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Calculation of Electromagnetic Fields in Three Dimensions using the Cauchy Integral

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This paper describes a method for calculating the three-dimensional monochromatic electromagnetic fields scattered by conducting and dielectric objects using the Cauchy integral cast in a multi-dimensional form based on Clifford algebra. Formal relationships to methods based on quaternions and vector calculus are presented. The characteristics of solutions based on the Cauchy method are described and its advantages over comparable methods involving Greens functions are discussed.

Index Terms—Boundary value problems, Cauchy integral, Clifford algebra, electromagnetic fields, integral equations.

I. INTRODUCTION

THERE are as many ways to write Maxwell's equations as one would care to invent. Historically, methods of choice include quaternions (by Maxwell), vector calculus (by engineers), tensors (by physicists), complex numbers (in two dimensions), Cartan's differential geometry, and Clifford algebra. In all cases the common features of these methods are a representation of 2, 3 or 4-dimensional geometry in space/time with an algebraic structure that supports spatial/temporal differential operators. As such, all of these methods can be perceived as one or another kind of differential geometric algebra.

For many problems which involve physical phenomena and their representation in mathematical structures there is no particularly strong reason for choosing one differential geometric algebra over another. For problems involving electromagnetism in the form of Maxwell's equations that has been but is no longer the case. There *is* a very definite advantage in one particular of these geometric algebras over the others. Here we review that particular geometric algebra in the context of Maxwell's equations, relate it to approaches using quaternions and vector calculus, and describe its advantages and the characteristics of the solutions it offers.

II. BACKGROUND

At the time Maxwell introduced the displacement current and formulated the set of equations named after him, he had three differential geometric algebras from which to choose: (i) Cartesian coordinates, (ii) complex numbers and (iii) quaternions. Gibbs's vector calculus, Cartan's differential geometry and tensors had not yet been invented.

Maxwell himself used quaternions [1], [2] but with his students, for whom all apart from Clifford [3] found quaternions too much of a challenge, wrote everything in Cartesian coordinates. Complex numbers were not sufficiently general because they were at that time restricted to problems in two dimensions only.

In developing quaternions Hamilton [4] had been intending to find a generalisation of complex numbers which carried them into three dimensions. However, the algebra he did create did not do so. That had to wait (not very long) for Clifford to take two independent and commutative sets of quaternions

written in terms of Grassmann's linear algebra [5], and from them construct what Clifford called his even 5 way algebra [6] and what we today call a four-dimensional Clifford algebra.

Clifford's approach to constructing his algebra from quaternions is not the only approach of relevance to electromagnetism. From quaternions it is also possible to construct first the algebra of octonions and then that of sedenions. Casting Maxwell's equations into sedenions offers some of the same advantages of Clifford algebra but also incorporates some of the disadvantages of Gibbs's vector calculus. In particular the sedenion product and vector cross product both lack an associate property. In contrast Clifford products are endowed with the associative property. That makes enough difference to render Clifford algebra the easier to use of these three alternatives.

Like both Cartan's differential geometry and Gibbs's vector calculus, the product in Clifford algebra is non-commutative. The non-commutativity in all three cases is associated with a vector–vector product. For vector calculus the vector–vector (cross \times) product yields another vector. In contrast the vector–vector (wedge \wedge) product of Cartan's differential geometry yields a bi-vector (a geometric object which behaves like an oriented area), and in further contrast the vector–vector product of Clifford algebra yields a compound geometric object containing both a bi-vector and a scalar. The bi-vectors arise from an exterior (outer) product, and in the Clifford case this is combined with an inner product. The outer product is the key which endows both Cartan's differential geometry and Clifford algebra with all the properties of a full geometric algebra. Vector calculus, as reflected in its name, is strictly not any kind of geometric algebra for the very reason that it lacks an outer product.

For electromagnetism the key here is not the similarity or dissimilarity of Clifford algebra with quaternions, sedenions, vector calculus or Cartan's differential geometry. Instead the key is in the solution using Clifford algebra of Hamilton's earlier problem of extending complex numbers into three dimensions. In particular, Clifford algebra provides for (i) the generalisation from *analytic* functions of a complex variable in two dimensions to *monogenic* functions of a Clifford variable in any number of dimensions, and (ii) the generalisation from the Cauchy integral of complex variables in two dimensions to the Cauchy integral of Clifford variables in any number of dimensions. These generalisations were properly formulated and documented only within the last few years [7]–[11]. The

$$\mathbb{X} = \left(\begin{array}{cc|cc} \mathcal{A} & -\mathcal{D} & -\mathcal{C} & -\mathcal{B} \\ \mathcal{D} & \mathcal{A} & \mathcal{B} & -\mathcal{C} \\ \hline \mathcal{C} & -\mathcal{B} & \mathcal{A} & \mathcal{D} \\ \mathcal{B} & \mathcal{C} & -\mathcal{D} & \mathcal{A} \end{array} \right) = \left(\begin{array}{cc|cc|cc|cc} a & -\mathbf{A} \cdot & -d & \mathbf{D} \cdot & -c & \mathbf{C} \cdot & -b & \mathbf{B} \cdot \\ \mathbf{A} & (aI + \mathbf{A} \times) & -\mathbf{D} & -(dI + \mathbf{D} \times) & -\mathbf{C} & -(cI + \mathbf{C} \times) & -\mathbf{B} & -(bI + \mathbf{B} \times) \\ \hline d & -\mathbf{D} \cdot & a & -\mathbf{A} \cdot & b & -\mathbf{B} \cdot & -c & \mathbf{C} \cdot \\ \mathbf{D} & (dI + \mathbf{D} \times) & \mathbf{A} & (aI + \mathbf{A} \times) & \mathbf{B} & (bI + \mathbf{B} \times) & -\mathbf{C} & -(cI + \mathbf{C} \times) \\ \hline c & -\mathbf{C} \cdot & -b & \mathbf{B} \cdot & a & -\mathbf{A} \cdot & d & -\mathbf{D} \cdot \\ \mathbf{C} & (cI + \mathbf{C} \times) & -\mathbf{B} & -(bI + \mathbf{B} \times) & \mathbf{A} & (aI + \mathbf{A} \times) & \mathbf{D} & (dI + \mathbf{D} \times) \\ \hline b & -\mathbf{B} \cdot & c & -\mathbf{C} \cdot & -d & \mathbf{D} \cdot & a & -\mathbf{A} \cdot \\ \mathbf{B} & (bI + \mathbf{B} \times) & \mathbf{C} & (cI + \mathbf{C} \times) & -\mathbf{D} & -(dI + \mathbf{D} \times) & \mathbf{A} & (aI + \mathbf{A} \times) \end{array} \right) \quad (1)$$

significance of these generalisations for electromagnetism is apparent once it is understood that electromagnetic waves in regions without sources are described by monogenic functions.

III. FORMULATION

A. Time Domain

Equation (1) shows a four-dimensional Clifford number \mathbb{X} (which has 16 independent complex components) in terms of an equivalent matrix of quaternions (centre) and in terms of an equivalent matrix written in the notation of vector calculus (right). Table I shows, with Clifford numbers $\mathbb{X} = \mathbb{D}, \mathbb{F}, \mathbb{S}, \mathbb{P}$ on the left and quaternions $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ on the right, the necessary substitutions to encode Maxwell's equations into the Clifford–quaternion–vector formulation of (1). The values in columns A, B, C, D are 3d Clifford vectors, *i.e.* regular vectors in which the unit vectors $\hat{x}, \hat{y}, \hat{z}$ are replaced by the Clifford units e_1, e_2, e_3 . The symbol ∇ is used to represent the Clifford vector equivalent of the vector differential operator ∇ . The other symbols in the body of the table represent electromagnetic parameters in common engineering notation, with i as the imaginary unit $\sqrt{-1}$.

The differential operator \mathbb{D} is fixed by the need to accommodate Einstein's theory of special relativity. The field \mathbb{F} is then fixed under the influence of the differential operator by the need to faithfully reproduce all the various parts of Maxwell's equations in regions void of sources. Finally the source \mathbb{S} is fixed to reproduce Maxwell's equations in the presence of sources. It then follows by simple algebraic verification (using the rules $e_j^2 = -1$ and $e_j e_{k \neq j} = -e_k e_j$ for the Clifford units) that Maxwell's equations are written:

$$\mathbb{D}\mathbb{F} = \mathbb{S} \quad (\text{field to source}) \quad (2)$$

$$\mathbb{D}\mathbb{P} = \mathbb{F} \quad (\text{potential to field}) \quad (3)$$

$$\mathbb{D}^2\mathbb{P} = \mathbb{S} \quad (\text{potential to source}) \quad (4)$$

The square of the gradient: $\mathbb{D}^2 = -(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2})$ is recognised the negative of the d'Alembertian (wave) operator.

B. Frequency Domain

For monochromatic fields $\mathbb{F}(\mathbb{R}, t) = \mathbb{F}_k(\mathbb{R})e^{i\omega t}$ with angular frequency ω and wavenumber $k = \omega/c$ Fourier transformation gives equations in the frequency domain similar in form to those in the time domain. In particular the gradient: $\mathbb{D}_k = \nabla + k e_0$ and its square: $\mathbb{D}_k^2 = -(\nabla^2 + k^2)$, the latter of which

is recognised as the negative of the Helmholtz operator. The Cauchy integral in multiple dimensions is then written [7]:

$$\mathbb{F}_k(\mathbb{R}) = \int_{\Sigma} \mathbb{E}_k(\mathbb{R}' - \mathbb{R}) \mathbb{N}(\mathbb{R}') \mathbb{F}_k(\mathbb{R}') d\sigma(\mathbb{R}') \quad (5)$$

Here \mathbb{F}_k is a Clifford-valued function which is monogenic at all points $\mathbb{R} = R_x e_1 + R_y e_2 + R_z e_3$ in a region Ω of three-dimensional space enclosed by a surface Σ , \mathbb{N} is a Clifford number representing the outward normal vector to the surface Σ , $d\sigma$ is the scalar elemental measure of surface and:

$$\mathbb{E}_k(\mathbb{R}) = - \left\{ \frac{\mathbb{R}}{|\mathbb{R}|^2} + ik \left(\frac{\mathbb{R}}{|\mathbb{R}|} - ie_0 \right) \right\} \frac{e^{-ik|\mathbb{R}|}}{4\pi|\mathbb{R}|} \quad (6)$$

is the three-dimensional Cauchy kernel.

Equation (5) plays the role of a direct inverse in the frequency domain to Maxwell's homogeneous (source free) equations: $\mathbb{D}_k \mathbb{F}_k = 0$ when cast in the Clifford formalism. Static solutions are recovered by setting $k=0$.

IV. DISCUSSION

The relationship between the Clifford, the quaternion and the vector representations in (1) makes explicit the additional structure of the Clifford over the quaternion and of the quaternion over the vector. To match with the Clifford, structure is imposed on the four quaternions by embedding them in a matrix and adopting the usual rules of matrix addition and multiplication. Similarly, to match with the quaternion additional structure is imposed on the scalars and vectors as used in vector calculus by embedding them in a matrix:

$$\mathcal{A} = \left(\begin{array}{cc} a & -\mathbf{A} \cdot \\ \mathbf{A} & (aI + \mathbf{A} \times) \end{array} \right) = \left(\begin{array}{cccc} a & -A_x & -A_y & -A_z \\ A_x & a & -A_z & A_y \\ A_y & A_z & a & -A_x \\ A_z & -A_y & A_x & a \end{array} \right) \quad (7)$$

where A_x, A_y, A_z are the Cartesian components of vector \mathbf{A} .

The additional structure afforded by the Clifford or the matrix-quaternion or the matrix-vector approaches properly represents all aspects of three-dimensional geometry, *i.e.* scalars, vectors, bi-vectors (oriented areas) and tri-vectors (oriented volumes). This eliminates any confusion between polar vectors (vectors) and axial vectors (bi-vectors), and by imposing additional constraints on which operations are and are not permitted prevents operations meaningless to electromagnetism such as the direct addition of the electric and magnetic fields. When Maxwell's equations are written in the conventional form involving four equations with vector calculus these constraints are missing.

TABLE I
ELECTROMAGNETIC PARAMETERS IN THE FORM OF CLIFFORD VARIABLES $\mathbb{D}, \mathbb{F}, \mathbb{S}, \mathbb{P}$ AND QUATERNIONS $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$.

	Clifford		Quaternion							
			\mathcal{A}	\mathcal{B}	\mathcal{C}	\mathcal{D}				
			a	A	b	B	c	C	d	D
gradient	$\mathbb{D} = \nabla - \frac{i}{c}e_0\frac{\partial}{\partial t}$		0	0	$-\frac{i}{c}\frac{\partial}{\partial t}$	0	0	$i\nabla$	0	0
field	$\mathbb{F} = \sqrt{\mu}H\sigma - i\sqrt{\epsilon}Ee_0$		0	$\sqrt{\mu}H$	0	0	0	0	0	$-\sqrt{\epsilon}E$
source	$\mathbb{S} = \sqrt{\mu}J + \frac{i}{\sqrt{\epsilon}}\rho e_0$		0	0	$\frac{i}{\sqrt{\epsilon}}\rho$	0	0	$i\sqrt{\mu}J$	0	0
potential	$\mathbb{P} = \frac{1}{\sqrt{\mu}}A + i\sqrt{\epsilon}\phi e_0$		0	0	$i\sqrt{\epsilon}\phi$	0	0	$\frac{i}{\sqrt{\mu}}A$	0	0

Whereas the additional structure imposed by Clifford algebra is beneficial, it is not fundamentally important; the same benefits can be obtained by embedding Maxwell's equations into a tensor formalism. The most important aspect of the Clifford formalism is in recognising the electromagnetic field in a region of uniform material properties devoid of sources as a monogenic function. From the properties of monogenic functions this leads to the ability to reconstruct the interior field from the full trace of the field on the boundary enclosing such a region. The reconstruction is via the multi-dimensional Clifford-valued Cauchy integral (5). It is a direct (forward) reconstruction without recourse to any kind of matrix inversion.

If one half of the trace of the electromagnetic field (say the normal component of the magnetic field and the tangential component of the electric field) is known on the boundary the other half can be reconstructed. The reconstruction is again via the multi-dimensional Clifford-valued Cauchy integral. It is an indirect (inverse) reconstruction which does require some kind of matrix or equivalent inversion.

In scattering problems involving conducting and dielectric objects, one half of the trace of the scattered field can be obtained from the incident field. This leads immediately to a new method for calculating the fields scattered from objects illuminated by electromagnetic waves. The new method can be applied to problems involving static or monochromatic fields in three spatial dimensions for regions of uniform material properties meeting at interfaces where the material properties are abruptly discontinuous.

For conducting objects (simpler than dielectrics) the new method reduces (somewhat surprisingly) to the intersection of two straight lines within a particular functional space (Banach space). The geometry of that space, lacking any measure of angle, behaves more like the affine rather than the familiar Euclidean geometry.

The geometric solution [12] is depicted in Fig. 1 by the intersection \mathbb{F}_k of the horizontal dashed line through the boundary data g and the axis OP' . That result holds regardless of the shape of the scatterer. Examples of applying the technique in solving for the fields scattered from a conducting cubic scatterer are found in [12], [13].

Any approximation \mathbb{F}'_k to the actual solution of the field \mathbb{F}_k can be measured in terms of two errors ϵ_1 and ϵ_2 , calculated from projections (\square) of \mathbb{F}'_k onto axes OP and OQ' . One measure, ϵ_2 , indicates how far the approximate solution deviates in reproducing the measured data g . The other measure ϵ_1 indicates how far the approximate solution deviates from being a solution to Maxwell's equations (any

position on the axis OP'). The latter situation could occur for example after reducing Maxwell's continuous equations to discontinuous (finite) difference equations.

Both measures can be calculated without knowledge of the solution \mathbb{F}_k . This makes it easy to compare two alternative solutions \mathbb{F}'_k and \mathbb{F}''_k : one is guaranteed better than the other only if both ϵ_1 and ϵ_2 are lower in value. It also makes it easy to determine whether any iterative method of solution which has ceased converging has actually reached the solution or not.

The new method follows a different approach and therefore behaves differently from other methods. In comparison to existing boundary integral formulations note that the Cauchy kernel and integral are used instead of the Greens function and its integral form. The formulation involves only first order differential and integral operators, not second order ones. The singular integrals involved are less demanding. Potentials and surface currents play no part in the solution. The latter point means the method is not restricted to conductors and applies equally well to dielectrics. Furthermore, solutions are formulated only in terms of fields. That means the solution is constructed only in a single functional space. There is no need to utilise multiple distinct functional spaces for potential, field and current, unless one particularly wishes to calculate the potential or current.

Finally, the solution is not formulated from the (Cauchy) integral equation by inner product with some chosen set of basis (weight or test) functions. That would lead to a Galerkin style of approach [14] and a weak solution, typically minimising the

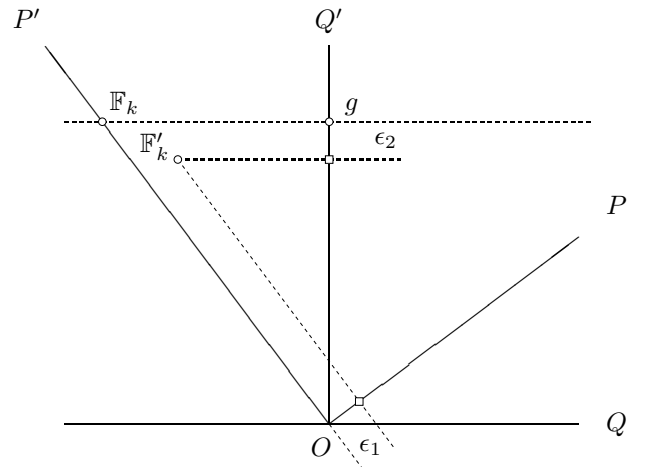


Fig. 1. Solution \mathbb{F}_k and error measures ϵ_1, ϵ_2 for approximate solution \mathbb{F}'_k .

TABLE II
CHARACTERISTICS OF TWO METHODS FOR CALCULATION OF ELECTROMAGNETIC FIELDS.

method	Clifford & Cauchy		Gibbs & Greens	
	1. Mathematical Foundation			
field functions	monogenic	(EM waves in 3 dimensions)	analytic	(static fields in 2 dimensions)
theorem for 2D	Cauchy	(single Clifford variable)	Cauchy	(single complex variable)
theorem for 3D	Cauchy	(single Clifford variable)	Greens	(multiple real variables)
	2. Algebraic Framework			
context	Clifford algebra	(any number of dimensions)	Gibbs's vector calculus	(strictly 3 dimensions)
product	associative	(easier to use)	not associative	(harder to use)
exterior product	present	(is a geometric algebra)	lacking	(not a geometric algebra)
geometric structure	complete	(polar & axial vectors distinct) (illegal field addition eliminated)	incomplete	(polar & axial vectors not distinct) (illegal field addition permitted)
	3. Computational Complexity			
equation	ordinary DE	(one equation)	partial DE	(system of equations)
variable	single	(multi-dimensional)	multiple components	(one-dimensional)
differential	1st order	(Maxwell-Dirac)	2nd order	(Helmholtz)
integral	1st order	(Cauchy)	2nd order	(Greens)
singularity	1st order	(numerically less demanding)	2nd order	(numerically more demanding)
	4. Practical Implementation			
linear system	original	(no weight/test functions) (minimise local error) (strong solution)	after inner product	(weighted residual method) (minimise global error) (weak solution)
reconstruction	direct	(from trace of field on boundary)	not direct	(no 3D Cauchy theorem)
surface currents	not needed	(dielectrics or conductors)	often used	(restricted to conductors when used)
basis functions	field only	(single functional space)	field, potential & currents	(multiple functional spaces)
interpretation	geometric	(for conducting objects)	no such interpretation	(must rely on algebra)

global error at the expense of higher errors in the vicinity of particular isolated points. The method reported in [12] results in a strong solution which if implemented properly should lead to higher accuracy, particularly in the vicinity of sharp corners where the gradient of the field exhibits high values.

V. CONCLUSION

The use of Clifford algebra opens the way to a new boundary integral method for calculating scattered fields. Of key importance to the new technique is the Cauchy integral as written in its multi-dimensional Clifford form (5). With the relationships in (1), the Cauchy integral can be recast into quaternion or vector calculus. The result is however a multitude of integral equations which require simultaneous solution. It appears that in practice any such approach would be unnecessarily complicated.

The new method is of similar complexity to comparable boundary integral techniques. Far from being more complicated in detail, the rules of Clifford algebra are simpler than the rules of vector calculus. Their regularity makes them well suited for effective implementation in algorithmic form.

As shown in table II there appear to be no overt disadvantages in the new method. A viable numerical implementation has already been demonstrated [12].

In favour of using the new technique (see table II) are various characteristics related to consistency, convenience and simplicity, but most importantly two characteristics which can deliver greater numerical accuracy: a low order solution (with less demanding singular integrals), and a strong solution of the original (Maxwell's) equations (with a correspondingly higher accuracy – particularly in the vicinity of sharp corners).

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