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Comment on "One-dimensional nonlinear steady infiltration" by H. A. Basha

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Basha [1999] has developed an interesting approach for steady infiltration when the soil-water conductivity obeys the relation

$$K = \frac{1}{1 + \psi^n}. \tag{1}$$

In this comment, Basha's notations are followed; K is a reduced conductivity so that $K = 1$ for $\psi = 0$, and the matric potential is also made dimensionless using some characteristic potential. Note that as a result, $\psi > 0$ for unsaturated soil. In general, the governing equation is

$$K(d\psi/dz + 1) = Q, \tag{2}$$

where Q is the dimensionless flux and z, positive downward, is a dimensionless distance. To integrate (2), an additional boundary condition is used:

$$\psi = \psi_b, \qquad z = z_b. \tag{3}$$

Basha [1999] then uses an ingenious expansion to solve the problem, assuming n to be sufficiently large.

Basha's [1999, equation (5)] approximation yields

$$\psi = 1 - \frac{1}{n-1}\ln(\alpha) + \frac{1}{(n-1)^2}[\ln(\alpha) + \tfrac{1}{2}\ln^2(\alpha)], \tag{4}$$

when $z_b \to \infty$, whereas the exact solution is [see *Basha*, 1999, equation (52)]

$$\Psi = (1/\alpha)^{1/n}, \tag{5}$$

with

$$\alpha = Q/(1 - Q). \tag{6}$$

Basha adds that his "Equation (51) compares well with the exact limit... especially for relatively high n values. The relative error ranges from 5% for $n = 3$ to less than 1% for higher n values." This cannot be true for all values of α. For instance, for a given n if $\alpha \gg 1$, that is, $Q \simeq 1$, $\psi \simeq 0$ (see (5)), whereas (4) gives $\psi \gg 1$. Although less striking, the error is also large for $\alpha \ll 1$. As an illustration, if we write $\alpha = \exp(-\lambda n)$, then (5) yields $\psi = \exp(\lambda)$, whereas (4) gives for $n \gg 1$, $\psi = 1 + \lambda + \lambda^2/2$, that is, the beginning of the expansion of $\exp \lambda$, so that the result is good for small λ only.

It is clear that it would be preferable if the approximate solution reduced to (5) when $z_b \gg 1$. This is satisfied in the following. To do so, (2) is approximated by replacing $K/(K - Q) = 1 + Q/(K - Q)$ using

$$K - Q \simeq C \exp[-\alpha(z_b - z)]. \tag{7}$$

This type of approximation has been used in the past and becomes exact when K is of the form prescribed by *Gardner* [1958] and it was used in the present context by *Warrick* [1974]. The same approach was also used by *Parlange* [1982] in a drainage problem. Here (7) is only an approximation, and C and α are two positive constants to be determined later. We note in passing that as $z_b \to \infty$, (7) yields $K = Q$, that is, the exact solution. Plugging the approximation into (2) yields by integration

$$(Q + C)\exp[-\alpha(\psi - \psi_b)] - Q = C\exp[-\alpha(z_b - z)], \tag{8}$$

which is identical to the approximation of (7) when $K = (Q + C)\exp[-\alpha(\psi - \psi_b)]$, that is, a Gardner-type behavior. For a different K, (8) should usually be more accurate than (7) since it results from one iteration. Here ψ_b is defined as the value of ψ where $z = z_b$, and $C = K_b - Q$, where K_b is the value of K at $\psi = \psi_b$.

We are now going to apply the above results to the two examples discussed by Basha. The first example follows the cases in Figure 1 of Basha, that is, $n = 3$ and 7 and $Q = 0.1$ and 0.01, with $\psi_b = 1$ and $z_b = 2$. Thus we know everything in (8) except α which has to be obtained judiciously. It is important to reiterate that this being a "comment" on Basha's work, the determination of α pertains to his examples. A more general theory based on the present approach will be published later. In particular, we are limiting ourselves to the case where K obeys (1) exactly. Hence $K_b = 1/2$. Equation (8) now becomes

$$\tfrac{1}{2}\exp[-\alpha(\psi - 1)] = Q + (\tfrac{1}{2} - Q)\exp[-\alpha(2 - z)]. \tag{9}$$

To find α, (9) can be forced to satisfy (1) and (2) at some point. It already satisfies $\psi = 1$ at $z = 2$. An obvious choice is the point $z = 0$, where $\psi = \psi_0$. Then

$$(\tfrac{1}{2} - Q)\exp(-2\alpha) + Q = 1/(1 + \psi_0^n) = \tfrac{1}{2}\exp(-\alpha(\psi_0 - 1)], \tag{10}$$

which provides two equations and thus yields α (and ψ_0). It is easy to solve (10) by iteration. Starting with $\alpha = n/2$ (the value obtained by crudely fitting (9) near $z = 2$), the left-hand side of (10) is used to obtain ψ_0. The right-hand side of (10) then provides a new α. Convergence is rapid, and the calculation easy. Figure 1 then repeats the four cases of Basha's Figure 1. Obviously, the results are very good by comparison with the exact solution of Basha. The accuracy of the present approximation is similar to that of Basha's, but the analytical form of (9) is neatly simpler than Basha's approximation.

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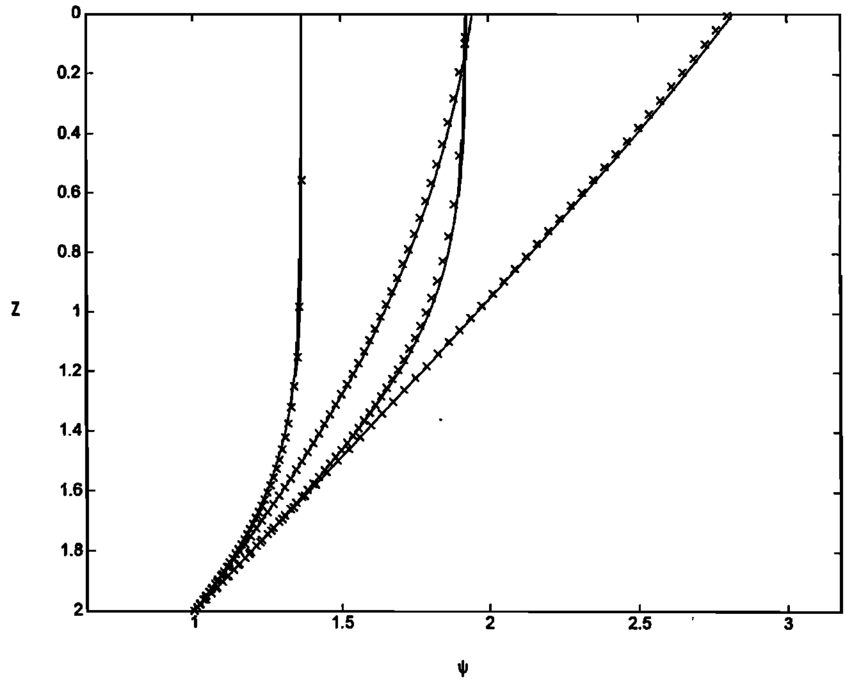

Figure 1. Pressure distribution obtained from the exact solution (crosses), as given by *Basha* [1999], and the present approximation (solid curve) as given by equation (9). The four cases from the left to the right correspond to $n = 7$, $Q = 0.1$; $n = 3$, $Q = 0.1$; $n = 7$, $Q = 0.01$; and $n = 3$, $Q = 0.01$. The values of α as given by equation (10) are $\alpha = 4.373$, 1.513, 4.215, and 1.352, respectively.

The second case considered by Basha is with root uptake for $z < D$, so that Q in the right-hand side of (2) is replaced by $Q - \bar{F}$, with

$$\bar{F} = Qz/D, \qquad z \leq D. \tag{11}$$

For $2 > z > D$, (9) applies for $Q = 0$ or, exactly,

$$\psi - 1 = 2 - z, \qquad z \geq D. \tag{12}$$

A similar result to (9) is now obtained for $z < D$ or, since $\psi = 3 - D$ at $z = D$ from (12),

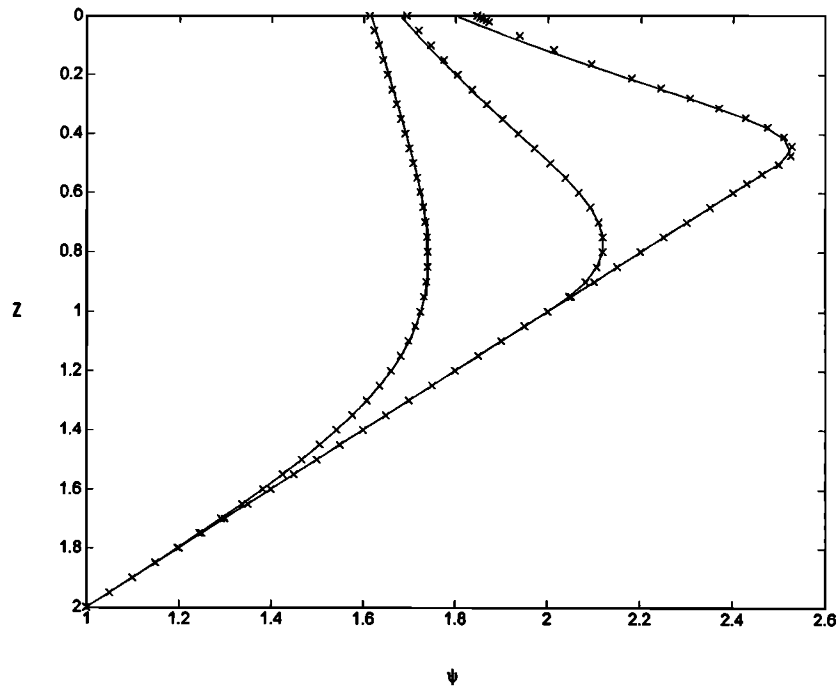

Figure 2. Pressure distribution for $n = 5$, $Q = 0.1$ corresponding to three rooting depths, $D = 2$, $D = 1$, and $D = 0.5$, from left to right. The present approximation (solid curve) is given by equation (12) for $z \geq D$ and by equation (15) for $z \leq D$. Values of α given by equation (16) are $\alpha = 2.888$, 2.364, and 1.971, respectively. The numerical solution (crosses) is the same as given by *Basha* [1999].

$$K_D \exp[-\alpha(\psi - \psi_D)] - Q = (K_D - Q + A)\exp[-\alpha(D - z)], \tag{13}$$

where $K_D = 1/[1 + (3 - D)^n]$ and

$$A = -\alpha \int_z^D \bar{F}(x) \exp[\alpha(D - x)]\, dx, \tag{14}$$

and from (11),

$$\frac{1}{1 + (3 - D)^n} \exp[-\alpha(\psi - 3 + D)] = \frac{1}{1 + (3 - D)^n}$$
$$\exp[-\alpha(D - z)] + Q\left(1 - \frac{z}{D} - \frac{1}{\alpha D}\right) + \frac{Q}{\alpha D} \exp[-\alpha(D - z)]. \tag{15}$$

Once again α has to be obtained. In this case it is easy to match (15) when $d\psi/dz = 0$ at $\psi = \psi^*$ or

$$\frac{Q}{\alpha D} \ln\left[1 + \frac{\alpha D/Q}{1 + (3 - D)^n}\right] = \frac{1}{1 + \psi^{*n}} = \frac{1}{1 + (3 - D)^n}$$
$$\cdot \exp[-\alpha(\psi^* + D - 3)]. \tag{16}$$

As with (10), we can start with $\alpha = n/2$ to calculate the left side of (16). Then the midterm gives an estimate of ψ^*, and the right side gives a new value of α to be used for reiteration. Rapid convergence results. Figure 2 shows the results for D = 2, 1, 0.5; $n = 5$; and $Q = 0.1$, reproducing Basha's Figure 2. Again the approximation of (15) is remarkably accurate by comparison with a numerical solution. For this example the analytical approximation is not only simpler than Basha's but, in addition, neatly more accurate.

In conclusion, using a very simple exponential approximation, we were able to provide simple and accurate results for Basha's examples. A general theory will be given later.

References

Basha, H. A., One-dimensional nonlinear steady infiltration, *Water Resour. Res.*, *35*, 1697–1704, 1999.

Gardner, W. R., Some steady-state solutions to the unsaturated moisture flow equations with application to evaporation from a water table, *Soil Sci.*, *85*, 228–232, 1958.

Parlange, J.-Y., Simple method for predicting drainage from field plot—Comment, *Soil Sci. Soc. Am. J.*, *46*, 887–888, 1982.

Warrick, A. W., Solution to the one-dimensional linear moisture flow equation with water extraction, *Soil Sci. Soc. Am. J.*, *38*, 573–576, 1974.

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