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ON THE RELATIVE GENERALIZED HAMMING WEIGHTS OF LINEAR CODES AND THEIR SUBCODES*

ZIHUI LIU[†], JIE WANG[‡], AND XIN-WEN WU[§]

Abstract. We first present an equivalent definition of relative generalized Hamming weights of a linear code and its subcodes, and we develop a method using finite projective geometry. Making use of the equivalent definition and the projective-geometry method, all of the relative generalized Hamming weights of a 3-dimensional q -ary linear code and its subcodes will be determined.

Key words. generalized Hamming weights, relative generalized Hamming weights, support weight, relative difference sequence

AMS subject classification. 94B05

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1. Introduction and basic notations. Motivated by a cryptographical application of linear codes to the wire-tap channel of type II [1], generalized Hamming weights (GHWs) of a linear code have been defined and studied in [2]. GHWs not only characterize the algebraic structure of linear codes, but also have a lot of applications; for instance, application to trellis decoding of linear codes. In the past two decades, GHWs of linear codes have been extensively studied (see [3], [4], [5], [6], [7], and [8]).

Extending GHWs to a two-code format, the authors of [9] have introduced relative generalized Hamming weights (RGHWs) of a linear code C and a subcode C^1 of C . When the subcode C^1 is zero, i.e., a code with zero vector as its unique codeword, the RGHWs of C are exactly the traditional GHWs of C . Therefore, RGHWs are a generalization of GHWs. RGHWs are useful in analyzing the coordinated multiparty wire-tap channel of type II as discussed in [9]. More precisely, for the multiparty wire-tap channel of type II with the coset coding scheme, the minimum uncertainty (also called equivocation) of the first parity's data bits to an adversary is characterized by the RGHWs (see Corollary 1 of [9]). Some upper and lower bounds on RGHWs have been given; conditions for achieving these bounds have been provided in [9]. Like the study of GHWs, the study of RGHWs is an interesting and important topic in algebraic coding theory. For a general linear code, it is difficult to determine all, or even part of, RGHWs, except for linear codes of small dimensions. In [10], all of the RGHWs of a 4-dimensional linear code with its 2-dimensional subcodes, and part of the RGHWs of a 4-dimensional linear code with its 1-dimensional subcodes, have been determined.

In this paper we first give an equivalent definition of the RGHWs. Based on the equivalent definition, we then develop a method using finite projective geometry,

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motivated by the application of finite projective geometry to the study of traditional GHWs (see [4] and [8]). Making use of the projective geometry method, we will determine all of the RGHWs of a 3-dimensional linear code with its subcodes.

Let C be an $[n, k]$ linear code and J be a subset of $I = \{1, \dots, n\}$. The subcode C_J of C is defined as $C_J = \{(c_1, \dots, c_n) \in C : c_t = 0 \text{ for } t \notin J\}$. For any subcode D of C , the support of D , denoted by $\chi(D)$, is defined as the set of positions where not all the codewords of D have zero coordinates. Define $w_S(D) = |\chi(D)|$.

DEFINITION 1 (see [2]). *Let C be an $[n, k]$ linear code. For any $r, 1 \leq r \leq k$, the r th GHW of C , denoted by $d_r(C)$ (or d_r for short), is defined as*

$$d_r(C) = \min\{w_S(D) : D \text{ is an } [n, r] \text{ subcode of } C\}.$$

Note that $d_1(C)$ equals to the traditional minimum Hamming weight of C .

DEFINITION 2 (see [9]). *Let C be an $[n, k]$ linear code and C^1 be a k_1 -dimensional subcode of C . For any $j, 1 \leq j \leq k - k_1$, the j th RGHW of C and C^1 , denoted by $M_j(C, C^1)$ (or M_j when C and C^1 are clear), is defined as*

$$M_j(C, C^1) = \min\{|J| : \dim(C_J) - \dim(C^1_J) \geq j\},$$

or equivalently

$$M_j(C, C^1) = \min\{|J| : \dim(C_J) - \dim(C^1_J) = j\}.$$

Obviously, when $C^1 = \{0\}$, we have $M_r(C, C^1) = d_r(C)$, the r th GHW.

DEFINITION 3. *Let $M_0 = 0$. Then the relative difference sequence (RDS) of the $[n, k]$ linear code C with the $[n, k_1]$ subcode C^1 is the sequence $(i_0, i_1, \dots, i_{k-k_1})$, where*

$$i_0 = n - M_{k-k_1}, \quad i_j = M_{k-k_1-j+1} - M_{k-k_1-j}, \quad 1 \leq j \leq k - k_1.$$

Obviously, the RDS $(i_0, i_1, \dots, i_{k-k_1})$ and the sequence $(M_1, M_2, \dots, M_{k-k_1}, n)$ can be determined from each other.

2. Finite projective geometry method. In this section, we give an equivalent definition of the RGHWs and some general results. We then develop a method using finite project geometry. These results and the projective geometry method will be critical to show our main results in next section. We first prove Lemma 1, part of which has been shown in [10].

LEMMA 1. *The RGHWs of C and C^1 can also be described as follows:*

$$\begin{aligned} M_j(C, C^1) &= \min\{w_S(D) : D \text{ is a subcode of } C, \dim D = j, D \cap C^1 = \{0\}\} \\ &= \min\{w_S(D) : D \text{ is a subcode of } C, \dim D - \dim D \cap C^1 = j\}, \end{aligned}$$

where $0 \leq j \leq k - k_1$.

Proof. For $0 \leq j \leq k - k_1$, let

$$\begin{aligned} r_j &= \min\{w_S(D) : D \text{ is a subcode of } C, \dim D = j, D \cap C^1 = \{0\}\}, \text{ and} \\ r'_j &= \min\{w_S(D) : D \text{ is a subcode of } C, \dim D - \dim(D \cap C^1) = j\}. \end{aligned}$$

We will show $r_j = r'_j = M_j$.

It is obvious that $r_j \geq r'_j$. We now show $r_j \leq r'_j$. Let D be a subcode of C , with $\dim D - \dim(D \cap C^1) = j$ and $r'_j = w_S(D)$. Suppose B_1 is a generator matrix of $D \cap C^1$. It can be extended as a generator matrix B of D , as

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

Then the subcode D_1 generated by \mathbf{B}_2 satisfies $\dim D_1 = j$, $D_1 \cap C^1 = \{0\}$, and $w_S(D_1) \leq w_S(D)$, which implies that $r_j \leq r'_j$.

Next we show $r'_j = M_j$. By the definition of the RGHWs, we have $M_0 = 0$ and

$$M_j(C, C^1) = \min\{|J| : \dim(C_J) - \dim(C^1_J) \geq (=)j\}, \quad 1 \leq j \leq k - k_1.$$

Since C_J is a subcode of C and $C^1_J = C_J \cap C^1$, we can get $r'_j \leq M_j$.

Now we show $r'_j \geq M_j$. Let D be a subcode of C , with $r'_j = w_S(D)$ and

$$\dim D - \dim D \cap C^1 = j.$$

We have

$$\begin{aligned} & \dim C_{\chi(D)} - \dim C_{\chi(D)} \cap C^1 \\ &= \dim C_{\chi(D)} - \dim C^1_{\chi(D)} \\ &= \dim (C_{\chi(D)} + C^1) - \dim C^1 \\ &\geq \dim (D + C^1) - \dim C^1 \\ &= \dim D - \dim D \cap C^1 = j. \end{aligned}$$

Therefore, $r'_j = w_S(D) = |\chi(D)| \geq M_j$ by Definition 2. \square

Let the RGHWs of C and C^1 be $(M_1, M_2, \dots, M_{k-k_1})$. Let the GHWs of C^1 be $(d_{11}, d_{12}, \dots, d_{1k_1})$ and the GHWs of C be (d_1, d_2, \dots, d_k) .

COROLLARY 1. For $0 \leq t_1 \leq k_1$ and $0 \leq t_2 \leq k - k_1$, it holds that $d_{t_1+t_2} \leq d_{1t_1} + M_{t_2}$. In particular, $d_1 = \min\{M_1, d_{11}\}$.

Proof. By Lemma 1, we can suppose that $M_{t_2} = w_S(D_{t_2})$, where $D_{t_2} \cap C^1 = \{0\}$ and $\dim D_{t_2} = t_2$. Let $w_S(D_{1t_1}) = d_{1t_1}$, where $\dim D_{1t_1} = t_1$ and $D_{1t_1} \subset C^1$. Then $\dim(D_{1t_1} + D_{t_2}) = t_1 + t_2$. Now the claim of the corollary follows from Definition 1. \square

Let $\mathbf{u} = (u_1, u_2, \dots, u_k) \in \text{GF}(q)^k$ be a row vector (or the column vector $(u_1, u_2, \dots, u_k)^T$). For $L \subseteq \{1, \dots, k\}$, define $P_L(\mathbf{u}) \in \text{GF}(q)^k$ such that the t th component of $P_L(\mathbf{u})$ is u_t if $t \in L$ and 0 if $t \notin L$. An example is as follows: Let $L = \{2, 4\}$ and $\mathbf{u} = (u_1, u_2, u_3, u_4, u_5) \in \text{GF}(q)^5$ (or the column vector $(u_1, u_2, u_3, u_4, u_5)^T$); then $P_L(\mathbf{u}) = (0, u_2, 0, u_4, 0)$ (or $P_L(\mathbf{u}) = (0, u_2, 0, u_4, 0)^T$).

For a subset $U \subseteq \text{GF}(q)^k$, we define $P_L(U) = \{P_L(\mathbf{u}) : \mathbf{u} \in U\}$. Obviously, if U is a subspace of $\text{GF}(q)^k$, $P_L(U)$ is a subspace of $\text{GF}(q)^k$.

Let \mathbf{A} be a generator matrix of the $[n, k]$ linear code C . For any $\mathbf{u} \in \text{GF}(q)^k$, the value of \mathbf{u} , denoted by $m(\mathbf{u})$, is defined as the number of occurrences of \mathbf{u} as a column in \mathbf{A} . Define the value of a subset U of $\text{GF}(q)^k$ as follows:

$$m(U) = \sum_{\mathbf{u} \in U} m(\mathbf{u}).$$

LEMMA 2 (see [10]). Let

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{k_1 \times n} \\ \mathbf{A}_{(k-k_1) \times n} \end{pmatrix}$$

be a generator matrix of C , where $\mathbf{A}_{k_1 \times n}$ is a generator matrix for the k_1 -dimensional subcode C^1 . Then there is a one-to-one correspondence between the set of the j -dimensional subcodes D satisfying $D \cap C^1 = \{0\}$ and the set of the $(k-j)$ -dimensional

subspaces $U \subset GF(q)^k$ satisfying $\dim(P_L(U)) = k_1$ such that if D corresponds to U , then $m(U) = n - w_S(D)$, where $1 \leq j \leq k - k_1$ and $L = \{1, 2, \dots, k_1\}$ represents the first k_1 coordinate positions of the vectors in $GF(q)^k$.

Let C, C^1, D , and \mathbf{A} be as in Lemma 2. If \mathbf{y} is a column of \mathbf{A} and $\mathbf{x} = \alpha\mathbf{y}$ for some nonzero $\alpha \in GF(q)$, then we may replace \mathbf{y} by \mathbf{x} without changing the support weight of the subcode D . Therefore, as in [4], we may view columns of \mathbf{A} as points of the projective space $PG(k - 1, q)$. Then $m(\mathbf{x})$ means the number of occurrences of \mathbf{x} in the columns of \mathbf{A} for any point \mathbf{x} of $PG(k - 1, q)$. Therefore, we can define a value function as

$$m : PG(k - 1, q) \rightarrow \mathbb{N} = \{0, 1, 2, \dots\}.$$

For any point $\mathbf{x} \in PG(k - 1, q)$, we call $m(\mathbf{x})$ the value of \mathbf{x} . Correspondingly, $m(U)$ in Lemma 2 is called the value of the projective subspace U , when U is a subspace, viewing the vectors in U as points in the projective space $PG(k - 1, q)$.

By using the value function $m(\cdot)$, we can define a generator matrix and a code (up to equivalence) as follows. By Lemma 1, we can assume $M_j(C, C^1) = w_S(D^*)$ for some j -dimensional subcode D^* satisfying $D^* \cap C^1 = \{0\}$. Then, from Lemma 2, D^* corresponds to a $(k - j)$ -dimensional subspace $U^* \subset GF(q)^k$ satisfying $\dim(P_L(U^*)) = k_1$ such that $m(U^*) = n - w_S(D^*)$. Now we consider U^* as a $(k - j - 1)$ -dimensional projective subspace of $PG(k - 1, q)$ and still denote it as U^* . Then $\dim(P_L(U^*)) = k_1 - 1$. Therefore, we have

$$\begin{aligned} (1) \quad & \max\{m(U) : U \subseteq PG(k - 1, q) \text{ with } \dim(U) = k - j - 1 \text{ and } \dim(P_L(U)) = k_1 - 1\} \\ & = m(U^*) = n - w_S(D^*) = n - M_j = \sum_{t=0}^{k-k_1-j} i_t, \end{aligned}$$

where $0 \leq j \leq k - k_1$.

Remark. We now summarize the finite projective geometry method as follows: to construct a linear k -dimensional code C and a k_1 -dimensional subcode C^1 with the parameters $(M_1, M_2, \dots, M_{k-k_1}, n)$, it is necessary to construct the value function $m(\cdot)$ satisfying (1).

In the following section, we will employ this method to study the RGHWs of 3-dimensional linear codes and their subcodes.

3. RGHWs of a 3-dimensional linear code and its subcodes. In this section we study the RGHWs of a 3-dimensional linear code C and its subcodes C^1 . We will determine all the RGHWs of C and C^1 , making use of our method given in the previous section.

We call the 0-dimensional, 1-dimensional, and 2-dimensional subspaces of $PG(k - 1, q)$ points, lines, and planes, respectively. We denote by \overline{PQ} the line spanned by points P and Q , and by \overline{PQR} the plane spanned by points P, Q , and R .

Let $E = (1, 0, 0)$, $F = (0, 1, 0)$, and $G = (0, 0, 1)$ denote the basis points in the projective plane $V = PG(2, q)$ (see Figure 1).

Now let us consider a 3-dimensional linear code C . Suppose C^1 is a 1-dimensional subcode of C . By Lemma 2, there is a one-to-one correspondence between the set of all j -dimensional ($1 \leq j \leq 2$) subcodes D satisfying $D \cap C^1 = \{0\}$ and the set of $(2 - j)$ -dimensional subspaces $U \subset PG(2, q)$ satisfying $\dim(P_L(U)) = k_1 - 1 = 0$, where $L = \{1\} \subset \{1, 2, 3\}$. It is clear that $\dim(P_L(U)) = 0$ for $L = \{1\}$ means $U \not\subseteq \overline{FG}$. Thus, from (1), the construction of C and C^1 with the parameters (M_1, M_2, n) is

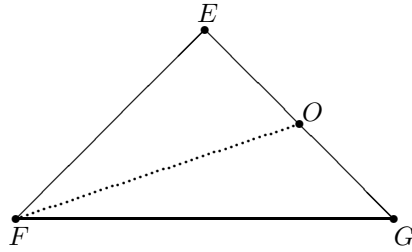


FIG. 1. Construction for the 3-dimensional codes.

equivalent to that of the value function $m(\cdot)$ such that

$$\begin{aligned} \max\{m(p) : p \text{ is a point of } \text{PG}(2, q) \text{ and } p \notin \overline{FG}\} &= i_0, \\ \max\{m(l) : l \text{ is a line of } \text{PG}(2, q) \text{ and } l \neq \overline{FG}\} &= i_0 + i_1, \\ m(V) = m(\text{PG}(2, q)) &= i_0 + i_1 + i_2. \end{aligned}$$

Define

$$\begin{aligned} \text{MU}_0 &= \{p : m(p) = i_0, p \text{ is a point and } p \notin \overline{FG}\}, \\ \text{MU}_1 &= \{l : m(l) = i_0 + i_1, l \text{ is a line and } l \neq \overline{FG}\}. \end{aligned}$$

By Lemma 1, we can assume that there exist a 1-dimensional subcode D_1 and a 2-dimensional subcode D_2 satisfying $w_S(D_1) = M_1, w_S(D_2) = M_2$, and $D_1 \cap C^1 = D_2 \cap C^1 = \{0\}$. In what follows, we will distinguish the discussion into two cases according to whether $D_1 \subset D_2$ or not.

Case A. $D_1 \subset D_2$. Equivalently, we have to construct a value function $m(\cdot)$ satisfying $p \in l$ for a point $p \in \text{MU}_0$ and a line $l \in \text{MU}_1$. We call the RDS for this case, i.e., Case A, ARDS.

THEOREM 1. *The sequence (i_0, i_1, i_2) is an ARDS if and only if it satisfies $i_0 \geq 1, i_1 \geq 1$, and $1 \leq i_2 \leq qi_1$.*

Proof. (\Rightarrow) Without loss of generality, we assume $E \in \text{MU}_0$ and $\overline{EF} \in \text{MU}_1$ (see Figure 1). Since the $q + 1$ lines passing through the point E in $V = \text{PG}(2, q)$ are all the possible ones in the set MU_1 , so we have

$$m(V) = m(\text{PG}(2, q)) = i_0 + i_1 + i_2 \leq i_0 + (q + 1)i_1,$$

that is, $i_2 \leq qi_1$.

(\Leftarrow) We first construct $m(\cdot)$ for $i_2 = qi_1$ as follows (see Figure 1):

$$m(p) = \begin{cases} i_0, & \text{the point } p = E, \\ i_1, & \text{the point } p \in \text{the line } \overline{FG}, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, we have $E \in \text{MU}_0$ and $\overline{EF} \in \text{MU}_1$. Note that $E \in \overline{EF}$. So, the value function $m(\cdot)$ satisfies the conditions stated in Case A. Thus, all the sequences (i_0, i_1, i_2) such that $i_0 \geq 1, i_1 \geq 1$, and $i_2 = qi_1$ are ARDSs.

If $i_2 < qi_1$, we can decrease the values of the points $p \in \text{PG}(2, q) \setminus \overline{EF}$ one by one until the value of i_2 is obtained. \square

Case B. We always have $D_1 \not\subset D_2$ for any 1-dimensional subcode D_1 and any 2-dimensional subcode D_2 such that $D_1 \cap C^1 = D_2 \cap C^1 = \{0\}, w_S(D_1) = M_1(C, C^1)$,

and $w_S(D_2) = M_2(C, C^1)$. Equivalently, we must find a value function $m(\cdot)$ such that $p \notin l$ for any point $p \in \text{MU}_0$ and any line $l \in \text{MU}_1$. We call the RDS for this case, i.e., Case B, BRDS.

THEOREM 2. (i) *If the sequence (i_0, i_1, i_2) is a BRDS, then $i_0 \geq 1$, $i_1 \geq 2$, $i_2 \leq qi_1 - (q + 1)$, and $i_2 \geq i_0$.*

(ii) *All the sequences (i_0, i_1, i_2) satisfying $i_0 \geq 2$ ($i_0 \geq 3$ if $q = 2$), $i_2 \leq qi_1 - (q + 1)$, and $i_2 \geq i_0$ are BRDSs.*

Proof. (i) Assume $\overline{FO} \in \text{MU}_1$ and $E \in \text{MU}_0$ (see Figure 1). Then, from the assumption that (i_0, i_1, i_2) is a BRDS, we can show similarly to Theorem 1 that $i_2 \leq qi_1 - (q + 1)$ and $i_2 \geq i_0$. Observe that if $i_1 = 1$, then $i_2 \leq qi_1 - (q + 1) < 0$. Thus, $i_1 \geq 2$.

To show (ii), let's construct $m(\cdot)$ for $i_2 = qi_1 - (q + 1)$ such that $m(\cdot)$ satisfies the conditions stated in Case B.

Assume $i_0 + 1 = q\alpha + \beta$ for $0 \leq \beta \leq q - 1$. Choose β points continuously on the line \overline{FO} and denote them $O_1 = O, O_2, \dots, O_\beta$. Suppose $\overline{EO_i} \cap \overline{FG} = G_i$, where $1 \leq i \leq \beta$ (see Figure 1). Note that $G_1 = G$. Then we can write $m(\cdot)$ as follows:

$$(2) \quad m(p) = \begin{cases} i_0, & p = E, \\ i_1 - 1, & p = F, \\ \alpha + 1, & p = O_j, 1 \leq j \leq \beta, \\ \alpha, & p \in \overline{FO}, p \neq F, p \neq O_j, 1 \leq j \leq \beta, \\ i_1 - 2 - \alpha, & p = G_j, 1 \leq j \leq \beta, \\ i_1 - 1 - \alpha, & p \in \overline{FG}, p \neq F, p \neq G_j, 1 \leq j \leq \beta, \\ 0, & \text{otherwise.} \end{cases}$$

Then $m(E) = i_0 = \max\{m(p) : p \notin \overline{FG}\}$ only if $i_0 > \alpha + 1$, which holds when $i_0 > 1 + \frac{2}{q-1}$ or when $i_0 \geq 2$ and $q \geq 4$.

Assume a line $l \neq \overline{FG}$. If $E \in l$ and $O_j \in l$ for some $1 \leq j \leq \beta$, then $m(l) = m(E) + m(O_j) + m(G_j) = i_0 + (\alpha + 1) + (i_1 - 2 - \alpha) = i_0 + i_1 - 1 < i_0 + i_1$. In addition, $m(\overline{EF}) = m(E) + m(F) = i_0 + i_1 - 1 < i_0 + i_1$.

For other lines l satisfying $l \neq \overline{FG}$ and $E \in l$ we can similarly obtain $m(l) = i_0 + i_1 - 1 < i_0 + i_1$.

If $l \neq \overline{FG}$, $F \notin l$, and $E \notin l$, then $m(l) \leq (i_1 - 1 - \alpha) + (\alpha + 1) = i_1 < i_0 + i_1$. For the lines l such that $E \notin l$, $F \in l$, and $l \neq \overline{FO}$, we have $m(l) = m(F) = i_1 - 1 < i_0 + i_1$. In addition, $m(\overline{FO}) = m(F) + (q - \beta)\alpha + \beta(\alpha + 1) = i_0 + i_1$. So we can obtain $m(\overline{FO}) = \max\{m(l) : l \neq \overline{FG}\} = i_0 + i_1$.

The arguments above show that $m(\cdot)$ satisfies the conditions stated in Case B. So all the sequences (i_0, i_1, i_2) such that $i_0 \geq 2$, $i_2 = qi_1 - (q + 1)$, and $i_2 \geq i_0$ are BRDSs when $q \geq 4$.

There remain the sequences (i_0, i_1, i_2) such that $i_2 \leq qi_1 - (q + 1)$, $i_2 \geq i_0$, $i_0 = 1$ for $q \geq 4$, $1 \leq i_0 \leq 2$ for $q = 3$, and $1 \leq i_0 \leq 3$ for $q = 2$ to cope with.

We first show that the sequences $i_0 = 1$, $i_2 \leq qi_1 - (q + 1)$, and $i_2 \geq i_0$ for any $q \geq 2$ are not BRDSs.

On the contrary, assume there exists a sequence such that $i_0 = 1$, $i_2 \leq qi_1 - (q + 1)$, and $i_2 \geq i_0$, which is a BRDS. Then there exists a value function $m(\cdot)$ satisfying the conditions stated in Case B. We suppose $m(p_0) = i_0 = 1 = \max\{m(p) : p \notin \overline{FG}\}$ and $m(l_0) = i_0 + i_1 = i_1 + 1 = \max\{m(l) : l \neq \overline{FG}\}$ (see Figure 1). Assume $l_0 \cap \overline{FG} = p'$. Since any point $p \in \text{MU}_0$ is not on the line l_0 , $m(p) = 0$ for any $p \in l_0 \setminus p'$. So $m(l_0) = m(p') = i_0 + i_1$. Then $m(\overline{p_0p'}) \geq m(p_0) + m(p') = 2i_0 + i_1 > \max\{m(l) : l \neq \overline{FG}\} = i_0 + i_1$, which is a contradiction. Thus, we can conclude that

the sequences $i_0 = 1, i_2 \leq qi_1 - (q + 1)$ and $i_2 \geq i_0$ for any $q \geq 2$ are not BRDSs.

Note that $q|(i_0 + 1)$ when $i_0 = 2$ for $q = 3$ and $i_0 = 3$ for $q = 2$. So $m(\cdot)$ in (2) becomes the following one:

$$m(p) = \begin{cases} i_0, & p = E, \\ i_1 - 1, & p = F, \\ \alpha, & p \in \overline{FO}, p \neq F, \\ i_1 - 1 - \alpha, & p \in \overline{FG}, p \neq F, \\ 0, & \text{otherwise.} \end{cases}$$

Then it is not difficult to check that the sequences (i_0, i_1, i_2) such that $i_2 = qi_1 - (q + 1), i_2 \geq i_0, i_0 = 2$ for $q = 3$ and $i_0 = 3$ for $q = 2$ are all BRDSs.

Lastly we show that the sequences $i_2 \leq qi_1 - (q + 1), i_2 \geq i_0$, and $i_0 = 2$ for $q = 2$ are not BRDSs. On the contrary, assume that there exists a sequence (i_0, i_1, i_2) which is a BRDS. Then there is a corresponding $m(\cdot)$. Assume $p_0 \in MU_0$ and $l_0 \in MU_1$. Suppose $l_0 \cap \overline{FG} = p'$ and denote the other two points on the line l_0 as p_1 and p_2 . Then $m(p_1) \leq i_0 - 1 = 1$ and $m(p_2) \leq i_0 - 1 = 1$ from the assumption that (i_0, i_1, i_2) is a BRDS. So, $m(p') = m(l_0) - m(p_1) - m(p_2) \geq (i_0 + i_1) - 1 - 1 = i_1$. Then we have $m(\overline{p_0p'}) \geq m(p_0) + m(p') \geq i_0 + i_1$. Since $\max\{m(l) : l \neq \overline{FG}\} = i_0 + i_1$ and $\overline{p_0p'} \neq \overline{FG}$, we have $m(\overline{p_0p'}) = i_0 + i_1$, which contradicts the assumption that (i_0, i_1, i_2) is a BRDS. Therefore, the sequences $i_2 \leq qi_1 - (q + 1), i_2 \geq i_0$, and $i_0 = 2$ for $q = 2$ are not BRDSs.

If $i_2 < qi_1 - (q + 1)$, it is necessary to decrease the values of the points $p \in PG(2, q) \setminus (\overline{FO} \cup \{E\})$ one by one until $i_2 = i_0$. \square

Now we consider the RGHWs of a 3-dimensional linear code C and a 2-dimensional subcode C^1 of C . The RDS is (i_0, i_1) , where $i_0 \geq 2$ and $i_1 \geq 1$. By Lemma 2 we can express the value function $m(\cdot)$ as follows (see Figure 1):

$$\max\{m(l) : G \notin l\} = i_0, m(V) = i_0 + i_1.$$

THEOREM 3. *For any $i_0 \geq 2, i_1 \geq 1$, there exist a 3-dimensional linear code C and a 2-dimensional subcode C^1 of C such that the RDS is (i_0, i_1) .*

Proof. Assume $i_0 = (q + 1)\alpha + \beta$ for $0 \leq \beta \leq q$, and denote the $q + 1$ points on the line \overline{EF} by $E_0 = E, E_1, \dots, E_q = F$, respectively. We then construct $m(\cdot)$ (see Figure 1) as follows:

$$m(p) = \begin{cases} \alpha, & p = E_i \text{ for } 0 \leq i \leq q - \beta, \\ \alpha + 1, & p = E_i \text{ for } q - \beta + 1 \leq i \leq q, \\ i_1, & p = G, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, we have

$$m(\overline{EF}) = \max\{m(l) : G \notin l\} = i_0 \text{ and } m(V) = i_0 + i_1.$$

Thus, any sequence (i_0, i_1) satisfying $i_0 \geq 2$ and $i_1 \geq 1$ is an RDS. \square

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