{\partial t}|\bar{\psi}_{\{r(Z_k,t)\}}(t)\rangle + |\bar{\psi}_{\{r(Z_k,t)\}}(t)\rangle\frac{\partial}{\partial t}\frac{1}{|\bar{\psi}_{\{r(Z_k,t)\}}(t)|} \\ &\sum_k\Big(\frac{1}{|\bar{\psi}_{\{r(Z_k,t)\}}(t)|}\frac{\partial}{\partial r(Z_k,t)}|\bar{\psi}_{\{r(Z_k,t)\}}(t)\rangle d_t r(Z_k,t) \\ &+|\bar{\psi}_{\{r(Z_k,t)\}}(t)\rangle\frac{\partial}{\partial r(Z_k,t)}\frac{1}{|\bar{\psi}_{\{r(Z_k,t)\}}(t)|}d_t r(Z_k,t)\Big). \end{aligned} \tag{9.17}$$

If this can be evaluated in terms of only one $|\psi_{\{r(Z_k,t)\}}(t)\rangle$ (no derivatives) then and only then we have generated a non-Markovian SSE for the unraveling into bath state $|\{z_k\}\rangle$. (Time dependent bath states, while mathematical more complex, should also be possible.) To determine this SSE we have assumed that we can generate the path $\{r(Z_k,t)\}$ (or equivalently $P(\{z_k\},t)$). To do this without calculating $|\Psi(t)\rangle$ we need to invoke the Girsanov transformation

$$P(\{z_k\},t) = \langle\bar{\psi}_{\{z_k\}}(t)|\bar{\psi}_{\{z_k\}}(t)\rangle\Lambda(\{z_k\}). \tag{9.18}$$

Differentiating this with respect to time and using Eq. (9.12), in some circumstances allows us to generate a FPE (without diffusion) for the probability density. Presently this has only been shown for the coherent-state, quadrature and position-state unraveling. Once we have this FPE we can easily obtain a set of differential equations for the random variables $\{r(Z_k,t)\}$. Thereby providing the terms needed to evaluate Eq. (9.17) (this in itself does not imply that Eq. (9.17) is a SSE but is an initial requirement).

Provided the derivatives with respect to $r(Z_k, t)$ in Eq. (9.17) can be replaced by an operator acting on $|\psi_{\{r(Z_k,t)\}}(t)\rangle$ we have a non-Markovian SSE, the solution being $|\psi_{\{r(Z_k,t)\}}(t)\rangle$. This state in the orthodox interpretation represents the conditioned system state (Eq. (9.4)) conditioned on a measurement of the bath at time t which yielded results $\{r(Z_k, t) = z_k\}$. Thus the non-Markovian SSE is just a numerical tool for generating what the system state at time t would be if we decided to measure the bath at that time and we observed the results $\{z_k\}$ (or the noise function $z(t, t)$). The linking of the states at different times to make a quantum trajectory is a fiction. By this I mean the solutions $|\psi_{\{r(Z_k,t)\}}(t)\rangle$ and $|\psi_{\{r(Z_k,t')\}}(t')\rangle$ correspond to two different physical events. Case one, the system and bath are left alone to interact for a time $t - t_0$ then a measurement on the bath is made at time t, with results $\{r(Z_k, t)\}$. The second case, the system and bath are left alone to interact for a time $t' - t_0$ then a measurement on the bath is made at time t', with results $\{r(Z_k, t')\}$.

To actually make a quantum trajectory (continuous-in-time measurements on the bath) we would simply let $t - t_0 = \delta t$, and find the system state $|\psi_{\{r(Z_k,t_0+\delta t)\}}(t_0 + \delta t)\rangle$ via the non-Markovian SSE. This gives the state of the system after the first measurement which yielded results $\{r(Z_k, t_0)\}$ or noise function $z(t_0, t_0)$ (this is determined by the unraveling, but simply corresponds to a weight sum of the κ results $\{r(Z_k, t_0)\}$). This state would then be used as the initial condition for the next interval, δt, and the same procedure employed for the first interval would be used to generate the solution at time $t_0 + 2\delta t$. We would then repeat this procedure until we have a trajectory for the required time of simulation. Continuous-in-time measurements are achieved when we let $\delta t \to dt$. To keep track of the measurements we would have to define a current $\mathbf{I}_{[t_0,t)}$ (as in chapter 7) which would be a string of the noise functions $z(t_0, t_0), ..., z(t - dt, t - dt)$. I would like to note that the ensemble average of this trajectory would not be the master equation, Eq. (6.50). This is due to the fact that in quantum mechanics a measurement disturbs the system, this disturbance is known as back action. In a non-Markovian open quantum system the back action arising from the bath measurement will be remembered the by the open quantum system. Thus the average dynamics after this measurement to be different to case when there is no measurement. Note that in the Markovian limit, the above type trajectories (quantum trajectories-see chapter 7) do average to the same master equation as the Markovian limit of non-Markovian SSEs (Markovian SSEs). This is because in the Markovian case the noise function in each interval dt is statistically independent (due to the delta correlation function-white noise). Because of this statistical independence we can not distinguish between the continuous noise function $z(t, t)$ for all t and the discrete (every dt) current $\mathbf{I}_{[t_0,t)} = \{z(t_0, t_0), ..., z(t - dt, t - dt)\}$ Thus if I had to make a choice I would say quantum trajectory theory is not equivalent to SSEs. It is only in the Markovian limit that it is possible to make a connection, which I feel is not strictly correct as no true system exhibits white noise; white noise is only an idealization.

9.1.2 Coherent-state unraveling

The coherent noise function

The first unraveling we consider is that associated with the bath being projected into a multimode coherent states (hence the name coherent unraveling), that is $|\{z_k\}\rangle = |\{a_k\}\rangle$ where

$$|\{a_k\}\rangle = \prod_k e^{-|a_k|^2/2} \sum_{n_k} \frac{a_k^{n_k}}{\sqrt{n_k!}} |n_k\rangle. \tag{9.19}$$

Thus the observable for this measurement scheme is

$$Z \equiv \{A_k\} = \Big\{ \Big(\{a_k\}, \hat{F}_{\{a_k\}}(t) \Big) \Big\}. \tag{9.20}$$

where

$$\hat{F}_{\{a_k\}}(t) = \frac{1}{\pi^\kappa} |\{a_k(t)\}\rangle\langle\{a_k(t)\}| \otimes \hat{1}_{\text{sys}}, \tag{9.21}$$

where κ corresponds to the number of modes. For this unraveling I will define the noise operator as

$$\hat{z}(s) = \sum_k g_k \hat{a}_k e^{-i\Omega_k(s-t_0)}, \tag{9.22}$$

where $\Omega_k = \omega_k - \omega_{\text{sys}}$. With this noise operator the noise function for a measurement at time t will be

$$z(t,s) = \sum_k g_k r(A_k, t) e^{-i\Omega_k(s-t_0)}. \tag{9.23}$$

The second time s is introduced so that we can define correlation functions for the environment that depend on the noise function (for a measurement at time t) at times s.

An important property of the bath is its correlation: how the noise function at time s is related to that at time s'. This is determined by the correlation function. For a complex noise function there are two important correlations, $E[z(t,s)z^*(t,s')]$ and $E[z(t,s)z(t,s')]$. This depends on the set of random variables $\{r(A_k,t) = a_k\}$ associated with the probability density $P(\{a_k\},t)$. In linear QMT this probability density is given by the ostensible distribution, denoted $\Lambda(\{a_k\})$, which I chose to be time-independent. It is convenient to choose $\Lambda(\{a_k\})$ to be equal to the actual probability that would arise when the bath is always in the vacuum state (the initial probability distribution). That is,

$$\Lambda(\{a_k\}) = \pi^{-\kappa} |\langle\{0_k\}|\{a_k\}\rangle|^2 = \pi^{-\kappa} \exp(-\sum_k |a_k|^2), \tag{9.24}$$

where $\kappa = \sum_k 1$. The correlation for the noise functions under this assumption is,

$$\bar{E}[z(t,s)] = 0, \tag{9.25}$$

$$\bar{E}[z(t,s)z(t,s')] = 0, \tag{9.26}$$

$$\bar{E}[z(t,s)z^*(t,s')] = \sum_{k,k'} g_k g_{k'}^* \bar{E}[\bar{r}(A_k)\bar{r}(A_{k'}^*)] e^{-i\Omega_k(s-t_0)} e^{i\Omega_{k'}(s'-t_0)} = \alpha(s-s'), \tag{9.27}$$

where

$$\alpha(s-s') = \sum_k |g_k|^2 \exp(-i\Omega_k(s-s')) \tag{9.28}$$

is the memory function.

The linear non-Markovian SSE for the coherent unraveling

In this section I will derive the linear non-Markovian SSEs, in particular the one associated with the ostensible distribution outlined in Eq. (9.24). I will use many of the same techniques as Diósi, Gisin and Strunz [48]. To calculate the linear SSE we write the Schrödinger equation (Eq. (6.12)) in terms of the operators $\{\hat{a}_k\}$ and $\{\hat{a}_k^\dagger\}$. Doing this we get

$$d_t|\Psi(t)\rangle = \Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + \hat{L}\sum_k g_k^*\hat{a}_k^\dagger \exp[i\Omega_k(t-t_0)] - \hat{L}^\dagger\sum_k g_k\hat{a}_k\exp[-i\Omega_k(t-t_0)]\Big\}|\Psi(t)\rangle. \quad (9.29)$$

Then Eq. (9.12) for this unraveling gives

$$\begin{aligned}\partial_t|\bar{\psi}_{\{a_k\}}(t)\rangle \;=\; & \Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + \sum_k g_k^* a_k^* e^{i\Omega_k(t-t_0)}\hat{L}\Big\}|\bar{\psi}_{\{a_k\}}(t)\rangle - \sum_k g_k e^{-i\Omega_k(t-t_0)} \\ & \times\hat{L}^\dagger\langle\{a_k\}|\hat{a}_k|\Psi(t)\rangle/\Lambda(\{a_k\}), \qquad (9.30)\end{aligned}$$

as $\hat{H}_{\rm int}(t)$ is a system-only operator and $\langle\{a_k\}|$ is the left-eigenstate of $\{\hat{a}_k^\dagger\}$. To satisfy the condition for a linear SSE we must evaluate the last term in this equation in terms of $|\bar{\psi}_{\{a_k\}}(t)\rangle$. To do this we use [110],

$$\langle\{a_k\}|\hat{a}_k|\Psi(t)\rangle = \Big(\frac{a_k}{2} + \partial_{a_k^*}\Big)\langle\{a_k\}|\Psi(t)\rangle \quad (9.31)$$

and

$$\partial_{a_k^*}|\bar{\psi}_{\{a_k\}}(t)\rangle = \frac{\partial_{a_k^*}\langle\{a_k\}|\Psi(t)\rangle}{\sqrt{\Lambda(\{a_k\})}} + \frac{a_k}{2}|\bar{\psi}_{\{a_k\}}(t)\rangle. \quad (9.32)$$

With these two expressions we can write,

$$\frac{\langle\{a_k\}|\hat{a}_k|\Psi(t)\rangle}{\sqrt{\Lambda(\{a_k\})}} = \partial_{a_k^*}|\bar{\psi}_{\{a_k\}}(t)\rangle. \quad (9.33)$$

This allows us to write equation (9.30) as

$$\partial_t|\bar{\psi}_{\{a_k\}}(t)\rangle = \Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + \hat{L}\sum_k g_k^* a_k^* e^{i\Omega_k(t-t_0)} - \hat{L}^\dagger\sum_k g_k e^{-i\Omega_k(t-t_0)}\partial_{a_k^*}\Big\}|\bar{\psi}_{\{a_k\}}(t)\rangle. \quad (9.34)$$

Thus by Eq. (9.13) for the set of random variables $\{a_k = \bar{r}(A_k)\}$ the linear non-Markovian SSE for this unraveling is

$$\begin{aligned}d_t|\bar{\psi}_{\{\bar{r}(A_k)\}}(t)\rangle \;=\; & \Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + \hat{L}\sum_k g_k^*\bar{r}(A_k^*)e^{i\Omega_k(t-t_0)} - \hat{L}^\dagger\sum_k g_k e^{-i\Omega_k(t-t_0)}\partial_{\bar{r}(A_k^*)}\Big\} \\ & \times|\bar{\psi}_{\{\bar{r}(A_k)\}}(t)\rangle. \qquad (9.35)\end{aligned}$$

This is what Diósi, Gisin and Strunz [48] call a linear SSE. However Wiseman and myself believe that this is not really a SSE, as the final term implies that the evolution of the state $|\bar{\psi}_{\{\bar{r}(A_k)\}}(t)\rangle$ depends not only on itself, but upon neighbouring states with different values of $\{\bar{r}(A_k)\}$. That is, we cannot simply choose (stochastically) a set of random variables $\{\bar{r}(A_k)\}$ from the ostensible distribution and then propagate forward the system state using that value [59]. However, we can make progress towards an equation where we can do this by rewriting the partial derivative in terms of a functional derivative with respect to the noise function (see for example [98]),

$$\frac{\partial}{\partial r(A_k^*,t)} = \int_{t_0}^t \frac{\delta}{\delta z^*(t,s)}\frac{\partial z^*(t,s)}{\partial r(A_k^*,t)}ds, \quad (9.36)$$

where t_0 is the initial time. This gives

$$d_t|\bar{\psi}_z(t)\rangle \quad = \quad \Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(t)+z^*(t,t)\hat{L}-\hat{L}^\dagger\int_{t_0}^t\alpha(t-s)\frac{\delta}{\delta z^*(t,s)}ds\Big\}|\bar{\psi}_z(t)\rangle, \tag{9.37}$$

where $\alpha(t-s)$ is defined in Eq. (9.28). By replacing the partial derivatives by the functional derivative we have enforced the initial condition $|\Psi(t_0)\rangle = |\{0_k\}\rangle|\psi(t_0)\rangle$ (which is fine for the dynamics I wish to consider). This is seen as follows; at $t = t_0$ the functional derivative term in the above equation will have zero contribution, from the definition (9.36). By comparison with the corresponding term in Eq. (9.34), it follows that $\partial_{a_k^*}|\bar{\psi}_{\{a_k\}}(t)\rangle|_{t=t_0} = 0$ for all k. From Eq. (9.10) this is only possible if the system and bath states initially (at time t_0) factorize, and if $\Lambda(\{a_k\}) = |\langle\{a_k\}|\psi_{\rm bath}\rangle|^2$. From our choice (9.24) of ostensible probability, this enforces $|\psi_{\rm bath}\rangle = |\{0_k\}\rangle$. This is physically acceptable as we may assume that at time t_0 the system and bath are uncoupled, and the bath is in the vacuum state.

Like Eq. (9.34), Eq. (9.37) is not really a SSE because the functional derivative means that it depends not upon a state $|\bar{\psi}_z(t)\rangle$ at all times for a single value of the function $z(t,s)$, but rather also upon states for other values of that function. That is, we cannot stochastically choose $z(t,s)$ in order to generate a trajectory independent of other trajectories. Instead, all possible trajectories would have to be calculated in parallel. This means that the amount of calculation involved in solving Eq. (9.37) would be comparable to that required for directly solving the Schrödinger equation (9.29). However in some circumstance we can make the following ansatz [48],

$$\frac{\delta}{\delta z^*(t,s)}|\bar{\psi}_z(t)\rangle = \hat{c}_z(t,s)|\bar{\psi}_z(t)\rangle, \tag{9.38}$$

where $\hat{c}_z(t,s)$ is some system operator which is a function of t, and s, and a functional of $z(t,s)$. With this ansatz the linear SSE becomes

$$d_t|\bar{\psi}_z(t)\rangle \quad = \quad \Big[-\frac{i}{\hbar}\hat{H}_{\rm int}(t)+z^*(t,t)\hat{L}-\hat{L}^\dagger\hat{C}_z(t)\Big]|\bar{\psi}_z(t)\rangle, \tag{9.39}$$

where

$$\hat{C}_z(t) = \int_{t_0}^t\alpha(t-s)\hat{c}_z(t,s)ds. \tag{9.40}$$

This is now a true SSE, where each trajectory can be evolved independently. It is the same as the linear SSE that Diósi, Gisin, and Strunz presented in Ref. [47, 48]. Note that it is non-Markovian because the noise $z^*(t,s)$ is non-white and because the operator $\hat{C}_z(t)$ may depend upon $z^*(t,s)$.

I would like to point out here that introducing the functional derivative is not necessary, the above anzatz could have been made on the partial derivatives

$$\sum_k g_k e^{-i\Omega_k(t-t_0)}\frac{\partial}{\partial a_k^*}|\bar{\psi}_{\{a_k\}(t)}\rangle\Big|_{\{a_k^*=\bar{r}(A_k^*)\}} = \hat{C}_{z(t)}|\bar{\psi}_z(t)\rangle. \tag{9.41}$$

The introduction of the functional derivative was made for two reasons, these being to illustrate the equivalence of this work with current literature and it also allows us to take the Markovian limit more easily (this will be seen later in this chapter).

The actual non-Markovian SSE for the coherent unraveling

In this section we will derive the non-Markovian SSEs for the actual probability distribution. Again, we use many of the same techniques as Diósi, Gisin, and Strunz [48].

As discussed in section 9.1.1, to find an actual (i.e. nonlinear) SSE for the normalized state we need to satisfy three conditions. The first was to derive a linear SSE, which we did in the preceding section (by making use of an anstaz). The second condition is to find the set of random variables for the actual measurement results. To work out these random variables, $\{r(a_k,t)\}$ we use the Girsanov transform (2.103) to find a first-order partial differential equation (PDE) for the probability, from which the characteristic equation generates the transformed variables.

To obtain the PDE we differentiate Eq. (9.18), with $|\{z_k\}\rangle = |\{a_k\}\rangle$ this gives

$$\partial_t P(\{a_k\},t) = \langle\bar{\psi}_{\{a_k\}}(t)|\partial_t|\bar{\psi}_{\{a_k\}}(t)\rangle\Lambda(\{a_k\}) + \text{c.c.} \tag{9.42}$$

By equation (9.34) the above becomes

$$\begin{aligned}\partial_t P(\{a_k\},t) \quad = \quad & \Big\{\langle\bar{\psi}_{\{a_k\}}(t)|\hat{L}|\bar{\psi}_{\{a_k\}}(t)\rangle\sum_k a_k^* g_k^* e^{i\Omega_k(t-t_0)} - \sum_k\langle\bar{\psi}_{\{a_k\}}(t)|\hat{L}^\dagger\partial_{a_k^*}|\bar{\psi}_{\{a_k\}}(t)\rangle \\ & \times g_k e^{-i\Omega_k(t-t_0)} + \text{c.c.}\Big\}\Lambda(\{a_k\}),\end{aligned} \tag{9.43}$$

Using the fact that $|\bar{\psi}_{\{a_k\}}(t)\rangle$ is analytical in a_k^* (so that $\partial_{a_k}|\bar{\psi}_{\{a_k\}}(t)\rangle = 0$) [47], and the product rule for differentiation, we can simplify the above to

$$\partial_t P(\{a_k\},t) \quad = \quad -\sum_k g_k e^{-i\Omega_k(t-t_0)}\partial_{a_k^*}\Big\{\langle\bar{\psi}_{\{a_k\}}(t)|\hat{L}^\dagger|\bar{\psi}_{\{a_k\}}(t)\rangle\Lambda(\{a_k\})\Big\} + \text{c.c.} \tag{9.44}$$

Defining

$$\langle\hat{L}^\dagger\rangle_t = \langle\psi_{\{a_k\}}(t)|\hat{L}^\dagger|\psi_{\{a_k\}}(t)\rangle = \frac{\langle\bar{\psi}_{\{a_k\}}(t)|\hat{L}^\dagger|\bar{\psi}_{\{a_k\}}(t)\rangle}{\langle\bar{\psi}_{\{a_k\}}(t)|\bar{\psi}_{\{a_k\}}(t)\rangle} \tag{9.45}$$

allows us to write

$$\partial_t P(\{a_k\},t) = -\sum_k g_k e^{-i\Omega_k(t-t_0)}\partial_{a_k^*}\Big\{\langle\hat{L}^\dagger\rangle_t P(\{a_k\},t)\Big\} + \text{c.c.} \tag{9.46}$$

This is the PDE for the probability distribution.

At $t = t_0$, we have from Eq. (9.18) that

$$P(\{a_k\},t_0) = \langle\bar{\psi}_{\{a_k\}}(t_0)|\bar{\psi}_{\{a_k\}}(t_0)\rangle\Lambda(\{a_k\}). \tag{9.47}$$

As noted above, to obtain Eq. (9.37) we had to assume that the bath was initially in the vacuum state, uncorrelated with the system. This enforces the equation of the initial probability distribution to be the ostensible distribution

$$P(\{a_k\},t_0) = \Lambda(\{a_k\}) = \pi^{-\kappa}\exp(-\sum_k|a_k|^2). \tag{9.48}$$

From this PDE we can find the characteristic equations

$$d_t r(A_k^*,t) = g_k e^{-i\Omega_k(t-t_0)}\langle\hat{L}^\dagger\rangle_t, \tag{9.49}$$

which integrates to give

$$r(A_k^*,t) = r(A_k^*,t_0) + \int_{t_0}^t g_k e^{-i\Omega_k(t'-t_0)}\langle\hat{L}^\dagger\rangle_{t'}dt'. \tag{9.50}$$

The random variable $r(A_k^*,t_0)$ corresponds to the probability distribution (9.48). With equation (9.50) and our noise function definition, Eq. (9.23), we can write $z(t,s)$ as

$$z^*(t,s) = \sum_k r(A_k^*,t_0)g_k^* e^{i\Omega_k(s-t_0)} + \int_{t_0}^t \alpha^*(s-t')\langle\hat{L}^\dagger\rangle_{t'}dt'. \tag{9.51}$$

The term $a_k^*(t_0)g_k^*e^{i\Omega_k(s-t_0)}$ is the noise function one would obtain if the bath were assumed to be in the vacuum state. This is our assumption for the ostensible distribution so we will label this term $z^*(t_0,s)$ (or $z^*_\Lambda(t,s)$). This allows us to write

$$z^*(t,s) = z^*(t_0,s) + \int_{t_0}^{t} \alpha^*(s-t')\langle \hat{L}^\dagger\rangle_{t'} dt', \tag{9.52}$$

where $z^*(t_0,s)$ obeys the correlations expressed in Eqs. (9.25) – (9.27).

The third condition was to show that we can write Eq. (9.17) in terms of only $|\psi_{\{r(A_k,t)\}}(t)\rangle$. Substituting Eq. (9.34) into Eq. (9.17) and evaluating for $\{a_k = r(A_k,t)\}$ gives (a mess)

$$\begin{aligned}
&d_t|\psi_{\{r(A_k,t)\}}(t)\rangle = \\
&\Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + \sum_k g_k^* r(A_k^*,t)e^{i\Omega_k(t-t_0)}\hat{L}\Big\}|\psi_{\{r(A_k,t)\}}(t)\rangle - \frac{\hat{L}^\dagger\sum_k g_k e^{-i\Omega_k(t-t_0)}}{|\bar{\psi}_{\{r(A_k,t)\}}(t)|}\frac{\partial}{\partial r(A_k^*,t)} \\
&\times|\bar{\psi}_{\{r(A_k^*,t)\}}(t)\rangle - \frac{|\psi_{\{r(A_k,t)\}}(t)\rangle}{2}\Big\{\langle\psi_{\{r(A_k,t)\}}(t)|\hat{L}^\dagger\frac{\sum_k g_k e^{-i\Omega_k(t-t_0)}}{|\bar{\psi}_{\{r(A_k,t)\}}(t)|}\frac{\partial}{\partial r(A_k^*,t)}|\bar{\psi}_{\{r(A_k^*,t)\}}(t)\rangle \\
&-\sum_k g_k^* r(A_k^*,t)e^{i\Omega_k(t-t_0)}\langle\hat{L}\rangle_t + c.c\Big\} + \frac{\sum_k g_k e^{-i\Omega_k(t-t_0)}\langle\hat{L}^\dagger\rangle_t}{|\bar{\psi}_{\{r(A_k,t)\}}(t)|}\frac{\partial}{\partial r(A_k^*,t)}|\bar{\psi}_{\{r(A_k^*,t)\}}(t)\rangle \\
&-\frac{|\psi_{\{r(A_k,t)\}}(t)\rangle}{2}\frac{\sum_k g_k e^{-i\Omega_k(t-t_0)}\langle\hat{L}^\dagger\rangle_t}{|\bar{\psi}_{\{r(A_k,t)\}}(t)|}\Big\{\langle\psi_{\{r(A_k,t)\}}(t)|\frac{\partial}{\partial r(A_k^*,t)}|\bar{\psi}_{\{r(A_k,t)\}}(t)\rangle + c.c\Big\},
\end{aligned} \tag{9.53}$$

as $\partial_{a_k}|\bar{\psi}_{\{a_k\}(t)}\rangle = 0$. This can be simplified by using the fact that if a SSE has the form $d_t|\psi\rangle = (\hat{A} + B/2 + B^*/2)|\psi\rangle$ then we can define a state $|\phi\rangle = \exp(\int(B-B^*)dt/2)|\psi\rangle$ (which is the same state as $|\psi\rangle$) that gives a equivalent SSE, of form $d_t|\phi\rangle = (\hat{A}+B)|\phi\rangle$. Applying this to the above gives

$$\begin{aligned}
d_t|\psi_{\{r(A_k,t)\}}(t)\rangle &= \Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + \sum_k g_k^* a_k^* e^{i\Omega_k(t-t_0)}(\hat{L}-\langle\hat{L}\rangle_t)\Big\}|\psi_{\{r(A_k,t)\}}(t)\rangle \\
&\quad -(\hat{L}^\dagger - \langle\hat{L}^\dagger\rangle_t)\frac{\sum_k g_k e^{-i\Omega_k(t-t_0)}}{|\bar{\psi}_{\{r(A_k^*,t)\}}|}\frac{\partial}{\partial r(A_k^*,t)}|\bar{\psi}_{\{r(A_k^*,t)\}}(t)\rangle + |\psi_{\{r(A_k,t)\}}(t)\rangle \\
&\quad \times\langle\psi_{\{r(A_k,t)\}}(t)|(\hat{L}^\dagger - \langle\hat{L}^\dagger\rangle_t)\frac{\sum_k g_k e^{-i\Omega_k(t-t_0)}}{|\bar{\psi}_{\{r(A_k^*,t)\}}|}\frac{\partial}{\partial r(A_k^*,t)}|\bar{\psi}_{\{r(A_k^*,t)\}}(t)\rangle
\end{aligned} \tag{9.54}$$

This is not yet a SSE as it still contains $|\bar{\psi}_{\{r(A_k^*,t)\}}(t)\rangle$ terms, however if we can make the ansatz described by Eq. (9.41), and use definition (9.23) we can write this as

$$d_t|\psi_z(t)\rangle = \Big[-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + z^*(t,t)(\hat{L}-\langle\hat{L}\rangle_t) - (\hat{L}^\dagger - \langle\hat{L}^\dagger\rangle_t)\hat{C}_z(t) + \Big\langle(\hat{L}^\dagger - \langle\hat{L}^\dagger\rangle_t)\hat{C}_z(t)\Big\rangle_t\Big]|\psi_z(t)\rangle, \tag{9.55}$$

which is a genuine SSE. This means that an actual SSE (generating normalized states with their actual probabilities) can only be found if we can make the ansatz describe in Eq. (9.41). Thus in the non-Markovian limit linear SSEs are essential as they provide a method for evaluating the operators $\hat{C}_z(t)$.

This SSE is the same as that presented in Refs. [48, 117]. As shown here, its solution gives us the state the system would be in if at time t we performed a measurement in the coherent basis on the bath, and the results are $\{r(A_k,t)\}$. We see that these results depend upon the system state at earlier times (the non-Markovian nature). I have argued above that this linking of states at different times is a convenient fiction, but we see here that it is mathematically necessary in order to generate measurement results for a particular time with the actual probability.

9.1.3 Quadrature unraveling

The quadrature noise function

The above unraveling corresponds to complex noise. In Ref. [59] Wiseman and myself showed that as in the Markovian case, non-Markovian SSE can be generalized. Here I am going to present an unraveling with real noise. This I call the quadrature unraveling. To obtain a SSE with real noise, it is natural to consider a quadrature noise operator,

$$\hat{z}(s) = [\sum_k g_k \hat{a}_k e^{-i\Omega_k(s-t_0)} e^{-i\phi} + \sum_k g_k^* \hat{a}_k^\dagger e^{i\Omega_k(s-t_0)} e^{i\phi}], \tag{9.56}$$

where ϕ is some arbitrary phase, which determines the quadrature being measured. Unless otherwise stated ϕ will be set to zero.

The basis for the bath measurement is $|\{q_k\}\rangle$ and must satisfy

$$\hat{z}(s)|\{q_k\}\rangle = z(t,s)|\{q_k\}\rangle, \tag{9.57}$$

where $z(t,s)$ is the noise function for the quadrature unraveling. The problem with this noise function is that it is hard (maybe impossible) to work out a time-independent eigenstate $|\{q_k\}\rangle$ in the interaction picture. However, we can find the eigenstate if we make the assumptions that for every mode k there exists another mode, which we can label $-k$, such that $\Omega_{-k} = -\Omega_k$ and $g_{-k} = g_k^*$. These assumptions simply mean that the modes coupled to the system come in symmetric pairs about the system frequency $\omega_{\rm sys}$. Without loss of generality we can take the g_k's to be real, absorbing any phases in the definitions of the bath operators. With all of these assumptions we can rewrite equation (9.56) as

$$\hat{z}(s) \quad = \quad \sum_k \left(g_k \hat{a}_k e^{-i\Omega_k(s-t_0)} + g_k \hat{a}_{-k} e^{i\Omega_k(s-t_0)} + g_k \hat{a}_k^\dagger e^{i\Omega_k(s-t_0)} + g_k \hat{a}_{-k}^\dagger e^{-i\Omega_k(s-t_0)} \right) \tag{9.58}$$

where the summation is only over half the modes ($k > 0$). This can be further simplified to

$$\hat{z}(s) = \sum_{k>0} 2g_k \{\hat{X}_k^+ \cos[\Omega_k(s-t_0)] + \hat{Y}_k^- \sin[\Omega_k(s-t_0)]\}. \tag{9.59}$$

Here we have introduced the two-mode quadrature operators

$$\hat{X}_k^\pm \quad = \quad (\hat{x}_k \pm \hat{x}_{-k})/\sqrt{2}\,, \tag{9.60}$$

$$\hat{Y}_k^\pm \quad = \quad (\hat{y}_k \pm \hat{y}_{-k})/\sqrt{2}\,, \tag{9.61}$$

where $\hat{x}_k$ and $\hat{y}_k$ are the quadratures of $\hat{a}_k$:

$$\hat{a}_k = (\hat{x}_k + i\hat{y}_k)/\sqrt{2}\,. \tag{9.62}$$

These operators have the commutators

$$[\hat{X}_k^-, \hat{Y}_k^-] = i, \qquad [\hat{X}_k^-, \hat{Y}_k^+] = 0, \tag{9.63}$$

$$[\hat{X}_k^+, \hat{Y}_k^-] = 0, \qquad [\hat{X}_k^+, \hat{Y}_k^+] = i. \tag{9.64}$$

Since $\{\hat{X}_k^+\}$ and $\{\hat{Y}_k^-\}$ form two mutually commuting sets of commuting operators, and thus have a common set of eigenstates. Since $\hat{z}(s)$ is a linear combination of these operators, the eigenstates of $\{\hat{X}_k^+\}$ and $\{\hat{Y}_k^-\}$ are the $|\{q_k\}\rangle$ we seek. Therefore we can write the two eigenvalue equations,

$$\hat{Y}_k^-|\{q_k\}\rangle \quad = \quad Y_k^-|\{q_k\}\rangle, \tag{9.65}$$

$$\hat{X}_k^+|\{q_k\}\rangle \quad = \quad X_k^+|\{q_k\}\rangle. \tag{9.66}$$

This suggest that we should write $|\{q_k\}\rangle$ as $|\{X_k^+, Y_K^-\}\rangle$, but for brevity we will continue to write it as $|\{q_k\}\rangle$. The form of the state that satisfies these equations, in the y_k-basis, for a particular k is

$$|\{q_k\}\rangle = \int \frac{dy'}{\sqrt{2\pi}} \Big|\frac{y' - Y_k^-}{\sqrt{2}}\Big\rangle_{-k} \Big|\frac{y' + Y_k^-}{\sqrt{2}}\Big\rangle_k e^{-iX_k^+ y'}, \tag{9.67}$$

while in the x_k-basis it is

$$|\{q_k\}\rangle = \int \frac{dx'}{\sqrt{2\pi}} \Big|\frac{X_k^+ - x'}{\sqrt{2}}\Big\rangle_{-k} \Big|\frac{X_k^+ + x'}{\sqrt{2}}\Big\rangle_k e^{iY_k^- x'} \tag{9.68}$$

which forms an orthogonal set.

That is for this unraveling the measurement corresponds to projection into a EPR-like state, with results $\{(X_k^+, Y_k^-)\}$ (or $\{q_k\}$ for short). That is the observable is

$$Z \equiv \{(X_k^+, Y_k^-)\} = \Big\{\Big(\{q_k\}, \hat{\pi}_{\{q_k\}} = |\{q_k(t)\}\rangle\langle\{q_k(t)\}| \otimes \hat{1}_{\rm sys}\Big)\Big\}. \tag{9.69}$$

and the noise function is

$$z(t,s) = \sum_{k>0} 2g_k\{r(X_k^+, t)\cos[\Omega_k(s - t_0)] + r(Y_k^-, t)\sin[\Omega_k(s - t_0)]\}. \tag{9.70}$$

Since $r(X_k^+, t)$ and $r(Y_k^-, t)$ are real, $z(t,s)$ is also.

We can define the correlation function for the noise functions as $E[z(t,s)z(t,s')]$, and again this depends on the probability distribution for the variables X_k^+ and Y_k^-. It is again convenient to choose the ostensible distribution to be that corresponding to the bath being in the vacuum state. Explicitly we then have

$$\Lambda(\{X_k, Y_k\}) = \pi^{-\kappa/2}\exp[-\sum_{k>0}(X_k^{+2} + Y_k^{-2})]. \tag{9.71}$$

With the usual ostensible distribution the correlation function is

$$\bar{E}[z(t,s)z(t,s')] = 2\sum_{k>0}|g_k|^2\cos(\Omega_k(s - s')) = \beta(s - s'), \tag{9.72}$$

while $\bar{E}[z(t,s)] = 0$ as before. Note when these assumptions about the bath are applied to $\alpha(s - s')$ we find that $\alpha(s - s') \equiv \beta(s - s')$.

The linear non-Markovian SSE for the quadrature unraveling

To find the linear non-Markovian SSE we start by applying our assumptions to the Schrödinger equation for the combined state

$$\begin{aligned} d_t|\Psi(t)\rangle &= \Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + \sum_{k>0} g_k\Big[\hat{L}(\hat{a}_k^\dagger e^{i\Omega_k(t-t_0)} + \hat{a}_{-k}^\dagger e^{-i\Omega_k(t-t_0)}) - \hat{L}^\dagger \\ &\quad \times(\hat{a}_k e^{-i\Omega_k(t-t_0)} + \hat{a}_{-k}e^{i\Omega_k(t-t_0)})\Big]\Big\}|\Psi(t)\rangle \end{aligned} \tag{9.73}$$

Using definitions (9.60), (9.61) and (9.62) we rewrite the above equation as

$$\begin{aligned} d_t|\Psi(t)\rangle &= \Big\{-i\hat{H}(t) + \hat{L}\sum_{k>0} 2g_k\{\hat{X}_k^+\cos[\Omega_k(t - t_0)] + \hat{Y}_k^-\sin[\Omega_k(t - t_0)]\} - \sum_{k>0} g_k(\hat{L} + \hat{L}^\dagger) \\ &\quad \times\Big(\hat{X}_k^+\cos[\Omega_k(t - t_0)] + i\hat{Y}_k^+\cos[\Omega_k(t - t_0)] - i\hat{X}_k^-\sin[\Omega_k(t - t_0)] \\ &\quad + \hat{Y}_k^-\sin[\Omega_k(t - t_0)]\Big)\Big\}|\Psi(t)\rangle. \end{aligned} \tag{9.74}$$

Then Eq. (9.12) for this unraveling gives

$$\begin{aligned}\partial_t|\bar{\psi}_{\{q_k\}}(t)\rangle &= \Big(-\frac{i}{\hbar}\hat{H}_{\rm int}(t)+\sum_{k>0}2g_k\{X_k^+\cos[\Omega_k(t-t_0)]+Y_k^-\sin[\Omega_k(t-t_0)]\}\hat{L}\Big)|\bar{\psi}_{\{q_k\}}(t)\rangle\\ &\quad -\sum_{k>0}g_k(\hat{L}+\hat{L}^\dagger)\Big\{\cos[\Omega_k(t-t_0)]\Big(i\frac{\langle\{q_k\}|Y_k^+|\Psi(t)\rangle}{\sqrt{\Lambda(\{X_k^+,Y_k^-\})}}+\hat{X}_k^+|\bar{\psi}_{\{q_k\}}(t)\rangle\Big)\\ &\quad +\sin[\Omega_k(t-t_0)]\Big(\hat{Y}_k^-|\bar{\psi}_{\{q_k\}}(t)\rangle-i\frac{\langle\{q_k\}|X_k^-|\Psi(t)\rangle}{\sqrt{\Lambda(\{X_k^+,Y_k^-\})}}\Big)\Big\}. \qquad (9.75)\end{aligned}$$

The inner products in the above equation can be simplified to

$$\langle\{q_k\}|\hat{X}_k^-|\Psi(t)\rangle = i\frac{\partial}{\partial Y_k^-}\langle\{q_k\}|\Psi(t)\rangle, \qquad (9.76)$$

$$\langle\{q_k\}|\hat{Y}_k^+|\Psi(t)\rangle = -i\frac{\partial}{\partial X_k^+}\langle\{q_k\}|\Psi(t)\rangle. \qquad (9.77)$$

as $\hat{X}_k^\pm$ and $\hat{Y}_k^\pm$ have the commutators listed in equations (9.63) and (9.64).

It can also be shown that

$$\frac{\partial}{\partial Y_k^-}|\bar{\psi}_{\{q_k\}}(t)\rangle = \frac{1}{\sqrt{\Lambda(\{X_k^+,Y_k^-\})}}\frac{\partial}{\partial Y_k^-}\langle\{q_k\}|\Psi(t)\rangle+Y_k^-|\bar{\psi}_{\{q_k\}}(t)\rangle, \qquad (9.78)$$

$$\frac{\partial}{\partial X_k^+}|\bar{\psi}_{\{q_k\}}(t)\rangle = \frac{1}{\sqrt{\Lambda(\{X_k^+,Y_k^-\})}}\frac{\partial}{\partial X_k^+}\langle\{q_k\}|\Psi(t)\rangle+X_k^+|\bar{\psi}_{\{q_k\}}(t)\rangle, \qquad (9.79)$$

and using equations (9.76) and (9.77) with the above two equations we can write the inner products in terms of their conjugate variables. This allows us to write the linear equation as

$$\begin{aligned}\partial_t|\bar{\psi}_{\{q_k\}}(t)\rangle &= \Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(t)+\hat{L}\sum_{k>0}2g_k\{X_k^+\cos[\Omega_k(t-t_0)]+Y_k^-\sin[\Omega_k(t-t_0)]\}\\ &\quad -\sum_{k>0}g_k\hat{L}_x\Big(\sin[\Omega_k(t-t_0)]\frac{\partial}{\partial Y_k^-}+\cos[\Omega_k(t-t_0)]\frac{\partial}{\partial X_k^+}\Big)\Big\}|\bar{\psi}_{\{q_k\}}(t)\rangle,\end{aligned} \qquad (9.80)$$

which is a linear equation solely in terms of the parameters $\{X_k^+\}$ and $\{Y_k^-\}$. As in the coherent case the linear SSE (Eq. (9.13)) is obtained by evaluating the above for $\{X_k^+=\bar{r}(X_k^+)\}$ and $\{Y_k^-=\bar{r}(Y_k^-)\}$. This gives

$$\begin{aligned}d_t|\bar{\psi}_{\{\bar{r}(X_k^+),\bar{r}(Y_k^-)\}}(t)\rangle &= \Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(t)+\hat{L}\sum_{k>0}2g_k\{\bar{r}(X_k^+)\cos[\Omega_k(t-t_0)]+\bar{r}(Y_k^-)\sin[\Omega_k(t-t_0)]\}\\ &\quad -\sum_{k>0}g_k(\hat{L}+\hat{L}^\dagger)\Big(\sin[\Omega_k(t-t_0)]\frac{\partial}{\partial\bar{r}(Y_k^-)}+\cos[\Omega_k(t-t_0)]\frac{\partial}{\partial\bar{r}(X_k^+)}\Big)\Big\}\\ &\quad \times|\bar{\psi}_{\{\bar{r}(X_k^+),\bar{r}(Y_k^-)\}}(t)\rangle, \qquad (9.81)\end{aligned}$$

As in the coherent case, to make progress towards a genuine SSE we wish to replace the partial derivatives by a functional derivative with respect to the noise function. To do this we note that,

$$\frac{\partial}{\partial r(X_k^+,t)} = \int_{t_0}^t\frac{\delta}{\delta z(t,s)}\frac{\partial z(t,s)}{\partial r(X_k^+,t)}ds, \qquad (9.82)$$

$$\frac{\partial}{\partial r(Y_k^-,t)} = \int_{t_0}^t\frac{\delta}{\delta z(t,s)}\frac{\partial z(t,s)}{\partial r(Y_k^-,t)}ds. \qquad (9.83)$$

Thus we obtain

$$\partial_t|\bar{\psi}_z(t)\rangle = \Big\{-i\hat{H}(t) + z(t,t)\hat{L} - (\hat{L}+\hat{L}^\dagger)\int_0^t \beta(t-s)\frac{\delta}{\delta z(t,s)}ds\Big\}|\bar{\psi}_z(t)\rangle, \tag{9.84}$$

where $\beta(t-s)$ is the memory function for the noise. As in the coherent state case, this enforces an initial vacuum state for the bath. I would like to point out here that this is the same equation as Eq. (5.58) if $\hat{L} = \hat{L}^\dagger$. That is, the Bassi and Ghirardi dynamical reduction model [5] can be derived under the orthodox theory. As in the coherent case Wiseman and myself believe this is not a SSE, the final step to obtaining the linear non-Markovian SSE with real noise is to assume that the functional derivative can be replaced by an operator

$$\frac{\delta}{\delta z(t,s)}|\bar{\psi}_z(t)\rangle = \hat{q}_z(t,s)|\bar{\psi}_z(t)\rangle, \tag{9.85}$$

where $\hat{q}_z(t,s)$ is some system operator which is a function of t, and s, and a functional of $z(t,s)$. With this ansatz the linear SSE becomes

$$d_t|\bar{\psi}_z(t)\rangle = \Big[-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + z(t,t)\hat{L} - (\hat{L}+\hat{L}^\dagger)\hat{Q}_z(t)\Big]|\bar{\psi}_z(t)\rangle, \tag{9.86}$$

where

$$\hat{Q}_z(t) = \int_0^t \beta(t-s)\hat{q}_z(t,s)ds. \tag{9.87}$$

The Actual non-Markovian SSE for the Quadrature Unraveling

As in the coherent case, to find an actual SSE (generating states with the actual probability) we need to find the random variables for the actual probabilities of measuring results $\{q_k\}$. To sort these out we use the Girsanov transform (9.18) to find a first-order partial differential equation (PDE) for the probability, from which the characteristic equation generates the transformed variables

$$\partial_t P(\{X_k^+, Y_k^-\},t) = \Big(\langle\bar{\psi}_{\{q_k\}}(t)|\partial_t|\bar{\psi}_{\{q_k\}}(t)\rangle + \text{c.c}\Big)\Lambda(\{X_k^+, Y_k^-\}). \tag{9.88}$$

Using equations (9.80) allows us to write

$$\begin{aligned}\partial_t P(\{X_k^+, Y_k^-\},t) = &-\sum_{k>0} g_k\frac{\partial}{\partial X_k^+}\Big(\cos[\Omega_k(t-t_0)]\langle\bar{\psi}_{\{q_k\}}(t)|\hat{L}+\hat{L}^\dagger|\bar{\psi}_{\{q_k\}}(t)\rangle\Lambda(\{X_k^+, Y_k^-\})\Big)\\ &-\sum_{k>0} g_k\frac{\partial}{\partial Y_k^-}\Big(\sin[\Omega_k(t-t_0)]\langle\bar{\psi}_{\{q_k\}}(t)|\hat{L}+\hat{L}^\dagger|\bar{\psi}_{\{q_k\}}(t)\rangle\Lambda(\{X_k^+, Y_k^-\})\Big).\end{aligned} \tag{9.89}$$

This can be simplified to

$$\begin{aligned}\partial_t P(\{X_k^+, Y_k^-\},t) = &-\sum_{k>0} g_k\frac{\partial}{\partial X_k^+}\Big(\cos[\Omega_k(t-t_0)]\langle\hat{L}+\hat{L}^\dagger\rangle_t P(\{X_k^+, Y_k^-\},t)\Big)\\ &-\sum_{k>0} g_k\frac{\partial}{\partial Y_k^-}\Big(\sin[\Omega_k(t-t_0)]\langle\hat{L}+\hat{L}^\dagger\rangle_t P(\{X_k^+, Y_k^-\},t)\Big).\end{aligned} \tag{9.90}$$

where $\langle\hat{L}\rangle_t$ is defined by equation (9.45).

The characteristic equations are

$$\frac{d}{dt}r(X_k^+,t) = g_k\cos[\Omega_k(t-t_0)]\langle\hat{L}+\hat{L}^\dagger\rangle_t, \tag{9.91}$$

$$\frac{d}{dt}r(Y_k^-,t) = g_k\sin[\Omega_k(t-t_0)]\langle\hat{L}+\hat{L}^\dagger\rangle_t. \tag{9.92}$$

Integrating these differential equation from time t_0 to t we get

$$r(X_k^+, t) = r(X_k^+, t_0) + \int_{t_0}^{t} g_k \cos(\Omega_k t')\langle \hat{L} + \hat{L}^\dagger\rangle_{t'} dt', \quad (9.93)$$

$$r(Y_k^-, t) = r(Y_k^-, t_0) + \int_{t_0}^{t} g_k \sin(\Omega_k t')\langle \hat{L} + \hat{L}^\dagger\rangle_{t'} dt'. \quad (9.94)$$

The distribution for $r(X_k^+, t_0)$ and $r(Y_k^-, t_0)$ is due to the quantum initial conditions. As before, the use of the functional derivative in Eq. (9.84) implies that the initial bath state is a vacuum state. Thus, the randomness in $r(X_k^+, t_0)$ and $r(Y_k^-, t_0)$ is that of the ostensible distribution:

$$P(\{X_k^+, Y_k^-\}, t_0) = \Lambda(\{X_k^+, Y_k^-\}) = \frac{\exp[-\sum_{k>0}(X_k^{+2} + Y_k^{-2})]}{\pi^{K/2}}. \quad (9.95)$$

With the above random variable equations for $r(X_k^+, t)$ and $r(Y_k^-, t)$ we can write the noise function for the actual probability as

$$z(t, s) = z(t_0, s) + \int_{t_0}^{t} \langle \hat{L} + \hat{L}^\dagger\rangle_{t'} \beta(s - t') dt', \quad (9.96)$$

where $z(t_0, t)$ is the random variable with statistics determined by the correlations in Eq. (9.72).

Now we have the correct noise function we can calculate the actual SSE. Following the same procedure as in the coherent case we obtain

$$\begin{aligned} d_t|\psi_z(t)\rangle = & \Big(-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + (\hat{L} - \langle\hat{L}\rangle_t) z(t,t)\Big)|\psi_z(t)\rangle \\ & -\frac{1}{|\tilde{\psi}_{\{q_k\}}(t)|}(\hat{L} + \hat{L}^\dagger - \langle \hat{L} + \hat{L}^\dagger\rangle_t)\int_{t_0}^{t} \beta(t-s)\frac{\delta}{\delta z(t,s)} ds |\bar{\psi}_z(t)\rangle \\ & + \frac{|\psi_z(t)\rangle}{|\tilde{\psi}_{\{q_k\}}(t)|}\langle\psi_z(t)|(\hat{L} + \hat{L}^\dagger - \langle \hat{L} + \hat{L}^\dagger\rangle_t)\int_{t_0}^{t} \beta(t-s)\frac{\delta}{\delta z(t,s)} ds |\bar{\psi}_z(t)\rangle. \end{aligned} \quad (9.97)$$

Again this is not a SSE until we make the ansatz defined in Eq. (9.85), which gives

$$\begin{aligned} d_t|\psi_z(t)\rangle = & \Big[-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + (\hat{L} - \langle\hat{L}\rangle_t) z(t,t) - (\hat{L} + \hat{L}^\dagger - \langle \hat{L} + \hat{L}^\dagger\rangle_t)\hat{Q}_z(t) \\ & + \Big\langle (\hat{L} + \hat{L}^\dagger - \langle \hat{L} + \hat{L}^\dagger\rangle_t)\hat{Q}_z(t)\Big\rangle_t\Big]|\psi_z(t)\rangle. \end{aligned} \quad (9.98)$$

This is the actual SSE for real-valued noise. All of the comments regarding the interpretation of the corresponding complex-valued noise SSE (9.55) carry over to this case.

9.1.4 Position-state unraveling

The position-state noise function

The last unraveling I am going to consider is the position-state unraveling. This unraveling is described by the observable

$$Z \equiv \{X_k\} = \Big\{\Big(\{x_k\}, \hat{\pi}_{\{x_k\}} = |\{x_k\}\rangle\langle\{x_k\}| \otimes \hat{1}_{\rm sys}\Big)\Big\}. \quad (9.99)$$

where $|\{x_k\}\rangle$ is the multi-mode x-state Eq. (4.130), that it is an eigenstate of the bath operators $\{\hat{X}_k\}$ where

$$\hat{X}_k = \frac{\hat{a}_k + \hat{a}_k^\dagger}{\sqrt{2}}. \quad (9.100)$$

Hence the name position unraveling. I would like to point out here that even through $|\{x_k\}\rangle$ is the eigenstate of the position operator it really is not a position state as we are in an interaction picture, and if we moved back to the Schrödinger picture this state would oscillate between position and momentum eigenstates. However for the purpose of this thesis the name position will suffice. For this unraveling I will define the noise operator as

$$\hat{z}(s) = \sum_k g_k\sqrt{2}\hat{X}_k e^{-i\Omega_k(s-t_0)}, \tag{9.101}$$

thus the noise function becomes

$$z(t,s) = \sum_k g_k\sqrt{2}r(X_k,t)e^{-i\Omega_k(s-t_0)}. \tag{9.102}$$

For the linear case I will assume the ostensible distribution is

$$\Lambda(\{x_k\}) = |\langle\{x_k\}|\{0_k\}\rangle|^2 = \prod_k \frac{\exp(-x_k^2)}{\sqrt{\pi}}, \tag{9.103}$$

which has the two bath correlation functions

$$\begin{aligned}\bar{E}[z(t,s)z^*(t,s')] &= \sum_{k,k'} g_k g_{k'}^* 2e^{-i\Omega_k(s-t_0)+i\Omega_{k'}(s'-t_0)}\bar{E}[\bar{r}(X_k,t)\bar{r}(X_{k'},t)] = \sum_k |g_k|^2 e^{-i\Omega_k(s-s')} \\ &= \alpha(s-s'), \qquad (9.104)\\ \bar{E}[z(t,s)z(t,s')] &= \sum_k g_k^2 e^{-i\Omega_k(s+s'-2t_0)} = \gamma(s+s'). \qquad (9.105)\end{aligned}$$

Here we see an example where both correlations are non-zero.

The linear non-Markovian SSE for the position-state unraveling

To derive the linear SSE for this unraveling we have to first rearrange the Schrödinger equation in terms of $\{\hat{X}_k\}$ and $\{\hat{Y}_k\}$. Doing this we get

$$d_t|\Psi(t)\rangle = \Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + \sum_k^K[\hat{L}g_k^* e^{i\Omega_k(t-t_0)}\frac{(\hat{X}_k - i\hat{Y}_k)}{\sqrt{2}} - \hat{L}^\dagger g_k e^{-i\Omega_k(t-t_0)}\frac{(\hat{X}_k + i\hat{Y}_k)}{\sqrt{2}}]\Big\}|\Psi(t)\rangle. \tag{9.106}$$

Then Eq. (9.12) for this unraveling gives

$$\begin{aligned}\partial_t|\bar{\psi}_{\{x_k\}}(t)\rangle = &\Big[-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + \sum_k \hat{L}g_k^* x_k\sqrt{2}e^{i\Omega_k(t-t_0)} - \sum_k \frac{(\hat{L}g_k^* e^{i\Omega_k(t-t_0)} + \hat{L}^\dagger g_k e^{-i\Omega_k(t-t_0)})}{\sqrt{2}} \\ &\times\partial_{x_k}\Big]|\bar{\psi}_{\{x_k\}}(t)\rangle, \qquad (9.107)\end{aligned}$$

where we have used $\langle\{x_k\}|\hat{X}_k|\Psi(t)\rangle = x_k\langle\{x_k\}|\Psi(t)\rangle$ and $\langle\{x_k\}|\hat{Y}_k|\Psi(t)\rangle = -i\partial_{x_k}\langle\{x_k\}|\Psi(t)\rangle$. Evaluating this for $\{x_k = \bar{r}(X_k)\}$ gives

$$\begin{aligned}\partial_t|\bar{\psi}_{\{\bar{r}(X_k)\}}(t)\rangle = &\Big[\sum_k \hat{L}g_k^*\bar{r}(X_k)\sqrt{2}e^{i\Omega_k(t-t_0)} - \sum_k \frac{(\hat{L}g_k^* e^{i\Omega_k(t-t_0)} + \hat{L}^\dagger g_k e^{-i\Omega_k(t-t_0)})}{\sqrt{2}}\frac{\partial}{\partial\bar{r}(X_k)} \\ &-\frac{i}{\hbar}\hat{H}_{\rm int}(t)\Big]|\bar{\psi}_{\{\bar{r}(X_k)\}}(t)\rangle. \qquad (9.108)\end{aligned}$$

To make it a SSE we need to replace the derivatives by operators, but before we do this, and to keep with past literature, it is convenient to define this in terms of the noise function $z(t,s)$ and functional derivatives. To do this we replace the κ partial derivatives by

$$\begin{aligned}\partial_{x_k} &= \int_{t_0}^{t}\Big[\frac{\delta}{\delta z(t,s)}\frac{\partial z(t,s)}{\partial r(X_k,t)}ds+\frac{\delta}{\delta z^*(t,s)}\frac{\partial z^*(t,s)}{\partial r(X_k,t)}ds\Big]\\ &= \int_{t_0}^{t}\sqrt{2}\Big[\frac{\delta}{\delta z(t,s)}g_k e^{-i\Omega_k(s-t_0)}ds+\frac{\delta}{\delta z^*(t,s)}g_k^* e^{i\Omega_k(s-t_0)}ds\Big].\end{aligned}\tag{9.109}$$

In terms of the noise function, Eq. (9.108) becomes

$$\begin{aligned}\partial_t|\bar{\psi}_z(t)\rangle &= \Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(t)+\hat{L}z^*(t,t)-\hat{L}\int_{t_0}^{t}\Big[\alpha^*(t-s)\delta_{z(t,s)}+\gamma^*(t+s)\delta_{z^*(t,s)}\Big]ds\\ &\quad -\hat{L}^\dagger\int_{t_0}^{t}\Big[\alpha(t-s)\delta_{z^*(t,s)}+\gamma(t+s)\delta_{z(t,s)}\Big]ds\Big\}|\bar{\psi}_z(t)\rangle\end{aligned}\tag{9.110}$$

This can be further simplified by noting that for an initial condition $|\Psi(t_0)\rangle = |\{0_k\}\rangle|\psi(t_0)\rangle$, the initial linear conditioned system state $|\bar{\psi}_z(t)\rangle$ is equal to $|\psi(t_0)\rangle$, independent of $z(t,s)$ and $z^*(t,s)$. Thus the functional derivatives with respect to $z(t,s)$ in the above of equation will always have zero contribution. Thus we can rewrite Eq. (9.110) as

$$\partial_t|\bar{\psi}_z(t)\rangle = \Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(t)+\hat{L}z^*(t,t)-\int_{t_0}^{t}\Big[\hat{L}\gamma^*(t+s)+\hat{L}^\dagger\alpha(t-s)\Big]\delta_{z^*(t,s)}ds\Big\}|\bar{\psi}_z(t)\rangle\tag{9.111}$$

This equation appears nicer than Eq. (9.108), but it is essentially the same equation and we still have the problem of representing the functional derivative by operators. To do this we make the two anstazen,

$$\hat{A}_z(t)|\bar{\psi}_z(t)\rangle = \int_{t_0}^{t} ds\alpha(t-s)\delta_{z^*(t,s)}|\bar{\psi}_z(t)\rangle,\tag{9.112}$$

$$\hat{B}_z(t)|\bar{\psi}_z(t)\rangle = \int_{t_0}^{t} ds\gamma^*(t+s)\delta_{z^*(t,s)}|\bar{\psi}_z(t)\rangle.\tag{9.113}$$

With these ansatzen the linear non-Markovian SSE for the position unraveling is

$$\partial_t|\bar{\psi}_z(t)\rangle = \Big[-\frac{i}{\hbar}\hat{H}_{\rm int}(t)+\hat{L}z^*(t,t)-\hat{L}\hat{B}_z(t)-\hat{L}^\dagger\hat{A}_z(t)\Big]|\bar{\psi}_z(t)\rangle.\tag{9.114}$$

The actual non-Markovian SSE for the position-state unraveling

To derive this equation we simply follow the same procedure as done for the other unravelings. Rather than presenting the full derivation I will simply outline only the important steps. To find the set of random variables $\{r(X_k,t)\}$ which obey the real distribution $P(\{x_k\},t)$, we simply differentiate the Girsanov transformation for this unraveling. Doing this it can be shown after some manipulating that

$$d_tP(\{x_k\},t) = -\sum_k \partial_{x_k}\Big\{\langle\psi_{\{x_k\}}(t)|[\hat{L}g_k^*e^{i\Omega_k(t-t_0)}+\hat{L}^\dagger g_k e^{-i\Omega_k(t-t_0)}]|\psi_{\{x_k\}}(t)\rangle P(\{x_k\},t)\Big\}/\sqrt{2}.\tag{9.115}$$

This is effectively a drift equation for the probability density. It has associated with it the following set of differential equations

$$d_tr(X_k,t) = [\langle\hat{L}\rangle_t g_k^* e^{i\Omega_k(t-t_0)}+\langle\hat{L}^\dagger\rangle_t g_k e^{-i\Omega_k(t-t_0)}]/\sqrt{2}\tag{9.116}$$

where $\langle \hat{L} \rangle_t$ is defined by Eq. (9.45).

Integrating this gives

$$r(X_k, t) = r(X_k, t_0) + \int_{t_0}^{t} dt' [\langle \hat{L} \rangle_{t'} g_k^* e^{i\Omega_k (t'-t_0)} + \langle \hat{L}^\dagger \rangle_{t'} g_k e^{-i\Omega_k (t'-t_0)}]/\sqrt{2} \tag{9.117}$$

where $r(X_k, t_0)$ is the random variable associated with the distribution $\langle \Psi(t_0)|\hat{\pi}_{\{x_k\}}|\Psi(t_0)\rangle$, which for an initial combined state of the form $|\Psi(t_0)\rangle = |\{0_k\}\rangle|\psi(t_0)\rangle$ is equivalent to the above ostensible distribution. In terms of the noise function defined in Eq. (9.102) the noise function for the real distribution becomes

$$z(t, s) = z(t_0, s) + \int_{t_0}^{t} dt' [\langle \hat{L} \rangle_{t'} \alpha(s - t') + \langle \hat{L}^\dagger \rangle_{t'} \gamma(s + t')] \tag{9.118}$$

where $z(t_0, s)$ obeys the correlations defined in Eqs. (9.104) and (9.105).

With this noise function and using Eq. (9.17) the actual non-Markovian SSE after some manipulation is

$$\begin{aligned} d_t|\psi_z(t)\rangle \;\; = \;\; & \Big[-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + (\hat{L} - \langle \hat{L} \rangle_t) z^*(t,t) - (\hat{L} - \langle \hat{L} \rangle_t)\hat{B}_z(t) + \Big\langle (\hat{L} - \langle \hat{L} \rangle_t)\hat{B}_z(t) \Big\rangle_t \\ & - (\hat{L}^\dagger - \langle \hat{L}^\dagger \rangle_t)\hat{A}_z(t) + \Big\langle (\hat{L}^\dagger - \langle \hat{L}^\dagger \rangle_t)\hat{A}_z(t) \Big\rangle_t \Big] |\psi_z(t)\rangle, \end{aligned} \tag{9.119}$$

where $|\psi_z(t)\rangle \equiv |\psi_{\{r(X_k,t)\}}(t)\rangle$. Here I have also used the anstazen defined in Eqs. (9.112) and (9.113).

9.2 Non-Markovian SSEs under the modal interpretation

Since orthodox quantum mechanics fails to give a satisfying interpretation for non-Markovian SSEs (something more than just numerical tools), in this section I will turn to a *non*-orthodox approach: the modal interpretation of Quantum Mechanics [11, 23, 126, 4, 121, 114, 62] (see chapter 4). This interpretation, unlike the orthodox interpretation, has as its basic goal to keep reality intact. That is, the values of some observables (the hidden variables) really exist before we measure them. Because of this, under this view I refer to these observables as "properties" or "beables" (after Bell [11]). Just as in the orthodox theory, where it is impossible to simultaneously measure all observables, in the modal theory it is impossible to give definite values to all properties (see chapters 3 and 4). In chapter 4 I came to the conclusion that under the modal view we have to accept choice, and say that when we have a particular situation we have to choose how we are going to decompose the wavefunction for the universe (system and bath). The best-known example of such an interpretation is Bohmian mechanics for particles [14, 15] in which position is the preferred decomposition.

I expect the modal interpretation to be applicable to non-Markovian SSEs because we can use it to assign definite properties to the bath for all t, *without* invoking a measurement. While the bath is ascribed definite properties, the system will be described as a purely quantum system. But, because of the entanglement between the system and the bath, we can define a system state associated with (or 'conditioned' on) a particular value for the bath property. I will show that the evolution of this system state is the non-Markovian SSE.

9.2.1 General derivation

In chapter 4 it was observed that to explain all of quantum measurement theory (projector and POM measurement) in terms of the modal interpretation of quantum mechanics, we need to enlarge the universe (add an auxiliary system). In this enlarged universe we could then define a general property by

$$Z(t) \equiv \{Z_k(t)\} = \Big\{\Big(\{z_k\}, \hat{\Pi}_{\{z_k\}}(t) = |\{z_k(t)\}\rangle\langle\{z_k(t)\}| \otimes \hat{1}_{\rm sys}\Big)\Big\}. \tag{9.120}$$

where the value of the property is denoted $\upsilon(Z_k, t)$. The probability that this set of values will be equal to $\{z_k\}$ is

$$P(\{z_k\}, t) = \langle\Phi(t)|\hat{\Pi}_{\{z_k\}}(t)|\Phi(t)\rangle. \tag{9.121}$$

where the guiding state $|\Phi(t)\rangle$ is defined by

$$|\Phi(t)\rangle = |\Psi(t)\rangle \otimes |\phi\rangle, \tag{9.122}$$

and $|\Psi(t)\rangle$ is still the solution to the Schrödinger equation. The property state (the actual state) is defined by

$$|\Phi_{\{z_k\}}(t)\rangle = \frac{\hat{\Pi}_{\{z_k\}}(t)|\Phi(t)\rangle}{\sqrt{P(\{z_k\}, t)}}. \tag{9.123}$$

As shown in chapter 4 the dynamics of this state (or the values $\upsilon(Z_k, t)$) can be calculated using the velocity operator technique [62] for certain Hamiltonians.

With $\hat{\Pi}_{\{z_k\}}(t) = |\{z_k(t)\}\rangle_{\rm env+aux}\langle\{z_k(t)\}| \otimes \hat{1}_{\rm sys}$ (or $\hat{\pi}_{\{z_k\}}(t) = |\{z_k(t)\}\rangle_{\rm env}\langle\{z_k(t)\}| \otimes \hat{1}_{\rm sys}$) the property states can be simplified to

$$|\Phi_{\{z_k\}}(t)\rangle = |\{z_k\}\rangle|\psi_{\{z_k\}}(t)\rangle, \tag{9.124}$$

where $|\psi_{\{z_k\}}(t)\rangle$ is called the conditioned system state. It receives this name because it lives entirely in $\mathcal{H}_{\rm sys}$ and is conditioned on the bath values $\{z_k\}$. Note conditioning under this now implies a conditioning on a property of the bath rather than an outcome of a measurement on the bath. The form of $|\psi_{\{z_k\}}(t)\rangle$ is

$$|\psi_{\{z_k\}}(t)\rangle = \langle\{z_k\}|\Psi(t)\rangle/\sqrt{N}, \tag{9.125}$$

where the normalization constant is defined as

$$N = \langle\Psi(t)|\{z_k\}\rangle\langle\{z_k\}|\Psi(t)\rangle. \tag{9.126}$$

This is completely equivalent to Eq. (9.4).

For an actual trajectory (in the sense of $|\psi_{\{r(Z_k,t)\}}(t)\rangle$), the bath values $\{\upsilon(Z_k, t)\}$ are time-dependent. This state becomes $|\psi_{\upsilon(Z_k,t)}(t)\rangle$ and represents the state of the system conditioned on the bath having this trajectory. That is, it is continuous in time and the differential equation of this state will represent its evolution. In section 9.1 it was shown that by starting with Eq. (9.125) (or Eq. (9.4)), the time derivative of this equation gives diffusive non-Markovian SSEs. Thus in this section I will not reproduce these derivations, but instead show that by using the velocity operator technique of section 4.4.3 we can re-derive the actual trajectories for $\{\upsilon(Z_k, t)\}$, which is $\{r(Z_k, t)\}$ in the orthodox view. That is I will derive the correct conditioning parameters without using the Girsanov transformation, thereby showing that diffusive non-Markovian SSEs have a modal interpretation. If fact, because the orthodox interpretation only gives an interpretation for the solutions of a non-Markovian SSE at a particular time (time of measurement), I believe that the only non-trivial interpretation of non-Markovian SSEs is a modal interpretation.

Before we consider specific decompositions (modal name for unravelings) I would like to note that the velocity field, Eq. (4.102), can be written in terms of the conditioned system state as

$$\mathrm{v}_j(\{z_k\}, t) = \mathrm{Re}[\langle\psi_{\{z_k\}}(t)|\overrightarrow{\hat{\mathrm{v}}_j(\{z_k\}, t)}|\psi_{\{z_k\}}(t)\rangle], \tag{9.127}$$

where

$$\overrightarrow{\hat{\mathrm{v}}_j(\{z_k\},t)}|\psi_{\{z_k\}}(t)\rangle \equiv \langle\{z_k\}|\hat{\mathrm{v}}_j(t)|\Psi(t)\rangle/\sqrt{N}. \tag{9.128}$$

and

$$\hat{\mathrm{v}}_j(t) = -\frac{i}{\hbar}[\hat{Z}_k, \hat{V}_{\mathrm{int}}(t)\otimes\hat{1}_{\mathrm{aux}}] \tag{9.129}$$

This results in the following differential equations for the bath values

$$d_t\upsilon(Z_j,t) = \mathrm{Re}[\langle\psi_{\{\upsilon(Z_k,t)\}}(t)|\overrightarrow{\hat{\mathrm{v}}_j(\{\upsilon(Z_k,t)\},t)}|\psi_{\{\upsilon(Z_k,t)\}}(t)\rangle], \tag{9.130}$$

where $|\psi_{\{\upsilon(Z_j,t)\}}(t)\rangle = |\psi_{\{z_k\}}(t)\rangle|_{\{z_k=\upsilon(Z_k,t)\}}$.

9.2.2 Position-state decomposition

The first decomposition I will consider under this view is the position-state. This is because this unraveling under this interpretation is the easiest to derive. It is effectively (in the interaction picture) Bohmian mechanics. To show that Eq. (9.130) does give the same trajectories for the values $\{r(X_k,t)\}$ as in Eq. (9.117), we apply the Hamiltonians defined in Eq. (6.14) to Eq. (9.129), with $\hat{Z}_k = \hat{X}_k$. This gives

$$\hat{\mathrm{v}}_k(t) = [g_k^* e^{i\Omega_k(t-t_0)}\hat{L} + g_k e^{-i\Omega_k(t-t_0)}\hat{L}^\dagger]/\sqrt{2}, \tag{9.131}$$

as $[\hat{x}_k,\hat{a}_k] = -1/\sqrt{2}$ and $[\hat{x}_k,\hat{a}_k^\dagger] = 1/\sqrt{2}$. Substituting this into Eq. (9.127) gives a velocity field of the form

$$\mathrm{v}_j(\{x_k\},t) = [g_k^* e^{i\Omega_k(t-t_0)}\langle\psi_{\{x_k\}}(t)|\hat{L}|\psi_{\{x_k\}}(t)\rangle + g_k e^{-i\Omega_k(t-t_0)}\langle\psi_{\{x_k\}}(t)|\hat{L}^\dagger|\psi_{\{x_k\}}(t)\rangle]/\sqrt{2}. \tag{9.132}$$

Thus Eq. (9.130) immediately reproduces Eq. (9.116) (with r replaced by υ), thereby confirming that the modal theory does give the same non-Markovian SSEs, as found with the orthodox theory.

9.2.3 Quadrature decomposition

To show that Eqs. (9.91) and (9.92) can be derived from the modal theory (velocity operator technique) we apply the Hamiltonians defined in Eq. (6.14) (with the required assumptions needed to generate the quadrature non-Markovian SSE) to Eq. (9.129). For this unraveling the set of velocity operators $\{\hat{\mathrm{v}}_k\}$ will be the union of $\{\hat{\mathrm{v}}_k^+\}$ and $\{\hat{\mathrm{v}}_k^-\}$, where

$$\hat{\mathrm{v}}_k^+(t) = -\frac{i}{\hbar}[\hat{X}_k^+, \hat{V}_{\mathrm{int}}] = g_k(\hat{L}+\hat{L}^\dagger)\cos[\Omega_k(t-t_0)], \tag{9.133}$$

$$\hat{\mathrm{v}}_k^-(t) = -\frac{i}{\hbar}[\hat{Y}_k^-, \hat{H}_{\mathrm{int}}] = g_k(\hat{L}+\hat{L}^\dagger)\sin[\Omega_k(t-t_0)], \tag{9.134}$$

which are both real by definition. Substituting these velocity operators into Eq. (9.127) gives

$$\mathrm{v}_k^+(\{X_k^+,Y_k^-\},t) = g_k\langle\psi_{\{X_k^+,Y_k^-\}}(t)|\hat{L}+\hat{L}^\dagger|\psi_{\{X_k^+,Y_k^-\}}(t)\rangle\cos(\Omega_k t), \tag{9.135}$$

$$\mathrm{v}_k^+(\{X_k^+,Y_k^-\},t) = g_k\langle\psi_{\{X_k^+,Y_k^-\}}(t)|\hat{L}+\hat{L}^\dagger|\psi_{\{X_k^+,Y_k^-\}}(t)\rangle\sin(\Omega_k t). \tag{9.136}$$

Thus Eq. (9.130) simply yields Eqs. (9.91) and (9.92). Thus the modal theory gives the correct non-Markovian SSE.

9.2.4 Coherent-state decomposition

To show that the modal theory can be used to describe the coherent non-Markovian SSE we have to find the projector in $\mathcal{K}$ which is equivalent to the POM elements defined in Eq. (9.21). In section 4.5.1 I showed that for a single mode this projector is $|x^+,y^-\rangle\langle x^+,y^-|$ where

$$|x^+,y^-\rangle = \int \frac{dx'}{\sqrt{2\pi}}\Big|\frac{x^+-x'}{\sqrt{2}}\Big\rangle_{\rm aux}\Big|\frac{x^++x'}{\sqrt{2}}\Big\rangle_{\rm uni} e^{iy^-x'} \tag{9.137}$$

(the states in the integrand are x-states). Thus the multi-mode projector used to define this unraveling is

$$\hat{\Pi}_{\{q_k\}} = \hat{\Pi}_{\{a_k\}} = |\{x_k^+,y_k^-\}\rangle\langle\{x_k^+,y_k^-\}|_{\rm bath+aux}\otimes\hat{1}_{\rm sys}, \tag{9.138}$$

where a_k is defined by

$$a_k = x_k^+ + iy_k^-. \tag{9.139}$$

This allows us to define the operator $\hat{A}_k$, such that

$$\hat{A}_k|x_k^+,y_k^-\rangle = a_k|x_k^+,y_k^-\rangle, \tag{9.140}$$

as

$$\hat{A}_k = \hat{x}_k^+ + i\hat{y}_k^- = \hat{a}_k + \hat{b}_k^\dagger. \tag{9.141}$$

which is a normal operator. Here $\hat{x}_k^+$ and $\hat{y}_k^-$ are defined as

$$\hat{x}_k^+ = [\hat{a}_k + \hat{a}_k^\dagger + \hat{b}_k + \hat{b}_k^\dagger]/2, \tag{9.142}$$
$$\hat{y}_k^- = [-i\hat{a}_k + i\hat{a}_k^\dagger + i\hat{b}_k - i\hat{b}_k^\dagger]/2, \tag{9.143}$$

where $\hat{b}_k$ and $\hat{b}_k^\dagger$ are annihilation and creation operators which act in $\mathcal{H}_{\rm aux}$. In this enlarged Hilbert space the velocity operators are

$$\hat{\rm v}_k^+(t) = -\frac{i}{\hbar}[\hat{x}_k^+,\hat{V}_{\rm int}\otimes\hat{1}_{\rm aux}] = [g_k^* e^{i\Omega_k(t-t_0)}\hat{L} + g_k e^{-i\Omega_k(t-t_0)}\hat{L}^\dagger]/2, \tag{9.144}$$
$$\hat{\rm v}_k^-(t) = -\frac{i}{\hbar}[\hat{y}_k^-,\hat{V}_{\rm int}\otimes\hat{1}_{\rm aux}] = [-ig_k^* e^{i\Omega_k(t-t_0)}\hat{L} + ig_k e^{-i\Omega_k(t-t_0)}\hat{L}^\dagger]/2\ . \tag{9.145}$$

With these velocity operators the velocity fields become

$${\rm v}_k^+(\{x_k^+,y_k^-\},t) = \langle\psi_{\{x_k^+,y_k^-\}}(t)|[g_k^* e^{i\Omega_k(t-t_0)}\hat{L} + g_k e^{-i\Omega_k(t-t_0)}\hat{L}^\dagger]|\psi_{\{x_k^+,y_k^-\}}(t)\rangle/2, \tag{9.146}$$
$${\rm v}_k^-(\{x_k^+,y_k^-\},t) = \langle\psi_{\{x_k^+,y_k^-\}}(t)|[-ig_k^* e^{i\Omega_k(t-t_0)}\hat{L} + ig_k e^{-i\Omega_k(t-t_0)}\hat{L}^\dagger]|\psi_{\{x_k^+,y_k^-\}}(t)\rangle/2. \tag{9.147}$$

Substituting these into Eq. (9.130) gives

$$d_t\upsilon(x_k^+,t) = [\langle\hat{L}\rangle_t g_k^* e^{i\Omega_k(t-t_0)} + \langle\hat{L}^\dagger\rangle_t g_k e^{-i\Omega_k(t-t_0)}]/2, \tag{9.148}$$
$$d_t\upsilon(y_k^-,t) = [-i\langle\hat{L}\rangle_t g_k^* e^{i\Omega_k(t-t_0)} + i\langle\hat{L}^\dagger\rangle_t g_k e^{-i\Omega_k(t-t_0)}]/2. \tag{9.149}$$

Since $a_k = x_k^+ + iy_k^-$, $\upsilon(A_k,t) = \upsilon(x_k^+,t) + i\upsilon(y_k^-,t)$, which once again easily yeilds Eq. (9.49).

9.3 The Markovian limit of non-Markovian SSEs

In this section I consider the Markovian limit of the above three non-Markovian SSE. I will show that the coherent state unraveling (or decomposition) corresponds to the heterodyne quantum trajectory (or Gisin's and Percival's CSL model), the quadrature state unraveling corresponds to the homodyne quantum trajectory (or Ghirardi's, Pearle's, and Rimini's CSL model) and the position-state unraveling has no Markovian limit.

9.3.1 Coherent-state case

In the Markovian limit the number of modes becomes continuous and the coupling constant $|g_k|$ becomes flat ($|g_k| = g$), this allows us to replace $\sum_k^\kappa$ by $(\gamma/2\pi)\int_0^\infty d\omega$. Making this replacement to $\alpha(t-s)$ gives

$$\begin{aligned}\alpha(t-s) &= \frac{\gamma}{2\pi}\int_0^\infty e^{-i(\omega-\omega_0)(t-s)}d\omega \\ &= \frac{\gamma}{2\pi}\int_{-\omega_{\rm sys}}^\infty e^{-i\Omega(t-s)}d\Omega, \end{aligned} \tag{9.150}$$

and for optical situations (high $\omega_{\rm sys}$ situations) with little error this can be written as

$$\alpha(t-s) = \frac{\gamma}{2\pi}\int_{-\infty}^\infty e^{-i\Omega(t-s)}d\Omega = \gamma\delta(t-s). \tag{9.151}$$

Therefore,

$$\bar{E}[z(t_0,s)z^*(t_0,s')] = \gamma\delta(s-s'), \tag{9.152}$$

$$\bar{E}[z(t_0,s)z(t_0,s')] = 0. \tag{9.153}$$

This implies that $z(t_0,s)$ is a complex gaussian random variable (GRV) of mean 0 and variance γ/dt. That is, $z(t_0,s) = \sqrt{\gamma}\,\xi(s)$, where $\xi(s)$ is the standard complex white noise function [64].

Taking the Markovian limit of the noise function (Eq. (9.52)), one obtains

$$z^*(t,t) = z^*(t_0,t) + \int_{t_0}^t \gamma\delta(t-t')\langle\hat{L}^\dagger\rangle_{t'}dt' = z^*(t_0,t) + \frac{\gamma}{2}\langle\hat{L}^\dagger\rangle_t. \tag{9.154}$$

To apply the Markovian limit to Eq. (9.55) we use $\alpha(t-s)\to\gamma\delta(t-s)$ and $\hat{c}_z(t,t) = \hat{L}$ (see appendix A), resulting in

$$d_t|\psi_z(t)\rangle = \Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + (\hat{L}-\langle\hat{L}\rangle_t)(z^*(t,t)+\frac{\gamma}{2}\langle\hat{L}^\dagger\rangle_t) - \frac{\gamma}{2}(\hat{L}^\dagger\hat{L}-\langle\hat{L}^\dagger\hat{L}\rangle_t)\Big\}|\psi_z(t)\rangle, \tag{9.155}$$

which is in Stratonovich form (as the natural limit of a delta function yields Stratonovich SSE). To convert this to an Itô SSE we have to calculate the Itô correction terms, using appendix B.2. For this equation the correction term is

$$\frac{dt\gamma}{2}\Big(-\langle\hat{L}^\dagger\hat{L}\rangle_t + \langle\hat{L}^\dagger\rangle_t\langle\hat{L}\rangle_t\Big)|\psi_z(t)\rangle, \tag{9.156}$$

which with Eq. (9.155) results in

$$d_t|\psi_z(t)\rangle = \Big\{-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + (\hat{L}-\langle\hat{L}\rangle_t)z^*(t,t) - \frac{\gamma}{2}(\hat{L}^\dagger\hat{L}-\hat{L}\langle\hat{L}^\dagger\rangle_t)\Big\}|\psi_z(t)\rangle. \tag{9.157}$$

This is the Itô version of the Markovian SSE. At first sight it appears to be different to the heterodyne quantum trajectory Eq. (7.90) (and Gisin's and Percival's CSL model Eq. (5.50)), but we have to remember that in quantum mechanics if $|\psi\rangle = e^{i\theta}|\phi\rangle$, $|\psi\rangle$ and $|\phi\rangle$ are the same physical state. One way to see if these states are equivalent is to work out the stochastic master equation (SME), as $\rho_z(t) = |\psi\rangle\langle\psi| = |\varphi\rangle\langle\varphi|$. To do this we use Itô calculus,

$$d\rho_z = d|\psi\rangle\langle\psi| + |\psi\rangle d\langle\psi| + d|\psi\rangle d\langle\psi|. \tag{9.158}$$

The SME for Eq. (9.157) is

$$\begin{aligned} d\rho_z(t) &= -\frac{i}{\hbar}[\hat{H}_{\rm int}(t), \rho_z(t)]dt + \hat{\mathcal{D}}\rho_z(t)dt + (\hat{L}\rho_z(t) - \langle\hat{L}\rangle_t\rho_z(t))(z^*(t,t) - \frac{\gamma}{2}\langle\hat{L}^\dagger\rangle_t)dt \\ &\quad +(\rho_z(t)\hat{L}^\dagger - \rho_z\langle\hat{L}^\dagger\rangle_t)(z(t,t) - \frac{\gamma}{2}\langle\hat{L}\rangle_t)dt, \end{aligned} \tag{9.159}$$

where the superoperator $\hat{\mathcal{D}}$ is defined in Eq. (6.33). Whereas the SME for Eq. (7.90)) is,

$$\begin{aligned} d\rho_{\mathbf{I}}(t) &= -\frac{i}{\hbar}[\hat{H}_{\rm int}(t), \rho_{\mathbf{I}}(t)]dt + \hat{\mathcal{D}}\rho_{\mathbf{I}}(t)dt + \sqrt{\gamma}\,(\hat{L}\rho_{\mathbf{I}}(t) - \langle\hat{L}\rangle_t\rho_{\mathbf{I}}(t))(r(I^*, t+dt) \\ &\quad -\sqrt{\gamma}\,\langle\hat{L}^\dagger\rangle_t)dt + \sqrt{\gamma}\,(\rho_{\mathbf{I}}(t)\hat{L}^\dagger - \rho_{\mathbf{I}}(t)\langle\hat{L}^\dagger\rangle_t)(r(I, t+dt) - \sqrt{\gamma}\,\langle\hat{L}\rangle_t)dt, \end{aligned} \tag{9.160}$$

which when we consider Eqs. (9.154) and (7.96) we see that both these equations give the same stochastic master equation.

To explain why the $z(t,t)$ and the heterodyne current $(\times\sqrt{\gamma}\,)$ differ by a factor of 1/2 multiplying the deterministic contribution, we have to consider the time of measurement. $z(t,t)$, according to the above theory, is the result of measuring the bath at time t in the coherent state basis. But in the usual quantum trajectory theory we must consider the measurement which conditions the state at time t as actually occurring at a time $t + dt$. That is, the δ-correlated bath must be given a chance to interact with the system before the measurement is made. By contrast, in the above theory the measurement occurs exactly at time t. For a non-Markovian bath (with a finite correlation time) the difference between t and $t + dt$ is infinitesimal. However in the Markovian limit, this infinitesimal difference in measurement time causes the finite difference between $z(t,t)$ and $r(I, t + dt)$.

The easiest way to see the difference is to note that $\sqrt{\gamma}\,r(I^*, t+dt) = z^*(t+dt, t)$. To show this we use Eq. (9.52) to write $z^*(t + dt, t)$ as

$$z^*(t+dt, s) = z^*(t_0, s) + \int_{t_0}^{t+dt} \alpha^*(s-t')\langle\hat{L}^\dagger\rangle_{t'}dt', \tag{9.161}$$

which in the Markovian limit gives

$$z^*(t+dt, s) = z^*(t_0, s) + \int_{t_0}^{t+dt} \gamma\delta(s-t')\langle\hat{L}^\dagger\rangle_{t'}dt' = z^*(t_0, s) + \gamma\langle\hat{L}^\dagger\rangle_s \tag{9.162}$$

which evaluated at time $s = t$ gives

$$z^*(t+dt, t) = z^*(t_0, t) + \gamma\langle\hat{L}^\dagger\rangle_t. \tag{9.163}$$

9.3.2 Quadrature case

For the quadrature case the symmetry assumptions needed to derive the quadrature noise function $z(t,s)$ (Eq. (9.70)) are compatible with the Markovian limit in which the modes become continuous and the coupling constant becomes flat in k-space (which of cause is symmetric around ω_{sys}). The memory function $\beta(t,s)$ under the Markovian limit becomes,

$$\beta(t,s) \quad = \quad 2\int_{\omega_0}^{\infty} g^2 \cos\Big((\omega-\omega_0)(t,s)\Big)d\omega = \int_0^{\infty} \frac{\gamma}{\pi}\cos\Big(\Omega(t,s)\Big)d\Omega = \gamma\delta(t-s). \quad (9.164)$$

Therefore in this limit the noise function is ostensibly given by $z(t_0,s) = \sqrt{\gamma}\,\xi(t)$ where $\xi(t)$ is a real-valued Gaussian white noise term [64].

Taking the Markovian limit of Eq. (9.96) gives

$$z(t,t) = z(t_0,t) + \int_{t_0}^{t} \gamma\delta(t-t')\langle \hat{L}+\hat{L}^\dagger\rangle_t = z(t_0,t) + \frac{\gamma}{2}\langle \hat{L}+\hat{L}^\dagger\rangle_t, \quad (9.165)$$

where $z(t_0,t) = \sqrt{\gamma}\,\xi(t)$. To apply the Markovian limit to Eq. (9.98) we use $\beta(t-s) \to \gamma\delta(t-s)$ and $\hat{q}_z(t,t) = \hat{L}$ (see appendix A), this results in

$$\begin{aligned} d_t|\psi_z(t)\rangle \quad = \quad & \Big\{ -\frac{i}{\hbar}\hat{H}_{\text{int}}(t) + (\hat{L}-\langle\hat{L}\rangle_t)[z(t,t) + \frac{\gamma}{2}\langle\hat{L}+\hat{L}^\dagger\rangle_t] - \frac{\gamma}{2}[(\hat{L}+\hat{L}^\dagger)\hat{L} \\ & -\langle(\hat{L}+\hat{L}^\dagger)\hat{L}\rangle_t]\Big\}|\psi_z(t)\rangle, \end{aligned} \quad (9.166)$$

This is in Stratonovich form. To compare it to the equivalent homodyne SSE we need to convert it to Itô form. The Itô correction term for this equation is (see appendix B.1)

$$\frac{dt}{2}\sum_l \Big(b_l\frac{\partial}{\partial\psi_l}b_j + b_l^*\frac{\partial}{\partial\psi_l^*}b_j\Big) = \frac{dt\gamma}{2}\Big(\hat{L}\hat{L} - 2\hat{L}\langle\hat{L}\rangle_t - \langle\hat{L}_x\hat{L}\rangle_t + \langle\hat{L}_x\rangle_t\langle\hat{L}\rangle_t + \langle\hat{L}\rangle_t\langle\hat{L}\rangle_t\Big)|\psi_z(t)\rangle. \quad (9.167)$$

Adding this to the Stratonovich SSE we get the following Itô SSE,

$$d|\psi_z(t)\rangle = dt\Big\{-\frac{i}{\hbar}\hat{H}_{\text{int}}(t) + (\hat{L}-\langle\hat{L}\rangle_t)z(t,t) - \frac{\gamma}{2}\Big(\hat{L}^\dagger\hat{L} - \hat{L}\langle\hat{L}^\dagger\rangle_t + \hat{L}\langle\hat{L}\rangle_t - \langle\hat{L}\rangle_t\langle\hat{L}\rangle_t\Big)\Big\}|\psi_z(t)\rangle. \quad (9.168)$$

At first glance this is different to Eq. (7.106), but as in the coherent case this does not mean these are not the same equation. To show that they are the same, we calculate the SME for both equations. The SME for equation (9.168) is

$$d\rho_z(t) \quad = \quad -\frac{i}{\hbar}[\hat{H}_{\text{int}}(t),\rho_z(t)]dt + \hat{\mathcal{D}}\rho_z(t)dt + dt\hat{\mathcal{H}}\rho_z(t)(z(t,t) - \frac{\gamma}{2}\langle\hat{L}+\hat{L}^\dagger\rangle_t), \quad (9.169)$$

where

$$\hat{\mathcal{H}}\rho_z(t) \quad = \quad \hat{L}\rho_z(t) + \rho_z(t)\hat{L}^\dagger - \langle\hat{L}+\hat{L}^\dagger\rangle\rho_z(t). \quad (9.170)$$

The SME for Eq. (7.106) is,

$$d\rho_{\mathbf{I}}(t) = -\frac{i}{\hbar}[\hat{H}_{\text{int}}(t),\rho_{\mathbf{I}}(t)]dt + \hat{\mathcal{D}}\rho_{\mathbf{I}}(t)dt + dt\sqrt{\gamma}\,\hat{\mathcal{H}}\rho_{\mathbf{I}}(r(I,t+dt) - \sqrt{\gamma}\,\langle\hat{L}+\hat{L}^\dagger\rangle_t), \quad (9.171)$$

which when we consider Eqs. (9.165) and (7.108) we see that these are the same stochastic master equation. As in the coherent case the difference between $z(t,t)$ and the homodyne current comes down to the fact the in the quantum trajectory theory the measurement occurs a time dt later.

9.3.3 Position-state case

The position-state non-Markovian SSE is an example of a non-Markovian SSE that does not have a Markovian limit. The reason for this is that the second correlation function $\gamma(s+s')$, defined in Eq. (9.105), in the Markovian limit is not well defined.

9.4 Applications

In this section I am going to apply the non-Markovian SSEs to two simple systems, these being a TLA coupled linearly to one and two single mode harmonic oscillators respectively.

9.4.1 A TLA coupled linear to one harmonic oscillators

The first application I am going to consider is a TLA coupled linearly to a single mode harmonic oscillator ($\kappa = 1$) with no detuning. That is, the interaction hamiltonian is

$$\hat{V}_{\text{int}} = i\hbar[g^* \hat{a}^\dagger \hat{\sigma} - g\hat{a}\hat{\sigma}^\dagger], \tag{9.172}$$

where $\hat{\sigma}$ and $\hat{a}$ are the lowering operators for the TLA (system) and the single mode (bath) respectively.

Exact dynamics for the non-Markovian master equation

To get an exact solution for comparison we first of all solve the total Schrödinger equation for the combined state. For this system the Schrödinger equation in the interaction frame is

$$d_t|\Psi(t)\rangle = (g^* \hat{\sigma}\hat{a}^\dagger - g\hat{\sigma}^\dagger \hat{a})|\Psi(t)\rangle. \tag{9.173}$$

The combined state can be written in terms of photon number states and atomic states as

$$|\Psi(t)\rangle = \sum_{s=e,b} \sum_{n} c_{s,n}|s\rangle|n\rangle, \tag{9.174}$$

where $|e\rangle$ and $|b\rangle$ are the excited and ground states of the TLA (thus $|e\rangle \equiv |\uparrow\rangle$ and $|b\rangle \equiv |\downarrow\rangle$ in the spin-1/2 model presented in chapter 3). Substituting this into Eq. (9.173) and using the fact that since the bath state is initially in a vacuum state the only nonzero amplitudes are $c_{e,0}(t)$, $c_{b,0}(t)$ and $c_{b,1}(t)$, we get the following differential equations

$$\begin{aligned} d_t c_{e,0}(t) &= -g c_{b,1}(t), & (9.175)\\ d_t c_{b,0}(t) &= 0, & (9.176)\\ d_t c_{b,1}(t) &= g^* c_{e,0}(t). & (9.177) \end{aligned}$$

These have the solutions

$$\begin{aligned} c_{e,0}(t) &= c_{e,0}(t_0)\cos[|g|(t-t_0)], & (9.178)\\ c_{b,0}(t) &= c_{b,0}(t_0), & (9.179)\\ c_{b,1}(t) &= c_{e,0}(t_0)\sin[|g|(t-t_0)]e^{-i\theta}, & (9.180) \end{aligned}$$

where θ is the argument of the coupling constant g and $|\psi(t_0)\rangle = c_{e,0}(t_0)|e\rangle + c_{b,0}(t_0)|b\rangle$. Thus the total state is

$$|\Psi(t)\rangle = c_{e,0}(t_0)\cos[|g|(t-t_0)]|e\rangle|0\rangle + c_{b,0}(t_0)|b\rangle|0\rangle + c_{e,0}(t_0)\sin[|g|(t-t_0)]e^{-i\theta}|b\rangle|1\rangle, \tag{9.181}$$

and the reduced state is simply

$$\begin{aligned} \rho_{\text{red}}(t) &= \Big[c_{e,0}^2(t_0)\cos^2[|g|(t-t_0)]|e\rangle\langle e| + \{c_{b,0}^2(t_0) + c_{e,0}^2(t_0)\sin^2[|g|(t-t_0)]\}|b\rangle\langle b| \\ &\quad + c_{e,0}(t_0)c_{b,0}^*(t_0)\cos[|g|(t-t_0)]|e\rangle\langle b| + c_{e,0}^*(t_0)c_{b,0}(t_0)\cos[|g|(t-t_0)]|b\rangle\langle e|\Big]. & (9.182)\\ & & (9.183) \end{aligned}$$

To graphically illustrate the reduced state the above was simulated for an excited state initial condition. This result in Bloch representation $x(t)$, $y(t)$, and $z(t)$ is shown in figure 9.1

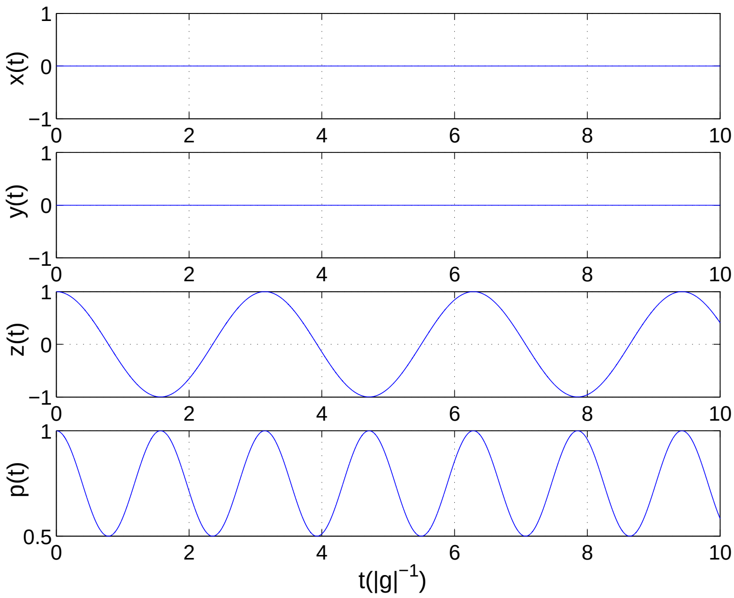

Figure 9.1: Solution of the Master equation for a TLA coupled to a single mode bath for an excited state initial condition.

Coherent-state case

The first unraveling I am going to consider is the coherent-state case. Here the noise function, Eq. (9.23), becomes

$$z(t,s) = gr(A,t), \tag{9.184}$$

and the memory function becomes

$$\alpha(t-s) = |g|^2. \tag{9.185}$$

Applying Eq. (9.55) to this model gives the following actual non-Markovian SSE,

$$d_t|\psi_z(t)\rangle = \Big[z^*(t,t)(\hat{\sigma} - \langle\hat{\sigma}\rangle_t) - (\hat{\sigma}^\dagger - \langle\hat{\sigma}^\dagger\rangle_t)\hat{C}_z(t) + \Big\langle(\hat{\sigma}^\dagger - \langle\hat{\sigma}^\dagger\rangle_t)\hat{C}_z(t)\Big\rangle_t\Big]|\psi_z(t)\rangle, \tag{9.186}$$

where

$$\hat{C}_z(t) = \int_0^t |g|^2 \hat{c}_z(t,s) ds. \tag{9.187}$$

To determine $\hat{c}_z(t,s)$ let's assume that we can rewrite the ansatz in Eq. (9.38) as

$$\frac{\delta}{\delta z^*(t,s)}|\bar{\psi}_z(t)\rangle = c_z(t,s)\hat{\sigma}|\bar{\psi}_z(t)\rangle, \tag{9.188}$$

That is we have assumed that $\hat{C}_z(t)$ is $C_z(t)\hat{\sigma}$, where

$$C_z(t) = \int_0^t |g|^2 c_z(t,s) ds. \tag{9.189}$$

The procedure used to evaluate this function is to rewrite the above in terms of a non-linear differential equation in terms of only $C_z(t)$. Differentiating Eq. (9.189) gives

$$d_t C_z(t) = |g|^2 c_z(t,t) + \int_0^t |g|^2 \partial_t c_z(t,s) ds, \tag{9.190}$$

where $c_z(t,t)=1$ (see appendix A).

To work out the function $\partial_t c_z(t,s)$ we have to use the linear non-Markovian SSE (Eq. (9.39)), which for this system is

$$\partial_t|\bar{\psi}_z(t)\rangle = \Big[z^*(t,t)\hat{\sigma} - \hat{\sigma}^\dagger\hat{C}_z(t)\Big]|\bar{\psi}_z(t)\rangle, \tag{9.191}$$

as well as the following consistency condition

$$\frac{\partial}{\partial t}\frac{\delta}{\delta z^*(t,s)}|\bar{\psi}_z(t)\rangle = \frac{\delta}{\delta z^*(t,s)}\frac{\partial}{\partial t}|\bar{\psi}_z(t)\rangle. \tag{9.192}$$

Evaluating the RHS of this condition gives

$$\frac{\delta}{\delta z^*(t,s)}\frac{\partial}{\partial t}|\bar{\psi}_z(t)\rangle = \Big[z^*(t,t)c_z(t,s)\hat{\sigma}\hat{\sigma} - c_z(t,s)\hat{\sigma}^\dagger\hat{C}_z(t)\hat{\sigma} - \hat{\sigma}^\dagger\frac{\delta}{\delta z^*(t,s)}\hat{C}_z(t)\Big]|\bar{\psi}_z(t)\rangle = 0, \tag{9.193}$$

Whereas the LHS gives

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\delta}{\delta z^*(t,s)}|\bar{\psi}_z(t)\rangle &= \frac{\partial}{\partial t}[c_z(t,s)]\hat{\sigma}|\bar{\psi}_z(t)\rangle + c_z(t,s)\hat{\sigma}\frac{\partial}{\partial t}[|\bar{\psi}_z(t)\rangle] \\ &= \Big[\partial_t[c_z(t,s)]\hat{\sigma} - C_z(t)c_z(t,s)\hat{\sigma}\hat{\sigma}^\dagger\hat{\sigma}\Big]|\bar{\psi}_z(t)\rangle. \end{aligned} \tag{9.194}$$

Thus

$$\partial_t[c_z(t,s)] = C_z(t)c_z(t,s). \tag{9.195}$$

Substituting this into Eq. (9.190) gives

$$d_tC_z(t) = |g|^2 + C_z(t)^2. \tag{9.196}$$

For the initial conditions $C_z(t_0)=0$, this has the solution $C_z(t)=|g|\tan[|g|(t-t_0)]$. Thus

$$\hat{C}_z(t) = |g|\tan[|g|(t-t_0)]\hat{\sigma}. \tag{9.197}$$

With this operator we can now evaluate both the linear and actual non-Markovian SSE, Eqs. (9.191) and (9.186) respectively. For the linear case the ostensible distribution is

$$\Lambda(a) = \frac{1}{\pi}\exp(-|a|^2). \tag{9.198}$$

Thus $r(A,t)=\bar{r}(A)$ for the linear case is a time independent complex Gaussian random variable of mean 0 and variance 1. For a TLA the general solution to $|\bar{\psi}_z(t)\rangle$ is,

$$|\bar{\psi}_z(t)\rangle = c_b(t)|b\rangle + c_e(t)|e\rangle. \tag{9.199}$$

Using this general expansion and Eq. (9.191), we can obtain two differential equations for the two complex amplitudes, they are,

$$d_tc_b(t) = z^*c_e(t) \tag{9.200}$$

$$d_tc_e(t) = -C_z(t)c_e(t), \tag{9.201}$$

which have solutions,

$$c_b(t) = c_b(t_0) + z^*c_e(t_0)\sin[|g|\ (t-t_0)] \tag{9.202}$$

$$c_e(t) = c_e(t_0)\cos[|g|\ (t-t_0)]. \tag{9.203}$$

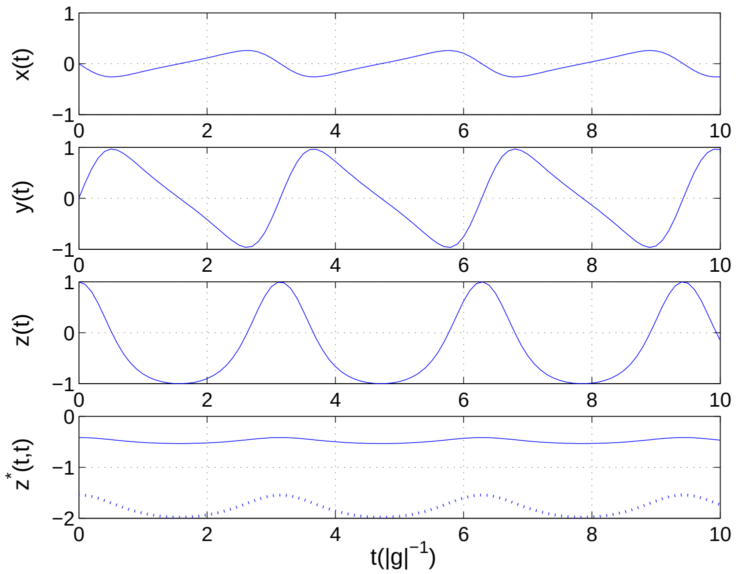

Figure 9.2: An example trajectory for the coherent-state non-Markovian SSE for a TLA in a single mode bath. Also shown is the real (solid) and imaginary (dotted) part of the noise function. Other details are as in figure 9.1.

To show that the ensemble average of the linear SSEs for this unraveling does average to the reduced state, 1000 trajectories were calculated and the difference between their ensemble average and the master equation is shown in figure 9.3 as a red line (with the standard error). Here we see that it agrees well with the master equation.

To work out the actual non-Markovian SSE for this unraveling we apply Eq. (9.199) to Eq. (9.186). Doing this we obtain the two (non-linear) differential equations

$$d_t c_b(t) = C_z(t) c_b(t) |c_e(t)|^2 [2 - |c_b(t)|^2] + c_e(t)[1 - |c_b(t)|^2] z^*(t,t), \tag{9.204}$$

$$d_t c_e(t) = C_z(t) c_e(t) [|c_e(t)|^2 - 1 - |c_b(t)|^2 |c_e(t)|^2] - c_b^*(t) c_e^2(t) z^*(t,t), \tag{9.205}$$

and the noise function $z^*(t,t) = g^* r(A^*, t)$ for the actual non-Markovian SSE is defined by (see Eq. (9.50))

$$r(A^*, t) = r(A^*, t_0) + g \int_{t_0}^{t} c_b(s) c_e^*(s) ds \tag{9.206}$$

where $r(A^*, t_0)$ is a random variable chosen from the initial probability distribution, which for initial conditions $|\Psi(0)\rangle = |\psi(0)\rangle|0\rangle$ is just Eq. (9.198), the ostensible distribution. Choosing an excited state initial conditions (and $dt = 0.0001$) an example trajectory for the coherent-state non-Markovian SSE is shown in figure 9.2. To show that the ensemble average of these trajectories is the reduced state, the average of 1000 trajectories, subtracted from the master equation, is shown in figure 9.3 as a blue line. Here, as in the linear case, we see that they do average to the reduced state.

Quadrature case

We can not apply this model to this unraveling as the symmetric requirements of the bath can not be met with only a single mode bath.

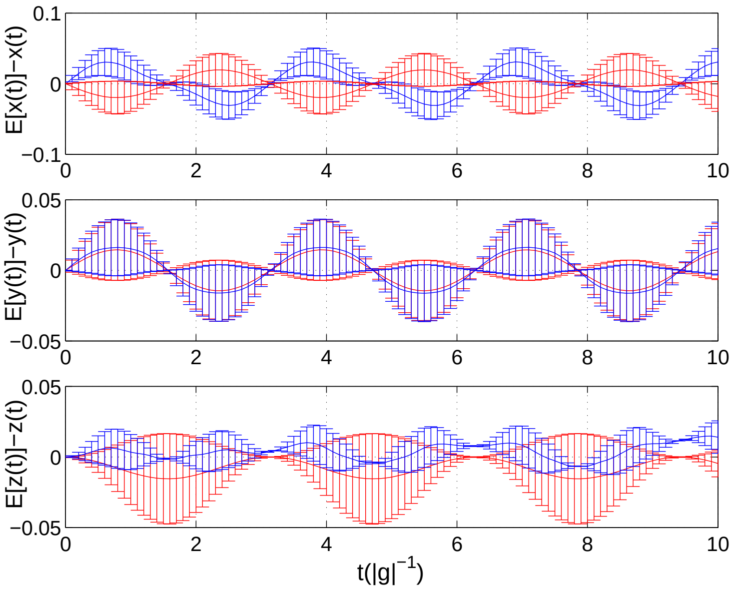

Figure 9.3: The difference between the ensemble average of 1000 trajectories (linear (red) and actual (blue)) and the master equation for the coherent unraveling. Other details are as in figure 9.1.

Position-state case

The general non-Markovian SSE for the position-state unraveling, defined in Eq. (9.119), for this simple system is

$$\begin{aligned} d_t|\psi_z(t)\rangle &= \Big\{(\hat\sigma-\langle\hat\sigma\rangle_t)z^*(t,t)-(\hat\sigma-\langle\hat\sigma\rangle_t)\hat B_z(t)+\langle(\hat\sigma-\langle\hat\sigma\rangle_t)\hat B_z(t)\rangle_t \\ &\quad -(\hat\sigma^\dagger-\langle\hat\sigma^\dagger\rangle_t)\hat A_z(t)+\langle(\hat\sigma^\dagger-\langle\hat\sigma^\dagger\rangle_t)\hat A_z(t)\rangle_t\Big\}|\psi_z(t)\rangle, \end{aligned} \tag{9.207}$$

where

$$z(t,t) = g\sqrt{2}r(X,t) \tag{9.208}$$

and the two memory functions become

$$\alpha(t-s) = |g|^2, \tag{9.209}$$

$$\gamma(t+s) = g^2. \tag{9.210}$$

To find the values of the two operators $\hat A_z(t)$ and $\hat B_z(t)$ we use Eqs. (9.112) and (9.113) where it is observed that a further ansatz is needed. We assume

$$\frac{\delta}{\delta z^*(t,s)}|\bar\psi_z(t)\rangle = f(t,s)\hat\sigma|\bar\psi_z(t)\rangle. \tag{9.211}$$

This results in $\hat A_z(t) = A_z(t)\hat\sigma$ and $\hat B_z(t) = B_z(t)\hat\sigma$ where

$$A_z(t) = \int_{t_0}^{t} ds|g|^2 f(t,s), \tag{9.212}$$

$$B_z(t) = \int_{t_0}^{t} dsg^{*2} f(t,s). \tag{9.213}$$

Taking the time derivative of these equations gives

$$d_t A_z(t) = |g|^2 f(t,t) + \int_{t_0}^t |g|^2 \partial_t f(t,s) ds, \quad (9.214)$$

$$d_t B_z(t) = g^{*2} f(t,t) + \int_{t_0}^t g^{*2} \partial_t f(t,s) ds, \quad (9.215)$$

where $f(t,t) = 1$ (this can be shown by using the same method as used in appendix A). To find the form of $\partial_t f(t,s)$ we use the linear non-Markovian SSE,

$$\partial_t |\bar{\psi}_z(t)\rangle = \Big[\hat{L} z^*(t,t) - \hat{\sigma}\hat{B}_z(t) - \hat{\sigma}^\dagger \hat{A}_z(t)\Big] |\bar{\psi}_z(t)\rangle, \quad (9.216)$$

and the following consistency conditions

$$\frac{\partial}{\partial t}\frac{\delta}{\delta z^*(t,s)}|\bar{\psi}_z(t)\rangle = \frac{\delta}{\delta z^*(t,s)}\frac{\partial}{\partial t}|\bar{\psi}_z(t)\rangle. \quad (9.217)$$

For this simple system this gives the following coupled differential equation

$$\partial_t f(t,s) = f(t,s) A_z(t), \quad (9.218)$$

which when substituted into Eqs. (9.214) and (9.215) gives

$$d_t A_z(t) = |g|^2 + A_z^2(t), \quad (9.219)$$

$$d_t B_z(t) = g^{*2} + B_z(t) A_z(t). \quad (9.220)$$

Solving this set of coupled differential equation with the initial condition $A_z(t_0) = B_z(t_0) = 0$ gives

$$A_z(t) = |g| \tan[|g|(t - t_0)], \quad (9.221)$$

$$B_z(t) = e^{-i2\theta} |g| \tan[|g|(t - t_0)]. \quad (9.222)$$

With these operators we can now evaluate both the linear and actual non-Markovian SSE, Eqs. (9.216) and (9.207) respectively. For the linear case the ostensible distribution is

$$\Lambda(x) = \frac{1}{\sqrt{\pi}} \exp(-x^2), \quad (9.223)$$

which implies that $r(X,t) = \bar{r}(X)$; a time independent Gaussian random variable of mean 0 and variance 1/2. Using Eq. (9.199) and Eq. (9.216), we can obtain two differential equations for the two complex amplitudes,

$$d_t c_b(t) = z^* c_e(t) \quad (9.224)$$

$$d_t c_e(t) = -A_z(t) c_e(t). \quad (9.225)$$

These have the same solution as the coherent case (namely Eqs. (9.202) and (9.203)). To show that the ensemble average of the linear SSE for the position-state unraveling does average to the reduced state, 1000 trajectories were calculated and the difference between their ensemble average and the master equation is shown in figure 9.5 as a red line, which to within statistical error agrees with the master equation.

To work out the actual non-Markovian SSE for this unraveling we apply Eq. (9.199) to Eq. (9.207). Doing this we obtain the two (non-linear) differential equations

$$\begin{aligned} d_t c_b(t) &= A_z(t) c_b(t) |c_e(t)|^2 [2 - |c_b(t)|^2] + c_e(t)[1 - |c_b(t)|^2] z^*(t,t) \\ &\quad + B_z(t) c_b^*(t) c_e^2(t) [1 - |c_b(t)|^2], \end{aligned} \quad (9.226)$$

$$\begin{aligned} d_t c_e(t) &= A_z(t) c_e(t) [|c_e(t)|^2 - 1 - |c_b(t)|^2 |c_e(t)|^2] - c_b^*(t) c_e^2(t) z^*(t,t) \\ &\quad - B_z(t) [c_b^*(t)]^2 c_e^3(t). \end{aligned} \quad (9.227)$$

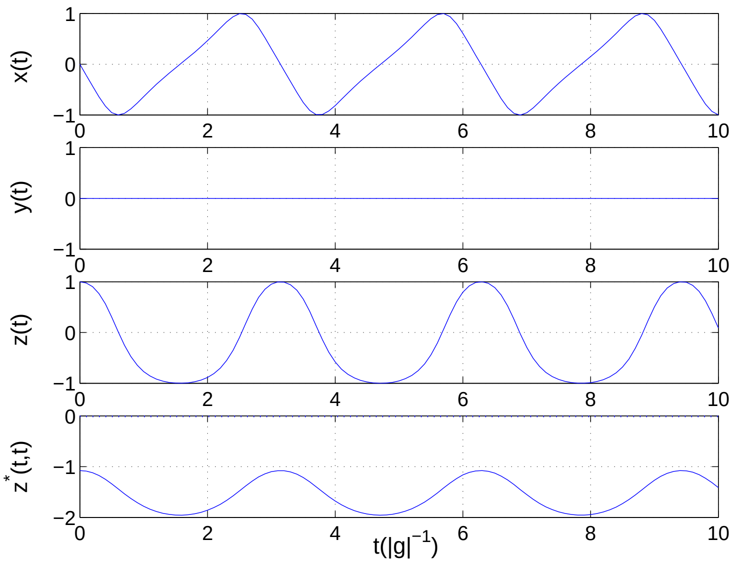

Figure 9.4: An example trajectory for the position-state non-Markovian SSE for a TLA in a single mode bath. Also shown is the real (solid) and imaginary (dotted) part of the noise function. Other details are as in figure 9.1.

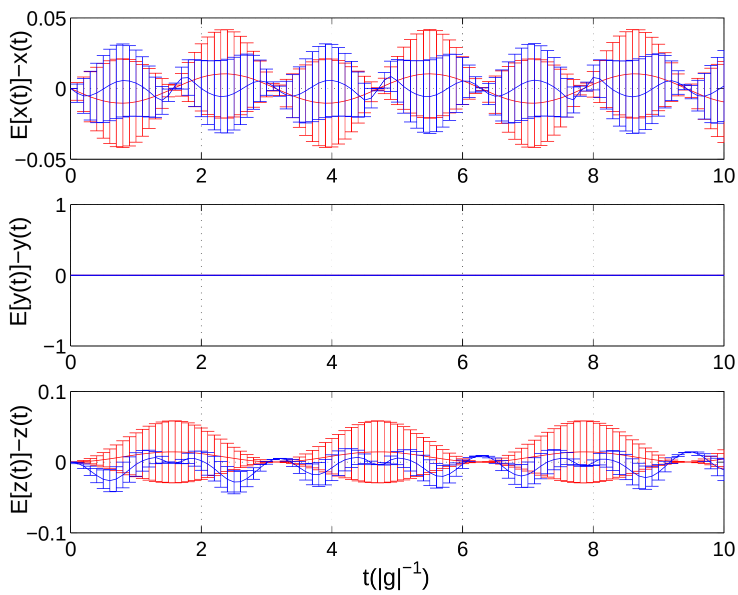

Figure 9.5: The difference between the ensemble average of 1000 trajectories for the position unraveling (linear (red) and actual (blue)) and the master equation for a TLA in a single mode bath. Other details are as in figure 9.1.

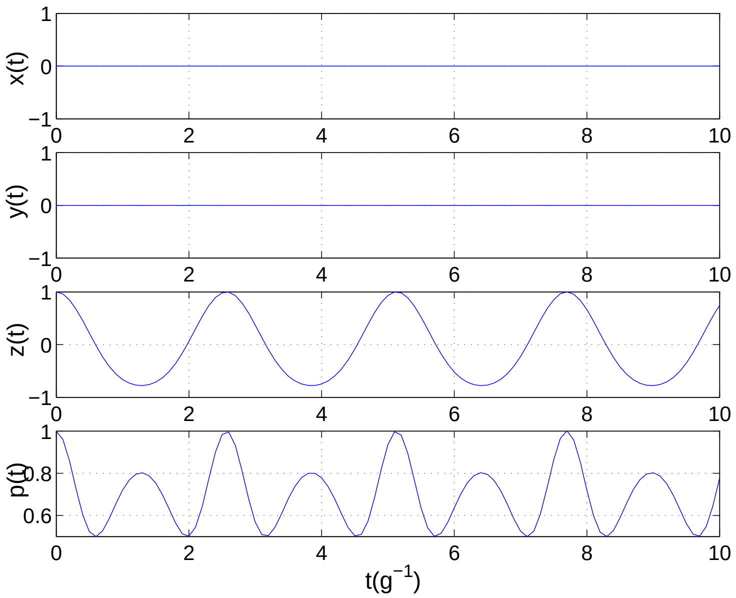

Figure 9.6: Solution of the Master equation for a TLA immersed in a two mode bath for $\Delta = 2g$, $dt = 0.0001$ and an excited state initial condition.

The noise function, $z^*(t,t) = g^* r(X,t)$, for the actual non-Markovian SSE is defined by (see Eq. (9.117))

$$r(X,t) = r(X,t_0) + \int_0^t ds[g^* c_b(s)c_e^*(s) + g c_b^*(s)c_e(s)]/\sqrt{2} \tag{9.228}$$

where $r(X,t_0)$ is a random variable chosen from the initial probability distribution, which is just Eq. (9.198). Choosing an excited state initial conditions (and $dt = 0.0001$) an example trajectory for the position-state non-Markovian SSE is shown in figure 9.4. To show that the ensemble average of these trajectories is the reduced state, the average of 1000 trajectories is shown in figure 9.5. Here as in the linear case we see that they do average to the reduced state.

9.4.2 A TLA coupled linear to two harmonic oscillators

In the above simple model because of the simplicity of the model the quadrature non-Markovian SSE could not be investigated. Here I am going to consider a slightly more complicate system: a TLA coupled linearly and with the same strength to two single mode fields (labelled by $k = \pm 1$) that are detuned from $\omega_{\rm sys}$ by $\pm\Delta$ respectively. Without loss of generality, we can take the coupling strength $g_1 = g$ to be real. Thus the interaction Hamiltonian is

$$\hat{V}(t) = g e^{i\Delta(t-t_0)}(\hat{a}_1^\dagger \hat{\sigma} - \hat{a}_{-1}\hat{\sigma}^\dagger) + g e^{-i\Delta(t-t_0)}(\hat{a}_{-1}^\dagger \hat{\sigma} - \hat{a}_1\hat{\sigma}^\dagger) \tag{9.229}$$

as $\Omega_1 = \Delta = -\Omega_{-1}$ and $g = g_{-1} = g_1$. I will also assume that $\hat{H}_{\rm int} = 0$ (there is no driving).

Exact dynamics for the non-Markovian master equation

To calculate the exact $\rho_{\rm red}(t)$ we need to solve the Schrödinger equation, which for this simple system is

$$d_t|\Psi(t)\rangle = [ge^{i\Delta(t-t_0)}(\hat{a}_1^\dagger\hat{\sigma} - \hat{a}_{-1}\hat{\sigma}^\dagger) + ge^{-i\Delta(t-t_0)}(\hat{a}_{-1}^\dagger\hat{\sigma} - \hat{a}_1\hat{\sigma}^\dagger)]|\Psi(t)\rangle. \tag{9.230}$$

Since initially the field is in the vacuum state ($|0_1\rangle\otimes|0_{-1}\rangle$) then the only non-zero complex amplitudes in $|\Psi(t)\rangle$ are

$$|\Psi(t)\rangle = c_1(t)|b00\rangle + c_2(t)|e00\rangle + c_3(t)|b01\rangle + c_4(t)|b10\rangle, \tag{9.231}$$

where $|b00\rangle$ is short hand for $|b\rangle\otimes|0_1\rangle\otimes|0_{-1}\rangle$ *etc.* Applying the above Hamiltonian to this state we get the following four differential equations for the complex amplitudes,

$$\dot{c}_1(t) = 0, \tag{9.232}$$
$$\dot{c}_2(t) = -c_3(t)ge^{i\Delta(t-t_0)} - c_4(t)ge^{-i\Delta(t-t_0)}, \tag{9.233}$$
$$\dot{c}_3(t) = c_2(t)ge^{-i\Delta(t-t_0)}, \tag{9.234}$$
$$\dot{c}_4(t) = c_2(t)ge^{i\Delta(t-t_0)}, \tag{9.235}$$

which can be solved numerically (for the initial state $|e00\rangle$, $c_2(t_0) = 1$ and the rest are zero). Once we have the amplitudes for all time we know $|\Psi(t)\rangle$ and by Eq. (6.1) we can then calculate $\rho_{\rm red}(t)$. Doing this gives

$$\rho_{\rm red}(t) = |c_2(t)|^2|e\rangle\langle e| + c_2(t)c_1^*(t)|e\rangle\langle b| + c_2^*(t)c_1(t)|b\rangle\langle e| + [|c_1(t)|^2 + |c_3(t)|^2 + |c_4(t)|^2]|b\rangle\langle b|, \tag{9.236}$$

which in Bloch representation is

$$I = |c_1(t)|^2 + |c_2(t)|^2 + |c_3(t)|^2 + |c_4(t)|^2, \tag{9.237}$$
$$x(t) = c_2(t)c_1^*(t) + c_2^*(t)c_1(t), \tag{9.238}$$
$$y(t) = -ic_2(t)c_1^*(t) + ic_2^*(t)c_1(t), \tag{9.239}$$
$$z(t) = |c_2(t)|^2 - |c_1(t)|^2 - |c_3(t)|^2 - |c_4(t)|^2. \tag{9.240}$$

To graphically illustrate the reduced state we numerically calculated the above for $\Delta = 2g$. The results are shown in Fig. 9.6.

Coherent-state case

For this system the noise function for the coherent unraveling is

$$z(t,s) = gr(A_1,t)e^{-i\Delta(s-t_0)} + gr(A_{-1},t)e^{i\Delta(s-t_0)}, \tag{9.241}$$

and the memory function becomes

$$\alpha(t-s) = g^2e^{-i\Delta(t-s)} + g^2e^{i\Delta(t-s)} = 2g^2\cos[\Delta(t-s)], \tag{9.242}$$

Note that this memory never decays (but is time dependent), indicating that the dynamics of the atom is extremely non-Markovian. This is different from all cases considered by Diósi, Gisin, and Strunz [48, 117] where the memory was taken to decay exponentially. It is thus interesting to see how the formalism copes with this case.

Applying Eq. (9.55) to this model gives the following actual non-Markovian SSE,

$$d_t|\psi_z(t)\rangle = \Big[z^*(t,t)(\hat{\sigma} - \langle\hat{\sigma}\rangle_t) - (\hat{\sigma}^\dagger - \langle\hat{\sigma}^\dagger\rangle_t)\hat{C}_z(t) + \Big\langle(\hat{\sigma}^\dagger - \langle\hat{\sigma}^\dagger\rangle_t)\hat{C}_z(t)\Big\rangle_t\Big]|\psi_z(t)\rangle, \tag{9.243}$$

where

$$\hat{C}_z(t) = \int_{t_0}^{t} 2g^2 \cos[\Delta(t-s)]\hat{c}_z(t,s)ds. \tag{9.244}$$

As in the last example the form of $\hat{C}_z(t)$ is determined by Eq. (9.38) and again let's assume that we can rewrite the ansatz in Eq. (9.38) as

$$\frac{\delta}{\delta z^*(t,s)}|\bar{\psi}_z(t)\rangle = c_z(t,s)\hat{\sigma}|\bar{\psi}_z(t)\rangle. \tag{9.245}$$

That is we have assumed that $\hat{C}_z(t)$ is $C_z(t)\hat{\sigma}$, where

$$C_z(t) = \int_{t_0}^{t} 2g^2 \cos[\Delta(t-s)]c_z(t,s)ds. \tag{9.246}$$

The best way to calculate $C_z(t)$ is to split it into to two terms, $C_z(t) = C_z^{(1)}(t) + C_z^{(-1)}(t)$ where,

$$C_z^{(1)}(t) = \int_{t_0}^{t} g^2 e^{-i\Delta(t-s)}c_z(t,s)ds, \tag{9.247}$$

$$C_z^{(-1)}(t) = \int_{t_0}^{t} g^2 e^{i\Delta(t-s)}c_z(t,s)ds. \tag{9.248}$$

Differentiating the above equations for $C_z^{(1)}(t)$ and $C_z^{(-1)}(t)$ yields

$$d_t C_z^{(1)}(t) = g^2 - i\Delta C_z^{(1)}(t) + \int_{t_0}^{t} g^2 e^{-i\Delta(t-s)}\partial_t[c_z(t,s)]ds, \tag{9.249}$$

$$d_t C_z^{(-1)}(t) = g^2 + i\Delta C_z^{(-1)}(t) + \int_{t_0}^{t} g^2 e^{i\Delta(t-s)}\partial_t[c_z(t,s)]ds, \tag{9.250}$$

as $c_z(t,t) = 1$ (see appendix A).

To work out the function $\partial_t c_z(t,s)$ we have to use the linear non-Markovian SSE (Eq. (9.39)), for this system it is

$$\partial_t|\bar{\psi}_z(t)\rangle = \Big[z^*(t,t)\hat{\sigma} - \hat{\sigma}^\dagger \hat{C}_z(t)\Big]|\bar{\psi}_z(t)\rangle, \tag{9.251}$$

which is the same as Eq. (9.191) (even through the $\hat{C}_z(t)$ will be different). Thus by the same procedure as used in the first example it can be shown that $\partial_t[c_z(t,s)]$ will equal $C_z(t)c_z(t,s)$. Substituting this into Eqs. (9.249) and (9.250) gives

$$d_t C_z^{(1)}(t) = g^2 - i\Delta C_z^{(1)}(t) + C_z^{(1)}(t)(t)C_z(t), \tag{9.252}$$

$$d_t C_z^{(-1)}(t) = g^2 + i\Delta C_z^{(-1)}(t) + C_z^{(-1)}(t)C_z(t), \tag{9.253}$$

which can be solved numerically. The initial conditions are $C_z(t_0) = C_z^{(1)}(t_0) = C_z^{(-1)}(t_0) = 0$.

With these operator we can now evaluate both the linear and actual non-Markovian SSE, Eqs. (9.251) and (9.243) respectively. For the linear case the ostensible distribution is

$$\Lambda(a_1, a_{-1}) = \pi^{-2}e^{-|a_1|^2 - |a_{-1}|^2}. \tag{9.254}$$

Thus $r(A_1, t) = \bar{r}(A_1)$ and $r(A_{-1}) = \bar{r}(A_{-1})$ for the linear case, are time independent complex Gaussian random variables of mean 0 and variance 1. Using Eq. (9.199) we can rewrite Eq. (9.251) as

$$d_t c_b(t) = z^*(t,t)c_e(t) \tag{9.255}$$

$$d_t c_e(t) = -C_z(t)c_e(t), \tag{9.256}$$

which for an excited-state initial condition can be solved numerically. Note that these solutions will not remain normalized, and the norm of most of them becomes very small. Nevertheless, the ensemble average of the unnormalized states is $\rho_{\rm red}(t)$. To show this I have simulated 1000 trajectories for different $z(t,t)$ and subtracted the reduced state solution. The results of this simulation are shown in figure 9.8 as a red line. Here it observed that within statistical error the ensemble average agrees with the master equation.

To work out the actual non-Markovian SSE for this unraveling we apply Eq. (9.199) to Eq. (9.243). Doing this the two (non-linear) differential equations are,

$$d_t c_b(t) = C_z(t)c_b(t)|c_e(t)|^2[2-|c_b(t)|^2] + c_e(t)[1-|c_b(t)|^2]z^*(t,t), \quad (9.257)$$

$$d_t c_e(t) = C_z(t)c_e(t)[|c_e(t)|^2 - 1 - |c_b(t)|^2|c_e(t)|^2] - c_b^*(t)c_e^2(t)z^*(t,t), \quad (9.258)$$

and noise function $z^*(t,t) = gr(A_1^*,t)e^{i\Delta(t-t_0)} + gr(A_{-1}^*,t)e^{-i\Delta(t-t_0)}$. For the actual non-Markovian SSE this is defined by (see Eq. (9.50))

$$r(A_1^*,t) = r(A_1^*,t_0) + g\int_0^t e^{-i\Delta(t'-t_0)}c_b(t')c_e^*(t')dt', \quad (9.259)$$

$$r(A_{-1}^*,t) = r(A_{-1}^*,t_0) + g\int_0^t e^{i\Delta(t'-t_0)}c_b(t')c_e^*(t')dt' \quad (9.260)$$

where $r(A_1^*,t_0)$ and $r(A_{-1}^*,t_0)$ are random variables chosen from the initial probability distribution, which for initial conditions $|\Psi(0)\rangle = |\psi(0)\rangle|0\rangle|0\rangle$ is just Eq. (9.254), the ostensible distribution. Choosing an excited state initial condition (and $dt = 0.0001$) an example trajectory for the coherent-state non-Markovian SSE is shown in figure 9.7. To show that the ensemble average of these trajectories is the reduced state, the average of 1000 trajectories (minus the reduced state solution) are shown in figure 9.8, where we once again observed that within statistical error they agree.

Quadrature case

The last unraveling I am going to consider for this system is the quadrature unraveling. If we apply the theory for the quadrature unraveling to this simple system, the quadrature noise function becomes

$$z(t,s) = 2g\{r(X^+,t)\cos[\Delta(s-t_0)] + r(Y^-,t)\sin[\Delta(s-t_0)]\}, \quad (9.261)$$

which is real. The memory function (Eq. (9.72)) becomes

$$\beta(t-s) = 2g^2\cos(\Delta(t-s)] = \alpha(t-s) \quad (9.262)$$

and the general non-Markovian SSE, defined in Eq. (9.98), for this simple system is

$$d_t|\psi_z(t)\rangle = \Big\{(\hat\sigma - \langle\hat\sigma\rangle_t)z(t,t) - (\hat\sigma_x - \langle\hat\sigma_x\rangle_t)\hat Q_z(t) + \langle(\hat\sigma_x - \langle\hat\sigma_x\rangle_t)\hat Q_z(t)\rangle_t\Big\}|\psi_z(t)\rangle, \quad (9.263)$$

where $\hat\sigma_x = \hat\sigma + \hat\sigma^\dagger$.

To find the values of the $\hat Q_z(t)$ we use Eq. (9.85) where, as in the other cases, we can make a further ansatz:

$$\frac{\delta}{\delta z(t,s)}|\bar\psi_z(t)\rangle = q_z(t,s)\hat\sigma|\bar\psi_z(t)\rangle. \quad (9.264)$$

This results in $\hat Q_z(t) = Q_z(t)\hat\sigma$, where

$$Q_z(t) = \int_{t_0}^t ds 2g^2\cos(\Delta(t-s)]q_z(t,s) \quad (9.265)$$

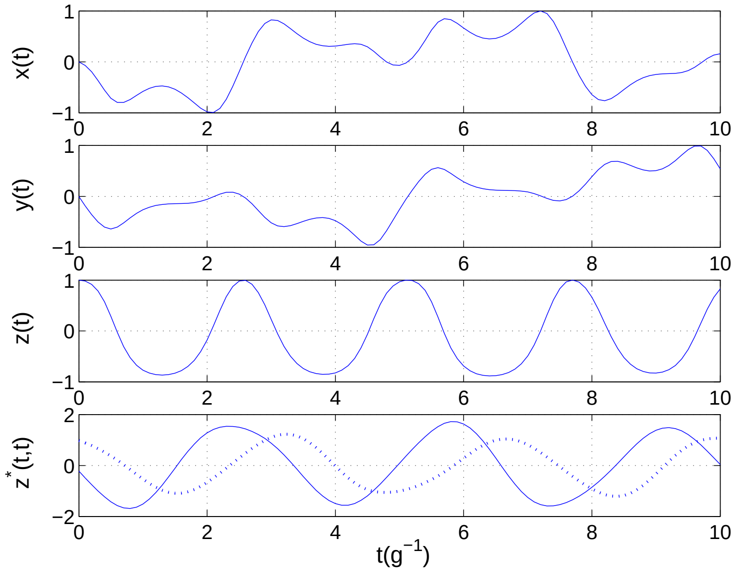

Figure 9.7: An example trajectory, for the coherent-state non-Markovian SSE, for a TLA immersed in a two mode bath. Also shown is the real (solid) and imaginary (dotted) parts of the noise function. Other details are as in figure 9.6.

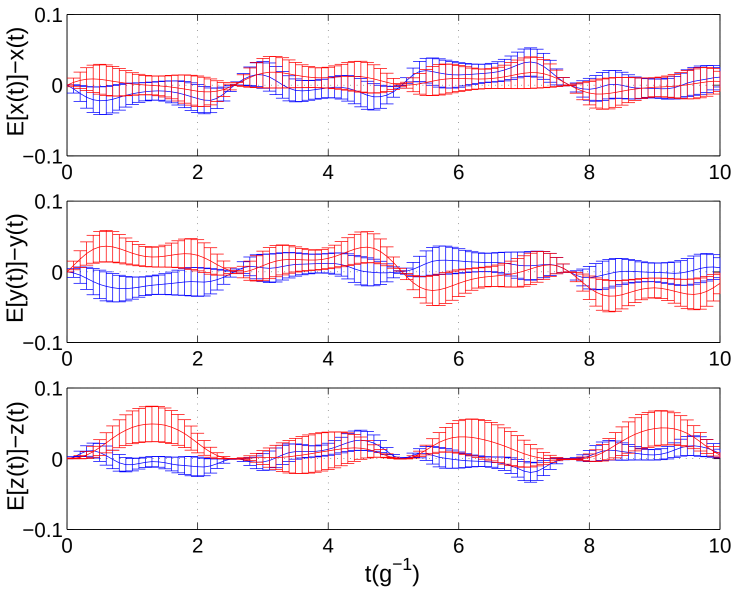

Figure 9.8: The difference between the ensemble average of 1000 trajectories for the coherent unraveling (linear (red) and actual (blue)) and the exact reduced state. Other details are as in figure 9.6.

It turns out that for this simple system $Q_z(t) \equiv C_z(t)$, because $\alpha(t-s) = \beta(t-s)$ and the linear SSE is

$$\partial_t|\bar{\psi}_z(t)\rangle = \Big(z(t,t)\hat{\sigma} - \hat{\sigma}^\dagger\hat{\sigma}Q(t)\Big)|\bar{\psi}_z(t)\rangle, \tag{9.266}$$

here I have used $\hat{\sigma}^\dagger\hat{\sigma} = \hat{\sigma}_x\hat{\sigma}$.

With this operators we can now evaluate both the linear and actual non-Markovian SSE, Eqs. (9.266) and (9.263) respectively. For the linear case the ostensible distribution is

$$\Lambda(X_1^+, Y_1^-) = \pi^{-1}e^{-X^{+2}-Y^{-2}}. \tag{9.267}$$

This implies that $r(X^+,t) = \bar{r}(X^+)$ and $r(Y^-,t) = \bar{r}(Y^-)$ are gaussian random variables of mean zero and variance 1/2. Using Eq. (9.199) and Eq. (9.266), we can obtain two differential equations for the two complex amplitudes, they are,

$$\begin{aligned} d_t c_b(t) &= z(t,t)c_e(t) && (9.268)\\ d_t c_e(t) &= -Q_z(t)c_e(t), && (9.269) \end{aligned}$$

These are the same as for the coherent case, except that $z(t,t)$ is generated differently. To show that the ensemble average of the solutions to the linear SSE for the quadrature unraveling converges to $\rho_{\text{red}}(t)$, 1000 trajectories for different $z(t,t)$ where simulated. The results of these simulations are shown in figure 9.10 as a red line, where it is seen that within statistical error these results agree with the exacted reduced state.

To work out the actual non-Markovian SSE for this unraveling we apply Eq. (9.199) to Eq. (9.263). This results in

$$\begin{aligned} d_t c_e(t) &= Q_z(t)c_e(t)(-1+|c_e(t)|^2 - |c_e(t)|^2|c_b(t)|^2) - Q_z(t)c_e^3(t)c_b^{*2}(t)\\ &\quad -c_e^2(t)c_b^*(t)z(t,t), && (9.270)\\ d_t c_b(t) &= Q_z(t)c_b(t)|c_e(t)|^2(2-|c_b(t)|^2) + Q_z(t)c_b^*(t)c_e^2(t)(1-|c_b(t)|^2)\\ &\quad +c_e(t)(1-|c_b(t)|^2)z(t,t). && (9.271) \end{aligned}$$

The random variables in the noise function, Eq. (9.261), are

$$\begin{aligned} r(X^+,t) &= r(X_k^+,t_0) + \int_{t_0}^t g\cos[\Omega_k(t'-t_0)]\langle\hat{\sigma}_x\rangle_{t'}dt', && (9.272)\\ r(Y^-,t) &= r(Y^-,t_0) + \int_{t_0}^t g\sin[\Omega_k(t'-t_0)]\langle\hat{\sigma}_x\rangle_{t'}dt'. && (9.273) \end{aligned}$$

where $r(X^+,t_0)$ and $r(Y^-,t_0)$ are random variables chosen from the initial probability distribution, which is just Eq. (9.267). Choosing an excited state initial conditions (and $dt = 0.0001$) an example trajectory for the quadrature non-Markovian SSE is shown in figure 9.9. Note that the noise is real and $y(t)$ is always zero, in contrast to the coherent state case. To show that the ensemble average of these trajectories is the reduced state, the average of 1000 trajectories is shown in figure 9.10 (blue line) where we see that with little error it agrees with the reduced state.

9.5 Summary of chapter

In this chapter I have derived three different non-Markovian SSEs, the coherent-state, quadrature, and position-state unraveling, under both the orthodox and modal interpretation of quantum mechanics. Doing this I concluded that under the orthodox interpretation non-Markovian SSEs have

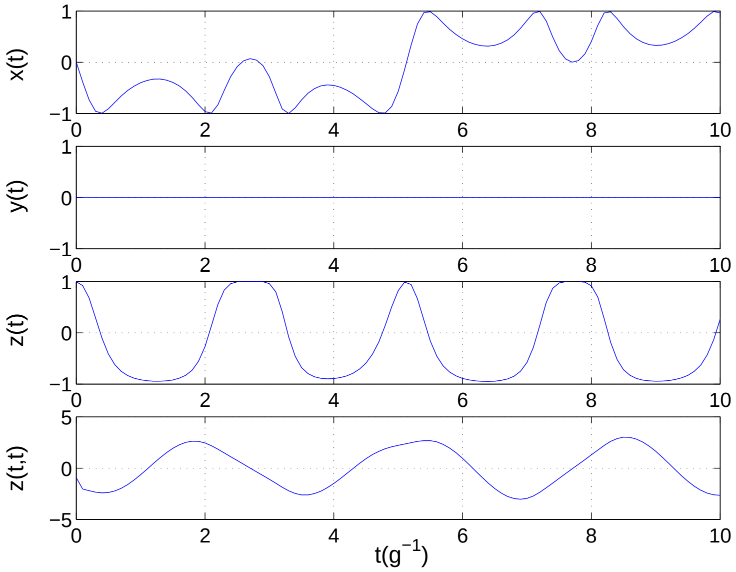

Figure 9.9: An example trajectory, for the quadrature non-Markovian SSE. for a TLA immersed in a two mode bath. Also shown is the noise function. Other details are as in figure 9.6.

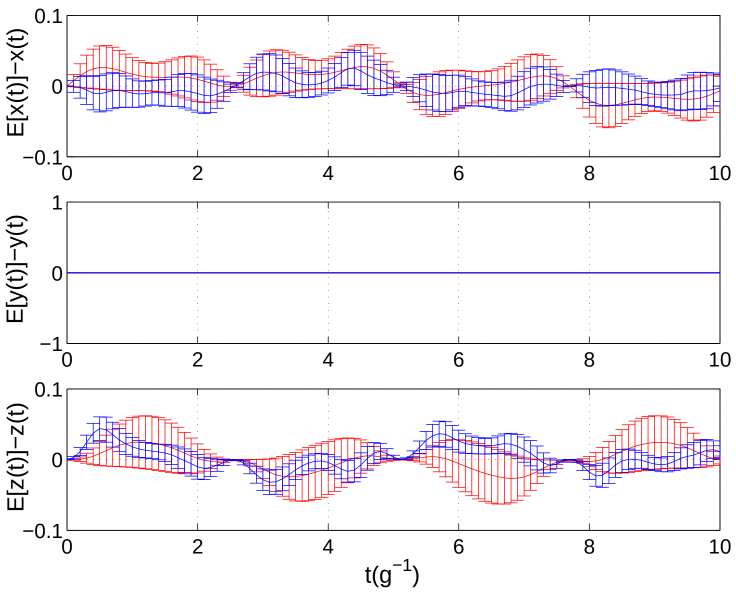

Figure 9.10: The difference between the ensemble average of 1000 trajectories for the quadrature unraveling (linear (red) and actual (blue)) and the exact reduced state. Other details are as in figure 9.6.

no physical meaning, they simply provide a numerical tool for calculating what the state of the system at time t will be if a measurement in basis $|\{z_k\}\rangle$ is performed on the bath at time t and yields results $\{z_k\}$. By contrast, in the modal interpretation non-Markovian SSEs are evolution equations for the system part of the property state of the universe when a property of the bath is given definite status. Thus the set of possible results $\{z_k\}$ under this view actually correspond to the possible values the bath properties can have.

In this chapter I have also shown that in the Markovian limit the coherent state SSE is equivalent to a heterodyne quantum trajectory (an evolution equation for the system state under the orthodox view) and the quadrature SSE is equivalent to a homodyne quantum trajectory. Thus the modal view of diffusive non-Markovian SSE also provides a new objective interpretation of diffusive Markovian SSEs (but not in the same line as CSL models).

Chapter 10

Perturbative Non-Markovian SSEs

In the last chapter we saw that it is only possible to generate non-Markovian SSE if we can replace the functional derivative by an operator. Presently this seems possible only for simple systems. However perturbative techniques do exist [139, 140, 60]. Recently Yu, Diósi, Gisin and Strunz (YDGS) have developed explicitly a 'post-Markovian' perturbation method to first order that allows solutions for systems that are close to the Markovian limit [139, 140]. In this chapter I will present a perturbation method (originally presented by Wiseman and myself in Ref. [60]) that can be carried to arbitrary order and so is not limited to the post Markovian regime. However we must place a requirement on the form of the memory functions. This requirement is that the memory function must take the form

$$\alpha(t-s) = \sum_{j=1}^{J} |G_j|^2 e^{-\kappa_j |t-s|/2 - i(\omega_j - \Omega)(t-s)}, \tag{10.1}$$

for some finite (and, in practice, relatively small) J. It should be noted also that convergence of our perturbation theory has not been proven and that this theory is only valid for a zero-temperature bath.

10.1 Added notation for non-Markovian SSEs

Here I am going to briefly rewrite the coherent and quadrature non-Markovian SSEs in a notation which will allow me to present the perturbation method in a manner which is easiest to understand.

10.1.1 The coherent-state unraveling

The first unraveling I will consider is the coherent-state unraveling. This unraveling arises when the bath is projected into a coherent state (see section 9.1.2 for more detail). The extra notation to be added occurs only on the operators used to replace the functional derivatives $\hat{c}_z(t,s) \to {}^{(0)}\hat{c}_z(t,s)$. That is Eq. (9.38) becomes

$$\frac{\delta}{\delta z^*(t,s)} |\bar{\psi}_z(t)\rangle = {}^{(0)}\hat{c}_z(t,s) |\bar{\psi}_z(t)\rangle, \tag{10.2}$$

and the linear SSE (Eq. (9.39)) becomes

$$\partial_t |\bar{\psi}_z(t)\rangle = \Big[-\frac{i}{\hbar} \hat{H}_{\rm int}(t) + z^*(t,t)\hat{L} - \hat{L}^{\dagger\,(0)}\hat{C}_z(t) \Big] |\bar{\psi}_z(t)\rangle, \tag{10.3}$$

where the functional operator ${}^{(0)}\hat{C}_z(t)$ is defined as

$$ {}^{(0)}\hat{C}_z(t) = \int_{t_0}^{t} \alpha(t-s)\, {}^{(0)}\hat{c}_z(t,s) ds. \tag{10.4} $$

The significance of the superscripts (0) proceeding these operators will become apparent in Sec. 10.2.

Under this notation the actual (non-linear) non-Markovian SSE for the coherent-state unraveling (Eq. (9.55)) becomes

$$ \begin{aligned} d_t|\psi_z(t)\rangle \;=\; & \Big[-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + z^*(t,t)(\hat{L}-\langle\hat{L}\rangle_t) - (\hat{L}^\dagger - \langle\hat{L}^\dagger\rangle_t)\, {}^{(0)}\hat{C}_z(t) \\ & + \Big\langle(\hat{L}^\dagger - \langle\hat{L}^\dagger\rangle_t)\, {}^{(0)}\hat{C}_z(t)\Big\rangle_t\Big]|\psi_z(t)\rangle, \end{aligned} \tag{10.5} $$

10.1.2 The quadrature unraveling

The added notation for the quadrature unraveling is to redefine $\hat{q}_z(t,s)$ as ${}^{(0)}\hat{q}_z(t,s)$ doing this simply means the ansatz defined in Eq. (9.85) becomes,

$$ \frac{\delta}{\delta z(t,s)}|\bar{\psi}_z(t)\rangle = {}^{(0)}\hat{q}_z(t,s)|\bar{\psi}_z(t)\rangle, \tag{10.6} $$

and the linear non-Markovian SSE, Eq. (9.86), becomes

$$ \partial_t|\bar{\psi}_z(t)\rangle \;=\; \Big[-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + z(t)\hat{L} - (\hat{L}+\hat{L}^\dagger)\, {}^{(0)}\hat{Q}_z(t)\Big]|\bar{\psi}_z(t)\rangle, \tag{10.7} $$

where

$$ {}^{(0)}\hat{Q}_z(t) = \int_{t_0}^{t} \beta(t-s)\, {}^{(0)}\hat{q}_z(t,s) ds. \tag{10.8} $$

Under this notation the actual (non-linear) non-Markovian SSE for the coherent-state unraveling (Eq. (9.98)) becomes

$$ \begin{aligned} d_t|\psi_z(t)\rangle \;=\; & \Big[-\frac{i}{\hbar}\hat{H}_{\rm int}(t) + z(t,t)(\hat{L}-\langle\hat{L}\rangle_t) - (\hat{L}+\hat{L}^\dagger - \langle\hat{L}+\hat{L}^\dagger\rangle_t)\, {}^{(0)}\hat{Q}_z(t) + \\ & \Big\langle(\hat{L}+\hat{L}^\dagger - \langle\hat{L}+\hat{L}^\dagger\rangle_t)\, {}^{(0)}\hat{Q}_z(t)\Big\rangle_t\Big]|\psi_z(t)\rangle. \end{aligned} \tag{10.9} $$

10.2 Perturbation method

To solve the non-Markovian SSE, and hence find $\rho_{\rm red}(t)$, for the coherent or quadrature unraveling we have to work out the operator functionals ${}^{(0)}\hat{C}_z(t)$ and ${}^{(0)}\hat{Q}_z(t)$ respectively. This has been done exactly only for systems for which an analytical solution for $\rho_{\rm red}(t)$ may be found by other means [48, 117, 33] or for systems with a small number of bath modes [59] (see section 9.4). In this section we propose a perturbation technique for working out these functionals when exact solutions are not possible.

10.2.1 Perturbation approach for the coherent-state unraveling

The perturbation that we are going to propose is only valid for memory functions of the form

$$ \alpha(t-s) = \sum_{j=1}^{J} \alpha^{(j)}(t-s), \tag{10.10} $$

where

$$\alpha^{(j)}(t-s) = |G_j|^2 e^{-\kappa_j|t-s|/2} e^{-i\Omega_j(t-s)}. \tag{10.11}$$

In principle this is always a valid decomposition for the memory function as in the $J \to \infty$ and $\kappa_j \to 0$ limit this memory function approaches the microscopic memory function displayed in Eq. (9.28). In Ref. [115] the authors suggest that in practice most environments can be simulated with J being quite small.

With this expansion for the memory function the functional ${}^{(0)}\hat{C}_z(t)$ can be written as

$${}^{(0)}\hat{C}_z(t) = \sum_j {}^{(0)}\hat{C}_z^{(j)}(t). \tag{10.12}$$

where

$${}^{(0)}\hat{C}_z^{(j)}(t) = \int_0^t \alpha^{(j)}(t-s)\,{}^{(0)}\hat{c}_z(t,s)ds. \tag{10.13}$$

To calculate these functional operators we set up a set of coupled nonlinear differential equations for ${}^{(0)}\hat{C}_z^{(j)}(t)$. Taking the time derivative of Eq. (10.13) we get

$$\begin{aligned}\partial_t\, {}^{(0)}\hat{C}_z^{(j)}(t) &= \alpha^{(j)}(0)\,{}^{(0)}\hat{c}_z(t,t) + \int_{t_0}^t [\partial_t \alpha^{(j)}(t-s)]\ {}^{(0)}\hat{f}_z(t,s)ds \\ &\quad + \int_{t_0}^t \alpha^{(j)}(t-s)\partial_t\, {}^{(0)}\hat{c}_z(t,s)ds. \end{aligned}\tag{10.14}$$

The first term is easily evaluated using

$${}^{(0)}\hat{c}_z(t,t) = \hat{L}, \tag{10.15}$$

as derived in Appendix A. The second term is where our earlier decomposition of $\alpha(t-s)$ is used. We chose $\alpha^{(j)}(t-s)$ such that $\partial_t\alpha^{(j)}(t-s) \propto \alpha^{(j)}(t-s)$. This results in the second term equaling

$$-(\frac{\kappa_j}{2} + i\Omega_j)\,{}^{(0)}\hat{C}_z^{(j)}(t). \tag{10.16}$$

The third term involves the partial derivative $\partial_t[{}^{(0)}\hat{c}_z(t,s)]$. To find this we use the fact that

$$\partial_t \frac{\delta}{\delta z^*(t,s)}|\bar{\psi}_z(t)\rangle = \frac{\delta}{\delta z^*(t,s)}\partial_t|\bar{\psi}_z(t)\rangle, \tag{10.17}$$

which is called the consistency condition in [48]. This consistency condition is only valid for $t \neq s$ this is because at time $t = s$ the functional derivative is not well defined (thus $\delta_{z^*(t,s)} z^*(t,t) = 0$) . Using Eq. (10.2), the LHS of the consistency condition can be evaluated as

$$\partial_t \frac{\delta}{\delta z^*(t,s)}|\bar{\psi}_z(t)\rangle = [\partial_t\, {}^{(0)}\hat{c}_z(t,s)]|\bar{\psi}_z(t)\rangle + {}^{(0)}\hat{c}_z(t,s)\partial_t|\bar{\psi}_z(t)\rangle. \tag{10.18}$$

Substituting Eq. (10.3) in for $\partial_t|\bar{\psi}_z(t)\rangle$ gives

$$\begin{aligned}\partial_t \frac{\delta}{\delta z^*(t,s)}|\bar{\psi}_z(t)\rangle &= \Big[\partial_t\, {}^{(0)}\hat{c}_z(t,s) - \frac{i}{\hbar}\,{}^{(0)}\hat{c}_z(t,s)\hat{H}_{\rm int}(t) + z^*(t,t)\,{}^{(0)}\hat{c}_z(t,s)\hat{L} \\ &\quad - {}^{(0)}\hat{c}_z(t,s)\hat{L}^\dagger\, {}^{(0)}\hat{C}_z(t)\Big]|\bar{\psi}_z(t)\rangle. \end{aligned}\tag{10.19}$$

Using Eqs. (10.3) and (10.2) the right-handed side (RHS) of the consistency condition gives

$$\begin{aligned}\frac{\delta}{\delta z^*(t,s)}\partial_t|\bar{\psi}_z(t)\rangle &= \Big[-\frac{i}{\hbar}\hat{H}_{\rm int}(t)\,{}^{(0)}\hat{c}_z(t,s) + z^*(t,t)\hat{L}\,{}^{(0)}\hat{c}_z(t,s) - \hat{L}^\dagger\,{}^{(0)}\hat{C}_z(t)\,{}^{(0)}\hat{c}_z(t,s) \\ &\quad -\hat{L}^\dagger \frac{\delta}{\delta z^*(t,s)}\,{}^{(0)}\hat{C}_z(t)\Big]|\bar{\psi}_z(t)\rangle. \end{aligned}\tag{10.20}$$

Equating the LHS with the RHS gives

$$\begin{aligned}\partial_t {}^{(0)}\hat{c}_z(t,s) &= -\frac{i}{\hbar}[\hat{H}_{\rm int}(t), {}^{(0)}\hat{c}_z(t,s)] + z^*(t,t)[\hat{L}, {}^{(0)}\hat{c}_z(t,s)] - [\hat{L}^\dagger {}^{(0)}\hat{C}_z(t), {}^{(0)}\hat{c}_z(t,s)] \\ &\quad -\hat{L}^\dagger \frac{\delta}{\delta z^*(t,s)} {}^{(0)}\hat{C}_z(t). \end{aligned} \tag{10.21}$$

Substituting this equation with Eqs. (10.15) and (10.16) into Eq. (10.14) we get

$$\begin{aligned}\partial_t {}^{(0)}\hat{C}_z^{(j)}(t) &= |G_j|^2\hat{L} - (\frac{\kappa_j}{2} + i\Omega_j){}^{(0)}\hat{C}_z^{(j)}(t) + z^*(t,t)[\hat{L}, {}^{(0)}\hat{C}_z^{(j)}(t)] - \frac{i}{\hbar}[\hat{H}_{\rm int}(t), {}^{(0)}\hat{C}_z^{(j)}(t)] \\ &\quad -[\hat{L}^\dagger {}^{(0)}\hat{C}_z(t), {}^{(0)}\hat{C}_z^{(j)}(t)] - \hat{L}^\dagger \sum_k {}^{(1)}\hat{C}_z^{(j,k)}(t), \end{aligned} \tag{10.22}$$

where ${}^{(1)}\hat{C}_z^{(j,k)}(t)$ is our first order functional. It has the form

$${}^{(1)}\hat{C}_z^{(j,k)}(t) = \int_{t_0}^t \alpha^{(j)}(t-s){}^{(1)}\hat{c}_z^{(k)}(t,s)ds, \tag{10.23}$$

where we have used the following Ansatz

$$\frac{\delta}{\delta z^*(t,s)}{}^{(0)}\hat{C}_z^{(k)}(t) = {}^{(1)}\hat{c}_z^{(k)}(t,s). \tag{10.24}$$

If we knew the form of ${}^{(1)}\hat{C}_z^{(j,k)}(t)$ then Eq. (10.22) could be solved numerically.

To find the form of ${}^{(1)}\hat{C}_z^{(j,k)}(t)$ we can take the time derivative of Eq. (10.23). Doing this we get

$$\begin{aligned}\partial_t {}^{(1)}\hat{C}_z^{(j,k)}(t) &= \alpha^{(j)}(0){}^{(1)}\hat{c}_z^{(k)}(t,t) + \int_{t_0}^t [\partial_t \alpha^{(j)}(t-s)]{}^{(1)}\hat{c}_z^{(k)}(t,s)ds \\ &\quad + \int_{t_0}^t \alpha^{(j)}(t-s)\partial_t {}^{(1)}\hat{c}_z^{(k)}(t,s)ds. \end{aligned} \tag{10.25}$$

The first term is easy to work out. From Eq. (10.22) it follows that

$${}^{(1)}\hat{c}_z^{(k)}(t,t) = [\hat{L}, {}^{(0)}\hat{C}_z^{(k)}(t)]. \tag{10.26}$$

The second term as before also simply evaluates to

$$-\Big(\frac{\kappa_j}{2} + i\Omega_j\Big){}^{(1)}\hat{C}_z^{(j,k)}(t). \tag{10.27}$$

The third term is worked out via a new consistency condition,

$$\partial_t \frac{\delta}{\delta z^*(t,s)}{}^{(0)}\hat{C}_z^{(k)}(t) = \frac{\delta}{\delta z^*(t,s)}\partial_t {}^{(0)}\hat{C}_z^{(k)}(t). \tag{10.28}$$

Substituting Eqs. (10.24) and (10.22) into this consistency condition gives

$$\begin{aligned}\partial_t {}^{(1)}\hat{c}_z^{(k)}(t,s) &= -(\frac{\kappa_k}{2} + i\Omega_k){}^{(1)}\hat{c}_z^{(k)}(t,s) - \frac{i}{\hbar}[\hat{H}_{\rm int}(t), {}^{(1)}\hat{c}_z^{(k)}(t,s)] + z^*(t,t)[\hat{L}, {}^{(1)}\hat{c}_z^{(k)}(t,s)] \\ &\quad -[\hat{L}^\dagger \sum_l {}^{(1)}\hat{c}_z^{(l)}(t,s), {}^{(0)}\hat{C}_z^{(k)}(t)] - [\hat{L}^\dagger \sum_l {}^{(0)}\hat{C}_z^{(l)}(t), {}^{(1)}\hat{c}_z^{(k)}(t,s)] \\ &\quad -\hat{L}^\dagger \sum_l \frac{\delta}{\delta z^*(s)}{}^{(1)}\hat{C}_z^{(k,l)}(t). \end{aligned} \tag{10.29}$$

Substituting all these terms into Eq. (10.25) gives

$$\begin{aligned}\partial_t {}^{(1)}\hat{C}_z^{(j,k)}(t) &= |G_j|^2[\hat{L}, {}^{(0)}\hat{C}_z^{(k)}(t)] - \Big(\frac{\kappa_j}{2} + i\Omega_j\Big){}^{(1)}\hat{C}_z^{(j,k)}(t) - \Big(\frac{\kappa_k}{2} + i\Omega_k\Big){}^{(1)}\hat{C}_z^{(j,k)}(t) \\ &-\frac{i}{\hbar}[\hat{H}_{\rm int}(t), {}^{(1)}\hat{C}_z^{(j,k)}(t)] + z^*(t,t)[\hat{L}, {}^{(1)}\hat{C}_z^{(j,k)}(t)] - [\hat{L}^\dagger \sum_l {}^{(1)}\hat{C}_z^{(j,l)}(t), {}^{(0)}\hat{C}_z^{(k)}(t)] \\ &-[\hat{L}^\dagger \sum_l {}^{(0)}\hat{C}_z^{(l)}(t), {}^{(1)}\hat{C}_z^{(j,k)}(t,s)] - \hat{L}^\dagger \sum_l {}^{(2)}\hat{C}_z^{(j,k,l)}(t). \end{aligned} \tag{10.30}$$

Where the last term is the second order functional, which equals

$${}^{(2)}\hat{C}_z^{(j,k,l)}(t) = \int_{t_0}^t \alpha^{(j)}(t-s)\frac{\delta}{\delta z^*(t,s)}{}^{(1)}\hat{C}_z^{(k,l)}(t)ds. \tag{10.31}$$

Here we see that we can develop a general way for setting up an $n^{\rm th}$ order set of differential equations. The $n^{\rm th}$ order functional is

$${}^{(n)}\hat{C}_z^{(j,k,\dots,l)}(t) = \int_{t_0}^t \alpha^{(j)}(t-s){}^{(n)}\hat{c}_z^{(k,\dots,l)}(t,s)ds, \tag{10.32}$$

where we have used the Ansatz

$$\frac{\delta}{\delta z^*(t,s)}{}^{(n-1)}\hat{C}_z^{(k,\dots,l)}(t) = {}^{(n)}\hat{c}_z^{(k,\dots,l)}(t,s). \tag{10.33}$$

The differential equation for the $n^{\rm th}$ order functional is

$$\begin{aligned}\partial_t {}^{(n)}\hat{C}_z^{(j,k,\dots,l)}(t) &= \alpha^{(j)}(0){}^{(n)}\hat{c}_z^{(k,\dots,l)}(t,t) + \int_{t_0}^t [\partial_t\alpha^{(j)}(t-s)]{}^{(n)}\hat{c}_z^{(k,\dots,l)}(t,s)ds \\ &+\int_{t_0}^t \alpha^{(j)}(t-s)\partial_t {}^{(n)}\hat{c}_z^{(k,\dots,l)}(t,s)ds.\end{aligned} \tag{10.34}$$

The first term can always be calculated by the $(n-1)^{\rm th}$ differential equation. The second term is always simple to calculate as $\partial_t\alpha^{(j)}(t-s) \propto \alpha^{(j)}(t-s)$ and the third term is always calculable by the $(n-1)^{\rm th}$ order consistency condition

$$\partial_t \frac{\delta}{\delta z^*(t,s)}{}^{(n-1)}\hat{C}_z^{(k,\dots,l)}(t) = \frac{\delta}{\delta z^*(t,s)}\partial_t {}^{(n-1)}\hat{C}_z^{(k,\dots,l)}(t). \tag{10.35}$$

The $n^{\rm th}$ order perturbation method we propose is to terminate this series by setting ${}^{(n)}\hat{C}_z^{(j,k,\dots,l)}(t)$ equal to an arbitrary operator. The simplest scheme would be to set this operator to zero, but to keep the theory consistent with the Markovian limit for all orders, we set ${}^{(n)}\hat{C}_z^{(j,k,\dots,l)}(t)$ in the following manner. The zeroth order perturbation arises when we use the approximation

$${}^{(0)}\hat{C}_z^{(j)}(t)|\bar{\psi}_z(t)\rangle \simeq \int_{t_0}^t \alpha^{(j)}(t-s)ds \lim_{s\to t}\Big[\frac{\delta}{\delta z^*(t,s)}|\bar{\psi}_z(t)\rangle\Big] = \int_{t_0}^t \alpha^{(j)}(t-s)ds\hat{L}|\bar{\psi}_z(t)\rangle. \tag{10.36}$$

Note that the approximation here is the replacement of $\delta/\delta z^*(t,s)$ by $\delta/\delta z^*(t,t)$. The first order perturbation arises when we use the approximation

$${}^{(1)}\hat{C}_z^{(j,k)}(t) \simeq \int_{t_0}^t \alpha^{(j)}(t-s)ds \lim_{s\to t}\Big[\frac{\delta {}^{(0)}\hat{C}_z^{(k)}(t)}{\delta z^*(t,s)}\Big] = \int_{t_0}^t \alpha^{(j)}(t-s)ds[\hat{L}, {}^{(0)}\hat{C}_z^{(k)}(t)] \tag{10.37}$$

and ${}^{(0)}\hat{C}_z^{(j)}(t)$ is calculated via Eq. (10.22). The n^{th} order perturbation arises when we use the approximation

$$\begin{aligned} {}^{(n)}\hat{C}_z^{(j,k,\ldots,l)}(t) &\simeq \int_{t_0}^{t} \alpha^{(j)}(t-s)ds \lim_{s\to t} \Big[\frac{\delta\,{}^{(n-1)}\hat{C}_z^{(k,\ldots,l)}(t)}{\delta z^*(t,s)}\Big] \\ &= \int_{t_0}^{t} \alpha^{(j)}(t-s)ds[\hat{L}, {}^{(n-1)}\hat{C}_z^{(k,\ldots,l)}(t)] \end{aligned} \tag{10.38}$$

and ${}^{(0)}\hat{C}_z^{(j)}(t), \ldots, {}^{(n-1)}\hat{C}_z^{(j,\ldots,k)}(t)$ are calculated via Eqs. (10.22), (10.30) and (10.34). The physical motivations for choosing this type of expansion are;

a) For most system the memory function will decay and thus the most dominant term in the functional derivative will be the value as $s \to t$.

b) Only ${}^{(0)}\hat{C}_z^{(j)}(t)$ affects the system directly, so the further removed the approximation the more accurate we expect the approximation to be.

c) In the Markovian limit, only the zeroth order term is needed.

To summarize this perturbation method, for environments which can be modeled by Eq. (10.10), it is possible to obtain a perturbative solution for the coherent non-Markovian SSE. From these SSEs it is possible to generate a perturbative solution for $\rho_{\text{red}}(t)$, which by definition will always be positive. The number of coupled complex differential equations that are required for this technique is

$$d^2(J^n + J^{n-1} + \ldots + J) + d + J = d^2 J\frac{J^n - 1}{J - 1} + d + J \tag{10.39}$$

where d is the system dimension, J is the number of exponentials required to simulate the memory function and n is the order of the perturbation. The first term represents the number of equations needed to simulate the functional derivative (or the n^{th} order perturbation for the operators ${}^{(n)}\hat{C}_z^{(j,k,\ldots,l)}(t)$). The next term d is for the d complex amplitudes of the system. The final term J is for the stochastic equations needed to generate the noise function $z(t,t)$.

10.2.2 Perturbation approach for the quadrature unraveling

The perturbation method in the quadrature case is essentially the same as the coherent case, but the memory function expressed in Eq. (10.11) is too general. This is because the memory function for the quadrature unraveling must be consistent with the assumptions stated below Eq. (9.57). The most general memory function that satisfies these requirements is

$$\beta(t-s) = \sum_j \beta^{(j,\cos)}(t-s), \tag{10.40}$$

where

$$\beta^{(j,\cos)}(t-s) = 2|G_j|^2 e^{-\kappa_j|t-s|/2}\cos[\Omega_j(t-s)]. \tag{10.41}$$

This presents a problem as $\partial_t\beta^{(j,\cos)}(t-s)$ is not proportional to $\beta^{(j,\cos)}(t-s)$. To get around this we define a new function $\beta^{(j,\sin)}(t-s)$ as

$$\beta^{(j,\sin)}(t-s) = 2|G_j|^2 e^{-\kappa_j|t-s|/2}\sin(\Omega_j(t-s)). \tag{10.42}$$

and two functionals

$${}^{(0)}\hat{Q}_z^{(j,\cos)}(t) = \int_{t_0}^{t} \beta^{(j,\cos)}(t-s)\hat{q}_z(t,s)ds, \tag{10.43}$$

$${}^{(0)}\hat{Q}_z^{(j,\sin)}(t) = \int_{t_0}^{t} \beta^{(j,\sin)}(t-s)\hat{q}_z(t,s)ds. \tag{10.44}$$

The functional ${}^{(0)}\hat{Q}_z(t)$ is then found by

$$ {}^{(0)}\hat{Q}_z(t) = \sum_j {}^{(0)}\hat{Q}_z^{(j,\cos)}(t). \tag{10.45} $$

Taking the time derivative of Eqs. (10.43) and (10.44) we get

$$ \begin{aligned} d_t\,{}^{(0)}\hat{Q}_z^{(j,\cos)}(t) &= \beta^{(j,\cos)}(0)\,{}^{(0)}\hat{q}_z(t,t) + \int_{t_0}^{t}[\partial_t\beta^{(j,\cos)}(t-s)]\,{}^{(0)}\hat{q}_z(t,s)ds \\ &\quad + \int_{t_0}^{t}\beta^{(j,\cos)}(t-s)\partial_t\,{}^{(0)}\hat{q}_z(t,s)ds, \end{aligned} \tag{10.46} $$

$$ d_t\,{}^{(0)}\hat{Q}_z^{(j,\sin)}(t) = \int_{t_0}^{t}[\partial_t\beta^{(j,\sin)}(t-s)]\,{}^{(0)}\hat{q}_z(t,s)ds + \int_{t_0}^{t}\beta^{(j,\sin)}(t-s)\partial_t\,{}^{(0)}\hat{q}_z(t,s)ds. \tag{10.47} $$

As in the coherent case it can be shown that ${}^{(0)}\hat{q}_z(t,t) = \hat{L}$. The two terms involving the derivative of $\beta^{(j,\cos)}(t-s)$ and $\beta^{(j,\sin)}(t-s)$ by definition give

$$ \int_0^t \partial_t\beta^{(j,\cos)}(t-s)\,{}^{(0)}\hat{q}_z(t,s)ds = -\frac{\kappa_j}{2}\,{}^{(0)}\hat{Q}_z^{(j,\cos)}(t) - \Omega_j\,{}^{(0)}\hat{Q}_z^{(j,\sin)}(t) \tag{10.48} $$

$$ \int_0^t \partial_t\beta^{(j,\sin)}(t-s)\,{}^{(0)}\hat{q}_z(t,s)ds = -\frac{\kappa_j}{2}\,{}^{(0)}\hat{Q}_z^{(j,\sin)}(t) + \Omega_j\,{}^{(0)}\hat{Q}_z^{(j,\cos)}(t). \tag{10.49} $$

The last two terms require finding $\partial_t{}^{(0)}\hat{q}_z(t,s)$. As in the coherent case this is found by the consistency condition

$$ \partial_t\frac{\delta}{\delta z(t,s)}|\bar{\psi}_z(t)\rangle = \frac{\delta}{\delta z(t,s)}\partial_t|\bar{\psi}_z(t)\rangle, \tag{10.50} $$

yielding

$$ \begin{aligned} \partial_t\,{}^{(0)}\hat{q}_z(t,s) &= -\frac{i}{\hbar}[\hat{H}_{\rm int}(t),\,{}^{(0)}\hat{q}_z(t,s)] + z(t,t)[\hat{L},\,{}^{(0)}\hat{q}_z(t,s)] - [(\hat{L}+\hat{L}^\dagger)\,{}^{(0)}\hat{Q}_z(t),\,{}^{(0)}\hat{q}_z(t,s)] \\ &\quad -(\hat{L}+\hat{L}^\dagger)\frac{\delta}{\delta z(t,s)}\,{}^{(0)}\hat{Q}_z(t). \end{aligned} \tag{10.51} $$

Substituting these terms into Eqs. (10.46) and (10.47) gives

$$ \begin{aligned} d_t\,{}^{(0)}\hat{Q}_z^{(j,\cos)}(t) &= 2|G_j|^2\hat{L} - \frac{\kappa_j}{2}\,{}^{(0)}\hat{Q}_z^{(j,\cos)}(t) - \Omega_j\,{}^{(0)}\hat{Q}_z^{(j,\sin)}(t) - \frac{i}{\hbar}[\hat{H}_{\rm int}(t),\,{}^{(0)}\hat{Q}_z^{(j,\cos)}(t)] \\ &\quad + z(t,t)[\hat{L},\,{}^{(0)}\hat{Q}_z^{(j,\cos)}(t)] - [(\hat{L}+\hat{L}^\dagger)\,{}^{(0)}\hat{Q}_z(t),\,{}^{(0)}\hat{Q}_z^{(j,\cos)}(t)] \\ &\quad -(\hat{L}+\hat{L}^\dagger)\sum_k {}^{(1)}\hat{Q}_z^{(j,k,\cos,\cos)}(t), \end{aligned} \tag{10.52} $$

$$ \begin{aligned} d_t\,{}^{(0)}\hat{Q}_z^{(j,\sin)}(t) &= -\frac{\kappa_j}{2}\,{}^{(0)}\hat{Q}_z^{(j,\sin)}(t) + \Omega_j\,{}^{(0)}\hat{Q}_z^{(j,\cos)}(t) - \frac{i}{\hbar}[\hat{H}_{\rm int}(t),\,{}^{(0)}\hat{Q}_z^{(j,\sin)}(t)] \\ &\quad + z(t,t)[\hat{L},\,{}^{(0)}\hat{Q}_z^{(j,\sin)}(t)] \\ &\quad -[(\hat{L}+\hat{L}^\dagger)\,{}^{(0)}\hat{Q}_z(t),{}^{(0)}\,{}^{(0)}\hat{Q}_z^{(j,\sin)}(t)] \\ &\quad -(\hat{L}+\hat{L}^\dagger)\sum_k {}^{(1)}\hat{Q}_z^{(j,k,\sin,\cos)}(t), \end{aligned} \tag{10.53} $$

where

$$ {}^{(1)}\hat{Q}_z^{(j,k,\cos,\cos)}(t) = \int_{t_0}^{t}\beta^{(j,\cos)}(t,s)\frac{\delta\,{}^{(0)}\hat{Q}_z^{(k,\cos)}(t)}{\delta z(t,s)}ds, \tag{10.54} $$

$$ {}^{(1)}\hat{Q}_z^{(j,k,\sin,\cos)}(t) = \int_{t_0}^{t}\beta^{(j,\sin)}(t,s)\frac{\delta\,{}^{(0)}\hat{Q}_z^{(k,\cos)}(t)}{\delta z(t,s)}ds. \tag{10.55} $$

The higher order functional differential equations are found in the same manner as in the coherent case, except the form of $\beta(t-s)$ results in 2^n as many equations for order n.

The perturbation expansion is similar for this unraveling, the only difference being that we have 2^n operators to approximate. The 0^{th} order approximation is to set the 0^{th} order functionals to

$$ {}^{(0)}\hat{Q}_z^{(j,\cos)}(t) = \int_{t_0}^{t} \beta^{(j,\cos)}(t,s)ds\hat{L} \tag{10.56} $$

$$ {}^{(0)}\hat{Q}_z^{(j,\sin)}(t) = \int_{t_0}^{t} \beta^{(j,\sin)}(t,s)ds\hat{L}. \tag{10.57} $$

The first order approximation is to set the four first order functionals to

$$ {}^{(1)}\hat{Q}_z^{(j,k,\cos,\cos)}(t) = \int_{t_0}^{t} \beta^{(j,\cos)}(t,s)ds[\hat{L}, {}^{(0)}\hat{Q}_z^{(k,\cos)}(t)], \tag{10.58} $$

$$ {}^{(1)}\hat{Q}_z^{(j,k,\sin,\cos)}(t) = \int_{t_0}^{t} \beta^{(j,\sin)}(t,s)ds[\hat{L}, {}^{(0)}\hat{Q}_z^{(k,\cos)}(t)], \tag{10.59} $$

$$ {}^{(1)}\hat{Q}_z^{(j,k,\cos,\sin)}(t) = \int_{t_0}^{t} \beta^{(j,\cos)}(t,s)ds[\hat{L}, {}^{(0)}\hat{Q}_z^{(k,\sin)}(t)], \tag{10.60} $$

$$ {}^{(1)}\hat{Q}_z^{(j,k,\sin,\sin)}(t) = \int_{t_0}^{t} \beta^{(j,\sin)}(t,s)ds[\hat{L}, {}^{(0)}\hat{Q}_z^{(k,\sin)}(t)]. \tag{10.61} $$

and we calculate the 0^{th} order functionals via Eqs. (10.52) and (10.53).

10.3 Enlarged system approach

To test the accuracy of our perturbation method we compare our results for the reduced state with the reduced state found via the enlarged system method of Imamoḡlu [88, 115]. An example of how this method is applied to a non-Markovian system can be found in Ref. [27].

For those who are not familiar with the enlarged system method, we provide a short proof that the reduced system dynamics are exactly reproduced by the enlarged system method provided that $\alpha(t-s)$, called $\Gamma(\tau)$ in Refs [88, 115], is of the form

$$ \alpha(t-s) = \sum_j |G_j|^2 e^{-\kappa_j|t-s|/2 - i\Omega_j(t-s)}, \tag{10.62} $$

which is the same as Eq. (10.10).

The total Hamiltonian for the enlarged system is

$$ \begin{aligned} \hat{H}_{\text{tot}} = {} & \hat{H}_{\text{sys}} + \hbar\sum_j \omega_j \hat{c}_j^\dagger \hat{c}_j + \hbar\sum_j \int_{-\infty}^{\infty} d\omega\omega\hat{\nu}_j(\omega)^\dagger\hat{\nu}_j(\omega) + i\hbar\sum_j [G_j^*\hat{L}\hat{c}_j^\dagger - G_j\hat{L}^\dagger\hat{c}_j] \\ & + i\hbar\sum_j \int_{-\infty}^{\infty} d\omega\sqrt{\frac{\kappa_j}{2\pi}}\,[\hat{\nu}_j^\dagger(\omega)\hat{c}_j - \hat{\nu}_j(\omega)\hat{c}_j^\dagger], \end{aligned} \tag{10.63} $$

where $\hat{H}_{\text{sys}} = \hat{H}_\Omega + \hat{H}$, $\hat{c}_j$ is the annihilation operator for the j^{th} added oscillator and $\hat{\nu}_j(\omega)$ is the Markovian bath operator with the correlation

$$ [\hat{\nu}_j(\omega), \hat{\nu}_k^\dagger(\omega)] = \delta_{j,k}\delta(\omega - \omega'). \tag{10.64} $$

If this is to be the same as Eq. (6.2), then the first two lines of Eq. (10.63) must give $\hat{H}_{\text{sys}} + \hat{H}_{\text{bath}}$ and the final line $\hat{V}$. Going to the same interaction picture as we did in Sec. 6.1, that is with respect to the Hamiltonians $\hat{H}_{\omega_{\rm sys}}$ and $\hat{H}_{\rm bath}$, we get

$$\hat{V}_{\rm int}(t) = i\hbar\sum_j [G_j^*\hat{L}e^{-i\Omega(t-t_0)}\hat{c}_j(t)^\dagger - G_j\hat{L}^\dagger e^{i\Omega(t-t_0)}\hat{c}_j(t)]. \tag{10.65}$$

Comparing with Eq. (6.14), for the enlarged system method to be correct we need

$$\sum_k g_k\hat{a}_k e^{-i\omega_k(t-t_0)} = \sum_j G_j\hat{c}_j(t). \tag{10.66}$$

To calculate $\hat{c}_j(t)$ we use the fact that

$$d_t\hat{c}_j(t) = -i\omega_j\hat{c}_j(t) - \frac{\kappa_j}{2}\hat{c}_j(t) - \sqrt{\kappa_j}\,\hat{\nu}_{{\rm in},j}(t) \tag{10.67}$$

where $\hat{\nu}_{{\rm in},j}(t)$ is the input field which has a time commutator $[\hat{\nu}_{{\rm in},j}(t), \hat{\nu}^\dagger_{{\rm in},k}(s)] = \delta_{j,k}\delta(t-s)$. For a derivation of equation Eq. (10.67) see Ref. [65]. This can be integrated to give

$$\hat{c}_j(t) \;=\; \sqrt{\kappa_j}\int_{t_0}^t e^{-\kappa_j(t-s)/2-i\omega_j(t-s)}\hat{\nu}_{{\rm in},j}(s)ds + \hat{c}_j(t_0)e^{-\kappa_j(t-t_0)/2-i\omega_j(t-t_0)}. \tag{10.68}$$

It is not obvious that $\sum_j G_j\hat{c}_j(t)$ is the same as Eq. (10.66). However the time commutator for the bath operators is

$$[\sum_j g_j\hat{a}_j e^{-i\omega_j(t-t_0)}, \sum_k g_k^*\hat{a}_k^\dagger e^{i\omega_k(t-t_0)}]e^{i\Omega(t-s)} = \alpha(t-s). \tag{10.69}$$

In terms of the enlarged system this means

$$\begin{aligned}
&\sum_{j,k} G_jG_k^*[\hat{c}_j(t), \hat{c}_k^\dagger(s)]e^{i\Omega(t-s)} \\
&= \sum_j |G_j|^2 e^{-\kappa_j(t+s)/2-i(\omega_j-\Omega)(t-s)}[1+\kappa_j\int_{t_0}^t\int_{t_0}^s e^{+\kappa_j(t'+s')/2+i\omega_j(t'-s')}\delta(t'-s')dt'ds'] \\
&= \sum_j |G_j|^2 e^{-\kappa_j|t-s|/2-i(\omega_j-\Omega)(t-s)} \\
&= \alpha(t-s),
\end{aligned} \tag{10.70}$$

provide $\alpha(t-s)$ has the form depicted in Eq. (10.62). It should noted that this result is exact. It is not necessary to discard initial transients as in the derivation in Ref [115].

Since we have shown that the total Hamiltonian for the enlarged system is equivalent to the standard non-Markovian one, then the total states $|\Psi(t)\rangle$ must be the same. We can define a reduced state (in the Schrödinger picture) for the enlarged system as $W_{\rm Sch}(t)$ which has the Markovian master equation

$$\begin{aligned}
d_tW_{\rm Sch}(t) \;=\; & -\frac{i}{\hbar}[\hat{H}_\Omega + \hat{H} + \hbar\sum_j\omega_j\hat{c}_j^\dagger\hat{c}_j + i\hbar\sum_j(G_j^*\hat{L}\hat{c}_j^\dagger - G_j\hat{L}^\dagger\hat{c}_j), W_{\rm Sch}(t)] \\
& + \sum_j\kappa_j\mathcal{D}[\hat{c}_j]W_{\rm Sch}(t).
\end{aligned} \tag{10.71}$$

The reduced state for the system in the Ω-interaction picture is

$$\rho_{\rm red}(t) \;=\; e^{\frac{i}{\hbar}\hat{H}_\Omega(t-t_0)}Tr_{\rm enl}[W_{\rm Sch}(t)]e^{-\frac{i}{\hbar}\hat{H}_\Omega(t-t_0)} = Tr_{\rm enl}[W_{\rm red}(t)], \tag{10.72}$$

where the trace is performed over the added oscillators and

$$W_{\rm red}(t) = e^{i\sum_j \omega_j \hat{c}_j^\dagger \hat{c}_j (t-t_0)+\frac{i}{\hbar}\hat{H}_\Omega (t-t_0)} W_{\rm Sch}(t) e^{-i\sum_j \omega_j \hat{c}_j^\dagger \hat{c}_j (t-t_0)-\frac{i}{\hbar}\hat{H}_\Omega (t-t_0)}. \tag{10.73}$$

This allows us to define a new master equation for the reduced state $W_{\rm red}(t)$ as

$$\begin{aligned} d_t W_{\rm red}(t) &= [-\frac{i}{\hbar}\hat{H}_{\rm int} + \sum_j [G_j^* \hat{L}\hat{c}_j^\dagger e^{i(\omega_j-\Omega)(t-t_0)} - G_j \hat{L}^\dagger \hat{c}_j e^{-i(\omega_j-\Omega)(t-t_0)}], W_{\rm red}(t)] \\ &\quad + \sum_j \kappa_j \mathcal{D}[\hat{c}_j] W_{\rm red}(t). \end{aligned} \tag{10.74}$$

which can be solved by standard Markovian techniques, for example quantum trajectories [28, 34, 50, 66, 67, 95]. This results in dM^J coupled differential equations, where d is the dimensions of the system, J is the number of added oscillators and M is the dimension (truncated) of the oscillator.

10.4 Numerical example: The driven TLA

In this section I will apply the above theory to a driven TLA with a simple non-Markovian memory function,

$$\alpha(t-s) = \frac{\gamma\kappa}{4} e^{i(\omega_{\rm env}-\omega_{\rm sys})(t-s)} e^{-\kappa|t-s|/2}, \tag{10.75}$$

where $\omega_{\rm env}$ is the central frequency of the environment, κ represent the exponential decay of bath memory and γ is the Markovian limit decay rate. That is, in the $\kappa \to \infty$ limit, $\alpha(t-s) = \gamma\delta(t-s)$, which is the Markovian limit of the memory function. We choose an interaction picture such the $\omega_{\rm sys} = \omega_{\rm env}$ so that this memory function simplifies to

$$\alpha(t-s) = \frac{\gamma\kappa}{4} e^{-\kappa|t-s|/2}, \tag{10.76}$$

which is consistent with the quadrature unravelings assumptions. This results in $\alpha(t-s) = \beta(t-s)$. However before we apply our theory to the TLA let us revise the standard TLA model.

10.4.1 The TLA

If we have a TLA driven by a classical electromagnetic field the system Hamiltonian for the TLA under the RWA approximation is

$$\hat{H}_{\rm sys} = \hbar\frac{\omega_0}{2}\hat{\sigma}_z + \hbar\frac{\Omega_{\rm dri}}{2}[\hat{\sigma}e^{i\omega_{\rm c}(t-t_0)} + \hat{\sigma}^\dagger e^{-i\omega_{\rm c}(t-t_0)}], \tag{10.77}$$

where $\Omega_{\rm dri}$ is the Rabi frequency and $\omega_{\rm c}$ is the oscillator frequency of the classical driving field and ω_0 is zero point energy of the free TLA. However as shown in Eq. (6.2) we can also write $\hat{H}_{\rm sys}$ as $\hat{H}_{\omega_{\rm sys}} + \hat{H}(t)$. If $\hat{H}_{\omega_{\rm sys}} = \omega_{\rm sys}\hat{\sigma}_z/2$, then in the $\omega_{\rm sys}$ interaction picture gives

$$\hat{H}_{\rm int}(t) = \hbar\frac{\omega_0-\omega_{\rm sys}}{2}\hat{\sigma}_z + \hbar\frac{\Omega_{\rm dri}}{2}[\hat{\sigma}e^{i(\omega_{\rm c}-\omega_{\rm sys})(t-t_0)} + \hat{\sigma}^\dagger e^{-i(\omega_{\rm c}-\omega_{\rm sys})(t-t_0)}], \tag{10.78}$$

For our purposes we assume $\omega_{\rm sys} = \omega_{\rm c}$. So

$$\hat{H}_{\rm int}(t) = \hbar\frac{\Delta}{2}\hat{\sigma}_z + \hbar\frac{\Omega_{\rm dri}}{2}\hat{\sigma}_x, \tag{10.79}$$

where $\Delta = \omega_0 - \omega_{\rm sys}$ is the detuning.

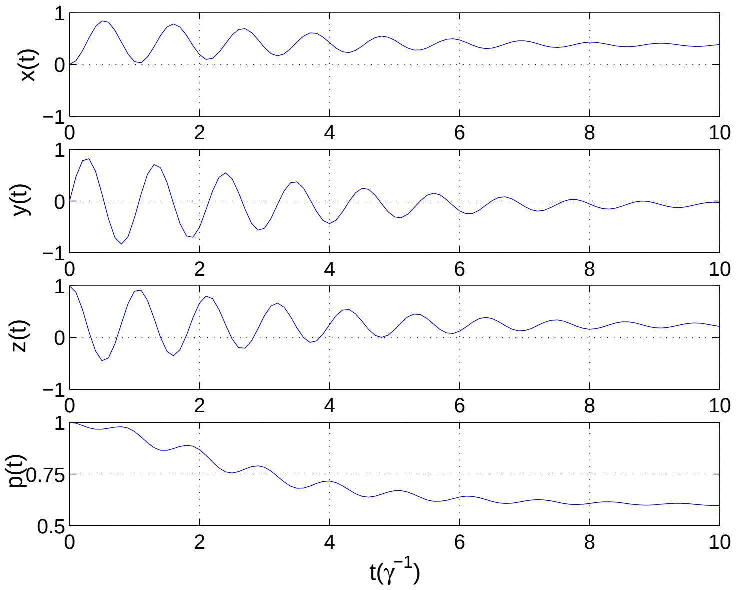

Figure 10.1: This figure depicts the Bloch vector components of the reduced state of a driven TLA calculated by the enlarged system method. In this figure all calculations were done using the initial system state $|\psi(0)\rangle = |e\rangle$ with system parameters $\kappa = \gamma$, $\Omega_{\rm dri} = 5\gamma$ and $\Delta = 3\gamma$. Time is measured in units γ^{-1}.

10.4.2 Enlarged system method

For the driven TLA with a memory function given by Eq. (10.75) the master equation for the enlarged systems is

$$d_t W_{\rm red}(t) = [-\frac{i\Delta}{2}\hat{\sigma}_z - \frac{i\chi}{2}\hat{\sigma}_x + \frac{\gamma\Omega_{\rm dri}}{4}(\hat{\sigma}\hat{c} - \hat{\sigma}^\dagger\hat{c}), W_{\rm red}(t)] + \kappa\mathcal{D}[\hat{c}]W_{\rm red}(t). \tag{10.80}$$

Using $\kappa = \gamma$, $\Omega_{\rm dri} = 5\gamma$ and $\Delta = 3\gamma$ the reduced state is shown in Fig. 10.1. For this simple case it was noted that the truncation error involved in the enlarged system state method was negligible. Because of this we use this reduced state for comparison with the ensemble average of the non-Markovian SSEs.

10.4.3 Coherent-state unraveling for the driven TLA

Applying the coherent non-Markovian SSE theory to the driven TLA, we find that we can rewrite the actual non-Markovian SSE as

$$\begin{aligned} d_t|\psi_z(t)\rangle = \Big[& -i\frac{\Delta}{2}\hat{\sigma}_z - i\frac{\Omega_{\rm dri}}{2}\hat{\sigma}_x - (\hat{\sigma}^\dagger - \langle\hat{\sigma}^\dagger\rangle_t)^{(0)}\hat{C}_z(t) + \Big\langle(\hat{\sigma}^\dagger - \langle\hat{\sigma}^\dagger\rangle_t)^{(0)}\hat{C}_z(t)\Big\rangle_t \\ & + z^*(t)(\hat{\sigma} - \langle\hat{\sigma}\rangle_t)\Big]|\psi_z(t)\rangle, \end{aligned} \tag{10.81}$$

and the noise function for the TLA becomes

$$z(t,s) = z(t_0,s) + \int_{t_0}^{t} \alpha(t-t')\langle\hat{\sigma}\rangle_{t'}dt'. \tag{10.82}$$

To calculate the complex amplitudes for the actual non-Markovian SSE we apply the system state $|\psi_z(t)\rangle = c_e(t)|e\rangle + c_b(t)|b\rangle$ to Eq. (10.81) and expand ${}^{(0)}\hat{C}_z(t)$ as

$$ {}^{(0)}\hat{C}_z(t) = \sum_m \hat{m}\,{}^{(0)}C_{m,z}(t) \tag{10.83} $$

where $m = \{\sigma,\, \sigma^\dagger,\, \sigma_z,\, I\}$. This results in

$$ \begin{aligned} d_t c_b(t) &= i\frac{\Delta}{2}c_b(t) - i\frac{\Omega_{\rm dri}}{2}c_e(t) + z^*(t,t)c_e(t)|c_e(t)|^2 - {}^{(0)}C_{\sigma^\dagger,z}c_b^3(t)c_e^{*2}(t) + {}^{(0)}C_{\sigma,z}c_b(t)|c_e(t)|^2 \\ &\quad \times(1+|c_e(t)|^2) - {}^{(0)}C_{\sigma_z,z}c_b^2(t)c_e^*(t)(1+2|c_e(t)|^2) + {}^{(0)}C_{I,z}c_b^2(t)c_e^*(t), \qquad (10.84) \\ d_t c_e(t) &= -i\frac{\Delta}{2}c_e(t) - i\frac{\Omega_{\rm dri}}{2}c_b(t) - z^*(t,t)c_e^2(t)c_b^*(t) - {}^{(0)}C_{\sigma,z}c_e(t)|c_b(t)|^2(1+|c_e(t)|^2) \\ &\quad + {}^{(0)}C_{\sigma^\dagger,z}c_b^2(t)c_e^*(t)|c_b(t)|^2 + {}^{(0)}C_{\sigma_z,z}c_b(t)|c_b(t)|^2(1+2|c_e(t)|^2) \\ &\quad - {}^{(0)}C_{I,z}c_b(t)|c_b(t)|^2. \qquad (10.85) \end{aligned} $$

In this equation the noise function is given by

$$ z^*(t,s) = z^*(t_0,s) + \frac{\gamma\kappa}{4}e^{-\kappa s/2}\int_{t_0}^{t} e^{\kappa t'/2}c_b(t')c_e^*(t')dt', \tag{10.86} $$

where $z^*(t_0,s)$ is defined by the correlation

$$ \tilde{E}[z(t_0,s)z^*(t_0,s')] = \frac{\gamma\kappa}{4}e^{-\kappa|s-s'|/2}. \tag{10.87} $$

This is generated by having $z^*(t_0,s)$ obey the following stochastic differential equation,

$$ d_t z^*(t_0,s) = -\frac{\kappa}{2}z^*(t_0,s) + \frac{\kappa}{2}\sqrt{\gamma}\,\xi^*(s), \tag{10.88} $$

with $z^*(t_0,s)$ being a Gaussian random variable (GRV) satisfying

$$ E[z(t_0,t_0)z^*(t_0,t_0)] = \frac{\kappa\gamma}{4} \tag{10.89} $$

Here $\xi(t)$ is standard complex white noise [64] and satisfies $E[\xi(t)\xi^*(s)] = \delta(t-s)$.

$0^{\rm th}$ Order Approximation

For the simple memory function, $J = 1$, which means ${}^{(0)}\hat{C}_z(t) = {}^{(0)}\hat{C}_z^{(j)}(t)$. The $0^{\rm th}$ order approximation occurs when we assume the form for ${}^{(0)}\hat{C}_z(t)$ in Eq. (10.36). From Eq. (10.76) this implies

$$ {}^{(0)}\hat{C}_z(t) = \frac{\gamma}{2}(1-e^{-\kappa(t-t_0)/2})\hat{\sigma}, \tag{10.90} $$

thus

$$ \begin{aligned} {}^{(0)}C_{\sigma,z}(t) &= \frac{\gamma}{2}(1-e^{-\kappa(t-t_0)/2}), \qquad (10.91) \\ {}^{(0)}C_{\sigma^\dagger,z}(t) &= {}^{(0)}C_{\sigma_z,z}(t) = {}^{(0)}C_{I,z}(t) = 0. \qquad (10.92) \end{aligned} $$

$1^{\rm st}$ Order Approximation

The $1^{\rm st}$ first order approximation occurs when we assume a form for ${}^{(1)}\hat{C}_z^{(j,k)}(t)$, by Eqs. (10.37) and (10.76) this means

$$ {}^{(1)}\hat{C}_z(t) = \frac{\gamma}{2}(1-e^{-\kappa(t-t_0)/2})[\hat{\sigma}, {}^{(0)}\hat{C}_z(t)], \tag{10.93} $$

thus

$$
\begin{aligned}
{}^{(1)}C_{\sigma,z}(t) &= \gamma(1-e^{-\kappa(t-t_0)/2})\,{}^{(0)}C_{\sigma_z,z}(t), && (10.94)\\
{}^{(1)}C_{\sigma_z,z}(t) &= -\frac{\gamma}{2}(1-e^{-\kappa(t-t_0)/2})\,{}^{(0)}C_{\sigma^\dagger,z}(t), && (10.95)\\
{}^{(1)}C_{\sigma^\dagger,z}(t) &= {}^{(1)}C_{I,z}(t)=0. && (10.96)
\end{aligned}
$$

The zeroth order functionals are found by applying the TLA operators to Eq. (10.22), giving

$$
\begin{aligned}
d_t\,{}^{(0)}\hat{C}_z(t) &= \frac{\gamma\kappa}{4}\hat{\sigma}-\frac{\kappa}{2}\,{}^{(0)}\hat{C}_z(t)+z^*(t,t)[\hat{\sigma},{}^{(0)}\hat{C}_z(t)]-i[\frac{\Delta}{2}\hat{\sigma}_z+\frac{\Omega_{\text{dri}}}{2}\hat{\sigma}_x,{}^{(0)}\hat{C}_z(t)]\\
&\quad-[\hat{\sigma}^{\dagger}\,{}^{(0)}\hat{C}_z(t),{}^{(0)}\hat{C}_z(t)]-\hat{\sigma}^{\dagger}\,{}^{(1)}\hat{C}_z(t). && (10.97)
\end{aligned}
$$

Using Eq. (10.83) this gives the following four coupled nonlinear equations

$$
\begin{aligned}
d_t\,{}^{(0)}C_{\sigma,z}(t) &= \tfrac{1}{4}\gamma\kappa-\frac{\kappa}{2}\,{}^{(0)}C_{\sigma,z}(t)+i\Delta\,{}^{(0)}C_{\sigma,z}(t)-i\Omega_{\text{dri}}\,{}^{(0)}C_{\sigma_z,z}(t)+2z^*(t,t)\,{}^{(0)}C_{\sigma_z,z}(t)\\
&\quad+{}^{(0)}C^2_{\sigma,z}(t), && (10.98)\\
d_t\,{}^{(0)}C_{\sigma^\dagger,z}(t) &= -\frac{\kappa}{2}\,{}^{(0)}C_{\sigma^\dagger,z}(t)+i\Omega_{\text{dri}}\,{}^{(0)}C_{\sigma_z,z}(t)-i\Delta\,{}^{(0)}C_{\sigma^\dagger,z}(t)+2\,{}^{(0)}C_{\sigma_z,z}(t)[{}^{(0)}C_{I,z}(t)\\
&\quad-{}^{(0)}C_{\sigma_z,z}(t)]-{}^{(0)}C_{\sigma^\dagger,z}(t)\,{}^{(0)}C_{\sigma,z}(t)-[{}^{(1)}C_{I,z}(t)-{}^{(1)}C_{\sigma_z,z}(t)], && (10.99)\\
d_t\,{}^{(0)}C_{\sigma_z,z}(t) &= -\frac{\kappa}{2}\,{}^{(0)}C_{\sigma_z,z}(t)+i\frac{\Omega_{\text{dri}}}{2}\,{}^{(0)}C_{\sigma^\dagger,z}(t)-i\frac{\Omega_{\text{dri}}}{2}\,{}^{(0)}C_{\sigma,z}(t)-{}^{(0)}C_{\sigma,z}(t)[{}^{(0)}C_{I,z}(t)\\
&\quad-{}^{(0)}C_{\sigma_z,z}(t)]-z^*(t,t)\,{}^{(0)}C_{\sigma^\dagger,z}(t)-\tfrac{1}{2}\,{}^{(1)}C_{\sigma,z}(t), && (10.100)\\
d_t\,{}^{(0)}C_{I,z}(t) &= -\frac{\kappa}{2}\,{}^{(0)}C_{I,z}(t)-\tfrac{1}{2}\,{}^{(1)}C_{\sigma,z}(t). && (10.101)
\end{aligned}
$$

which can be solved in parallel with Eq. (10.84).

2^{nd} Order Approximation

The 2^{nd} order approximation occurs when we assume a form for ${}^{(2)}\hat{C}_z^{(j,k,l)}(t)$, by Eqs. (10.38) and (10.76) this means

$$
{}^{(2)}\hat{C}_z(t)=\frac{\gamma}{2}(1-e^{-\kappa(t-t_0)/2})[\hat{\sigma},{}^{(1)}\hat{C}_z(t)], \tag{10.102}
$$

thus

$$
\begin{aligned}
{}^{(2)}C_{\sigma,z}(t) &= \gamma(1-e^{-\kappa(t-t_0)/2})\,{}^{(1)}C_{\sigma_z,z}(t), && (10.103)\\
{}^{(2)}C_{\sigma_z,z}(t) &= -\frac{\gamma}{2}(1-e^{-\kappa(t-t_0)/2})\,{}^{(1)}C_{\sigma^\dagger,z}(t), && (10.104)\\
{}^{(2)}C_{\sigma^\dagger,z}(t) &= {}^{(2)}C_{I,z}(t)=0. && (10.105)
\end{aligned}
$$

The zeroth order functionals are given by Eqs. (10.98) – (10.101), however we now need equations for ${}^{(1)}\hat{C}_z(t)$. The fist order functionals are found applying TLA operators to Eq. (10.30). With a memory function specified by Eq. (10.76) we get

$$
\begin{aligned}
d_t\,{}^{(1)}\hat{C}_z(t) &= \frac{\gamma\kappa}{4}[\hat{\sigma},{}^{(0)}\hat{C}_z(t)]-\kappa\,{}^{(1)}\hat{C}_z(t)-i[\frac{\Delta}{2}\hat{\sigma}_z+\frac{\Omega_{\text{dri}}}{2}\hat{\sigma}_x,{}^{(1)}\hat{C}_z(t)]+z^*(t,t)[\hat{\sigma},{}^{(1)}\hat{C}_z(t)]\\
&\quad-[\hat{\sigma}^{\dagger}\,{}^{(1)}\hat{C}_z(t),{}^{(0)}\hat{C}_z(t)]-[\hat{\sigma}^{\dagger}\,{}^{(0)}\hat{C}_z(t),{}^{(1)}\hat{C}_z(t)]-\hat{\sigma}^{\dagger}\,{}^{(2)}\hat{C}_z(t). && (10.106)
\end{aligned}
$$

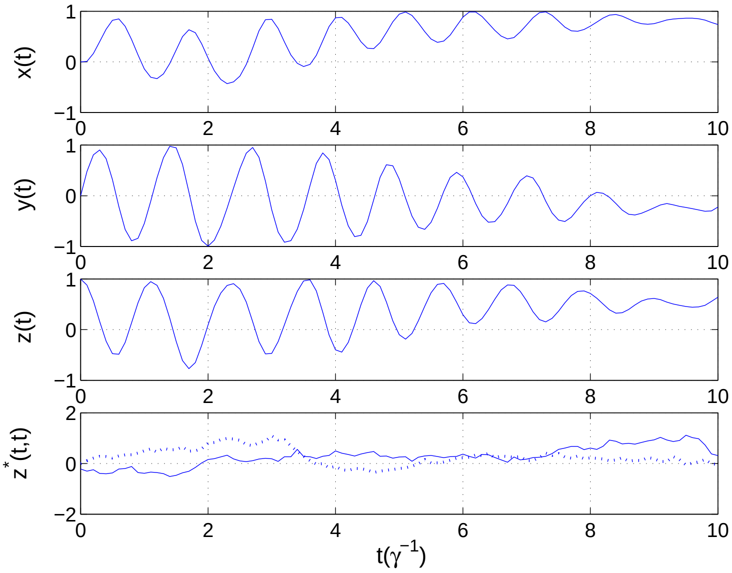

Figure 10.2: An example trajectory for the coherent-state non-Markovian SSE (1^{st} order perturbation) for a driven TLA immersed in a non-Markovian bath. Other details are as in Fig. 10.1.

Using Eq. (10.103) this turns into the four equations

$$
\begin{aligned}
d_t\,{}^{(1)}C_{\sigma,z}(t) &= \tfrac{1}{2}\gamma\kappa\,{}^{(0)}C_{\sigma_z,z}(t) - \kappa\,{}^{(1)}C_{\sigma,z}(t) + i\Delta\,{}^{(1)}C_{\sigma,z}(t) - i\Omega_{\mathrm{dri}}\,{}^{(1)}C_{\sigma_z,z}(t) \\
&\quad + 2z^*(t,t)\,{}^{(1)}C_{\sigma_z,z}(t) + 2\,{}^{(0)}C_{\sigma,z}(t)\,{}^{(1)}C_{\sigma,z}(t), && (10.107)\\
d_t\,{}^{(1)}C_{\sigma^\dagger,z}(t) &= -\kappa\,{}^{(1)}C_{\sigma^\dagger,z}(t) + i\Omega_{\mathrm{dri}}\,{}^{(1)}C_{\sigma_z,z}(t) - i\Delta\,{}^{(1)}C_{\sigma^\dagger,z}(t) + 2\,{}^{(1)}C_{\sigma_z,z}(t)[{}^{(0)}C_{I,z}(t) \\
&\quad - {}^{(0)}C_{\sigma_z,z}(t)] + 2\,{}^{(0)}C_{\sigma_z,z}(t)[{}^{(1)}C_{I,z}(t) - {}^{(1)}C_{\sigma_z,z}(t)] - [{}^{(1)}C_{\sigma^\dagger,z}(t)\,{}^{(0)}C_{\sigma,z}(t) \\
&\quad + {}^{(0)}C_{\sigma^\dagger,z}(t)\,{}^{(1)}C_{\sigma,z}(t)] - {}^{(2)}C_{I,z}(t) + {}^{(2)}C_{\sigma_z,z}(t), && (10.108)\\
d_t\,{}^{(1)}C_{\sigma_z,z}(t) &= -\frac{\gamma\kappa}{4}\,{}^{(0)}C_{\sigma^\dagger,z}(t) - \kappa\,{}^{(1)}C_{\sigma_z,z}(t) + i\frac{\Omega_{\mathrm{dri}}}{2}\,{}^{(1)}C_{\sigma^\dagger,z}(t) - i\frac{\Omega_{\mathrm{dri}}}{2}\,{}^{(1)}C_{\sigma,z}(t) \\
&\quad - {}^{(1)}C_{\sigma,z}(t)[{}^{(0)}C_{I,z}(t) - {}^{(0)}C_{\sigma_z,z}(t)] - {}^{(0)}C_{\sigma,z}(t)[{}^{(1)}C_{I,z}(t) - {}^{(1)}C_{\sigma_z,z}(t)] \\
&\quad - z^*(t,t)\,{}^{(1)}C_{\sigma^\dagger,z}(t) - \tfrac{1}{2}\,{}^{(2)}C_{\sigma,z}(t), && (10.109)\\
d_t\,{}^{(1)}C_{I,z}(t) &= -\kappa\,{}^{(1)}C_{I,z}(t) - \tfrac{1}{2}\,{}^{(2)}C_{\sigma,z}(t). && (10.110)
\end{aligned}
$$

To illustrate how accurate our perturbation method is, the difference between the reduced state calculated via the enlarged system method and the ensemble average from the coherent non-Markovian SSE is plotted in Fig. 10.3. The blue line corresponds to the 0^{th} order perturbation, the red is the 1^{st} and the green is the 2^{nd}. It is observed that the 1^{st} and 2^{nd} order perturbation are a lot more accurate then the 0^{th} order perturbation. However, it can be seen that the 2^{nd} order perturbation is not necessarily more accurate than the 1^{st} order perturbation. This suggest that this perturbation method is an asymptotic expansion rather than a convergent series.

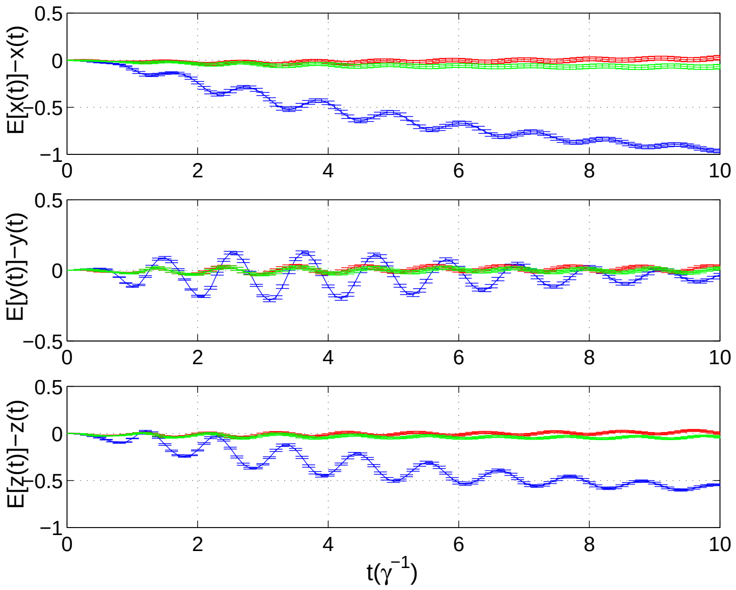

Figure 10.3: This figure depicts the difference between the reduced state calculated form our perturbative coherent non-Markovian SSE (1000 trajectories) and the enlarged system method. The blue line corresponds to the 0^{th} order perturbation, the red is the 1^{st} and the green is the 2^{nd}. Other details are as in Fig. 10.1.

10.4.4 Quadrature unraveling for the driven TLA

For the quadrature unraveling the actual non-Markovian SSE is

$$\begin{aligned} d_t|\psi_z(t)\rangle \quad &= \quad \Big[-i\frac{\Delta}{2}\hat{\sigma}_z - i\frac{\Omega_{\text{dri}}}{2}\hat{\sigma}_x - (\hat{\sigma}_x - \langle\hat{\sigma}_x\rangle_t)\ {}^{(0)}\hat{Q}_z(t) + \Big\langle (\hat{\sigma}_x - \langle\hat{\sigma}_x\rangle_t)\ {}^{(0)}\hat{Q}_z(t) \Big\rangle_t \\ &\quad + z(t,t)(\hat{\sigma} - \langle\hat{\sigma}\rangle_t)\Big]|\psi_z(t)\rangle, \end{aligned} \tag{10.111}$$

and the noise function for the TLA is

$$z(t,s) = z(t_0,s) + \int_0^t \beta(s-t')\langle\hat{\sigma}_x\rangle_{t'}dt'. \tag{10.112}$$

Again, as in the coherent case, we can calculate the complex amplitude equation via applying the state $|\psi_z(t)\rangle = c_e(t)|e\rangle + c_g(t)|g\rangle$ to Eq. (10.111) and expanding ${}^{(0)}\hat{Q}_z(t)$ as

$${}^{(0)}\hat{Q}_z(t) = \sum_m \hat{m}\,{}^{(0)}Q_{m,z}(t) \tag{10.113}$$

where $m = \{\sigma,\, \sigma^\dagger,\, \sigma_z,\, I\}$. This results in a coupled set of differential equations for $c_e(t)$ and $c_b(t)$ that depend on ${}^{(0)}Q_{m,z}(t)$ and $z(t,s)$. In these equations the real-valued noise is given by

$$z(t,s) \quad = \quad z(t_0,s) + \frac{\gamma\kappa}{4}e^{-\kappa s/2}\int_{t_0}^t e^{\kappa t'/2}[c_b(t')c_e^*(t') + c_b^*(t')c_e(t')]dt', \tag{10.114}$$

where $z(t_0,s)$ is found by

$$\tilde{E}[z(t_0,s)z(t_0,s')] = \frac{\gamma\kappa}{4}e^{-\kappa|s-s'|/2}. \tag{10.115}$$

This is generated by

$$d_t z(t_0, s) = -\frac{\kappa}{2} z(t_0, s) + \frac{\kappa}{2}\sqrt{\gamma}\,\xi(s) \tag{10.116}$$

with $z(t_0, t_0)$ being a gaussian random variable satisfying $E[z(t_0, t_0)z(t_0, t_0)] = \kappa\gamma/4$. Here $\xi(s)$ is standard white noise and satisfies $E[\xi(t)\xi(s)] = \delta(t - s)$ [64].

0^{th} Order Approximation

The situation is greatly simplified with the memory function in Eq. (10.75), as $\beta(t, s) = \beta^{(j,cos)}(t, s) = \beta^{(j,cos)}(t, s)$, which in turn implies ${}^{(0)}\hat{Q}_z(t) = {}^{(0)}\hat{Q}_z^{(j,\cos)}(t) = {}^{(0)}\hat{Q}_z^{(j,\sin)}(t)$.

The 0^{th} order approximation is to set

$${}^{(0)}\hat{Q}_z(t) = \frac{\gamma}{2}(1 - e^{-\kappa(t-t_0)/2})\hat{\sigma}, \tag{10.117}$$

thus

$$\begin{aligned} {}^{(0)}Q_{\sigma,z}(t) &= \frac{\gamma}{2}(1 - e^{-\kappa(t-t_0)/2}), && (10.118\text{a}) \\ {}^{(0)}Q_{\sigma^\dagger,z}(t) &= {}^{(0)}Q_{\sigma_z,z}(t) = {}^{(0)}Q_{I,z}(t) = 0. && (10.118\text{b}) \end{aligned}$$

1^{th} Order Approximation

The first order approximation is to set

$${}^{(1)}\hat{Q}_z(t) = \frac{\gamma}{2}(1 - e^{-\kappa(t-t_0)/2})[\hat{\sigma}, {}^{(0)}\hat{Q}_z(t)] \tag{10.119}$$

thus

$$\begin{aligned} {}^{(1)}Q_{\sigma,z}(t) &= \gamma(1 - e^{-\kappa(t-t_0)/2})\,{}^{(0)}Q_{\sigma_z,z}(t), && (10.120\text{a}) \\ {}^{(1)}Q_{\sigma_z,z}(t) &= -\frac{\gamma}{2}(1 - e^{-\kappa(t-t_0)/2})\,{}^{(0)}Q_{\sigma^\dagger,z}(t), && (10.120\text{b}) \\ {}^{(1)}Q_{\sigma^\dagger,z}(t) &= {}^{(1)}Q_{I,z}(t) = 0. && (10.120\text{c}) \end{aligned}$$

The 0^{th} order functionals are found by applying TLA operators to Eqs. (10.52) and (10.53). With the simple memory function this gives

$$\begin{aligned} d_t\,{}^{(0)}\hat{Q}_z(t) &= \frac{\gamma\kappa}{4}\hat{\sigma} - \frac{\kappa}{2}\,{}^{(0)}\hat{Q}_z(t) + z(t,t)[\hat{\sigma}, {}^{(0)}\hat{Q}_z(t)] - i[\frac{\Delta}{2}\hat{\sigma}_z + \frac{\Omega_{\text{dri}}}{2}\hat{\sigma}_x, {}^{(0)}\hat{Q}_z(t)] \\ &\quad -[\hat{\sigma}_x\,{}^{(0)}\hat{Q}_z(t), {}^{(0)}\hat{Q}_z(t)] - \hat{\sigma}_x\,{}^{(1)}\hat{Q}_z(t). && (10.121) \end{aligned}$$

Using Eq. (10.113) this gives,

$$\begin{aligned} d_t\,{}^{(0)}Q_{\sigma,z}(t) &= \tfrac{1}{4}\gamma\kappa - \frac{\kappa}{2}\,{}^{(0)}Q_{\sigma,z}(t) + i\Delta\,{}^{(0)}Q_{\sigma,z}(t) - i\Omega_{\text{dri}}\,{}^{(0)}Q_{\sigma_z,z}(t) + 2z(t,t)\,{}^{(0)}Q_{\sigma_z,z}(t) \\ &\quad + {}^{(0)}Q^2_{\sigma,z}(t) - 2\,{}^{(0)}Q_{\sigma_z,z}(t)[{}^{(0)}Q_{I,z}(t) + {}^{(0)}Q_{\sigma_z,z}(t)] - {}^{(0)}Q_{\sigma^\dagger,z}(t)\,{}^{(0)}Q_{\sigma,z}(t) \\ &\quad -[{}^{(1)}Q_{I,z}(t) + {}^{(1)}Q_{\sigma_z,z}(t)], && (10.122) \\ d_t\,{}^{(0)}Q_{\sigma^\dagger,z}(t) &= -\frac{\kappa}{2}\,{}^{(0)}Q_{\sigma^\dagger,z}(t) + i\Omega_{\text{dri}}\,{}^{(0)}Q_{\sigma_z,z}(t) - i\Delta\,{}^{(0)}Q_{\sigma^\dagger,z}(t) + 2\,{}^{(0)}Q_{\sigma_z,z}(t)[{}^{(0)}Q_{I,z}(t) \\ &\quad - {}^{(0)}Q_{\sigma_z,z}(t)] - {}^{(0)}Q_{\sigma^\dagger,z}(t)\,{}^{(0)}Q_{\sigma,z}(t) + {}^{(0)}Q^2_{\sigma^\dagger,z}(t) - {}^{(1)}Q_{I,z}(t) \\ &\quad + {}^{(1)}Q_{\sigma_z,z}(t), && (10.123) \\ d_t\,{}^{(0)}Q_{\sigma_z,z}(t) &= -\frac{\kappa}{2}\,{}^{(0)}Q_{\sigma_z,z}(t) + i\frac{\Omega_{\text{dri}}}{2}\,{}^{(0)}Q_{\sigma^\dagger,z}(t) - i\frac{\Omega_{\text{dri}}}{2}\,{}^{(0)}Q_{\sigma,z}(t) - {}^{(0)}Q_{\sigma,z}(t)[{}^{(0)}Q_{I,z}(t) \\ &\quad - {}^{(0)}Q_{\sigma_z,z}(t)] + {}^{(0)}Q_{\sigma^\dagger,z}(t)[{}^{(0)}Q_{I,z}(t) + {}^{(0)}Q_{\sigma_z,z}(t)] - z(t,t)\,{}^{(0)}Q_{\sigma^\dagger,z}(t) \\ &\quad -\tfrac{1}{2}[{}^{(1)}Q_{\sigma,z}(t) - {}^{(1)}Q_{\sigma^\dagger,z}(t)], && (10.124) \\ d_t\,{}^{(0)}Q_{I,z}(t) &= -\frac{\kappa}{2}\,{}^{(0)}Q_{I,z}(t) - \tfrac{1}{2}[{}^{(1)}Q_{\sigma,z}(t) + {}^{(1)}Q_{\sigma^\dagger,z}(t). && (10.125) \end{aligned}$$

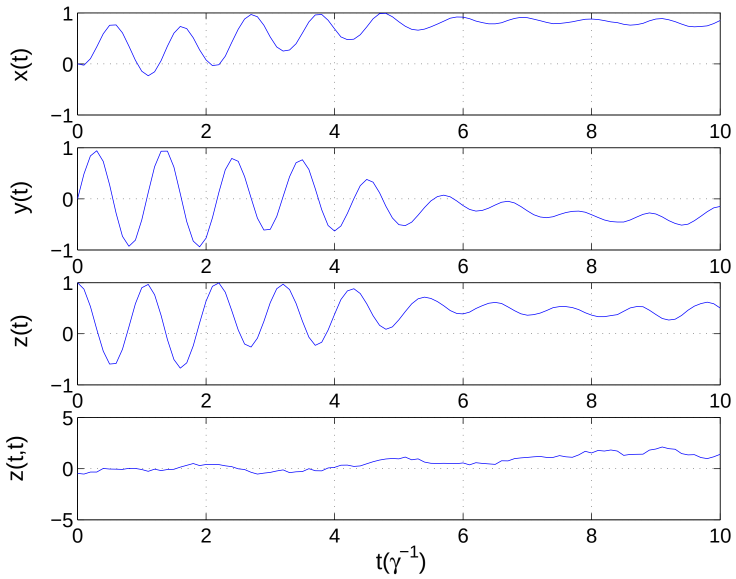

Figure 10.4: An example trajectory for the quadrature non-Markovian SSE ($1^{\rm st}$ order perturbation) for a driven TLA immersed in a non-Markovian bath. Other details are as in Fig. 10.1.

which can be solved in parallel with $c_e(t)$ and $c_b(t)$.

To illustrate how accurate this perturbation method is for the quadrature unraveling. Fig. 10.5 shows the difference between the reduced state calculated via the enlarged system method and the ensemble average from the quadrature non-Markovian SSEs for the $0^{\rm th}$ (blue) and $1^{\rm st}$ (red) order perturbation. As in the coherent case we find the $1^{\rm st}$ order perturbation is more accurate then the $0^{\rm th}$.

10.5 Post-Markovian perturbation

In this section I will extend the Yu, Diósi Gisin and Strunz (YDGS) post-Markovian perturbation [139, 140] to include the quadrature unraveling and compare the post-Markovian method with the above perturbation method.

The basic idea behind their perturbation method is to expand the operators ${}^{(0)}\hat{c}_z(t,s)$ in powers of $(t-s)$ around the point $t=s$ (this is why it is called the post Markovian perturbation). That is

$$ {}^{(0)}\hat{c}_z(t,s) = {}^{(0)}\hat{c}_z(s,s) + [\partial_t {}^{(0)}\hat{c}_z(t,s)|_{t=s}](t-s) + \tfrac{1}{2}[\partial_t^2 {}^{(0)}\hat{c}_z(t,s)|_{t=s}](t-s)^2 + ..., \quad (10.126) $$

where ${}^{(0)}\hat{c}_z(s,s) = \hat{L}$. To find the first order term we simply evaluate Eq. (10.21) at $t=s$

$$ \partial_t {}^{(0)}\hat{c}_z(t,s)|_{t=s} = -\frac{i}{\hbar}[\hat{H}_{\rm int}(s),\hat{L}] - [\hat{L}^\dagger {}^{(0)}\hat{C}_z(s),\hat{L}] - \hat{L}^\dagger[\hat{L},{}^{(0)}\hat{C}_z(s)]. \quad (10.127) $$

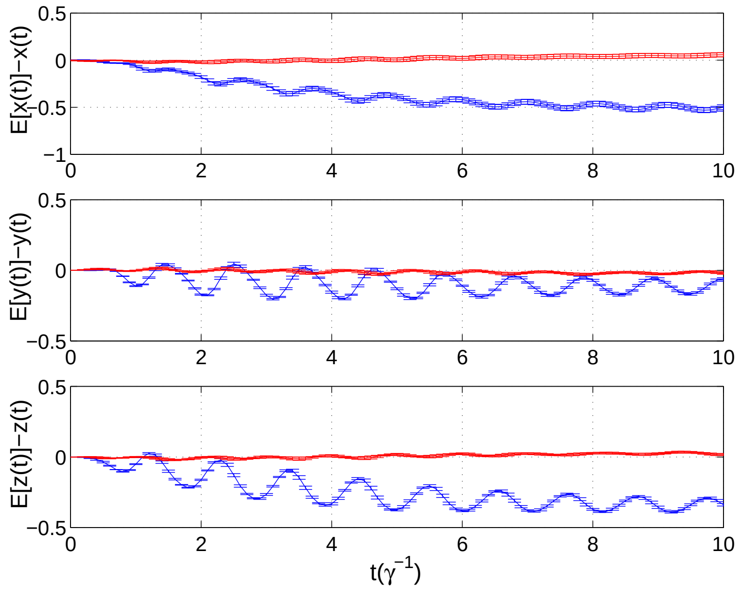

Figure 10.5: This figure depicts the difference between the reduced state calculated form our perturbative quadrature non-Markovian SSE (1000 trajectories) and the enlarged system method. The blue line corresponds to the 0^{th} and the red is the 1^{st} order perturbation. Other details are as in Fig. 10.1.

Thus the functional ${}^{(0)}\hat{C}_z(t)$ for this perturbation is given by

$$\begin{aligned} {}^{(0)}\hat{C}_z(t) &= g_0(t)\hat{L} - g_1(t)\frac{i}{\hbar}[\hat{H}_{\text{int}}(t),\hat{L}] - \int_{t_0}^{t}\alpha(t-s)(t-s)[\hat{L}^{\dagger\,(0)}\hat{C}_z(s),\hat{L}]ds \\ &\quad - \int_{t_0}^{t}\alpha(t-s)(t-s)\hat{L}^\dagger[\hat{L},{}^{(0)}\hat{C}_z(s)]ds, \end{aligned} \tag{10.128}$$

where

$$g_0(t) = \int_{t_0}^{t}\alpha(t-s)ds, \tag{10.129}$$

$$g_1(t) = \int_{t_0}^{t}\alpha(t-s)(t-s)ds. \tag{10.130}$$

This equation can not be solved. However, if we make a further approximation

$${}^{(0)}\hat{C}_z(s) = \int_{t_0}^{s}\alpha(s-u)\hat{L}du, \tag{10.131}$$

for $s>t$. Substituting this into Eq. (10.128) gives

$${}^{(0)}\hat{C}_z(t) = g_0(t)\hat{L} - g_1(t)\frac{i}{\hbar}[\hat{H}_{\text{int}}(t),\hat{L}] - g_2(t)[\hat{L}^\dagger\hat{L},\hat{L}], \tag{10.132}$$

where

$$g_2(t) = \int_{t_0}^{t}\int_{t_0}^{s}\alpha(t-s)\alpha(s-u)(t-s)duds, \tag{10.133}$$

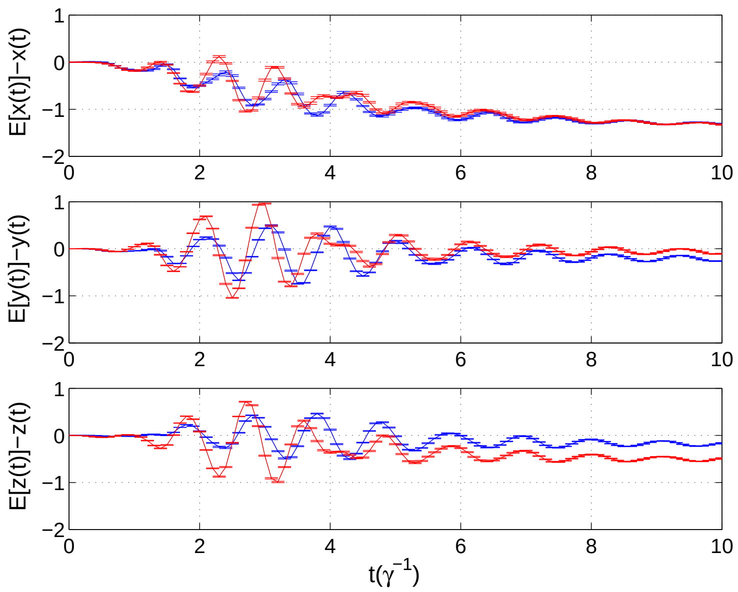

Figure 10.6: This figure shows the difference between the reduced state calculated from the YDGS post-Markovian non-Markovian SSE method and the enlarged system method, for both the coherent (red line) and quadrature (blue line) unraveling. Other details are as in Fig. 10.1.

which can be solved. The same could be done for the second order terms, but as well as making an approximation for ${}^{(0)}\hat{C}_z(s)$ we would need to approximate $d_s{}^{(0)}\hat{C}_z(s)$. For the purpose of this paper we will only go to first order.

To extend the idea to the quadrature case we Taylor expand the operator ${}^{(0)}\hat{q}_z(t,s)$ in powers of $(t-s)$ around the point $t=s$. To find the first order term we simply evaluate Eq. (10.51) at $t=s$. With the approximation

$$ {}^{(0)}\hat{Q}_z(s) = \int_{t_0}^{s} \beta(s-u)\hat{L}du \tag{10.134}$$

we get

$$ {}^{(0)}\hat{Q}_z(t) = h_0(t)\hat{L} - h_1(t)\frac{i}{\hbar}[\hat{H}_{\text{int}}(t),\hat{L}] - h_2(t)[\hat{L}_x\hat{L},\hat{L}]. \tag{10.135}$$

where

$$\begin{aligned} h_0(t) &= \int_{t_0}^{t} \beta(t-s)ds, & (10.136)\\ h_1(t) &= \int_{t_0}^{t} \beta(t-s)(t-s)ds, & (10.137)\\ h_2(t) &= \int_{t_0}^{t}\int_{t_0}^{s} \beta(t-s)\beta(s-u)(t-s)duds. & (10.138)\end{aligned}$$

For the simple TLA system it is easy to generate these approximate expressions for ${}^{(0)}\hat{C}_z(t)$ and ${}^{(0)}\hat{Q}_z(t)$ for all time, hence we can obtain solution to the non-Markovian SSE. To compare the YDGS post-Markovian non-Markovian SSE method with our perturbation method, we again plot the difference between the YDGS method (when 1000 trajectories where used) and the enlarged systems method. The results of this are shown in Fig. 10.6, where it is observed that the YDGS first order perturbation has a greater error than Wiseman and my first order perturbation method (Figures 10.3 and 10.5). This is perhaps not surprising, as the system we modeled has $\kappa = 1$, which implies it is very non-Markovian. Since one of the requirements of YDGS perturbation method is for the environment to be close the Markovian regime one would expect their method to fail in this regime.

In Ref. [139] YDGS suggest an alternative perturbation method. The functional operator ${}^{(0)}\hat{C}_z(t)$, which equals $\bar{O}_z(t)$ in their notation, is expanded by the functional expansion

$$\begin{aligned}\bar{O}_z(t) &= \bar{O}^{(0)}(t) + \int_{t_0}^{t} \bar{O}^{(1)}(t,v)z(v)dv + \int_{t_0}^{t}\int_{t_0}^{t} \bar{O}^{(2)}(t,v_1,v_2)z(v_1)z(v_2)dv_1dv_2 + ... \\ &\quad + \int_{t_0}^{t} ... \int_{t_0}^{t} \bar{O}^{(n)}(t,v_1,...,v_n)z(v_1)...z(v_n)dv_1...dv_n + ..., \end{aligned} \tag{10.139}$$

It can be shown that one can establish a set of coupled differential equations for these operators provided $\alpha(t-s)$ is given by Eq. (10.10). To truncated this perturbation at $\bar{O}^{(n)}$ one has to assume a value $\bar{O}^{(n+1)}$. It turns out that for all operators $\bar{O}^{(n)}$ other then $\bar{O}^{(0)}$ the only reason the operators change from their initial value 0 at t_0 is if the assumed $\bar{O}^{(n+1)}$ is nonzero. This suggest that this method is highly dependent on the assumed value for $\bar{O}^{(n+1)}$.

10.6 Summary of chapter

In this chapter I have presented a perturbation method for solving the coherent and quadrature non-Markovian SSEs. This perturbation method is easily extended to any order and is not limited to the post Markovian regime. However, the environment is restricted such that it has a correlation function satisfying Eq. (10.10). As shown in Ref. [115] most non-Markovian environments can be simulated via this correlation function with a relative small J. This suggest that this perturbation method might be useful for simulating non-Markovian evolution for $\rho_{\rm red}(t)$.

One appealing feature of this method is that it provides a perturbative solution for $\rho_{\rm red}(t)$ which is positive by definition. However there is another method, namely Imamoḡlu's enlarged system method [88, 115], which provides a (better) solution for $\rho_{\rm red}(t)$. Imamoḡlu's enlarged system method requires, generally fewer coupled differential equations to solve (dM^J compared to $d^2(J^n+J^{n-1}+...+J)+d+J$ for our method) and the only approximation comes in by a truncation of the Hilbert space of the fictitious modes (M). As one increases the basis size for these modes this method will converge to the correct solution. By contrast, convergence has not been shown for our method.

This does not mean that this method is useless, as the primary interest in this method is not to simulate $\rho_{\rm red}(t)$, but to simulate the non-Markovian SSEs.

Chapter 11

Conclusions and Future Work

11.1 Conclusion

In this thesis I investigate both interpretations of quantum mechanics and non-Markovian SSEs. In particular I have answered the question: which interpretation of quantum mechanics provides the best understanding of non-Markovian SSEs? I find this to be the modal or hidden variable interpretation. To those familiar with quantum trajectory theory (an application of the orthodox interpretation to continuous-in-time measurements on the bath, see chapter 7), the predominant view of Markovian SSEs, this may come as a surprise. Why should a relaxation of an assumption about the bath drastically affect the physical understanding of an equation? The brief answer to this question is that in the non-Markovian limit the bath *remembers* what has happened at earlier times and if a measurement was performed on the bath then the bath is physically different (due to the quantum mechanical disturbance) to what it would be if there had been no measurement.

To illustrate my claim that the modal theory of quantum mechanics does provide the best interpretation of non-Markovian SSEs I am going to briefly summarizes the four current interpretations of non-Markovian SSEs, these being: the numerical, under the orthodox theory, under the modal theory, and as a new dynamical reduction model. The numerical interpretation simply states that non-Markovian SSEs are nothing more then numerical tools used to generate the correct non-Markovian master equation. This is the view of Diósi, Gisin, Strunz in Refs. [48, 47, 117]. However, they do suggest in [48] that an interpretation should exist.

In Ref. [59] Wiseman and myself showed that, under the orthodox interpretation of quantum mechanics, the solution of a non-Markovian SSEs, at time t, is the conditioned system state for a bath measurement at that time. That is, it is the state the system would be in given a measurement on the bath (of observables $\{Z_k\}$) at that time which yielded results $\{r(Z_k, t) = z_k\}$. Thus the solutions at different times correspond to different physical events and can not be logically linked to form a quantum trajectory. In light of this we have to conclude that under the orthodox interpretation non-Markovian SSEs do not have any physical interpretation they are simply a numerical tool which allows us to calculate the correct conditioned system state. I would also like to note that the different unravelings, under this view, simply corresponds to a different bath measurements.

In the modal theory (see chapter 4), we have to assume the quantum mechanics is not complete and an extra state (or hidden variable) has to be added to the theory. This state is known as the property state and is the actual state of the universe. Doing this allows us to keep the traditional concept of reality (things exists even when we are not looking at them) intact. However to agree with the orthodox theory we have to accept that only some properties (these are effectively observables in the orthodox theory) can be given definite status. This is related to the contextual nature of quantum mechanics (see chapter 3). In the standard modal theory I have accounted for this by simply accepting the arbitrariness of choice. I have assumed that when assigning definite values only one complete set of projectors (which determine the set of property states - see Eq. (4.8)), can be given definite status. This is known as choosing the decomposition.

When applying this theory to open quantum systems (the universe consists of a system and bath) we find that non-Markovian SSEs correspond to the evolution equation for the system part of the property state of the universe when the bath is given definite values, denoted $\{v(Z_k, t)\}$. Thus under this view a physical interpretation can be given to non-Markovian SSEs. The different unravelings arise by simply choosing a different decomposition for the bath. However to derive the coherent-state unraveling the modal theory must be generalized to include POM measurements. This was first done by Wiseman and myself [62] and is also presented in chapter 4. The basic principle behind this generalization is to use Naimark's theorem to find a set of projectors (in a larger Hilbert space) which are equivalent to the set of POM elements.

The last view is that of Bassi and Ghirardi [5, 7]. They interpret non-Markovian SSEs as follows: their real-valued non-Markovian SSE (which is essentially Wiseman's and my quadrature non-Markovian SSE [59]) forms a new interpretation of quantum mechanics. That is, it is a new dynamical reduction model with non-white gaussian noise for the reduction process. While there is nothing wrong, mathematically, with this view I believe the motivation behind this view is limited. To be more specific, Bassi and Ghirardi motivate this dynamical modal by a generalization of the CSL (continuous spontaneous localization) dynamical reduction model to include non-white noise. However since the original motivation behind CSL, that there is some unknown process which continuously localizes the wavefunction, by definition must have white noise (if the evolution equation is to be a pure state), I am forced to conclude that this view of non-Markovian SSEs is unsatisfactory.

If we conclude that non-Markovian SSE are best understood under the modal view then I believe Markovian SSEs should also be given this interpretation. Thus in the Markovian limit we have both the quantum trajectory and this modal interpretation of Markovian SSEs (actually a third if we include the CSL model and forth if we include numerical tools). In my opinion this is not a problem as both views are valid and it depends on the nature of the problem which interpretation is best.

An example of a scenario which requires the quantum trajectory view of Markovian SSE was presented in chapter 8. Here I considered the following simple problem: if we have a system with an unknown dynamical parameter how can we obtain knowledge about this parameter, and with this knowledge what is our best estimate of the quantum state? To answer this question I considered a TLA driven by a classical field of unknown Rabi frequency, $\Omega_{\rm dri}$. The five different detection schemes: direct, an adaptive technique, homodyne-x, homodyne-y, and heterodyne were investigated by calculating the four different measures of the knowledge gain: Shannon information about $\Omega_{\rm dri}$, variance in $\Omega_{\rm dri}$, long-time system purity, and short-time system purity. It was observed that a high gain in purity does not necessarily correspond to a high information gain (knowledge about $\Omega_{\rm dri}$). This is because in some detection schemes, namely the adaptive technique, the positioning of the state is approximately independent of $\Omega_{\rm dri}$. The scheme which had the best information gain and highest long-time purity was homodyne y. This is expected as this scheme involves a weak measurement of the y quadrature, which for $\hat{\sigma}_x$ driving is most affected by an unknown Rabi frequency. That is, as time goes on the repeated measurements (continuous-in-time) will hone-in on the correct $\Omega_{\rm dri}$.

11.2 Future Work

In this thesis I have derive two new non-Markovian SSEs and presented what I believe is their best interpretation. However there is still plenty of work to be done in this new field. The obvious extension is to derive more unravelings and answer questions such as:

(1) Are there non-Markovian unravelings which are equivalent to the jump-like Markovian SSEs?

(2) Can a complete parameterization be derived for diffusive non-Markovian SSE (as done by Wiseman and Diósi in Ref. [135] for diffusive Markovian SSEs)? Bassi in Ref. [6] has provided some work along this line of questioning.

(3) What is the unraveling that corresponds to the Schmidt decomposition? This is interesting for the modal view as this unraveling will correspond to the case when both the bath and the system can be given definite status.

Other questions which need addressing concern the derivation of non-Markovian SSEs. In deriving these equations it is assumed that the functional derivative can be replaced by an operator. This leads to questions such as:

(4) Is it possible to show that this replacement is true in general? If this can be shown then criticism against interpretations of non-Markovian SSEs (if they can not be derived in general then how can a general interpretation exists?) would be void.

(5) If this operator exists can it be found in general and if not do better perturbative techniques exist? That is, do there exist techniques which are not limited to certain memory function [60] or post-Markovian approximations [139].

Other directions for future work are applications of this theory to systems which experimentalists are currently working with. These include:

(6) Bose Einstein condensates.

(7) Photonic band gaps (or other strongly non-Markovian open quantum systems).

In this thesis I also presented an application of quantum trajectory theory: quantum state and parameter estimation. This work also has many areas that still require investigation. These include:

(8) To apply this theory to measurement schemes which take into account realistic detectors.

(9) To first develop non-Markovian quantum trajectories (repeated-in-time measurements), then to apply these equations to quantum state and parameter estimation

Other questions which have arisen from the work presented in this thesis concern the modal view of quantum mechanics, in particular Wiseman's and my generalization to include POM measurements [62]. These questions are:

(10) Is there a physical reason for enlarging the Hilbert space of the universe and if so what is it?

(11) Does this generalization provide any insight to the problem of choice (how is one decomposition of the state chosen over other decompositions)? This is currently the biggest limitation of the modal theory and is an area that I find very interesting.

(12) Since Bell's solution for $J_{nm}(t)$ and $T_{nm}(t)$ does not reproduce Brown and Hiley's [19] continuous dynamics for cubic potentials (see chapter 4), then what is the solution for $J_{nm}(t)$ and $T_{nm}(t)$ which reproduce these dynamics?

(13) In chapter 4 it was shown that provided the universe was at most quadratic in the conjugate variable to the hidden variable (beable) then the continuous dynamics can be evaluated using Wiseman and my velocity operator technique [62]. However presently there is no physical reason why this technique only works for quadratic or lower orders.

In closing I would like to say that there is clearly much work to be done on non-Markovian SSEs and the interpretations of quantum mechanics.

Appendix A

Evaluation of operators required for non-Markovian SSEs

A.1 Derivation of $\hat{c}_z(t,t) = \hat{L}$

To show that $\hat{c}_z(t,t) = \hat{L}$ we start by discretizing the functional derivative. We divide the range $[0,t)$ into N intervals of width Δt, so the change in $|\bar{\psi}_z(t)\rangle$ is

$$\int_0^t \frac{\delta|\bar{\psi}_z(t)\rangle}{\delta z^*(t,s)}\delta z^*(s)ds = \sum_{i=0}^{N-1} \Delta t\Big[\frac{\partial|\bar{\psi}_z(t_N)\rangle}{\partial z^*(t_i)\Delta t}\Big]dz^*(t_N,t_i), \tag{A.1}$$

thus

$$\frac{\delta}{\delta z^*(t,s)}|\bar{\psi}_z(t)\rangle = \frac{\partial|\bar{\psi}_z(t_N)\rangle}{\partial z^*(t_i)\Delta t}, \tag{A.2}$$

if s (t_i) is less than t (t_N), which is the only situation we are interested in, then taking the limit that $s \to t$ ($t_i = t_{N-1}$) this becomes

$$\lim_{s\to t}\frac{\delta|\bar{\psi}_z(t)\rangle}{\delta z^*(t,s)} = \frac{\partial[|\bar{\psi}_z(t_{N-1})\rangle + \Delta t\partial_t|\bar{\psi}_z(t_{N-1})\rangle]}{\partial z^*(t_{N-1})\Delta t}. \tag{A.3}$$

Discretizing Eq. (9.37) we get

$$\partial_t|\bar{\psi}_z(t_{N-1})\rangle = \Big[-\frac{i}{\hbar}\hat{H}_{\rm int}(t_{N-1}) + z^*(t_{N-1})\hat{L} - \hat{L}^\dagger\sum_{j=0}^{N-2}\alpha(t_{N-1}-t_j)\frac{\partial}{\partial z^*(t_j)}\Big]|\bar{\psi}_z(t_{N-1})\rangle. \tag{A.4}$$

Substituting this into Eq. (A.3) and using the fact that the state at time t_{N-1} only depends on the noise at time less then t_{N-1}, we get the limit

$$\frac{\delta|\bar{\psi}_z(t)\rangle}{\delta z^*(t,t)} \to \hat{L}|\bar{\psi}_z(t)\rangle. \tag{A.5}$$

Thus by Eq. (9.38) $\hat{c}_z(t,t) = \hat{L}$.

A.2 Derivation of $\hat{q}_z(t,t) = \hat{L}$

To show that $\hat{q}_z(t,t) = \hat{L}$ we use the same procedure as above except we have a real noise function and we use Eq. (9.84) in place of Eq. (9.37). Discretizing Eq. (9.84) gives

$$\partial_t|\bar{\psi}_z(t_{N-1})\rangle = \Big[\frac{-i}{\hbar}\hat{H}_{\rm int}(t_{N-1}) + z(t_{N-1})\hat{L} - (\hat{L}+\hat{L}^\dagger)\sum_{j=0}^{N-2}\beta(t_{N-1}-t_j)\frac{\partial}{\partial z^*(t_j)}\Big]|\bar{\psi}_z(t_{N-1})\rangle. \quad \text{(A.6)}$$

Substituting this into Eq. (A.3) (but with real noise) and using the fact that the state at time t_{N-1} only depends on the noise at times less then t_{N-1}, we get the limit

$$\frac{\delta|\bar{\psi}_z(t)\rangle}{\delta z(t,t)} \to \hat{L}|\bar{\psi}_z(t)\rangle. \quad \text{(A.7)}$$

Thus by Eq. (9.85), $\hat{q}_z(t,t) = \hat{L}$.

Appendix B

Stratonovich-Itô conversion for Markovian SSEs

In this appendix I will demonstrate how to convert a Stratonovich Markovian SSE to an Itô Markovian SSE. Whilst Itô and Stratonovich conversions are well known for SDE of real parameters (see [64]) in my reading I have never come across the conversion equations for complex parameters. Here I will show how they can be derived.

B.1 Complex variable and real white noise

Lets consider the following general Markovian SSE,

$$d_t|\psi(t)\rangle = \hat{A}|\psi(t)\rangle + \hat{B}|\psi(t)\rangle\xi(t) \tag{B.1}$$

where $\xi(t)$ represents white noise. That is it has the following correlations

$$E[\xi(t)] = 0, \tag{B.2}$$
$$E[\xi(t)\xi(t')] = \delta(t-t'). \tag{B.3}$$

This equation can be written in component form as,

$$d_t\psi_j = a_j(\{\psi_k(t)\},\{\psi_k^*(t)\}) + b_j(\{\psi_k(t)\},\{\psi_k^*(t)\})\xi(t) \tag{B.4}$$

Integrating this over time we get,

$$\psi_j(t)-\psi_j(t_0) = \int_{t_0}^{t} a_j(\{\psi_k(s)\},\{\psi_k^*(s)\})ds + \int_{t_0}^{t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw(s), \tag{B.5}$$

where $dw(s) = \xi(s)ds$ is a Wiener increment, it must satisfy

$$E[dw(t)] = 0, \tag{B.6}$$
$$E[dw(t)^2] = dt. \tag{B.7}$$

Defining the integral as a Riemann-Stieltjes integral, namely we divide the interval $[t_0, t]$ into N subintervals such that $t_0 \leq t_1 \leq t_2 ... \leq t_N$, and define intermediate points τ_i such that $t_{i-1} \leq \tau_i \leq t_i$. It turns out that to order dt the method used to give the first integral is irrelevant, as both methods give the same results; this is simple calculus. This is not the case for the second integral as $dw(s) = \xi(s)ds$ is effectively of order $\sqrt{dt}$ and thus any dw^2 terms will need inclusion in the final result when we take the dt limit. We define this integral as,

$$\int_{t_0}^{t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw(s) = \sum_{i=1}^{N} b_j(\{\psi_k(\tau_i)\},\{\psi_k^*(\tau_i)\})dw(t_{i-1}), \tag{B.8}$$

where $dw(t_{i-1}) = W(t_i) - W(t_{i-1})$. There are many ways we can define the midpoints, however, for this appendix we are only going to consider two of them.

The first method is when $\tau_i = t_{i-1}$. This allows us to write the integral as

$$\mathcal{I}\int_{t_0}^{t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw(s) = \sum_{i=1}^{N} b_j(\{\psi_k(t_{i-1})\},\{\psi_k^*(t_{i-1})\})dw(t_{i-1}). \tag{B.9}$$

and is called the Itô integral. It has the following unique properties $dw^2 = dt$ and $dw^{2+n} = 0$ where $n = 1, 2....$ Further properties of this method are $E[dw(t)] = 0$ where $E[...]$ represents the average value and $E[f(\psi(t_i), \psi^*(t_i))dw(t_i)] = 0$ that is the state at time t_i is independent of the noise at that time.

The second method which is commonly referred to as the Stratonovich method is defined as,

$$\begin{aligned}\mathcal{S}\int_{t_0}^{t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw(s) &= \sum_{i=1}^{N} b_j[\{(\psi_k(t_{i-1})+\psi_k(t_i))/2\},\{(\psi_k^*(t_{i-1})+\psi_k^*(t_i))/2\}] \\ &\quad \times dw(t_{i-1}).\end{aligned} \tag{B.10}$$

or as

$$\begin{aligned}\mathcal{S}\int_{t_0}^{t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw(s) &= \sum_{i=1}^{N} b_j[\{\psi_k(t_{i-1})+\tfrac{1}{2}d\psi(t_{i-1})\},\{\psi_k^*(t_{i-1})+\tfrac{1}{2}d\psi_k^*(t_{i-1})\}] \\ &\quad \times dw(t_{i-1}).\end{aligned} \tag{B.11}$$

Thus the state at time t_i is not independent of the noise at this time,

$$E[f(\psi(t_i), \psi^*(t_i))dw(t_i)] \neq 0. \tag{B.12}$$

To work out the stochastic integral with this method we have to Taylor expand $b_j[\{\psi_k(t_{i-1}) + \frac{1}{2}d\psi(t_{i-1})\}, \{\psi_k^*(t_{i-1}) + \frac{1}{2}d\psi_k^*(t_{i-1})\}]$ around the point $[\{\psi_k(t_{i-1})\}, \{\psi_k^*(t_{i-1})\}]$,

$$\begin{aligned}&\mathcal{S}\int_{t_0}^{t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw(s) = \\ &\quad \sum_{i=1}^{N}\Big[b_j(\{\psi_k(t_{i-1})\},\{\psi_k^*(t_{i-1})\}) + \sum_l \tfrac{1}{2}d\psi_l(t_{i-1})\frac{\partial}{\partial\psi_l}b_j(\{\psi_k(t)\},\{\psi_k^*(t)\})\Big|_{t=t_{i-1}} \\ &\quad + \sum_l \tfrac{1}{2}d\psi_l^*(t_{i-1})\frac{\partial}{\partial\psi_l^*}b_j(\{\psi_k(t)\},\{\psi_k^*(t)\})\Big|_{t=t_{i-1}} + \text{H.O.T}\Big]dw(t_{i-1})\end{aligned} \tag{B.13}$$

Using equation (B.4) and $\xi(s)dt = dw(s)$ to order dt this can be written as

$$\begin{aligned}\mathcal{S}\int_{t_0}^{t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw(s) &= \sum_{i=1}^{N}\Big[b_j(\{\psi_k(t_{i-1})\},\{\psi_k^*(t_{i-1})\})dw(t_{i-1}) \\ &+ \sum_l \tfrac{1}{2}b_j(\{\psi_k(t_{i-1})\},\{\psi_k^*(t_{i-1})\})dt_{i-1}\frac{\partial}{\partial\psi_l}b_j(\{\psi_k(t)\},\{\psi_k^*(t)\})\Big|_{t=t_{i-1}} \\ &+ \sum_l \tfrac{1}{2}b_j^*(\{\psi_k(t_{i-1})\},\{\psi_k^*(t_{i-1})\})dt_{i-1}\frac{\partial}{\partial\psi_l^*}b_j(\{\psi_k(t)\},\{\psi_k^*(t)\})\Big|_{t=t_{i-1}}\Big],\end{aligned} \tag{B.14}$$

this is the Stratonovich integral.

Thus with white noise and the order dt limit, we obtain two different results for an integral containing this noise, we can relate the Stratonovich to the Itô by,

$$\begin{aligned}\mathcal{S}\int_{t_0}^{t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw(s) &= \mathcal{I}\int_{t_0}^{t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw(s) + \\ &\frac{1}{2}\int_{t_0}^{t}\sum_l \left[b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})\frac{\partial}{\partial\psi_l}b_j(\{\psi_k(s')\},\{\psi_k^*(s')\})\Big|_{s'=s}\right]ds + \\ &\frac{1}{2}\int_{t_0}^{t}\sum_l \left[b_j^*(\{\psi_k(s)\},\{\psi_k^*(s)\})\frac{\partial}{\partial\psi_l^*}b_j(\{\psi_k(s')\},\{\psi_k^*(s')\})\Big|_{s'=s}\right]ds, \end{aligned} \tag{B.15}$$

where S and I refers to a Stratonovich and Itô integral respectively.

If we know assume that the Eq. (B.4) is a Stratonovich Markovian SSE I will now show have to derive the we Itô Markovian SSE which is equivalent to it. To do this we integrate the Stratonovich equation over time for a range δt. This gives,

$$\psi_j(t_0+\delta t)-\psi_j(t_0) = a_j(\{\psi_k(t_0)\},\{\psi_k^*(t_0)\})\delta t + \mathcal{S}\int_{t_0}^{\delta t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw(s), \tag{B.16}$$

and by equation (B.15) we can write this as,

$$\begin{aligned}\delta\psi_j(t_0) &= a_j(\{\psi_k(t_0)\},\{\psi_k^*(t_0)\})\delta t + \mathcal{I}\int_{0}^{\delta t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw(s) \\ &+\frac{\delta t}{2}\sum_l b_j(\{\psi_k(t_0)\},\{\psi_k^*(t_0)\})\frac{\partial}{\partial\psi_l}b_j(\{\psi_k(t)\},\{\psi_k^*(t)\})\Big|_{t=t_0} \\ &+\frac{\delta t}{2}\sum_l b_j^*(\{\psi_k(t_0)\},\{\psi_k^*(t_0)\})\frac{\partial}{\partial\psi_l^*}b_j(\{\psi_k(t)\},\{\psi_k^*(t)\})\Big|_{t=t_0}, \end{aligned} \tag{B.17}$$

where $\delta\psi_j(t_0) = \psi_j(t_0+\delta t) - \psi_j(t_o 0)$. By the definition of an Itô integral (Eq. (B.9)), in the limit $\delta t \to dt$, we can write it as

$$\lim_{\delta t\to dt}\mathcal{I}\int_0^{\delta t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw(s) = b_j(\{\psi_k(t_0)\},\{\psi_k^*(t_0)\})dw(t_0) \tag{B.18}$$

Thus the Itô Markovian SSE is,

$$\begin{aligned}d\psi_j(t) &= a_j(\{\psi_k(t)\},\{\psi_k^*(t)\})dt + b_j(\{\psi_k(t)\},\{\psi_k^*(t)\})dw(t) \\ &+\frac{dt}{2}\sum_l b_j(\{\psi_k(t)\},\{\psi_k^*(t)\})\frac{\partial}{\partial\psi_l}b_j(\{\psi_k(t)\},\{\psi_k^*(t)\}) \\ &+\frac{dt}{2}\sum_l b_j^*(\{\psi_k(t)\},\{\psi_k^*(t)\})\frac{\partial}{\partial\psi_l^*}b_j(\{\psi_k(t)\},\{\psi_k^*(t)\}). \end{aligned} \tag{B.19}$$

This is the key result of this appendix and it shows that when our dependent variable (ψ) is complex two terms make up the Itô corrections. For a derivation when we have a non complex variable see reference [64]. The next section deals with the case that arrives when we have a complex variable and complex white noise.

B.2 Complex variable and complex white noise

In this section we consider the following general stochastic equation,

$$d_t|\psi(t)\rangle = \hat{A}|\psi(t)\rangle + \hat{B}|\psi(t)\rangle\xi^*(t), \tag{B.20}$$

where $\xi(t)$ is now a complex white noise function, satisfying

$$E[\xi(t)] = 0, \tag{B.21}$$
$$E[\xi(t)\xi^*(t')] = \delta(t-t'). \tag{B.22}$$

This can be written in terms of to real white noise process ($\xi_1(t)$ and $\xi_2(t)$) as $\xi^*(t) = (\xi_1(t) + i\xi_2(t))/\sqrt{2}$. With these two real white noises the component form of the above general Markovian SSE is,

$$d_t\psi_j = a_j(\{\psi_k(t)\},\{\psi_k^*(t)\})+b_j(\{\psi_k(t)\},\{\psi_k^*(t)\})\xi_1(t)/\sqrt{2}+ib_j(\{\psi_k(t)\},\{\psi_k^*(t)\})\xi_2(t)/\sqrt{2}\,. \tag{B.23}$$

The Itô integral for this equation is,

$$\begin{aligned}\mathcal{I}\int_{t_0}^{t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw^*(s) &= \sum_{i=1}^{N} b_j(\{\psi_k(t_{i-1})\},\{\psi_k^*(t_{i-1})\})dw_1(t_{i-1})/\sqrt{2}\\ &\quad +i\sum_{i}^{n} b_j(\{\psi_k(t_{i-1})\},\{\psi_k^*(t_{i-1})\})dw_2(t_{i-1})/\sqrt{2}\,,\end{aligned} \tag{B.24}$$

where $dw^*(t) = \xi^*(t)dt$.

Like in the prior case we can write a Stratonovich integral, and after the same procedures as before we obtain the following Stratonovich-Itô relation for the integral,

$$\begin{aligned}\mathcal{S}\int_{t_0}^{t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw^*(s) &= \mathcal{I}\int_{t_0}^{t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw^*(s) + \\ &\quad \frac{1}{2}\int_{t_0}^{t}\sum_{l} b_j^*(\{\psi_k(s)\},\{\psi_k^*(s)\})\frac{\partial}{\partial\psi_l^*}b_j(\{\psi_k(s')\},\{\psi_k^*(s')\})\Big|_{s'=s}ds.\end{aligned} \tag{B.25}$$

This was obtain by using the fact that $dw_1^2 = dt$, $dw_1dw_2 = 0$ etc. If equation (B.23) is a Stratonovich SSE then to obtain the Itô equivalent we have to integrate it from time t_0 to $t_0+\delta t$, and substitute equation (B.25) for the integral. Doing this we obtain,

$$\begin{aligned}\psi_j(t_0+\delta t)-\psi_j(t_0) &= a_j(\{\psi_k(t_0)\},\{\psi_k^*(t_0)\})\delta t + \mathcal{I}\int_{t_0}^{t_0+\delta t} b_j(\{\psi_k(s)\},\{\psi_k^*(s)\})dw^*(s)\\ &\quad +\frac{\delta t}{2}\sum_{l} b_j^*(\{\psi_k(t_0)\},\{\psi_k^*(t_0)\})\frac{\partial}{\partial\psi_l^*}b_j(\{\psi_k(t)\},\{\psi_k^*(t)\})\Big|_{t=t_0}.\end{aligned} \tag{B.26}$$

By definition of an Itô integral when $\delta t \to dt$ we obtain the following Itô equation,

$$\begin{aligned}d\psi_j(t) &= a_j(\{\psi_k(t)\},\{\psi_k^*(t)\})dt + b_j(\{\psi_k(t)\},\{\psi_k^*(t)\})dw^*(t)\\ &\quad +\frac{dt}{2}\sum_{l} b_j^*(\{\psi_k(t)\},\{\psi_k^*(t)\})\frac{\partial}{\partial\psi_l^*}b_j(\{\psi_k(t)\},\{\psi_k^*(t)\}).\end{aligned} \tag{B.27}$$

Thus we see that if the variable and noise are complex then there is only one Itô correction term.

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